

On Bi-Decompositions of Logic Functions

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Abstract

A logic function f has a disjoint bi-decomposition iff f can be represented as $f = h(g_1(X_1), g_2(X_2))$, where X_1 and X_2 are disjoint set of variables, and h is an arbitrary two-variable logic function. f has a non-disjoint bi-decomposition iff f can be represented as $f(X_1, X_2, x) = h(g_1(X_1, x), g_2(X_2, x))$, where x is the common variable. In this paper, we show a fast method to find bi-decompositions. Also, we enumerate the number of functions having bi-decompositions.

I Introduction

Functional decomposition is a basic technique to realize economical networks. If the function f is represented as $f(X_1, X_2) = h(g(X_1), X_2)$, then f can be realized by the network shown in Fig. 1.1. To find such a decomposition,

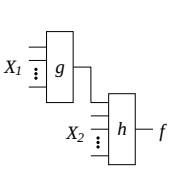


Figure 1.1: A simple disjoint bi-decomposition.

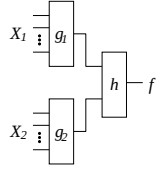


Figure 1.2: A disjoint bi-decomposition.

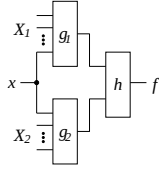


Figure 1.3: A non-disjoint bi-decomposition.

a decomposition chart with 2^{n_1} columns and 2^{n_2} rows are used, where n_i is the number of variables in X_i ($i = 1, 2$). When n is large, the decomposition chart is too large to build. Recently, a method using BDDs has been developed [13]. This greatly reduces memory requirements and computation time. However, it is still time consuming, since we have to check all the $\binom{n_1+n_2}{n_1}$ partitions of $n = n_1 + n_2$. In this paper, we consider bi-decompositions of logic functions, a restricted class of functional decompositions that have the form $f(X_1, X_2) = h(g_1(X_1), g_2(X_2))$. Fig. 1.2 shows the realization of this decomposition.

The reasons we consider bi-decompositions are as follows:

- 1) If f has no bi-decomposition, then the computation time is quite small.

- 2) Some programmable logic devices have two-input logic elements in the outputs.
- 3) If f has a bi-decomposition, then the optimization of the expression is relatively easy.

A restricted class of bi-decompositions has been considered by [8]. The goals of this paper are

- 1) Present a fast method for finding bi-decompositions.
- 2) Enumerate the functions that have bi-decompositions.

Most of the proofs are omitted. They can be available from authors.

II Disjoint Bi-Decomposition

Definition 2.1 Let $X = (X_1, X_2)$ be a partition of the variables. A logic function f has a disjoint bi-decomposition iff f can be represented as $f(X_1, X_2) = h(g_1(X_1), g_2(X_2))$, where h is any two-variable logic function.

If f has a disjoint bi-decomposition, then f can be realized by the network shown in Fig. 1.2.

Definition 2.2 Let $X = (X_1, X_2)$ be a partition of the variables. Let n_1 and n_2 be the number of variables in X_1 and X_2 , respectively. A decomposition chart of the function f for a partition (X_1, X_2) consists of 2^{n_1} columns and 2^{n_2} rows of 0s and 1s. The 2^{n_1} distinct binary numbers for X_1 are listed across the top, and the 2^{n_2} distinct binary numbers for X_2 are listed down the side. The entry for the chart corresponds to the value of $f(X_1, X_2)$.

Example 2.1 Two decomposition charts for the function $f(x_1, x_2, x_3, x_4) = x_1x_2 \oplus x_3x_4$ are shown in Fig. 2.1 (a) and (b). $\square$

Note that the decomposition chart is similar to the Karnaugh map with a different ordering for the cell locations.

Definition 2.3 The number of distinct column (row) patterns in the decomposition chart is called column (row) multiplicity.

