Network Science Center at West Point

A Mathematical Model of Network Communication

James R. Gatewood
United States Military Academy, West Point
Rensselaer Polytechnic Institute

Donald Drew
Rensselaer Polytechnic Institute

Report Documentation Page

Form Approved
OMB No. 0704-0188

Public reporting burden for the collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden, to Washington Headquarters Services, Directorate for Information Operations and Reports, 1215 Jefferson Davis Highway, Suite 1204, Arlington VA 22202-4302. Respondents should be aware that notwithstanding any other provision of law, no person shall be subject to a penalty for failing to comply with a collection of information if it does not display a currently valid OMB control number.

1. REPORT DATE **15 MAR 2012**	2. REPORT TYPE **N/A**	3. DATES COVERED
4. TITLE AND SUBTITLE **A Mathematical Model of Network Communication**		5a. CONTRACT NUMBER 5b. GRANT NUMBER 5c. PROGRAM ELEMENT NUMBER
6. AUTHOR(S)		5d. PROJECT NUMBER 5e. TASK NUMBER 5f. WORK UNIT NUMBER
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) **US Military Academy West Point, NY 10996**		8. PERFORMING ORGANIZATION REPORT NUMBER
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)		10. SPONSOR/MONITOR’S ACRONYM(S) 11. SPONSOR/MONITOR’S REPORT NUMBER(S)
12. DISTRIBUTION/AVAILABILITY STATEMENT **Approved for public release, distribution unlimited.**		
13. SUPPLEMENTARY NOTES		
14. ABSTRACT		
15. SUBJECT TERMS		

16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
a. REPORT **unclassified**	b. ABSTRACT **unclassified**	c. THIS PAGE **unclassified**	**UU**	**34**	

Standard Form 298 (Rev. 8-98)
Prescribed by ANSI Std Z39-18

Outline of Presentation

- Motivation
- Communication Network
- Discrete Network Model
- Discrete Conservation of Packets Equations
- Continuum Network Model
- Example: One Dimensional Flow Model
- Current Work

Motivation

- Rapid Communication is essential in today's world.
- Understand the dynamics of flow by creating a communication network model.
- Modeling flow on a communication network will allow us to:
  - Describe normal and congested flow on large communication networks.
  - Predict changes in flow pattern due to changes in the spatial density and per-link traffic.

Communication Networks

- A communication network is a global system of interconnected networks, both big and small.
- Packet switching network because all data traffic is broken down into data chunks called packets.
- Everything traveling on a communication network is called a packet.

Discrete Communication Network

- Use graph theory to describe the connectivity of a network, where a graph is composed of nodes and links.
- Information travels along links connecting nodes.
- The graph is undirected; information travels in both directions.
- The nodes are routers in this model
- Routers will act as both host and switch computers.
- Host computers is where information enters (source), and exits (destination) a network.

Route Matrix

- The Route matrix depicts the global state of a network.
- Describes how to direct packets from source to destination.
- Each entry describes the next appropriate router a packet will take along its path.
- Routes are pre-determined by an optimal path algorithm.

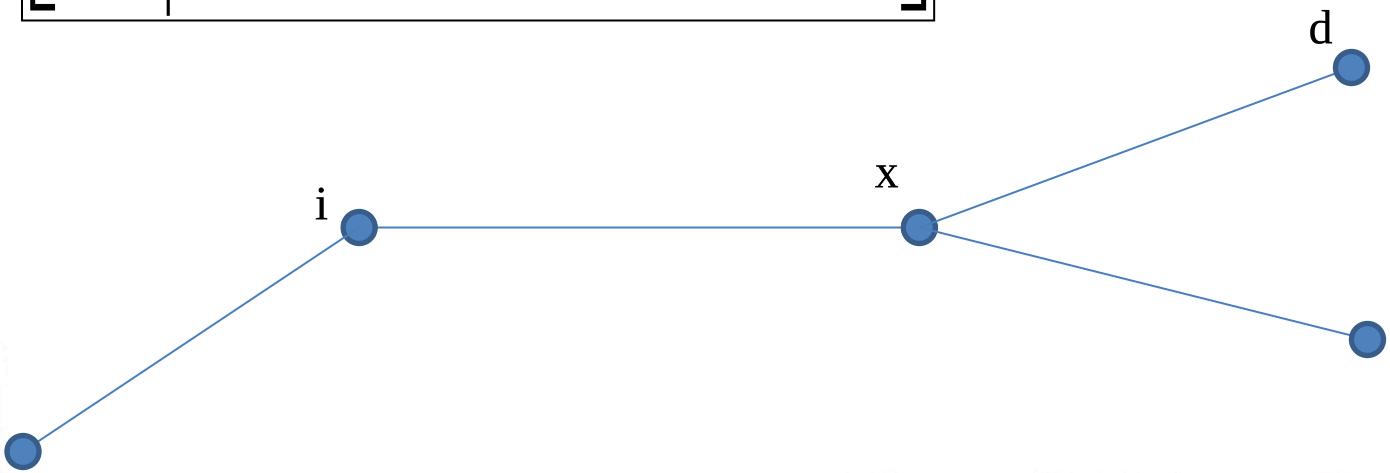
$$\left[\begin{array}{c|cccc} & 1 & 2 & 3 & \dots \\ \hline 1 & r_{1,1} & r_{1,2} & r_{1,3} & \dots \\ 2 & r_{2,1} & r_{2,2} & r_{2,3} & \dots \\ 3 & r_{3,1} & r_{3,2} & r_{3,3} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right]$$

$$r_{i,d} = x$$

$i =$ Current position

$d =$ Destination

$x =$ Next position



Queue Dynamics

- Each router contains a queue of buffered packets.
- FIFO unlimited memory buffer
  - Packets enter queue in one of two ways
    - Packets flowed from another router along a connecting link.
    - Generated at a router according to some packet generator rate appropriate to the router.
- Upon being generated, a packet is given a destination by its originator.
- Once a packet reaches its destination it is delivered and exits immediately.

