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Dynamic Stiffness Modeling of Composite Plate and Shell Assemblies

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14. ABSTRACT
This grant sought to develop the dynamic stiffness method for composite shell assemblies. In the first part an exact dynamic stiffness element based on higher order shear deformation theory and extensive use of symbolic algebra is developed for the first time to carry out buckling analysis of composite plate assemblies. The principle of minimum potential energy is applied to derive the governing differential equations and natural boundary conditions. The effects of significant parameters such as thickness-to-length ratio, orthotropy ratio, number of layers, lay-up and stacking sequence and boundary conditions on the critical buckling loads and mode shapes are investigated. In the second part of the grant an exact free vibration analysis of laminated composite doubly-curved shallow shells was carried out by combining the dynamic stiffness method (DSM) and a higher order shear deformation theory (HSDT) for the first time. The Wittrick-Williams algorithm is used as a solution technique to compute the eigenvalues of the overall DS matrix.

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Abbreviations

1D, 2D, 3D	One dimensional, Two dimensional, Three dimensional
CPT	Classical plate theory
CLT	Classical lamination theory
DS	Dynamic stiffness
DSM	Dynamic stiffness method
FEM	Finite element method
FSDT	First order shear deformation theory
HSDT	Higher order shear deformation theory
LW	Layer-wise

Nomenclature

A	Area
$\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{D}$, $\boldsymbol{E}$, $\boldsymbol{F}$, $\boldsymbol{H}$	Membrane, coupling and bending stiffness
b	Width of the place
$\overline{\boldsymbol{C}}$	Constitutive matrix of laminate
E	Elastic modulus
f	Elements of the dynamic stiffness matrix
G	Shear modulus
h	Thickness of plate
I_0, I_1, I_2, I_3, I_4, I_5, I_6	Inertias of plate
j	Number of natural frequencies lying below the trial frequency
j_0	Number of clamped-clamped natural frequencies
k	Shear correction factor
$\boldsymbol{K}$	Dynamic stiffness matrix
L	Length of the plate
M, N, Q, P	Moment and force resultants using HSDT
$\mathcal{M}$, $\mathcal{N}$, $\mathcal{Q}$, $\mathcal{P}$	Amplitude of moment and force resultants using HSDT
N_l	Number of layers
r_p	Roots of the polynomial for the in-plane case
r_o	Roots of the polynomial for the out-of-plane case
s	Elements of the dynamic stiffness matrix
T	Kinetic Energy
t	Time
U	Potential Energy
U, V, W	Displacements amplitudes
u, v, w	Displacements in the x, y, and z direction respectively
$\boldsymbol{\eta}$	Displacement vector
x, y, z	Rectangular Cartesian coordinate system

Symbols

α	Half sin-wave
ε, $\boldsymbol{\varepsilon}$	Strain and strain vector
ϕ	Rotations
Φ	Amplitudes of rotation
ρ	Density of plate material
σ, $\boldsymbol{\sigma}$	Stress and stress vector
ν	Poisson's ratio

Super and Subscripts

0	Membrane
$1, 2$	Fiber direction and orthogonal direction
11, 12, ij	Matrix element positions for rows and columns
m	Number of half sin-waves in the y directions
n	Number of half sin-waves in the x directions
k	Layer number
x, y, z	Directions

Part I

Buckling Analysis of Composite Plates Using Higher Order Shear Deformation Theory

Summary

An exact dynamic stiffness element based on higher order shear deformation theory and extensive use of symbolic algebra is developed for the first time to carry out buckling analysis of composite plate assemblies. The principle of minimum potential energy is applied to derive the governing differential equations and natural boundary conditions. Then by imposing the geometric boundary conditions in algebraic form the dynamic stiffness matrix, which includes contributions from both stiffness and initial pre-stress terms, is developed. The Wittrick-Williams algorithm is used as solution technique to compute the critical buckling loads and mode shapes for a range of laminated composite plates. The effects of significant parameters such as thickness-to-length ratio, orthotropy ratio, number of layers, lay-up and stacking sequence and boundary conditions on the critical buckling loads and mode shapes are investigated. The accuracy of the method is demonstrated by comparing results whenever possible with those available in the literature.

1 Introduction

Several methodologies have been developed over the years to solve the elastic stability problem. A simplified approach to calculate the ith critical load, is to consider the critical load as the load at which more than one infinitesimally adjacent equilibrium configurations exist that can be identified with the ith bifurcation point (Euler's method) [1]. In a linearized structural stability analysis, the determination of the critical load leads to a linear eigenvalues problem. The bifurcation method can be successfully used particularly for plates, when the critical equilibrium configuration shows a slight geometry change as the critical buckling load is reached. However, as explained by Leissa [2], linearized stability analysis is meaningful, if and only if, the initial in-plane loading does not produce an out-of-plane deformation. Furthermore, there are many cases in which Euler's method may fail, in particular when thin-walled structures like shells exhibit the snap-buckling phenomenon. In such cases, the most general approach, based on the solution of the complete equilibrium and stability equations [3, 4] is preferred.
Amongst a wide class of methodologies employed to analyze the elastic stability of advanced composite structures, the DSM is probably the most accurate and computationally efficient option. The DSM based on Lèvy-type closed form solution for plates [5] is indeed an exact approach to the solution procedure. Wittick [6] laid the groundwork of the DSM for plates. The basic assumption in this work is that the deformation of any component plate varies sinusoidally in the longitudinal direction. Using this assumption, a stiffness matrix may be derived that relates the amplitudes of the edge forces and moments to the corresponding edge displacements and rotations for a single component plate. For the exact DSM, this stiffness matrix is derived directly from the equations of equilibrium that describe the buckling behavior of the plate. Essentially, Wittrick [6] developed an exact stiffness matrix for a single isotropic, long flat plate subject to uniform axial compression. His analysis basically used classical plate theory (CPT). Wittrick and Curzon [7] later extended this analysis to account for the spatial phase difference between the perturbation forces and displacements which occur at the edges of the plate during buckling due to the presence of in-plane shear loading. This phase difference was accounted for by defining the magnitude of these quantities using complex quantities. Wittrick [8] then extended his analysis further to consider flat isotropic plates under any general state of stress that remains uniform in the longitudinal direction (i.e., combinations of bi-axial direct stress and in-plane shear). A method very similar to that described in [6] was also presented by Smith in [9] for the bending, buckling, and vibration of plate-beam structures. Following these developments, Williams [10] presented two computer programs, GASVIP and VIPAL to compute the initial buckling stress of prismatic plate assemblies subjected to uniform longitudinal stress or uniform longitudinal compression, respectively. GASVIP was used to set up the overall stiffness matrix for the structure, and VIPAL demonstrated the use of substructuring. Next, Wittrick and Williams [11] reported on the VIPASA computer code for the buckling analysis of prismatic plate assemblies. This code allowed for analysis of isotropic or anisotropic plates using a general state of stress (including in-plane shear). The complex stiffnesses