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$X_1 = (x_1, x_2)$

	00	01	10	11
00	0	0	0	1
01	0	0	0	1
10	0	0	0	1
11	1	1	1	0

$X_2 = (x_3, x_4)$

(a)

$X_1 = (x_1, x_3)$

	00	01	10	11
00	0	0	0	0
01	0	1	0	1
10	0	0	1	1
11	0	1	1	0

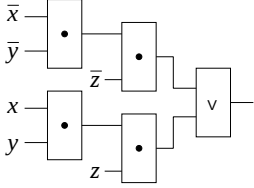
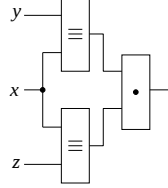
$X_2 = (x_2, x_4)$

(b)

(a)

(b)

Figure 2.1: Decomposition chart.

Figure 3.1: A realization of $f(x, y, z) = \bar{x}\bar{y}\bar{z} \vee xyz$.Figure 3.2: Non-disjoint bi-decomposition for $f(x, y, z) = \bar{x}\bar{y}\bar{z} \vee xyz$.

Example 2.2 In Fig. 2.1 (a), the row and column multiplicities are two. In Fig. 2.1 (b), the row and column multiplicities are four. $\square$

Definition 2.4 Let $\mu(f : X_1, X_2)$ be the column multiplicities for f with respect to X_1 and X_2 . Let $\mu(f : X_2, X_1)$ be the row multiplicities for f with respect to X_1 and X_2 .

Theorem 2.1 f has a disjoint bi-decomposition of form $f(X_1, X_2) = h(g_1(X_1), g_2(X_2))$ iff $\mu(f : X_1, X_2) \leq 2$ and $\mu(f : X_2, X_1) \leq 2$.

III Non-Disjoint Bi-Decomposition

Definition 3.1 Let X_1 and X_2 be disjoint sets of variables, and let x be disjoint from X_1 and X_2 . A logic function f has a non-disjoint bi-decomposition iff f can be represented as $f(X_1, X_2, x) = h(g_1(X_1, x), g_2(X_2, x))$, where h is a two-variable logic function. In this case, x is called the common variable.

A function f with a non-disjoint bi-decomposition can be realized by the network shown in Fig. 1.3.

Lemma 3.1 Let $X = (X_1, X_2, x)$ be a partition of the input variables. Let $h(g_1, g_2)$ be an arbitrary logic function of two variables. Then,

$$h(g_1(X_1, x), g_2(X_2, x)) = \bar{x}h(g_1(X_1, 0), g_2(X_2, 0)) \vee xh(g_1(X_1, 1), g_2(X_2, 1)).$$

Definition 3.2 Let x be the common variable of the non-disjoint bi-decomposition. Let $f(X_1, X_2, a)$ be a sub-function, where x is set to a 0 or 1.

	$X_1 = (x_1, x_2)$					$X_1 = (x_1, x_3)$			
	00	01	10	11		00	01	10	11
$X_2 = (x_3, x_4)$	00	1	0	0	0	00	1	0	0
	01	1	0	0	0	01	1	1	0
	10	1	0	0	0	10	1	1	0
	11	0	1	1	1	11	1	1	0

(a) $f_0 = \bar{x}_1 \bar{x}_2 \oplus x_3 x_4$ (b) $f_1 = x_1 x_2 \oplus (x_3 \vee x_4)$

Figure 3.3: Functions in Example 3.2

Theorem 3.1 $f(X_1, X_2, x)$ has a non-disjoint bi-decomposition of the form $h(g_1(X_1, x), g_2(X_2, x))$ iff $f(X_1, X_2, 0)$ and $f(X_1, X_2, 1)$ have disjoint bi-decompositions $h(g_{01}(X_1), g_{02}(X_2))$ and $h(g_{11}(X_1), g_{12}(X_2))$, respectively.