Flow equation (Arrivals Age zero)

$$b(j,d,0,\tau+1)=\sum_{a=0}^{\infty}\sum_{\substack{i=1\\ j\neq d}}^{N}\delta_{j,R_{i,d}}\,\beta(i,d,a,\tau)+\nu(j,d,\tau)$$

$b(j,d,a,\tau)$, number of buffered packets at node j, with destination d, age a at time t

$\beta(i,d,a,\tau)$ is the sending rate of packets being sent from router i to router j toward its destination d at time τ

$\nu(j,d,\tau)$ is the number of new packets entering the network at node j

Network Science Center
West Point

Evolution Equation

- Describes how packets are aging in the buffer.

$$b(j,d,a+1,\tau+1) = b(j,d,a,\tau) - \beta(j,d,a,\tau)$$

$\beta(j,d,a,\tau)$ determines the number of packets that are sent outward from node j with destination d

Network Science Center
West Point

Discrete Conservation of Packet Equation

- The number of buffered messages at router *j* with destination *d* at time *t*

$$n(j,d,\tau) = \sum_{a=0}^{\infty} b(j,d,a,\tau)$$

$$n(j,d,\tau+1) - n(j,d,\tau) = \sum_{a=0}^{N} b(j,d,a,\tau+1) - \sum_{a=0}^{N} b(j,d,a,\tau)$$

Network Science Center
West Point

Discrete Conservation of Packet Equations

$$n(j,d,\tau+1)-n(j,d,t)=-\sum_{a=0}^{\infty}\beta(j,d,a,\tau)+\sum_{a=0}^{\infty}\sum_{i=1}^{N}\delta_{i,R_{i,d}}\beta(i,d,a,\tau)$$

$$+\nu(j,d,\tau)-\sum_{a=0}^{\infty}\beta(j,j,a,\tau)$$

Network Science Center
West Point

Continuum Network Model

- To view this model as a flow model, well discuss the collection of routers as opposed to one. The collection of routers create a Voronoi Diagram.

Voronoi Diagram

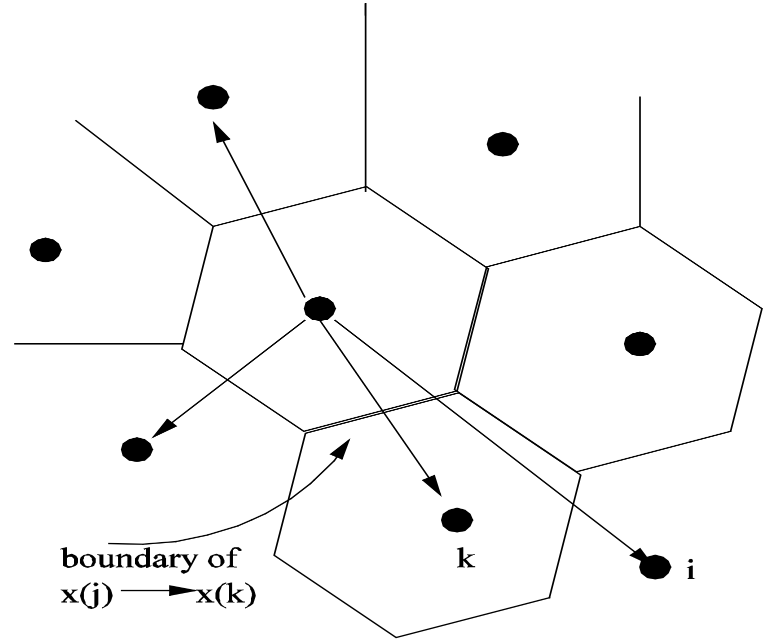
- The spatial location of each router is of importance.
- Each router serves a particular coverage area of users who are sending and receiving packets in an area
- Associate each router with a physical spatial location
- Each router shares physical boundaries with another set of routers so that packets moving through the network will pass through physical boundaries around routers.

Network Science Center
West Point

Voronoi Diagram

x_j = Location of router j

y = Location of destination d



Each x_j generates a Voronoi polygon $V(x_j)$, a tessellation in the plane. The Voronoi polygon $V(x_j)$ is the set of points closer to the point (router) x_j, than to all other routers in the diagram. A collection of Voronoi polygons is called a Voronoi Diagram.

Network Science Center
West Point

Continuum Description

Let V_P be a Voronoi Diagram and ∂V_P be its boundary

The density of packets for Voronoi polygon $V(x_j)$

$$\rho(x_j, y, t) = \frac{n(j, d, \tau)}{\left|V(x_j)\right|}$$

The number of packets buffered in the Vorono Diagram V_P

$$\sum_{j \in P} n(j, d, \tau)$$

Continuum Description Cont.

