
Newton's Rotating Water Bucket: A Simple Model

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Abstract

Isaac Newton proposed hanging a bucket of water by a cord in the *Principia*. If the cord is twisted and the bucket is then released, it begins to spin and the surface of the water acquires a paraboloidal shape. In this paper, the parabolic profile as a function of the angle of rotation is derived, as well as the period of the torsional oscillations as a function of the initial parameters of the system.

1. Introduction

IF A VESSEL, hung by a long cord, is so often turned about that the cord is strongly twisted, then filled with water, and held at rest together with the water... [then] while the cord is untwisting itself... the vessel, by gradually communicating its motion to the water, will make it begin sensibly to revolve, and recede by little and little from the middle, and ascend to the sides of the vessel, forming itself into a concave figure (as I have experienced) and the swifter the motion becomes, the higher will the water rise ... This ascent of the water shows its endeavor to recede from the axis of its motion; and the true and absolute circular motion of the water ... becomes known, and may be measured by this endeavor. [1]

Newton's bucket is well known to philosophers of science, who have pondered the metaphysics of why the liquid in a rotating container adopts a curved surface. Ernst Mach, for example, postulated that the parabolic shape must be due to the existence of matter in the universe,

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specifically of the distant “fixed stars” relative to which the bucket rotates [2]. However, the simple physical questions are: How does the shape of the water surface vary as the cord supporting the bucket unwinds? What is the period of the torsional oscillations?

2. Surface Profile From a Force Analysis

Suppose water partly fills a right cylindrical bucket (of radius R) that is rotating about its vertical z axis of symmetry at angular speed ω . Make the key assumption (to be discussed later) that the liquid instantaneously follows the motion of the bucket. (The speed of the bucket is restricted to be less than some value $\omega_{\max}$ that prevents both the spinning water from spilling over the top edge, and the bottom of the bucket from being exposed at the axis of rotation.) Denote the cylindrical coordinates of any point on the surface of the water as (r, ϕ, z) where the origin lies on the axis at the bottom of the bucket. The axes are fixed in the laboratory frame and do not rotate with the bucket. The angular speed of the water and bucket is $\omega = d\phi/dt$. Two forces act on a bit of water at the surface. One is gravity vertically downward. The other can be alternatively described as being due to the pressure from the surrounding water [3], as a buoyant force [4], or simply as a normal force [5]. The resultant of the two forces is a radially inward centripetal force [6] in the inertial laboratory frame, or equivalently a radially outward centrifugal force [7] in the noninertial frame of the bucket. By considering the vertical and horizontal components of Newton’s second law [8], one finds that the water surface adopts the paraboloidal shape

$$z = z_0 + \frac{\omega^2 r^2}{2g} \quad (1)$$

where $g = 9.80 \text{ m/s}^2$ is Earth’s gravitational field strength and z_0 is the height of the water at the center of the spinning bucket. (Another way to derive this result is to note that the surface of the water must be an equipotential relative to the sum of the gravitational and centrifugal potential energies [9].) The value of z_0 can be related to the total mass m of water in the bucket,

$$m = 2\pi\rho \int_0^R r z dr \quad (2)$$

where $\rho = 1000 \text{ kg/m}^3$ is the density of water. Substituting Eq. (1) into (2) and performing the integral gives

$$m = \pi \rho R^2 \left(z_0 + \frac{\omega^2 R^2}{4g} \right). \quad (3)$$

Denote the height of the water in the bucket when it is stationary by h . Then putting $\omega = 0$ in Eq. (3) implies

$$m = \pi \rho R^2 h, \quad (4)$$

so that Eq. (3) can be rearranged as

$$z_0 = h - \frac{\omega^2 R^2}{4g}. \quad (5)$$

Substituting this result into Eq. (1) leads to the normalized expression

$$\frac{z}{h} = 1 - \frac{\omega^2}{\omega_{\max}^2} \left(1 - \frac{2r^2}{R^2} \right) \quad (6)$$

where $\omega_{\max} \equiv 2R^{-1}\sqrt{gh}$. Note that $z = 0$ at $r = 0$ when $\omega = \omega_{\max}$, in agreement with the parenthetical discussion of $\omega_{\max}$ above Eq. (1). Also note that $z/h = 2$ at $r = R$ when $\omega = \omega_{\max}$, which implies that the bucket must initially be filled no more than halfway with water, to prevent liquid from spilling out at the maximum angular speed. Equation (6) is plotted in Figure 1 for three different values of $\omega/\omega_{\max}$. For any angular speed, $z = h$ when $r/R = 2^{-1/2}$. As the bucket spins faster, the water level drops in the center and rises up near the walls, as Newton noted.