Example 3.1 Consider the three-variable function: $f(x, y, z) = \bar{x}\bar{y}\bar{z} \vee xyz$. Suppose modules that realizes any function of two variables can be used. The straightforward realization shown in Fig. 3.1 requires five modules. The Shannon expansion with respect to x is $f(x, y, z) = \bar{x}f(0, y, z) \vee xf(1, y, z)$, where $f(0, y, z) = \bar{y}\bar{z}$, and $f(1, y, z) = yz$. Note that both $f(0, y, z)$ and $f(1, y, z)$ have bi-decompositions with $h(x, y) = xy$. Since, $g_1(x, y) = \bar{x}g_{01}(X_1) \vee xg_{11}(X_1) = \bar{x}\bar{y} \vee xy$, and $g_2(x, y) = \bar{x}g_{02}(X_2) \vee xg_{12}(X_2) = \bar{x}\bar{z} \vee xz$. We have $f(x, y, z) = g_1(x, y)g_2(x, z) = (\bar{x}\bar{y} \vee xy)(\bar{x}\bar{z} \vee xz)$. From this expression, we have the network in Fig. 3.2. This network requires only three modules. $\square$

Example 3.2 Consider the five-variable function $f = \bar{x}_5 f_0 \vee x_5 f_1$, where f_0 and f_1 are shown in Fig. 3.3. Since both f_0 and f_1 have disjoint bi-decompositions of the form $h(g_1(X_1), g_2(X_2))$, $f = \bar{x}_5 f_0 \vee x_5 f_1$ has a non-disjoint bi-decomposition as follows:

$$f = \bar{x}_5 \{ \bar{x}_1 \bar{x}_2 \oplus x_3 x_4 \} \vee x_5 \{ x_1 x_2 \oplus (x_3 \vee x_4) \} \\ = \{ \bar{x}_5 (\bar{x}_1 \bar{x}_2) \vee x_5 (x_1 x_2) \} \oplus \{ \bar{x}_5 (x_3 x_4) \vee x_5 (x_3 \vee x_4) \}.$$

The converse is true also. $\square$

Up to now, we only considered the case where there is a single common variable. However, the theorem can be extended to k common variables, where $k \geq 2$.

Definition 3.3 Let X_1 , X_2 , and X_3 be disjoint sets of variables. Let $f(X_1, X_2, \mathbf{a})$ be the sub-functions, where X_3 is set to $\mathbf{a} \in \{0, 1\}^k$, and k denotes the number of variables in X_3 .

Theorem 3.2 Let X_1 , X_2 , and X_3 be disjoint sets of variables. Then, f has a non-disjoint bi-decomposition of form $f(X_1, X_2, X_3) = h(g_1(X_1, X_3), g_2(X_2, X_3))$ iff $f(X_1, X_2, \mathbf{a})$ has a decomposition of the form $h(g_1 \mathbf{a}(X_1), g_2 \mathbf{a}(X_2))$ for all possible $\mathbf{a} \in \{0, 1\}^k$, where k denotes the number of variables in X_3 .

IV A Fast Method for Bi-Decompositions

In this section, we show necessary and sufficient conditions for a function to have a disjoint bi-decomposition. Then, we show efficient algorithms to find disjoint bi-decompositions. In the previous sections, $h(g_1, g_2)$ is an arbitrary two-variable logic function. To find a disjoint bi-decomposition, we need to consider only three types:

- 1) OR type: $f = g_1(X_1) \vee g_2(X_2)$,
- 2) AND type: $f = g_1(X_1)g_2(X_2)$, and
- 3) EXOR type: $f = g_1(X_1) \oplus g_2(X_2)$.

Since f has an AND type disjoint bi-decomposition iff $\bar{f}$ has OR type disjoint bi-decomposition, we only consider the OR type and EXOR type bi-decompositions.

Definition 4.1 x and $\bar{x}$ are literals of a variable x . A logical product which contains at most one literal for each variable is called a product term or a product. Product terms combined with OR operators form a sum-of-products expression (SOP).

Definition 4.2 A prime implicant (PI) p of a function f is a product term which implies f , such that the deletion of any literal from p results in a new product which does not imply f .

Definition 4.3 An irredundant sum-of-products expression (ISOP) is an SOP, where each product is a PI, and no product can be deleted without changing the function represented by the expression.