The evolution equation for the density of packets in a Voronoi Diagram

$$\int_{V_P} \frac{\rho(x,y,t+\Delta t)-\rho(x,y,t)}{\Delta t}\,dV = \sum_{j\in P}\frac{n(j,d,\tau+1)-n(j,d,\tau)}{\Delta t}$$

Take the limit as Δt goes to zero, first term becomes

$$\int_{V_P} \frac{\partial \rho}{\partial t}\,dV$$

Network Science Center
West Point

Flux

The flow vector from router x_j to router x_i is

$$\frac{x_i - x_j}{\left|x_i - x_j\right|}\beta(j,d,a,\tau)$$

The outflow through boundary element $\partial V_{l,m}$

$$\sum_{a=0}^{\infty}\sum_{\substack{j\in P\\ i\notin P}}\delta_{(j,i)|(l,m)}\delta_{i,R_{j,d}}\frac{-1}{\Delta t}\frac{x_i - x_j}{\left|x_i - x_j\right|}\beta(j,d,a,\tau)\cdot n_{l,m} = \Phi^{O}(x_{l,m},y,t)\cdot n_{l,m}\left|\partial V_{l,m}\right|$$

Similar derivation fo the incoming flow terms

$$\sum_{a=0}^{\infty}\sum_{\substack{j\in P\\ i\notin P}}\delta_{(j,i)|(l,m)}\delta_{j,R_{i,d}}\frac{1}{\Delta t}\frac{x_i - x_j}{\left|x_i - x_j\right|}\beta(i,d,a,\tau)\cdot n_{l,m} = \Phi^{I}(x_{l,m},y,t)\cdot n_{l,m}\left|\partial V_{l,m}\right|$$

Network Science Center
West Point

Flux Cont.

The total flux of packets entering and exiting a boundary $\partial V_{l,m}$

$$\Phi(x_{l,m}, y, t) \cdot n_{l,m} \left|\partial V_{l,m}\right| = \Phi^{O}(x_{l,m}, y, t) \cdot n_{l,m} \left|\partial V_{l,m}\right| - \Phi^{I}(x_{l,m}, y, t) \cdot n_{l,m} \left|\partial V_{l,m}\right|$$

Total flux of packets in Voronoi Diagram V_P with destination y

$$\sum_{\substack{l \in \partial P \\ m \notin \partial P}} \Phi(x_{l,m}, y, t) \cdot n_{l,m} \left|\partial V_{l,m}\right|$$

Written in the continuum limit and using the divergence theorem the total flux becomes

$$\sum_{\substack{l \in \partial P \\ m \notin \partial P}} \Phi(x_{l,m}, y, t) \cdot n_{l,m} \left|\partial V_{l,m}\right| \to \int_{\partial V_P} \Phi(x, y, t) \cdot n \, ds$$

Network Science Center
West Point

Source and Sink

The source and sink in their continuum limit

$$\sum_{j \in P} \frac{1}{\Delta t} \nu(j,d,\tau) \rightarrow \int_{V_P} \gamma(x,y,t)\, dv$$

$$\sum_{a=0}^{\infty} \sum_{j \in P} \frac{1}{\Delta t} \beta(j,d,a,\tau) \rightarrow \int_{V_P} \sigma(x,t)\, dV$$

Continuity Equation

Putting the continuum limits together we have the conservation of packets equation in a Voronoi Diagram V_P

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \Phi(x, y, t) = \gamma(x, y, t) - \sigma(x, t)$$

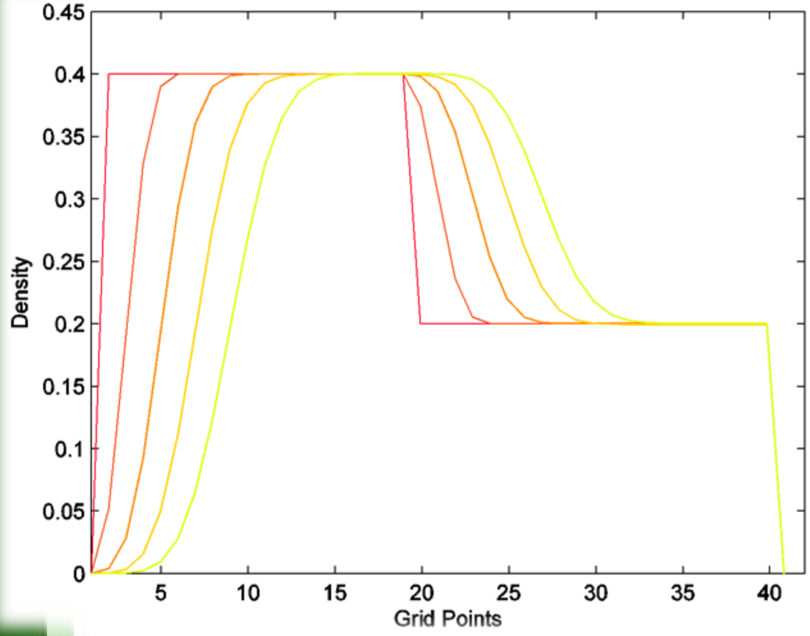
One Dimensional Network Flow Model

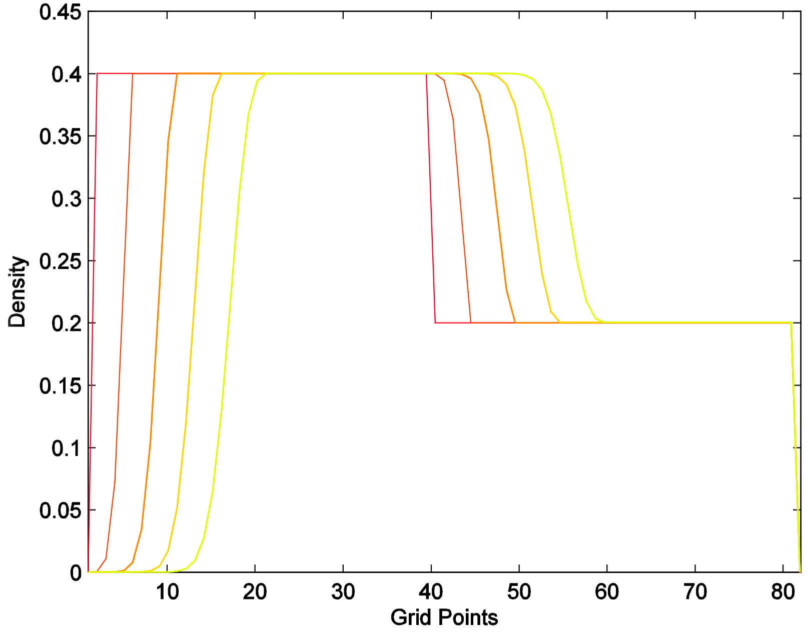
- One dimensional Network flow model with x=0 and a destination x=y.
- Analyze inner nodes
- Continuity equation in one dimensions