described in [12] were incorporated in VIPASA, as well as allowances were made for eccentric connections between component plates. This code also implemented an algorithm, referred to as the Wittrick-Williams algorithm [13] for determining any buckling load for any given wavelength. The development of this algorithm was necessary because the complex stiffnesses described above are transcendental functions of the load factor and half wavelength of the buckling modes of the structure which make a determinant plot cumbersome and unfeasible. Viswanathan and Tamekuni [14, 15] presented an exact DSM based upon CPT for the elastic stability analysis of composite stiffened structures subjected to biaxial in-plane loads. The structure was idealized as an assemblage of laminated plate elements (flat or curved) and beam elements. Tamekuni, and Baker extended this analysis in [16] considering long curved plates subject to any general state of stress, together with in-plane shear loads. Anisotropic material properties were also allowed. This analysis utilized complex stiffnesses as described in [12]. The works described in [9, 13, 16] are more or less similar. The differences are discussed in [11]. Williams and Anderson [17] presented modifications to the eigenvalue algorithm described in [13]. Further modifications presented in [17] allowed the buckling mode corresponding to a general loading to be represented as a series of sinusoidal modes in combination with Lagrangian multipliers to apply point constraints at any location on edges. the DS matrix for laminated composite plates for buckling analysis. This useful extension is of considerable theoretical and computational complexity as will be shown later. The research is particularly relevant when analysing thick composite plates for their buckling characteristics.

2 Theoretical formulation

2.1 Displacement field and derivation of governing differential equations

In the derivation that follows, the hypotheses of straightness and normality of a transverse normal after deformation are assumed to be no longer valid for the displacement field which is now considered to be a cubic function in the thickness coordinate, and hence the use of higher order shear deformation theory (HSDT). This development is in sharp contrast to earlier developments based on CPT and FSDT and no doubt a significant step forward. The deformation pattern through thickness of the plate is shown in Fig. 1. A laminated composite plate composed of N_l layers is considered in order to make the theory sufficiently general. The integer k is used as a superscript denoting the layer number which starts from the bottom of the plate. The kinematics of deformation of a transverse normal using both first order and higher order shear deformation are shown in Fig. 1. After imposing the transverse shear stress homogeneous conditions [18, 19] at the top/bottom surface of the plate, the displacements field are given below in the usual form:

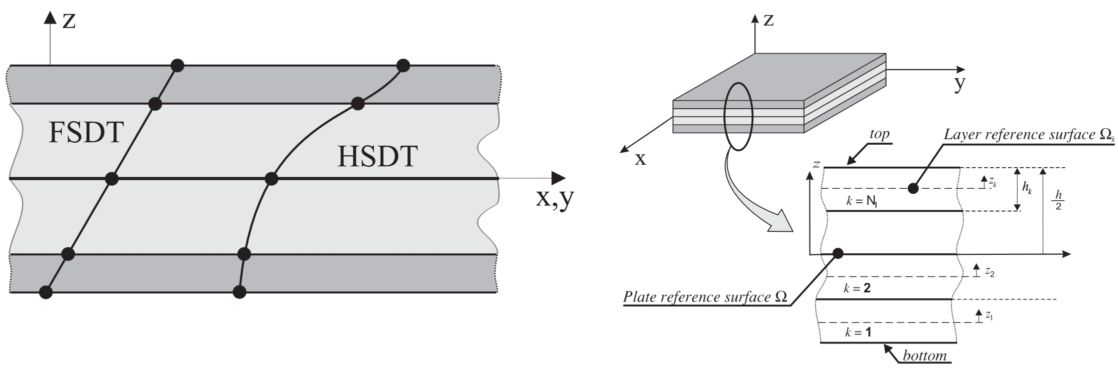


Figure 1: Kinematic descriptions of FSDT and HSDT for a multilayered plates.

$$\begin{aligned} u\,(x,y,z,t) &= u_0\,(x,y,t) + z\,\phi_x\,(x,y,t) + c_1\,z^3\left(\phi_x\,(x,y,t) + \frac{\partial w_0\,(x,y,t)}{\partial x}\right) \\ v\,(x,y,z,t) &= v_0\,(x,y,t) + z\,\phi_y\,(x,y,t) + c_1\,z^3\left(\phi_y\,(x,y,t) + \frac{\partial w_0\,(x,y,t)}{\partial y}\right) \\ w\,(x,y,z,t) &= w_0\,(x,y,t) \end{aligned} \tag{1}$$

where u, v, w are the plate displacement components of the displacement vector,

$$\boldsymbol{\eta} = \left\{\begin{array}{ccc} u & v & w \end{array}\right\}^T \tag{2}$$