Definition 4.4 Let $f(X)$ be a function and p be a product of literal(s) in X . The restriction of f to p , denoted by $f(X|p)$ is obtained as follows: If x_i appears in p , then set x_i in 1 in f , and if $\bar{x}_i$ appears in p , then set x_i in 0 in f .

Example 4.1 Let $f(x_1, x_2, x_3) = x_1x_2 \vee \bar{x}_2x_3$ and $p = x_1x_3$. $f(X|p)$ is obtained as follows: Set $x_1 = x_3 = 1$ in f , and we have $f(X|x_1x_3) = f(1, x_2, 1) = x_2 \vee \bar{x}_2 = 1$. $\square$

Lemma 4.1 p is an implicant of $f(X)$, iff $f(X|p) = 1$.

Example 4.2 By Lemma 4.1, x_1x_3 is an implicant of $x_1x_2 \vee \bar{x}_2x_3$, shown in Example 4.1. $\square$

Theorem 4.1 (OR type disjoint bi-decomposition) f has a disjoint bi-decomposition of form $f(X_1, X_2) = g_1(X_1) \vee g_2(X_2)$ iff every product in an ISOP for f consists of literals from X_1 only or X_2 only.

Definition 4.5 $x^0 = \bar{x}$. $x^1 = x$.

Corollary 4.1 If $f(x_1, x_2, \dots, x_n)$ has a PI of the form $x_1^{a_1}x_2^{a_2}\dots x_n^{a_n}$, where $a_i \in \{0, 1\}$, then f has no OR type disjoint bi-decomposition.

Let $x_i (i = 1, 2, \dots, n)$ be the input variables of f . Let $p_1 \vee p_2 \vee \dots \vee p_t$ be an irredundant sum-of-products expression for f , where $p_i (i = 1, 2, \dots, t)$ are PIs of f . Let Π_0 be the trivial partition of $\{1, 2, \dots, n\}$, $\Pi_0 = [\{1\}, \{2\}, \dots, \{n\}]$.

Algorithm 4.1 (OR type disjoint bi-decomposition: $f(X_1, X_2) = g_1(X_1) \vee g_2(X_2)$).

1. For $i = 1$ to t , form Π_i from Π_{i-1} by merging two blocks Ω_1 and Ω_2 of Π_{i-1} if at least one literal in p_i occurs in both Ω_1 and Ω_2 .
2. If Π_t has at least two blocks, then $f(X_1, X_2)$ has a disjoint bi-decomposition of the form $f(X_1, X_2) = g_1(X_1) \vee g_2(X_2)$, with X_1 the union of one or more blocks of Π_t and X_2 the union of the remaining blocks.

Example 4.3 Consider the ISOP: $f(x_1, x_2, \dots, x_6) = x_1x_2 \vee x_2x_3 \vee x_4x_5 \vee x_5x_6$. The products x_1x_2 and x_2x_3 show that x_1, x_2 , and x_3 are in the same block. Also, the products x_4x_5 and x_5x_6 show that x_4, x_5 , and x_6 are in the same block. Thus, we have the partition $[\{1, 2, 3\}, \{4, 5, 6\}]$. The corresponding OR type disjoint bi-decomposition is $f(X_1, X_2) = g_1(X_1) \vee g_2(X_2)$, where $X_1 = (x_1, x_2, x_3)$ and $X_2 = (x_4, x_5, x_6)$. $\square$

Example 4.4 Consider the function f with an ISOP: $f(x_1, x_2, x_3, x_4, x_5) = x_1x_2x_3 \vee x_3x_4x_5$.

- 1) The product $x_1x_2x_3$ shows that x_1, x_2 , and x_3 belong to the same block.
- 2) The product $x_3x_4x_5$ shows that x_3, x_4 , and x_5 belong to the same block.

Thus, all the variables belong to the same block. From this, it follows that f has no OR type decomposition. $\square$

Theorem 4.2 (AND type disjoint bi-decomposition) f has a disjoint bi-decomposition of form $f(X_1, X_2) = g_1(X_1)g_2(X_2)$ iff every product in an ISOP for f consists of literals from X_1 only or X_2 only.