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x} = 0$$

$$\Phi(x,y,t) = \begin{cases} \rho(x,y,t) & \text{if } \rho(x,y,t) < \Phi_{\max}(x,y,t) \\ \Phi_{\max}(x,y,t) & \text{if } \rho(x,y,t) \geq \Phi_{\max}(x,y,t) \end{cases}$$

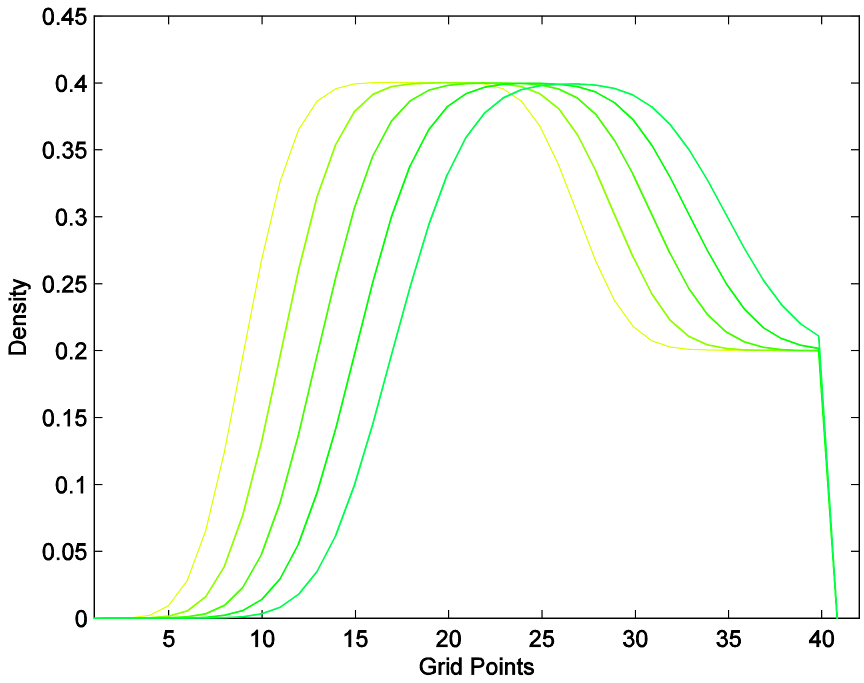
Non-Saturated Flow

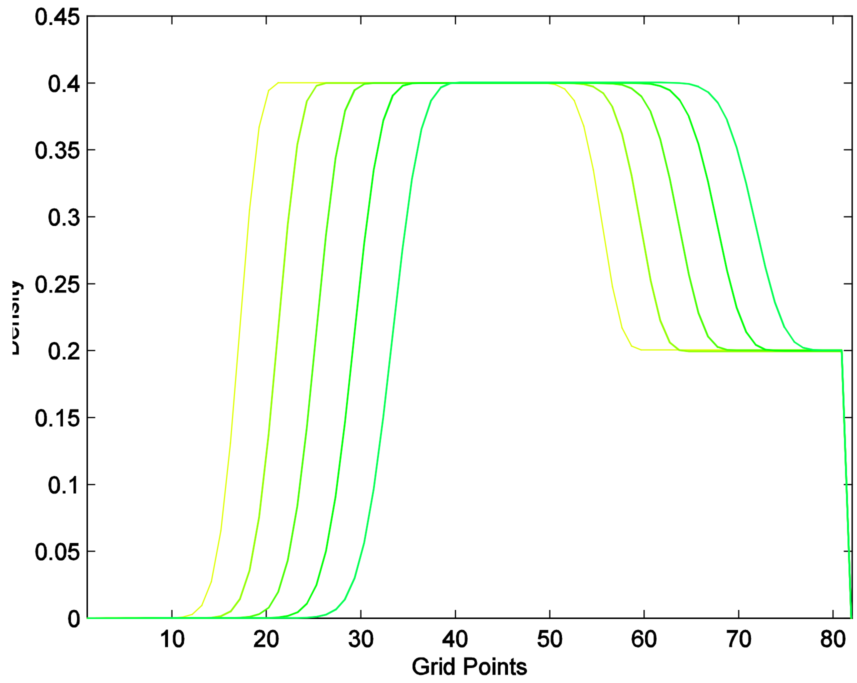




Flow Movement In Time

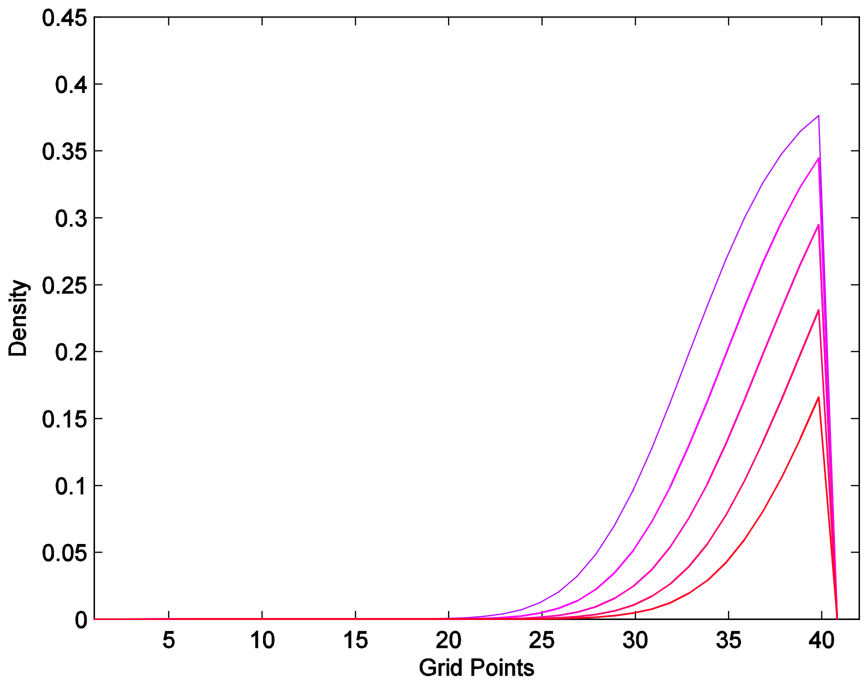