$c_1 = -\frac{4}{3\,h^2}$ whereas u_0, v_0, w_0 are the displacement components defined on the plate middle surface Ω in the directions x, y and z. The principle of minimum potential energy is now applied. The variational statement at multilayer level is:

$$\sum_{k=1}^{N_l} \delta\,\Pi^k = 0 \tag{3}$$

where Π^k is the total potential energy for the kth layer of the composite plate. The first variation can be expressed as:

$$\delta\Pi^k = \delta U^k + \delta V^k \tag{4}$$

where δU^k is the virtual potential strain energy, δV^k is the virtual potential energy due to external loadings, and assume the following form:

$$\delta U^k = \int_{\Omega^k}\int_{z^k} \left(\boldsymbol{\delta\varepsilon}^{k^T}\,\boldsymbol{\sigma}^k\right) d\Omega^k\,dz, \quad \delta V^k = \int_{\Omega^k}\int_{z^k} \left(\delta\varepsilon^{nl}_{xx}\,\tilde{\sigma}_{x_0} + \delta\varepsilon^{nl}_{yy}\,\tilde{\sigma}_{y_0}\right) d\Omega^k\,dz \tag{5}$$

the stresses, $\boldsymbol{\sigma}$ and the strains, $\boldsymbol{\varepsilon}$ vectors are expressed as follows:

$$\boldsymbol{\sigma} = \left\{\begin{matrix} \sigma_{xx} & \sigma_{yy} & \tau_{xy} & \tau_{xz} & \tau_{yz}\end{matrix}\right\}^T, \quad \boldsymbol{\varepsilon} = \left\{\begin{matrix} \varepsilon_{xx} & \varepsilon_{yy} & \gamma_{xy} & \gamma_{xz} & \gamma_{yz}\end{matrix}\right\}^T \tag{6}$$

$\tilde{\sigma}_{x_0}$ and $\tilde{\sigma}_{y_0}$ denote the in-plane initial stresses. The non-linear strains ε^{nl}_{xx} and ε^{nl}_{yy} are approximated with the Von Karman's non-linearity:

$$\varepsilon^{nl}_{xx} = \frac{1}{2}\,(w,x)^2 \qquad \varepsilon^{nl}_{yy} = \frac{1}{2}\,(w,y)^2 \tag{7}$$

The subscript T signifies an array transposition and δ the variational operator. Constitutive and geometrical relationships are defined respectively as:

$$\boldsymbol{\sigma}^k = \tilde{\boldsymbol{C}}^k\,\boldsymbol{\varepsilon}^k \qquad \boldsymbol{\varepsilon} = \boldsymbol{\mathcal{D}}\,\boldsymbol{\eta} \tag{8}$$

where $\tilde{\boldsymbol{C}}^k$ is the plane stress constitutive matrix and $\boldsymbol{\mathcal{D}}$ is the differential matrix (see Appendix A for details). Substituting Eq. (42) into the Eq. (48) and imposing the condition in Eq. (46), the equations of motion are obtained after extensive algebraic manipulation as:

$$\begin{aligned}
\delta u_0:\quad & A_{11}\,u_{0,xx} + A_{12}\,v_{0,yx} + A_{16}\,(u_{0,yx} + v_{0,xx}) + B_{11}\,\phi_{x,xx} + B_{12}\,\phi_{y,yx} + B_{16}\,(\phi_{x,yx} + \phi_{y,xx}) + E_{11}\,\mathrm{c}_2\,\phi_{x,xx}\\
& + E_{11}\,\mathrm{c}_2\,w_{0,xxx} + E_{12}\,\mathrm{c}_2\,\phi_{y,yx} + E_{12}\,\mathrm{c}_2\,w_{0,yyx} + E_{16}\,\mathrm{c}_2\,\phi_{x,yx} + E_{16}\,\mathrm{c}_2\,\phi_{y,xx} + 2\,E_{16}\,\mathrm{c}_2\,w_{0,xyx} + A_{16}\,u_{0,xy}\\
& + A_{26}\,v_{0,yy} + A_{66}\,(u_{0,yy} + v_{0,xy}) + B_{16}\,\phi_{x,xy} + B_{26}\,\phi_{y,yy} + B_{66}\,(\phi_{x,yy} + \phi_{y,xy}) + E_{12}\,\mathrm{c}_2\,(\phi_{x,xy} + w_{0,xxy})\\
& + E_{26}\,\mathrm{c}_2\,(\phi_{y,yy} + w_{0,yyy}) + E_{66}\,\mathrm{c}_2\,(\phi_{x,yy} + \phi_{y,xy} + 2\,w_{0,xyy}) = 0\\[2ex]
\delta v_0:\quad & A_{16}\,u_{0,xx} + A_{26}\,v_{0,yx} + A_{66}\,(u_{0,yx} + v_{0,xx}) + B_{16}\,\phi_{x,xx} + B_{26}\,\phi_{y,yx} + B_{66}\,(\phi_{x,yx} + \phi_{y,xx}) + E_{16}\,\mathrm{c}_2\,\phi_{x,xx}\\
& + E_{16}\,\mathrm{c}_2\,w_{0,xxx} + E_{26}\,\mathrm{c}_2\,\phi_{y,yx} + E_{26}\,\mathrm{c}_2\,w_{0,yyx} + E_{66}\,\mathrm{c}_2\,\phi_{x,yx} + E_{66}\,\mathrm{c}_2\,\phi_{y,xx} + 2\,E_{66}\,\mathrm{c}_2\,w_{0,xyx} + A_{12}\,u_{0,xy}\\
& + A_{22}\,v_{0,yy} + A_{26}\,(u_{0,yy} + v_{0,xy}) + B_{12}\,\phi_{x,xy} + B_{22}\,\phi_{y,yy} + B_{26}\,(\phi_{x,yy} + \phi_{y,xy}) + E_{12}\,\mathrm{c}_2\,(\phi_{x,xy} + w_{0,xxy})\\
& + E_{22}\,\mathrm{c}_2\,(\phi_{y,yy} + w_{0,yyy}) + E_{26}\,\mathrm{c}_2\,(\phi_{x,yy} + \phi_{y,xy} + 2\,w_{0,xyy}) = 0
\end{aligned} \tag{9}$$