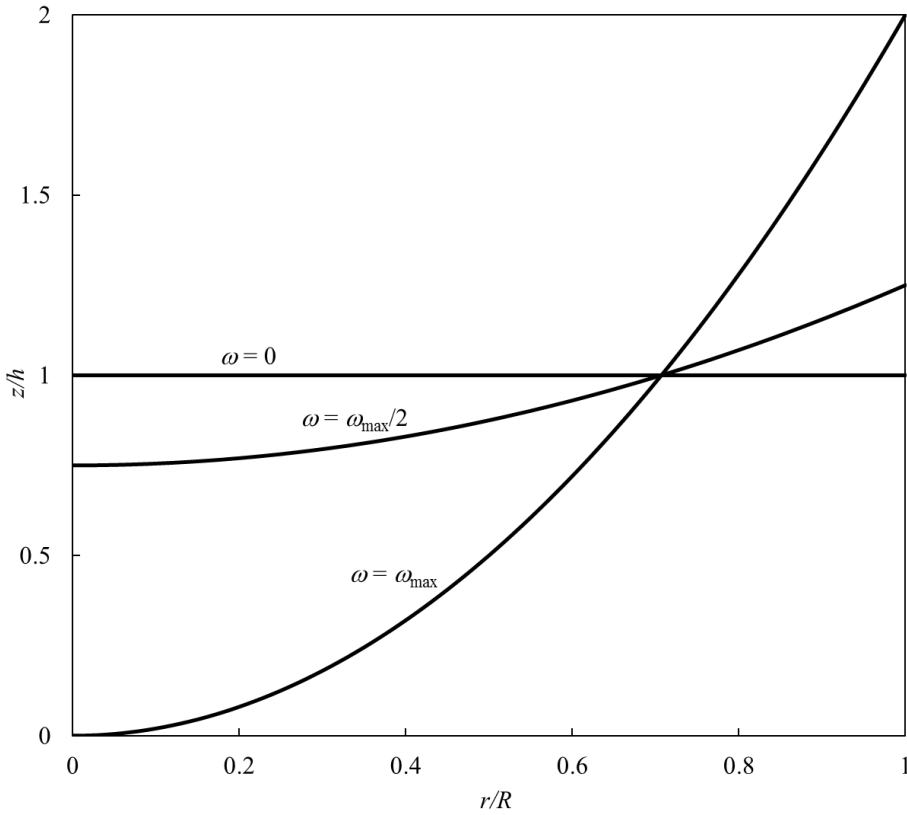


Figure 1. Profiles of the water surface for three different angular speeds.

3. Angular Speed From an Energy Analysis

The moment of inertia of the water in the spinning bucket is

$$I = \int_0^R r^2 2\pi \rho r z dr = I_0 \left(1 + \frac{\omega^2}{3\omega_{\max}^2} \right) \quad (7)$$

using Eqs. (4) and (6), where the moment of inertia of the water when the bucket is at rest is $I_0 = mR^2 / 2$. The moment of inertia increases as water is flung farther away from the axis of rotation with increasing angular speed, up to a maximum value of $4I_0 / 3$.

Just as the elastic potential energy of a spring with a particle attached to its end is $kx^2 / 2$ where x is the translational displacement of

the particle and k is the spring constant, so the torsional potential energy of the rope with the bucket attached to its end is

$$U_T = \frac{1}{2} c \phi^2 \quad (8)$$

where ϕ is the angular displacement of the bucket and c is the torsional constant of the rope. The gravitational potential energy of the water relative to the bottom of the bucket is

$$U_G = \int_0^R \int_0^z 2\pi p r dz dr g z = \pi \rho g \int_0^R r h^2 \left(1 - \frac{\omega^2}{\omega_{\max}^2} + \frac{2\omega^2 r^2}{\omega_{\max}^2 R^2} \right)^2 dr \quad (9)$$

using Eq. (6). With the help of Eq. (4), the integral simplifies to

$$U_G = \frac{mgh}{2} \left(1 + \frac{\omega^4}{3\omega_{\max}^4} \right) \quad (10)$$

which reduces to the expected result if $\omega = 0$. Note the seemingly paradoxical fact that even though the water is cylindrically symmetric and U_G is therefore independent of ϕ as measured in the *rotating frame of the bucket*, U_G is a function of ω which in turn depends on the angle ϕ as measured in the *inertial frame of the laboratory*. The resolution of this paradox is that the angular acceleration is presumed to be small enough that z can be taken to be independent of ϕ over any 2π range, and yet the height of the water at a given radius varies over the course of many revolutions of the bucket. As noted in Ref. [7], water in a 9-cm-diameter Lucite cylinder spinning at a constant rate of 300 rpm takes about 1 minute to attain its equilibrium paraboloidal shape, indicating that the coupling between z and ϕ is weak but nonzero.