Lemma 4.2 [15] An arbitrary n -variable function can be uniquely represented as

$$\begin{aligned} f(x_1, x_2, \dots, x_n) = & a_0 \oplus (a_1x_1 \oplus a_2x_2 \oplus \dots \oplus a_nx_n) \\ & \oplus (a_{12}x_1x_2 \oplus a_{13}x_1x_3 \oplus \dots \oplus a_{n-1n}x_{n-1}x_n) \\ & \oplus \dots \oplus a_{12\dots n}x_1x_2\dots x_n, \end{aligned} \quad (4.1)$$

where $a_i \in \{0, 1\}$. The above expression is called a positive polarity Reed-Muller expression (PPRM).

For a given function f , the coefficients $a_0, a_1, a_2, \dots, a_{12\dots n}$ are uniquely determined. Thus, the PPRM is a canonical representation. The number of products in (4.1) is at most 2^n , and all the literals are positive (uncomplemented).

Theorem 4.3 (*EXOR type disjoint bi-decomposition*) f has a disjoint bi-decomposition of the form $f(X_1, X_2) = g_1(X_1) \oplus g_2(X_2)$ iff every product in the PPRM for f consists of literals from X_1 only or X_2 only.

Corollary 4.2 If the PPRM of an n -variable function has the product $x_1 x_2 \cdots x_n$, then f has no EXOR type disjoint bi-decomposition.

Theorem 4.4 When f has an EXOR type disjoint bi-decomposition, the number of true minterms of f is an even number.

Corollary 4.3 When the number of true minterms of f is an odd number, then f does not have an EXOR type disjoint bi-decomposition.

The significance of this observation is that at least one half of the functions can be quickly rejected as candidates for EXOR type disjoint bi-decomposition.

Let x_i ($i = 1, 2, \dots, n$) be the input variables of f . Let $p_1 \oplus p_2 \oplus \cdots \oplus p_t$ be PPRM for f , where p_i ($i = 1, 2, \dots, t$) are products. Let, Π_0 be the trivial partition of $\{1, 2, \dots, n\}$, $\Pi_0 = [\{1\}, \{2\}, \dots, \{n\}]$.

Algorithm 4.2 (*EXOR type disjoint bi-decomposition*: $f(X_1, X_2) = g_1(X_1) \oplus g_2(X_2)$).

1. For $i = 1$ to t , form Π_i from Π_{i-1} by merging two blocks Ω_1 and Ω_2 of Π_{i-1} if at least one literal in p_i occurs in both Ω_1 and Ω_2 .
2. If Π_i has at least two blocks, then $f(X_1, X_2)$ has a disjoint bi-decomposition of form $f(X_1, X_2) = g_1(X_1) \oplus g_2(X_2)$, with X_1 the union of one or more blocks of Π_i and X_2 the union of the remaining blocks.

Example 4.5 Consider the PPRM: $f(x_1, x_2, \dots, x_6) = x_1 x_2 \oplus x_2 x_3 \oplus x_4 x_5 \oplus x_5 x_6$. The products $x_1 x_2$ and $x_2 x_3$ show that x_1, x_2 , and x_3 are in the same block. Also, the products $x_4 x_5$ and $x_5 x_6$ show that x_4, x_5 , and x_6 are in the same block. Thus, we have the partition $[\{1, 2, 3\}, \{4, 5, 6\}]$. The corresponding EXOR type disjoint bi-decomposition is $f(X_1, X_2) = g_1(X_1) \oplus g_2(X_2)$, where $X_1 = (x_1, x_2, x_3)$ and $X_2 = (x_4, x_5, x_6)$. $\square$

Algorithm 4.3 (*Non-disjoint bi-decomposition*). $f(X_1, X_2, x_i) = g_1(X_1, x_i) \otimes g_2(X_2, x_i)$, where $\otimes$ denotes either OR, AND, or EXOR. Let (X_1, X_2, x_i) be a partition of the variables $x_1, x_2, \dots$, and x_n . For $i = 1$ to n , do

- i) Let $f_{0i} = f(X_1, X_2, 0)$. (Set x_i to 0). Let $f_{1i} = f(X_1, X_2, 1)$. (Set x_i to 1).
- ii) If both f_{0i} and f_{1i} have the same type of disjoint bi-decompositions with the same partition, then f has a non-disjoint bi-decomposition.