$$
\begin{aligned}
\delta w_0 : \quad & A_{44}\,(\phi_{y,y} + w_{0,yy}) + A_{45}\,(\phi_{x,y} + w_{0,xy}) + D_{44}\,\mathrm{c}_1\,(\phi_{y,y} + w_{0,yy}) + D_{45}\,\mathrm{c}_1\,(\phi_{x,y} + w_{0,xy}) \\
& + A_{45}\,(\phi_{y,x} + w_{0,xy}) + A_{55}\,(\phi_{x,x} + w_{0,xx}) + D_{45}\,\mathrm{c}_1\,(\phi_{y,x} + w_{0,xy}) + D_{55}\,\mathrm{c}_1\,(\phi_{x,x} + w_{0,xx}) \\
& + D_{44}\,\mathrm{c}_1\,(\phi_{y,y} + w_{0,yy}) + D_{45}\,\mathrm{c}_1\,(\phi_{x,y} + w_{0,xy}) + F_{44}\,\mathrm{c}_1^2\,(\phi_{y,y} + w_{0,yy}) + F_{45}\,\mathrm{c}_1^2\,(\phi_{x,y} + w_{0,xy}) \\
& + D_{45}\,\mathrm{c}_1\,(\phi_{y,x} + w_{0,xy}) + D_{55}\,\mathrm{c}_1\,(\phi_{x,x} + w_{0,xx}) + F_{45}\,\mathrm{c}_1^2\,(\phi_{y,x} + w_{0,xy}) + F_{55}\,\mathrm{c}_1^2\,(\phi_{x,x} + w_{0,xx}) \\
& - E_{11}\,\mathrm{c}_2\,u_{0,xxx} - E_{12}\,\mathrm{c}_2\,v_{0,xxy} - E_{16}\,\mathrm{c}_2\,(u_{0,xxy} + v_{0,xxx}) - F_{11}\,\mathrm{c}_2\,\phi_{x,xxx} - F_{12}\,\mathrm{c}_2\,\phi_{y,xxy} \\
& - F_{16}\,\mathrm{c}_2\,(\phi_{x,xxy} + \phi_{y,xxx}) - H_{11}\,\mathrm{c}_2^2\,(\phi_{x,xxx} + w_{0,xxxx}) - H_{12}\,\mathrm{c}_2^2\,(\phi_{x,xxy} + w_{0,xxyy}) \\
& - H_{16}\,\mathrm{c}_2^2\,(\phi_{x,xxy} + \phi_{y,xxx} + 2w_{0,xxxy}) - 2\,E_{16}\,\mathrm{c}_2\,u_{0,xxy} - 2\,E_{26}\,\mathrm{c}_2\,v_{0,xyy} - 2\,E_{66}\,\mathrm{c}_2\,(u_{0,xyy} + v_{0,xxy}) \\
& - 2\,F_{16}\,\mathrm{c}_2\,\phi_{x,xxy} - 2\,F_{26}\,\mathrm{c}_2\,\phi_{y,xyy} - 2\,F_{66}\,\mathrm{c}_2\,(\phi_{x,xyy} + \phi_{y,xxy}) - 2\,H_{16}\,\mathrm{c}_2^2\,(\phi_{x,xxy} + w_{0,xxxy}) \\
& - 2\,H_{26}\,\mathrm{c}_2^2\,(\phi_{y,xyy} + w_{0,xyyy}) - 2\,H_{66}\,\mathrm{c}_2^2\,(\phi_{x,xyy} + \phi_{y,xxy} + 2\,w_{0,xxyy}) - E_{12}\,\mathrm{c}_2\,u_{0,xyy} - E_{22}\,\mathrm{c}_2\,v_{0,yyy} \\
& - E_{26}\,\mathrm{c}_2\,(u_{0,yyy} + v_{0,xyy}) - F_{12}\,\mathrm{c}_2\,\phi_{x,xyy} - F_{22}\,\mathrm{c}_2\,\phi_{y,yyy} - F_{26}\,\mathrm{c}_2\,(\phi_{x,yyy} + \phi_{y,xyy}) \\
& - H_{12}\,\mathrm{c}_2^2\,(\phi_{x,xyy} + w_{0,xxyy}) - H_{22}\,\mathrm{c}_2^2\,(\phi_{y,yyy} + w_{0,yyyy}) - 2\,H_{26}\,\mathrm{c}_2^2\,(\phi_{x,yyy} + \phi_{y,xyy} + 2\,w_{0,xyyy}) \\
& = \tilde{N}_{x_0}\,w_{0,xx} + \tilde{N}_{y_0}\,w_{0,yy}
\end{aligned}
$$