Assume that the mass of the bucket is negligible compared to that of the water. Then the total potential energy U of the system is the sum of U_T and U_G . Suppose that the rope is twisted through an initial angle ϕ_0 and the bucket is released from rest, so that the initial kinetic energy is $K_i = 0$ and the initial potential energy is

$$U_i = \frac{1}{2} c \phi_0^2 + \frac{1}{2} mgh. \quad (11)$$

When the rope has untwisted to some angle ϕ so that the bucket is rotating at angular speed ω , the kinetic energy is

$$K_f = \frac{1}{2} I \omega^2 = \frac{1}{2} I_0 \omega^2 + \frac{I_0}{6\omega_{\max}^2} \omega^4 \quad (12)$$

according to Eq. (7), and the potential energy is

$$U_f = \frac{1}{2} c \phi^2 + \frac{1}{2} mgh + \frac{mgh}{6\omega_{\max}^4} \omega^4. \quad (13)$$

Conservation of energy now implies that

$$c(\phi_0^2 - \phi^2) = I_0 \omega^2 + \frac{I_0 \omega_{\max}^2 + mgh}{3\omega_{\max}^4} \omega^4. \quad (14)$$

Noting that $I_0 \omega_{\max}^2 = 2mgh$, we can rearrange Eq. (14) into the normalized form

$$\frac{\omega^4}{\omega_{\max}^4} + 2 \frac{\omega^2}{\omega_{\max}^2} = \frac{c\phi_0^2}{mgh} \left(1 - \frac{\phi^2}{\phi_0^2} \right). \quad (15)$$

Solving this biquadratic equation gives

$$\frac{\omega^2}{\omega_{\max}^2} = \sqrt{1 + \beta \left(1 - \frac{\phi^2}{\phi_0^2} \right)} - 1 \quad (16)$$

where

$$\beta \equiv \frac{c\phi_0^2}{mgh}. \quad (17)$$

The dimensionless constant β is the ratio of the initial torsional potential energy $c\phi_0^2/2$ to the initial gravitational potential energy $mgh/2$. The square root of Eq. (16) is plotted in Figure 2 for three different values of β . Note that the maximum value of β is 3 if ω is not to exceed $\omega_{\max}$ when the cord has fully unwound at $\phi = 0$. Furthermore, even at the midpoint of the bucket's oscillations when $\phi = 0$, increasing β from 1 to 3 increases ω by only 55%.