V Complexity Analysis of the Algorithms

5.1 OR type disjoint bi-decomposition

We assume that the function is given as an ISOP with t products. Note that $t \leq 2^{n-1}$. The time to form the partition of variables is $O(n \cdot t)$.

5.2 EXOR type disjoint bi-decomposition

A PPRM can be represented by a functional decision diagram (FDD [5, 15]). Each path from the root node to the constant 1 node corresponds to a product in the PPRM. Thus, the partition of the input variables is directly generated from the FDD. The number of paths in an FDD is $O(2^n)$, where n is the number of the input variables. However, we can avoid exhaustive generation of paths as follows: Let p_1 and p_2 be products in a PPRM. If all the literals in p_1 also appear in p_2 , then p_2 need not be generated in the Algorithm, since the product p_1 that contains more literals than p_2 is more important. By searching the paths with more literals first, we can efficiently detect functions with no disjoint bi-decomposition.

Example 5.1 Consider the function $f(X)$ given as a PPRM: $f(X) = x_1 \oplus x_1 x_2 \oplus x_3 x_4 \oplus x_1 x_2 x_5 x_6$. In constructing the partition of X , we need not consider the products x_1 or $x_1 x_2$, since $x_1 x_2 x_5 x_6$ has the literals of x_1 and $x_1 x_2$. In this case, the product $x_1 x_2 x_5 x_6$ shows that x_1, x_2, x_5 , and x_6 belong to the same group. Also, the product $x_3 x_4$ shows that x_3 and x_4 belong to the same group. Thus, X is partitioned as $X = (X_1, X_2)$, where $X_1 = (x_1, x_2, x_5, x_6)$ and $X_2 = (x_3, x_4)$. $\square$

Definition 5.1 Let p be a product. The set of variables in p is denoted by $V(p) = \{x_i | x_i \text{ or } \bar{x}_i \text{ appears in } p\}$. For example, $V(x_1 x_2 \bar{x}_4) = \{x_1, x_2, x_4\}$

Definition 5.2 Let F be a PPRM. A product p is said to have maximal variable set $V(p)$ if there is no other product p' such that $V(p) \subset V(p')$.

Example 5.2 For the PPRM, $F = x_1 x_2 \oplus x_1 x_3 \oplus x_1 x_2 x_3 \oplus x_4$, $V(x_1 x_2) = \{x_1, x_2\}$, $V(x_1 x_3) = \{x_1, x_3\}$, $V(x_1 x_2 x_3) = \{x_1, x_2, x_3\}$, and $V(x_4) = \{x_4\}$. Thus, $x_1 x_2 x_3$ and x_4 have maximal variable sets. $\square$

Theorem 5.1 A function f has an EXOR type disjoint bi-decomposition if a function f' from the PPRM of f by eliminating implicants not having maximal variable sets has an EXOR type disjoint bi-decomposition.

The following theorem says that if a function has an EXOR type disjoint bi-decomposition, then the number of products in the PPRM is relatively small.

Theorem 5.2 If f has a disjoint bi-decomposition of the form $f(X_1, X_2) = g_1(X_1) \oplus g_2(X_2)$, then the number of products in the PPRM is at most $2^{n_1} + 2^{n_2} - 1$, where n_i is the number of variables in X_i ($i = 1, 2$).

VI Number of Functions with Bi-Decompositions

6.1 Functions with a small number of variables

In the previous sections, we showed that disjoint bi-decompositions are easy to find. In this section, we will enumerate the functions with disjoint bi-decompositions.

Definition 6.1 A function f is said to be nondegenerate if for all x_i $f(|\bar{x}_i) \neq f(|x_i)$.

Definition 6.2 Two functions f and g are NP-equivalent, denoted by $f \stackrel{\text{NP}}{\sim} g$, iff g is derived from f by the following operations:

- 1) Permutation of the input variables.
- 2) Negations of the input variables.