$$
\begin{aligned}
\delta \phi_x : \quad & B_{11}\,u_{0,xx} + B_{12}\,v_{0,yx} + B_{16}\,(u_{0,yx} + v_{0,xx}) + D_{11}\,\phi_{x,xx} + D_{12}\,\phi_{y,xy} + D_{16}\,(\phi_{x,yx} + \phi_{y,xx}) \\
& + F_{11}\,\mathrm{c}_2\,(\phi_{x,xx} + w_{0,xxx}) + F_{12}\,\mathrm{c}_2\,(\phi_{y,yx} + w_{0,yyx}) + F_{16}\,\mathrm{c}_2\,(\phi_{x,yx} + \phi_{y,xx} + 2\,w_{0,xyx}) \\
& + B_{16}\,u_{0,xy} + B_{26}\,v_{0,yy} + B_{66}\,(u_{0,yy} + v_{0,xy}) + D_{16}\,\phi_{x,xy} + D_{26}\,\phi_{y,yy} + D_{66}\,(\phi_{x,yy} + \phi_{y,xy}) \\
& + F_{16}\,\mathrm{c}_2\,(\phi_{x,xy} + w_{0,xxy}) + F_{26}\,\mathrm{c}_2\,(\phi_{y,yy} + w_{0,yyy}) + F_{66}\,\mathrm{c}_2\,(\phi_{x,yy} + \phi_{y,xy} + 2\,w_{0,xyy}) \\
& + E_{11}\,\mathrm{c}_2\,u_{0,xx} + E_{12}\,\mathrm{c}_2\,v_{0,yx} + E_{16}\,\mathrm{c}_2\,(u_{0,yx} + v_{0,xx}) + F_{11}\,\mathrm{c}_2\,\phi_{x,xx} + F_{12}\,\mathrm{c}_2\,\phi_{y,xy} + F_{16}\,\mathrm{c}_2\,(\phi_{x,yx} + \phi_{y,xx}) \\
& + H_{11}\,\mathrm{c}_2^2\,(\phi_{x,xx} + w_{0,xxx}) + H_{12}\,\mathrm{c}_2^2\,(\phi_{y,yx} + w_{0,yyx}) + H_{16}\,\mathrm{c}_2^2\,(\phi_{x,yx} + \phi_{y,xx} + 2\,w_{0,xyx}) \\
& + E_{16}\,\mathrm{c}_2\,u_{0,xy} + E_{26}\,\mathrm{c}_2\,v_{0,yy} + E_{66}\,\mathrm{c}_2\,(u_{0,yy} + v_{0,xy}) + F_{16}\,\mathrm{c}_2\,\phi_{x,xy} + F_{26}\,\mathrm{c}_2\,\phi_{y,yy} + F_{66}\,\mathrm{c}_2\,(\phi_{x,yy} + \phi_{y,xy}) \\
& + H_{16}\,\mathrm{c}_2^2\,(\phi_{x,xy} + w_{0,xxy}) + H_{26}\,\mathrm{c}_2^2\,(\phi_{y,yy} + w_{0,yyy}) + H_{66}\,\mathrm{c}_2^2\,(\phi_{x,yy} + \phi_{y,xy} + 2\,w_{0,xyy}) \\
& - A_{45}\,(\phi_y + 2\,w_{0,y}) - A_{55}\,(\phi_x + 2\,w_{0,x}) - 2\,D_{45}\,\mathrm{c}_1\,(\phi_y + 2\,w_{0,y}) - 2\,D_{55}\,\mathrm{c}_1\,(\phi_x + 2\,w_{0,x}) \\
& - F_{45}\,\mathrm{c}_1^2\,(\phi_y + 2\,w_{0,y}) - F_{55}\,\mathrm{c}_1^2\,(\phi_x + 2\,w_{0,x}) = 0
\end{aligned}
$$

$$
\begin{aligned}
\delta \phi_y : \quad & B_{16}\,u_{0,xx} + B_{26}\,v_{0,yx} + B_{66}\,(u_{0,yx} + v_{0,xx}) + D_{16}\,\phi_{x,xx} + D_{26}\,\phi_{y,xy} + D_{66}\,(\phi_{x,yx} + \phi_{y,xx}) \\
& + F_{16}\,\mathrm{c}_2\,(\phi_{x,xx} + w_{0,xxx}) + F_{26}\,\mathrm{c}_2\,(\phi_{y,yx} + w_{0,yyx}) + F_{66}\,\mathrm{c}_2\,(\phi_{x,yx} + \phi_{y,xx} + 2\,w_{0,xyx}) \\
& + B_{12}\,u_{0,xy} + B_{22}\,v_{0,yy} + B_{26}\,(u_{0,yy} + v_{0,xy}) + D_{12}\,\phi_{x,xy} + D_{22}\,\phi_{y,yy} + D_{26}\,(\phi_{x,yy} + \phi_{y,xy}) \\
& + F_{12}\,\mathrm{c}_2\,(\phi_{x,xy} + w_{0,xxy}) + F_{22}\,\mathrm{c}_2\,(\phi_{y,yy} + w_{0,yyy}) + F_{26}\,\mathrm{c}_2\,(\phi_{x,yy} + \phi_{y,xy} + 2\,w_{0,xyy}) \\
& + E_{16}\,\mathrm{c}_2\,u_{0,xx} + E_{26}\,\mathrm{c}_2\,v_{0,yx} + E_{66}\,\mathrm{c}_2\,(u_{0,yx} + v_{0,xx}) + F_{16}\,\mathrm{c}_2\,\phi_{x,xx} + F_{26}\,\mathrm{c}_2\,\phi_{y,xy} + F_{66}\,\mathrm{c}_2\,(\phi_{x,yx} + \phi_{y,xx}) \\
& + H_{16}\,\mathrm{c}_2^2\,(\phi_{x,xx} + w_{0,xxx}) + H_{26}\,\mathrm{c}_2^2\,(\phi_{y,yx} + w_{0,yyx}) + H_{66}\,\mathrm{c}_2^2\,(\phi_{x,yx} + \phi_{y,xx} + 2\,w_{0,xyx}) \\
& + E_{12}\,\mathrm{c}_2\,u_{0,xy} + E_{22}\,\mathrm{c}_2\,v_{0,yy} + E_{26}\,\mathrm{c}_2\,(u_{0,yy} + v_{0,xy}) + F_{12}\,\mathrm{c}_2\,\phi_{x,xy} + F_{22}\,\mathrm{c}_2\,\phi_{y,yy} + F_{26}\,\mathrm{c}_2\,(\phi_{x,yy} + \phi_{y,xy}) \\
& + H_{12}\,\mathrm{c}_2^2\,(\phi_{x,xy} + w_{0,xxy}) + H_{22}\,\mathrm{c}_2^2\,(\phi_{y,yy} + w_{0,yyy}) + H_{26}\,\mathrm{c}_2^2\,(\phi_{x,yy} + \phi_{y,xy} + 2\,w_{0,xyy}) \\
& - A_{44}\,(\phi_y + 2\,w_{0,y}) - A_{45}\,(\phi_x + 2\,w_{0,x}) - 2\,D_{44}\,\mathrm{c}_1\,(\phi_y + 2\,w_{0,y}) - 2\,D_{45}\,\mathrm{c}_1\,(\phi_x + 2\,w_{0,x}) \\
& - F_{44}\,\mathrm{c}_1^2\,(\phi_y + 2\,w_{0,y}) - F_{45}\,\mathrm{c}_1^2\,(\phi_x + 2\,w_{0,x}) = 0
\end{aligned}
$$