Network Science Center
West Point



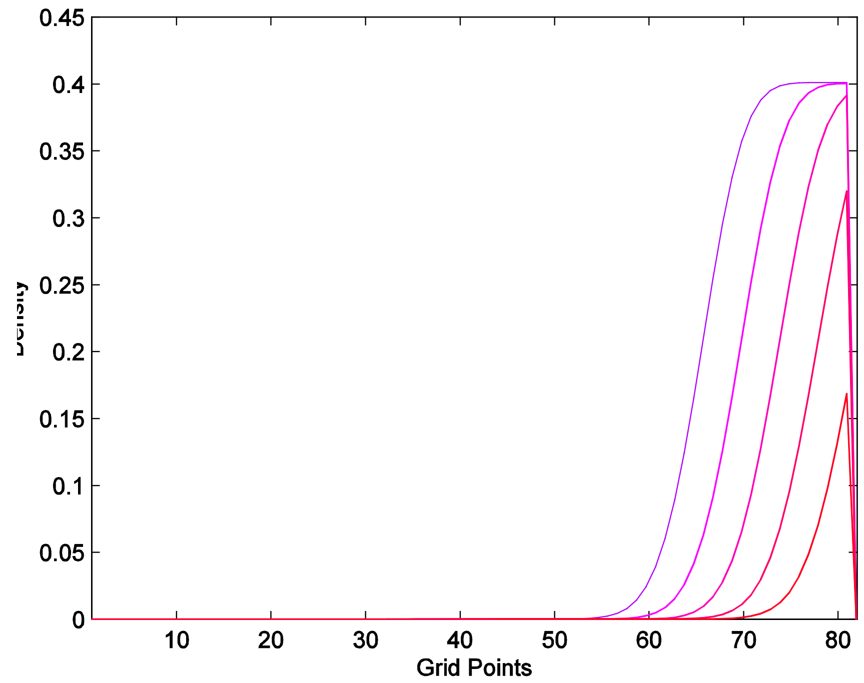


Network Science Center
West Point

End of Flow

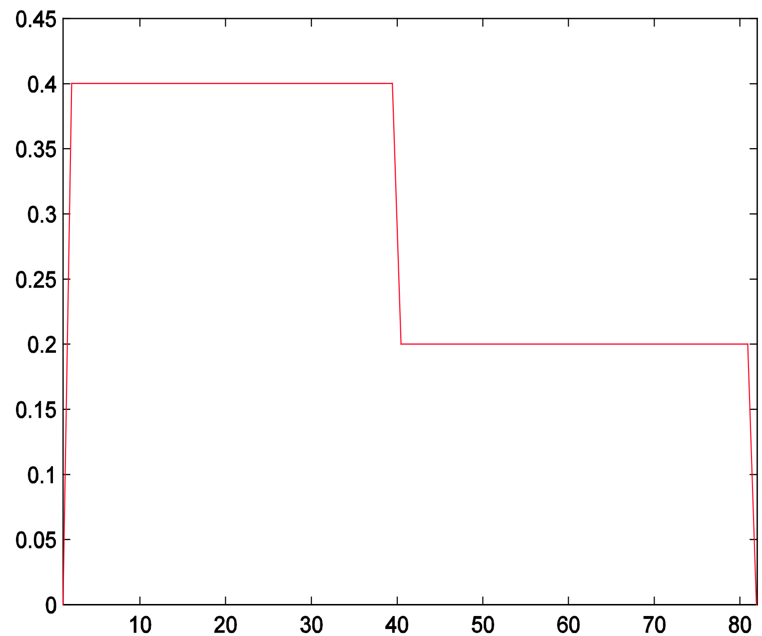


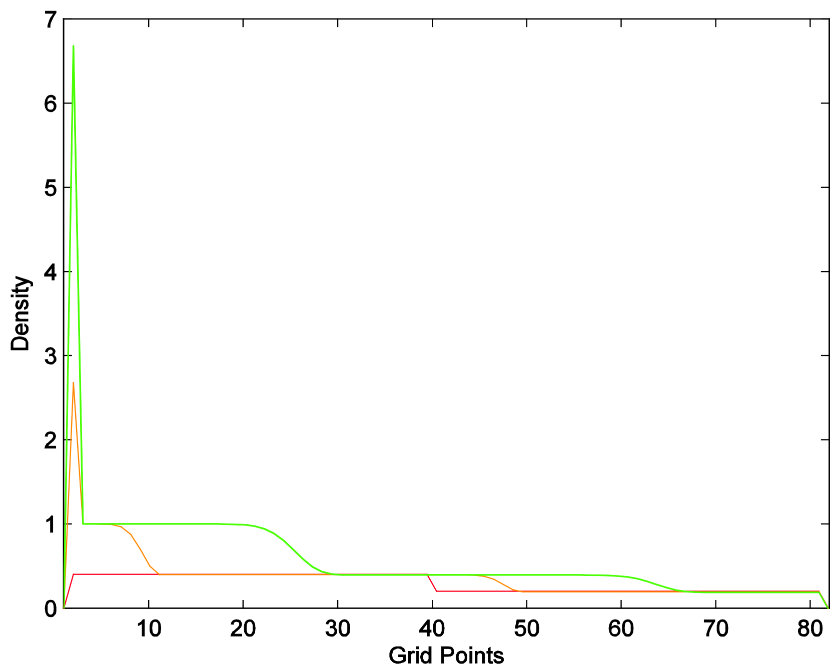
Density
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
5
10
15
20
25
30
35
40
Grid Points



0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
10
20
30
40
50
60
70
80
Grid Points