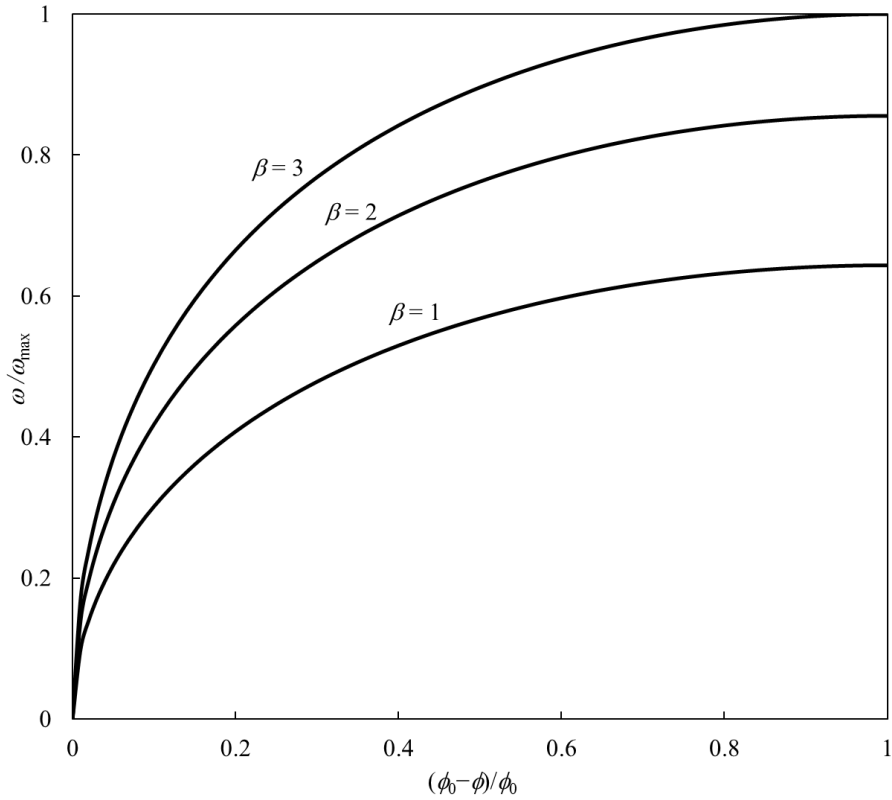


Figure 2. Angular speed of the water as a function of the fractional unwinding of the rope for three different initial numbers of twists of the bucket.

4. Period of Torsional Oscillations

Take the square root of Eq. (16), substitute $\omega = d\phi/dt$, and separate variables. Then integrate over a quarter period $T/4$ as the bucket passes through its equilibrium position and the cord fully winds back up, to get

$$\int_0^{\phi_0} \left[\sqrt{1 + \beta \left(1 - \frac{\phi^2}{\phi_0^2} \right)} - 1 \right]^{-1/2} d\phi = \omega_{\max} \int_0^{T/4} dt. \quad (18)$$

Multiply both sides of this equation by $\phi_0^{-1} \beta^{1/2}$. Then make the change of variable in the left-hand integral to θ where $\sin \theta \equiv \phi/\phi_0$. Perform the right-hand integral and substitute Eq. (17) into it to eliminate β . Using

$\omega_{\max} = \sqrt{2mgh/I_0}$, Eq. (18) gives the period of the bucket's torsional oscillation as

$$T = \sqrt{\frac{8I_0}{c}} \int_0^{\pi/2} \left[\frac{\beta \cos^2 \theta}{\sqrt{1 + \beta \cos^2 \theta} - 1} \right]^{1/2} d\theta. \quad (19)$$

The square root in the denominator inside the square brackets is approximately $1 + 0.5\beta \cos^2 \theta$ in the limit as $\beta \rightarrow 0$. Denoting the period as T_0 in this small-angular-amplitude limit, one immediately obtains

$$T_0 = 2\pi \sqrt{\frac{I_0}{c}} \quad (20)$$

as expected, since $I \rightarrow I_0$ for small ω according to Eq. (7). The integral in Eq. (19) can be numerically evaluated for nonzero β , but it is found to only increase slowly with β . Even at the maximum value of $\beta = 3$, the period of oscillation is merely 12% larger than T_0 .

In any case, Eq. (16) gives an exact solution in phase space, whereby quantities are expressed in terms of the twist angle ϕ rather than in terms of the elapsed time t . For example, substituting Eq. (16) into (6) gives the height of the water at any point in the bucket as a function of the angle that the cord has unwound. In particular, at the walls of the bucket where $r = R$, let Z denote the height of the water. Then the fractional rise in the height of the water at the walls above the stationary level is

$$\frac{Z-h}{h} = \sqrt{1 + \beta \left(1 - \frac{\phi^2}{\phi_0^2} \right)} - 1 \quad (21)$$

which is equal to the normalized square of the angular speed of the bucket, according to Eq. (16). As already mentioned, $Z = 2h$ when $\beta = 3$ and $\phi = 0$.

5. Closing Remarks

Why is the period 12% longer for large-angle oscillations of this torsional pendulum than it is for small amplitudes? The reason is *not* the same as for a simple pendulum. For a simple pendulum, the period increases because the approximation $\sin \theta \approx \theta$ breaks down at large

angles. Instead, the reason here is the increase in the moment of inertia of the water, in accordance with Eq. (7). In particular, if we froze the water, then the period would be independent of amplitude, just as it is for a mass on a Hookean spring.

Finally, let's return to the key assumption underlying the analysis. The viscosity of the water must be high at the walls and bottom of the bucket if the fluid is to instantaneously adjust to the motion of the solid container. At the same time, the viscosity needs to be low within the bulk of the fluid to prevent differences in angular speed between one region of the water and another. Fortunately, simulations for the spin of an incompressible fluid in a cylindrical container suggest that there are viscous boundary layers in the water near the solid surfaces of the cylinder, accompanied by an inner inviscid core [10]. The situation is similar to laminar flow over an airplane wing, with drag motion close to the wing and potential flow far away from it.

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Bios

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