The natural boundary conditions are:

$$
\begin{aligned}
\delta u_0:\quad \mathcal{N}_{xx} &= A_{11}\,u_{0,x} + B_{11}\,\phi_{x,x} + E_{11}\,\mathrm{c}_2\,\phi_{x,x} + E_{11}\,\mathrm{c}_2\,w_{0,xx} + A_{12}\,v_{0,y} + B_{12}\,\phi_{y,y} + E_{12}\,\mathrm{c}_2\,\phi_{y,y} + E_{12}\,\mathrm{c}_2\,w_{0,yy}\\
&\quad + A_{16}\,u_{0,y} + A_{16}\,v_{0,x} + B_{16}\,\phi_{x,y} + B_{16}\,\phi_{y,x} + E_{16}\,\mathrm{c}_2\,\phi_{x,y} + E_{16}\,\mathrm{c}_2\,\phi_{y,x} + 2\,E_{16}\,\mathrm{c}_2\,w_{0,xy}\\[2ex]
\delta v_0:\quad \mathcal{N}_{xy} &= A_{16}\,u_{0,x} + B_{16}\,\phi_{x,x} + E_{16}\,\mathrm{c}_2\,\phi_{x,x} + E_{16}\,\mathrm{c}_2\,w_{0,xx} + A_{26}\,v_{0,y} + B_{26}\,\phi_{y,y} + E_{26}\,\mathrm{c}_2\,\phi_{y,y} + E_{26}\,\mathrm{c}_2\,w_{0,yy}\\
&\quad + A_{66}\,u_{0,y} + A_{66}\,v_{0,x} + B_{66}\,\phi_{x,y} + E_{66}\,\mathrm{c}_2\,\phi_{y,x} + E_{66}\,\mathrm{c}_2\,\phi_{x,y} + E_{66}\,\mathrm{c}_2\,\phi_{y,x} + 2\,E_{66}\,\mathrm{c}_2\,w_{0,xy}\\[2ex]
\delta w_0:\quad \mathcal{Q}_{x} &= H_{11}\,\mathrm{c}_2^2\,\phi_{x,xx} + H_{11}\,\mathrm{c}_2^2\,w_{0,xxx} + E_{11}\,\mathrm{c}_2\,u_{0,xx} + F_{11}\,\mathrm{c}_2\,\phi_{x,xx} + E_{12}\,\mathrm{c}_2\,v_{0,yx} + F_{12}\,\mathrm{c}_2\,\phi_{y,yx}\\
&\quad + H_{12}\,\mathrm{c}_2^2\,\phi_{y,yx} + H_{12}\,\mathrm{c}_2^2\,w_{0,yyx} + 2\,E_{16}\,\mathrm{c}_2\,u_{0,xy} + 2\,F_{16}\,\mathrm{c}_2\,\phi_{x,xy} + 2\,H_{16}\mathrm{c}_2^2\,\phi_{x,xy} + E_{16}\,\mathrm{c}_2\,u_{0,yx}\\
&\quad + E_{16}\,\mathrm{c}_2\,v_{0,xx} + F_{16}\,\mathrm{c}_2\,\phi_{x,yx} + H_{16}\,\mathrm{c}_2^2\,\phi_{x,yx} + H_{16}\,\mathrm{c}_2^2\,\phi_{y,xx} + 2\,H_{16}\,\mathrm{c}_2^2\,w_{0,xxy} + 2\,E_{26}\,\mathrm{c}_2\,v_{0,yy}\\
&\quad + 2\,F_{26}\,\mathrm{c}_2\,\phi_{y,yy} + 2\,H_{26}\mathrm{c}_2^2\,w_{0,yyy} + 4\,H_{66}\mathrm{c}_2^2\,w_{0,xyy} + 2\,H_{26}\,\mathrm{c}_2^2\,\phi_{x,yy} + 2\,H_{26}\,\mathrm{c}_2^2\,\phi_{y,xy} + 2\,E_{66}\,\mathrm{c}_2\,u_{0,yy}\\
&\quad + 2\,E_{66}\,\mathrm{c}_2\,v_{0,xy} + 2\,F_{66}\,\mathrm{c}_2\,\phi_{x,yy} + 2\,F_{66}\,\mathrm{c}_2\,\phi_{y,xy} - 2\,D_{45}\,\mathrm{c}_1\,\phi_y - 2\,D_{45}\,\mathrm{c}_1\,w_{0,y} - F_{45}\,\mathrm{c}_1^2\,\phi_y\\
&\quad - F_{45}\,\mathrm{c}_1^2\,w_{0,y} - A_{55}\,\phi_x - A_{55}\,w_{0,x} - D_{55}\,\mathrm{c}_1\,\phi_x - 2\,\mathrm{c}_1\,w_{0,x} - F_{55}\,\mathrm{c}_1^2\,\phi_x - F_{55}\,\mathrm{c}_1^2\,w_{0,x}\\[2ex]