The following is easy to prove.

Lemma 6.1 If f has a disjoint bi-decomposition and if $f \stackrel{\text{NP}}{\sim} g$, then g has also the same type of disjoint bi-decomposition.

Lemma 6.2 All the two-variable functions have disjoint bi-decompositions.

Example 6.1 There are $2^{2^3} = 256$ three-variable logic functions of which 218 are nondegenerate. These non-degenerate functions are grouped into 16 NP-equivalence classes as shown in Table 6.1 [9]. In this table, the column headed by N denotes the number of functions in that equivalence class. Eight classes have disjoint bi-decompositions, and three have non-disjoint bi-decompositions. Note that 194 functions have bi-decompositions. $\square$

The number of functions with AND type disjoint bi-decompositions is equal to the number of functions with OR type disjoint bi-decompositions.

In the case of disjoint bi-decompositions, a function has exactly one type of decomposition (Lemma 6.4). On the other hand, in the case of non-disjoint bi-decompositions, a function may have more than one type of bi-decompositions.

Example 6.2 Consider the three-variable function $f = \bar{x}_1 x_3 \vee x_1 x_2$. This function has three types of non-disjoint bi-decompositions:

$$\begin{aligned} f &= \bar{x}_1 x_3 \vee x_1 x_2 && (\text{OR type bi-decomposition}) \\ &= \bar{x}_1 x_3 \oplus x_1 x_2 && (\text{EXOR type bi-decomposition}) \\ &= (x_1 \vee x_3)(\bar{x}_1 \vee x_2) && (\text{AND type bi-decomposition}) \end{aligned} \quad \square$$

Table 6.1: NP-representative functions of three variables.

	Representative functions	N	Type	Property
1	$x_1 \oplus x_2 \oplus x_3$	2	EXOR	Disjoint Bi-Decomposition
2	$x_1 x_2 x_3$	8	AND	
3	$x_1 \vee x_2 \vee x_3$	8	OR	
4	$x_1(x_2 \vee x_3)$	24	AND	
5	$x_1 \vee x_2 x_3$	24	OR	
6	$x_1(x_2 \oplus x_3)$	12	AND	
7	$x_1 \vee (x_2 \oplus x_3)$	12	OR	
8	$x_1 \oplus x_2 x_3$	24	EXOR	
9	$x_1 x_2 x_3 \vee \bar{x}_1 \bar{x}_2 \bar{x}_3$	4		Non-Disjoint Bi-Decomposition
10	$(x_1 \vee x_2 \vee x_3)(\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3)$	4		
11	$\bar{x}_1 x_3 \vee x_1 x_2$	24		
12	$x_1 \bar{x}_2 \bar{x}_3 \vee x_2 x_3$	24		
13	$(x_1 \vee \bar{x}_2 \vee \bar{x}_3)(x_2 \vee x_3)$	24		No Bi-Decomposition
14	$x_1 x_2 \vee x_2 x_3 \vee x_3 x_1$	8		
15	$x_1 x_2 \vee x_2 x_3 \vee x_1 x_3 \vee \bar{x}_1 \bar{x}_2 \bar{x}_3$	8		
16	$\bar{x}_1 x_2 x_3 \vee x_1 \bar{x}_2 x_3 \vee x_1 x_2 \bar{x}_3$	8		

N : Number of the functions in the class.

Table 6.2: Number of functions.

			$n = 2$	$n = 3$	$n = 4$
All the functions			16	256	65536
Nondegenerate functions			10	218	64594
Functions with bi-decomposition	Disjoint	AND	4	44	1660
		OR	4	44	1660
		EXOR	2	26	914
	Non-disjoint		0	80	3860
	Total		10	194	8094

6.2 The number of functions with bi-decompositions

Lemma 6.3 [4]: Let $\alpha(n)$ be the number of nondegenerate functions on n variables. Then,

$$\alpha(n) = \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} 2^{2^k} \sim 2^{2^n},$$

where $a(n) \sim b(n)$ means $\lim_{n \rightarrow \infty} \frac{a(n)}{b(n)} = 1$.