Example: Flow with Saturation

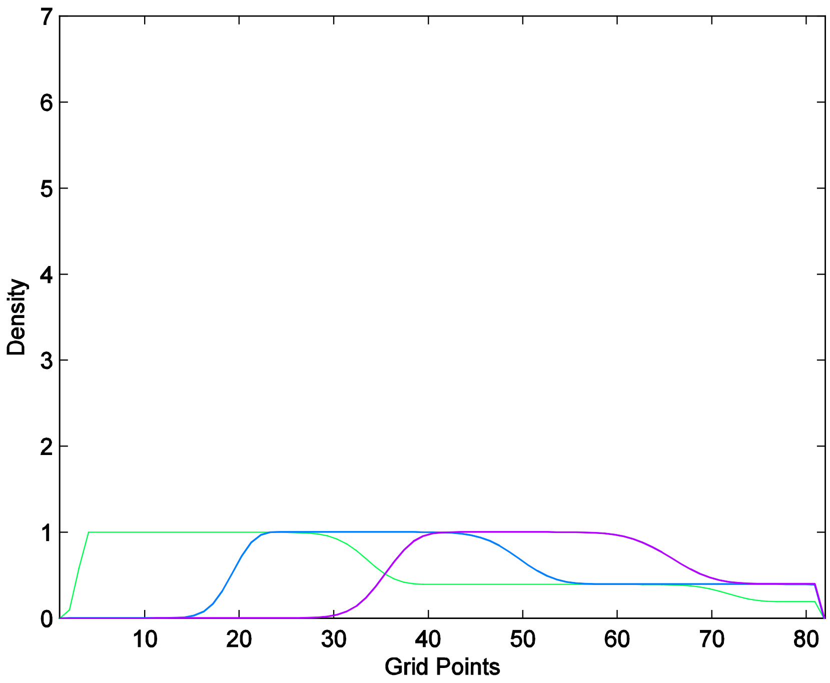


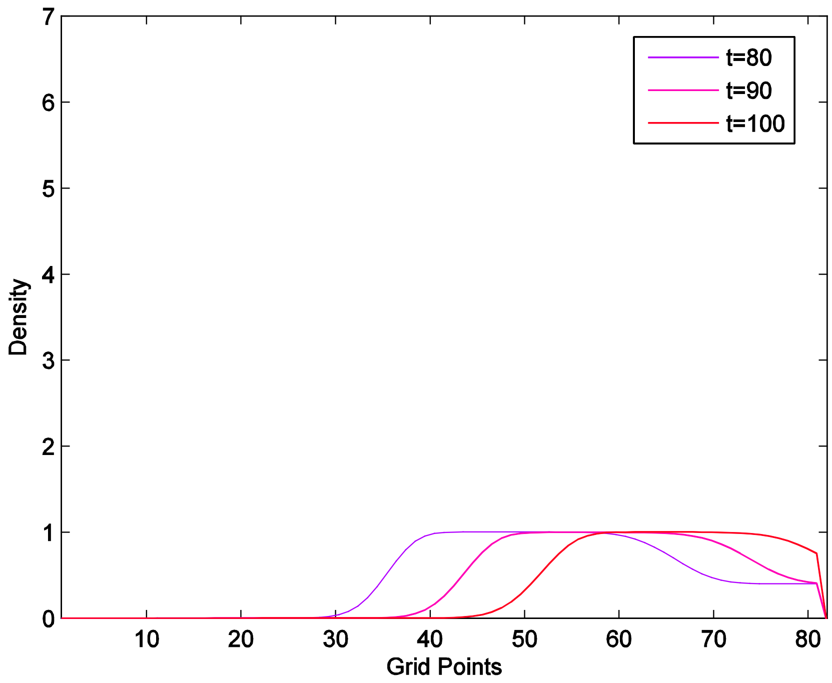


Initial Condition

Network Science Center
West Point

End of Flow





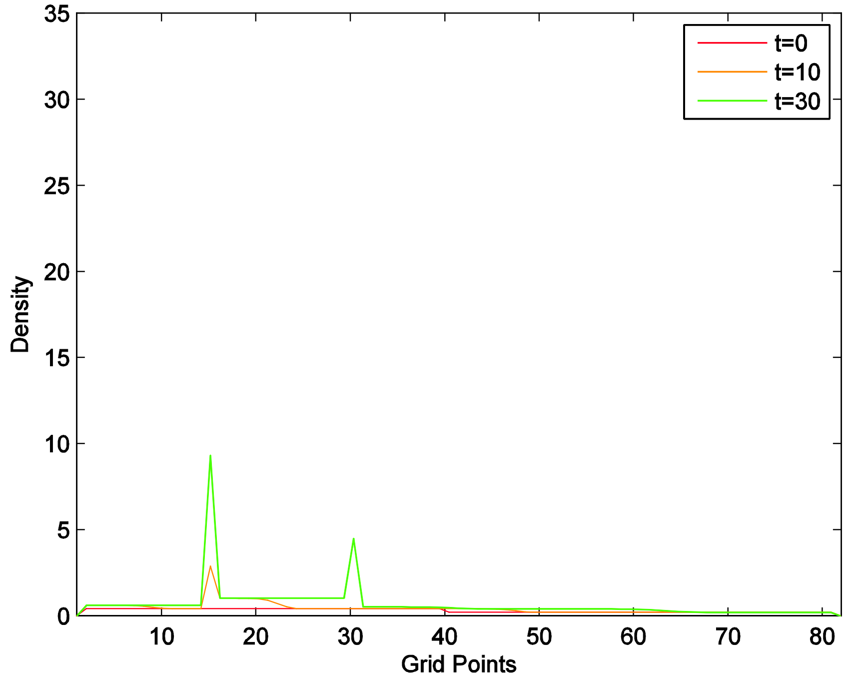
Network Science Center
West Point

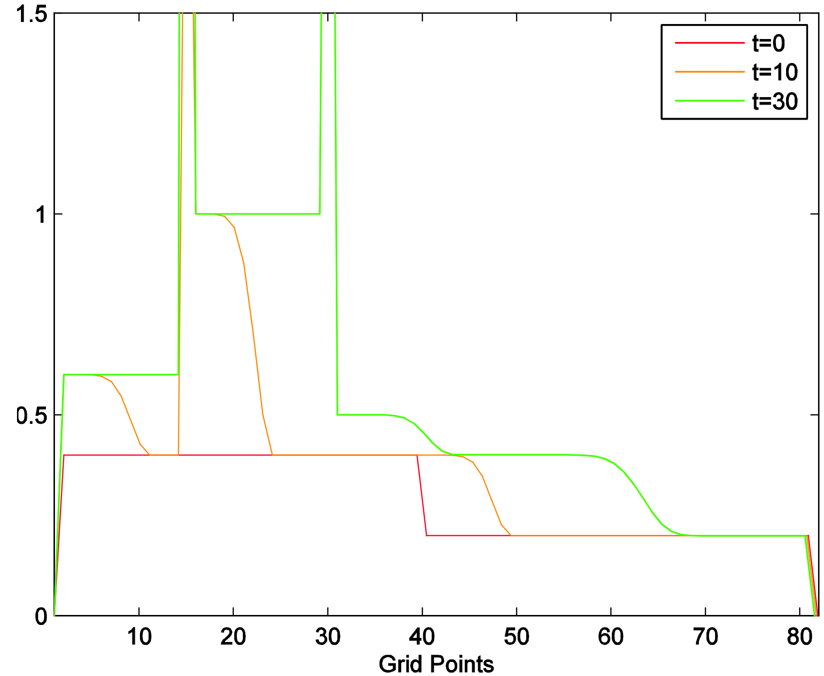
Interruptions

- Disturbance in the flow
  - Limited bandwidth, link capacity drops to a lower value
  - Router (grid point) is down for some time.

- Example
  - Source node has constant flux of packets below link capacity.
  - One of the grid points has limited bandwidth for a while.
  - One of the inner grid points has a source term.