\delta \phi_y:\quad \mathcal{M}_{xx} &= D_{11}\,\phi_{x,x} + H_{11}\,\mathrm{c}_2^2\,\phi_{x,x} + H_{11}\,\mathrm{c}_2^2\,w_{0,xx} + B_{11}\,u_{0,x} + E_{11}\,\mathrm{c}_2\,u_{0,x} + 2\,F_{11}\,\mathrm{c}_2\,\phi_{x,x} + F_{11}\,\mathrm{c}_2\,w_{0,xx}\\
&\quad + F_{11}\,\mathrm{c}_2\,w_{0,xx} + B_{12}\,v_{0,y} + D_{12}\,\phi_{y,y} + F_{12}\,\mathrm{c}_2\,\phi_{y,y} + F_{12}\,\mathrm{c}_2\,w_{0,yy} + E_{12}\,\mathrm{c}_2\,v_{0,y} + F_{12}\,\mathrm{c}_2\,\phi_{y,y} + H_{12}\,\mathrm{c}_2^2\,\phi_{y,y}\\
&\quad + H_{12}\,\mathrm{c}_2^2\,w_{0,yy} + B_{16}\,u_{0,y} + B_{16}\,v_{0,x} + D_{16}\,\phi_{x,y} + D_{16}\,\phi_{y,x} + F_{16}\,\mathrm{c}_2\,\phi_{x,y} + F_{16}\,\mathrm{c}_2\,\phi_{y,x} + 2\,F_{16}\,\mathrm{c}_2\,w_{0,xy}\\
&\quad + E_{16}\,\mathrm{c}_2\,u_{0,y} + E_{16}\,\mathrm{c}_2\,v_{0,x} + F_{16}\,\mathrm{c}_2\,\phi_{x,y} + F_{16}\,\mathrm{c}_2\,\phi_{y,x} + H_{16}\,\mathrm{c}_2^2\,\phi_{x,y} + H_{16}\,\mathrm{c}_2^2\,\phi_{y,x} + 2\,H_{16}\,\mathrm{c}_2^2\,w_{0,xy}\\[2ex]
\delta \phi_x:\quad \mathcal{M}_{xy} &= D_{16}\,\phi_{x,x} + H_{16}\,\mathrm{c}_2^2\,\phi_{x,x} + H_{16}\,\mathrm{c}_2^2\,w_{0,xx} + B_{16}\,u_{0,x} + E_{16}\,\mathrm{c}_2\,u_{0,x} + 2\,F_{16}\,\mathrm{c}_2\,\phi_{x,x} + F_{16}\,\mathrm{c}_2\,w_{0,xx}\\
&\quad + F_{16}\,\mathrm{c}_2\,w_{0,xx} + B_{26}\,v_{0,y} + D_{12}\,\phi_{y,y} + F_{26}\,\mathrm{c}_2\,\phi_{y,y} + F_{26}\,\mathrm{c}_2\,w_{0,yy} + E_{26}\,\mathrm{c}_2\,v_{0,y} + F_{26}\,\mathrm{c}_2\,\phi_{y,y} + H_{26}\,\mathrm{c}_2^2\,\phi_{y,y}\\
&\quad + H_{26}\,\mathrm{c}_2^2\,w_{0,yy} + B_{66}\,u_{0,y} + B_{66}\,v_{0,x} + D_{66}\,\phi_{x,y} + D_{66}\,\phi_{y,x} + F_{66}\,\mathrm{c}_2\,\phi_{x,y} + F_{66}\,\mathrm{c}_2\,\phi_{y,x} + 2\,F_{66}\,\mathrm{c}_2\,w_{0,xy}\\
&\quad + E_{66}\,\mathrm{c}_2\,u_{0,y} + E_{66}\,\mathrm{c}_2\,v_{0,x} + F_{66}\,\mathrm{c}_2\,\phi_{x,y} + F_{66}\,\mathrm{c}_2\,\phi_{y,x} + H_{66}\,\mathrm{c}_2^2\,\phi_{x,y} + H_{66}\,\mathrm{c}_2^2\,\phi_{y,x} + 2\,H_{66}\,\mathrm{c}_2^2\,w_{0,xy}\\[2ex]
\delta w_{0,x}:\quad \mathcal{P}_{xx} &= H_{11}\,\mathrm{c}_2^2\,\phi_{x,x} + H_{11}\,\mathrm{c}_2^2\,w_{0,xx} + E_{11}\,\mathrm{c}_2\,u_{0,x} + F_{11}\,\mathrm{c}_2\,\phi_{x,x} + E_{12}\,\mathrm{c}_2\,v_{0,y} + F_{12}\,\mathrm{c}_2\,\phi_{y,y} + H_{12}\,\mathrm{c}_2^2\,\phi_{y,y}\\
&\quad + H_{12}\,\mathrm{c}_2^2\,w_{0,yy} + E_{16}\,\mathrm{c}_2\,u_{0,y} + E_{16}\,\mathrm{c}_2\,v_{0,x} + F_{16}\,\mathrm{c}_2\,\phi_{x,y} + F_{16}\,\mathrm{c}_2\,\phi_{y,x} + H_{16}\,\mathrm{c}_2^2\,\phi_{x,y} + H_{16}\,\mathrm{c}_2^2\,\phi_{y,x}\\
&\quad + 2\,H_{16}\,\mathrm{c}_2^2\,w_{0,xy}
\end{aligned}
\tag{10}
$$