Lemma 6.4 A nondegenerate function f has at most one type of disjoint bi-decomposition:

1. $f(X_1, X_2) = g_1(X_1) \cdot g_2(X_2)$,
2. $f(X_1, X_2) = g_1(X_1) \vee g_2(X_2)$, or
3. $f(X_1, X_2) = g_1(X_1) \oplus g_2(X_2)$,

where g_1 and g_2 are nondegenerate functions on one or more variables.

Theorem 6.1 The number of functions $N_{\text{disjoint}}(n)$ with disjoint bi-decompositions is $N_{\text{disjoint}}(n) = A_{\text{dis}}(n) + O_{\text{dis}}(n) + E_{\text{dis}}(n)$, where

$$A_{\text{dis}}(n) = n! \sum_{\substack{k_1, k_2, \dots, k_n \geq 0 \\ 1k_1 + 2k_2 + \dots + nk_n = n}} \prod_{i=1}^n \left(\frac{\alpha(i) - A_{\text{dis}}(i)}{i!} \right)^{k_i} \frac{1}{k_i!}$$

Table 7.1: Number of functions with bi-decompositions.

Decomposition Type		Number of Functions
Disjoint	AND	853
	OR	264
	EXOR	73
Non-disjoint	AND	162
	OR	91
	EXOR	42

$$O_{dis}(n)=n! \sum_{\substack{k_1, k_2, \dots, k_n \geq 0 \\ 1k_1 + 2k_2 + \dots + nk_n = n}} \prod_{i=1}^n \left(\frac{\alpha(i) - O_{dis}(i)}{i!} \right)^{k_i} \frac{1}{k_i!}$$

$$E_{dis}(n)=2n! \sum_{\substack{k_1, k_2, \dots, k_n \geq 0 \\ 1k_1 + 2k_2 + \dots + nk_n = n}} \prod_{i=1}^n \left(\frac{\alpha(i) - E_{dis}(i)}{i!} \right)^{k_i} \frac{1}{2^{k_i} k_i!}$$

where the sums are over all partitions of n except the trivial partition $n = 0 \cdot 1 + 0 \cdot 2 + \dots + 0 \cdot (n-1) + 1 \cdot n$ (i.e. the sum is over all partitions where $k_n = 0$), and where $A_{dis}(1) = O_{dis}(1) = E_{dis}(1) = 0$.

Table 6.2 shows the number of functions with disjoint bi-decompositions up to $n = 4$.

VII Experimental Results

We analyzed the bi-decomposability of 136 benchmark functions. Over these multiple-output functions, the total number of outputs (functions) is 1908. For each function, we determined whether there exists a disjoint bi-decomposition. If none existed, we determined if there exists a non-disjoint bi-decomposition (with a single common variable). Table 7.1 summarizes our results. It is interesting that 1190 out of 1908 functions, or 62 percent, have disjoint bi-decompositions. Of the remaining 718 functions, 295 have non-disjoint decompositions. It should be noted that more than 295 functions have non-disjoint decompositions, since a function with a disjoint bi-decomposition may also have a non-disjoint bi-decomposition.

VIII Conclusions and Comments

In this paper, we presented the bi-decomposition, a special case of functional decomposition. Disjoint bi-decompositions have the following features:

- 1) They are easy to detect; we use ISOPs or PPRMs rather than decomposition charts.
- 2) Programmable logic devices exist that realize bi-decompositions.
- 3) If the function has an OR (AND) type bi-decomposition, then we can optimize the expression separately.

We enumerated functions with bi-decompositions. Among 218 nondegenerate functions of 4 variables, 194 have bi-

decompositions. Also, we derived formulae for the number of disjoint bi-decompositions.

Since the fraction of functions with decompositions approaches to zero as n increase [4], the fraction of functions with bi-decompositions also approaches to zero as n increases. However, for 1908 functions we analyzed about 78% of them had either disjoint or non-disjoint bi-decompositions.

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