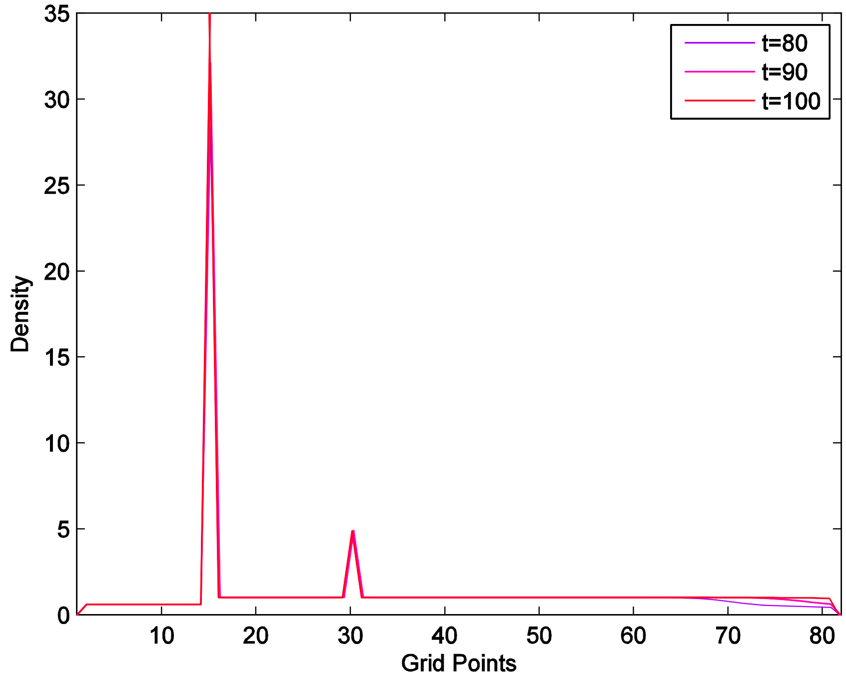
Interruptions

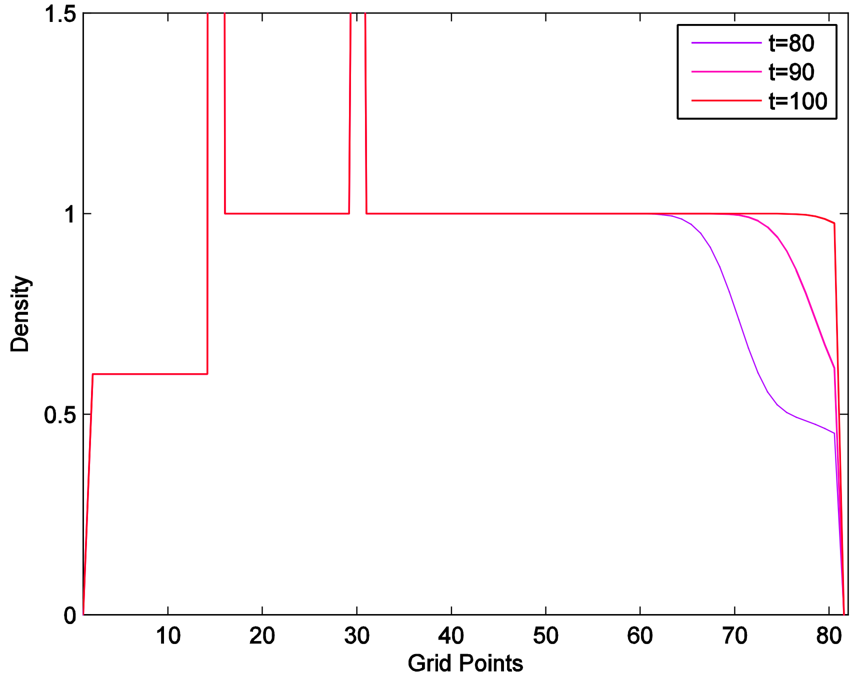




Network Science Center
West Point

End of Flow





Network Science Center
West Point

Current Work

- Currently route matrix is static.
- Make route matrix dynamic at every node (changes because of link weight, upstream traffic, etc.

$$R_{j,d} = i$$ ← **Before**

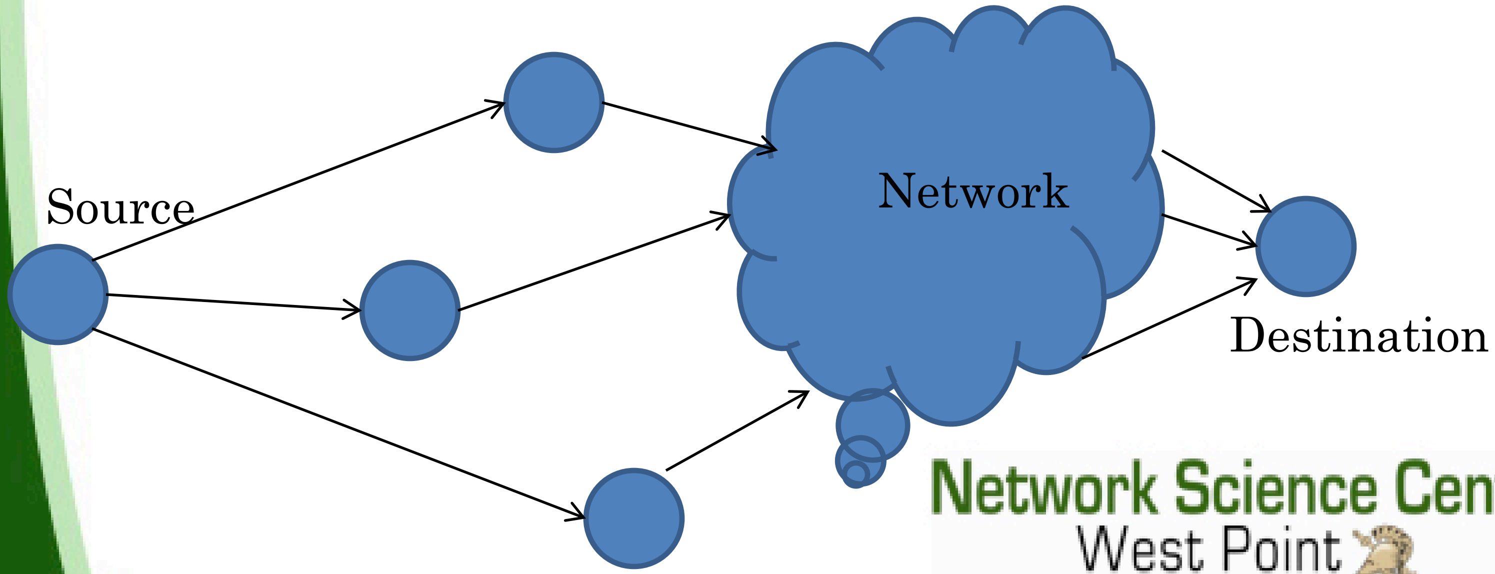
$$R_{j,d}(t) \neq R_{j,d}(t+1)$$ ← **Now**

Network Science Center
West Point

Current Work

- Also leads to probabilities in which nodes packets will take next.
- Probability of going to node *i* to node *j* with destination *d*.

$$p(j,d,t)$$



Network Science Center
West Point

Questions!!!!!!!!!!!!!

Network Science Center
West Point