where the suffix after the comma denotes the partial derivative with respect to that variable and

$$
\begin{aligned}
(A_{ij}, B_{ij}, D_{ij}, E_{ij}, F_{ij}, H_{ij}) &= \sum_{k=1}^{N_l} \int_{z^k} \tilde{C}_{ij}^k \left(1\ z,\ z^2,\ z^3,\ z^4,\ z^6\right)\, dz\\
(I_0, I_1, I_2, I_3, I_4, I_6) &= \sum_{k=1}^{N_l} \int_{z^k} \rho^k \left(1\ z,\ z^2,\ z^3,\ z^4,\ z^6\right)\, dz
\end{aligned}
\tag{11}
$$

are laminate stiffnesses and rotatory inertial terms, respectively with i and j varying form 1 to 6. The in-plane loadings can be defined as $\tilde{N}_{x_0} = \lambda\, N_{x_0}$ and $\tilde{N}_{y_0} = \lambda\, N_{y_0}$, where N_{x_0}, N_{y_0} are the initial in-plane loadings and λ is a scalar load factor, c_1 has already been defined (see Eq. (37)) and $c_2 = -\frac{4}{h^2}$.

2.2 Dynamic stiffness method

Once the equations of motion and the natural boundary conditions, i.e., Eqs. (9) and (10) above are obtained, the classical method to carry out an exact buckling analysis of a plate consists of (i) solving the system of differential equations in Navier or Lèvy-type closed form in an exact manner, (ii) applying particular boundary conditions on the edges and finally (iii) obtaining the stability equation by eliminating the integration constants [20–23]. This method, although extremely useful for analysing an individual plate, it lacks generality and cannot be easily applied to complex structures assembled from plates for which researchers usually resort to approximate methods such as the FEM. In this respect, the dynamic stiffness method (DSM), which is, in many ways, analogous to FEM has no such limitations and importantly it always retains the exactness of the solution even when applied to complex structures. This is because once the dynamic stiffness matrix of a structural element is obtained from the exact solution of the governing differential equations and it can be offset and/or rotated and assembled in a global DS matrix in the same way as the FEM. This global DS matrix thus contains implicitly all the exact critical buckling loads of the structure which can be computed by using the well established algorithm of Wittrick-Williams [13].
A general procedure to develop the dynamic stiffness matrix of a structural element is generally summarized as follows:

(i) Seek a closed form analytical solution of the governing differential equations of the structural element.

(ii) Apply a number of general boundary conditions in algebraic forms that are equal to twice the number of integration constants; these are usually nodal displacements and forces.

(iii) Eliminate the integration constants by relating the amplitudes of the harmonically varying nodal forces to those of the corresponding displacements which essentially generates the dynamic stiffness matrix, providing the force-displacement relationship at the nodes of the structural element.

Referring to the equations of motion Eqs.(9), an exact solution can be found in Lèvy's form for symmetric, cross ply laminates. For such laminates $\boldsymbol{B} = \boldsymbol{E} = 0$, and $\tilde{C}^k_{16} = \tilde{C}^k_{26} = \tilde{C}^k_{45} = 0$ and the out-of-plane displacements are uncoupled from the in-plane ones.

2.3 Lèvy-type closed form exact solution and DS formulation

The solution of Eqs. (9) related to the out-of-plane displacements is sought as:

$$\begin{aligned} w^0(x,y,t) &= \sum_{m=1}^{\infty} W_m(x)\, \mathrm{e}^{i\omega t} \sin(\alpha\, y), \qquad \phi_x(x,y,t) = \sum_{m=1}^{\infty} \Phi_{x_m}(x)\, \mathrm{e}^{i\omega t} \sin(\alpha\, y), \\ \phi_y(x,y,t) &= \sum_{m=1}^{\infty} \Phi_{y_m}(x)\, \mathrm{e}^{i\omega t} \cos(\alpha\, y) \end{aligned} \tag{12}$$

where ω is the unknown circular or angular frequency, $\alpha = \frac{m\pi}{L}$ and $m = 1, 2, \ldots, \infty$. Equation (12) is the so-called Lèvy's solution which assumes that two opposite sides of the plate are simply supported (S-S), i.e. $w = \phi_x = 0$ at $y = 0$ and $y = L$. Substituting Eq. (12) into Eqs. (9) a set of three ordinary differential equations is derived which can be written in matrix form as follows:

$$\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix} \begin{bmatrix} W_m \\ \Phi_x \\ \Phi_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{13}$$

where $\mathcal{L}_{ij}$ $(i, j = 1, 2, 3)$ are differential operators and given by:

$$\begin{aligned}
\mathcal{L}_{11} &= -c_1^2\mathcal{D}_x^4 H_{11} + \mathcal{D}_x^2 \left(A_{55} + 2c_2D_{55} + c_2^2F_{55} + 2\alpha^2c_1^2H_{12} + 4\alpha^2c_1^2H_{66} + \lambda N_{x0}\right) - \alpha^2\Big(A_{44} + 2c_2D_{44} \\
&\quad + c_2^2F_{44} + \alpha^2c_1^2H_{22} + \lambda N_{y0}\Big) \\
\mathcal{L}_{12} &= \mathcal{D}_x^3\left(-c_1F_{11} - c_1^2H_{11}\right) + \mathcal{D}_x\left(A_{55} + 2c_2D_{55} + \alpha^2c_1F_{12} + c_2^2F_{55} + 2\alpha^2c_1F_{66} + \alpha^2c_1^2H_{12} + 2\alpha^2c_1^2H_{66}\right) \\
&\quad + \left(-c_1\,F_{11} - c_1^2\,H_{11}\right)\mathcal{D}_x^3 \\
\mathcal{L}_{13} &= -\alpha\left(A_{44} + c_2(2D_{44} + c_2F_{44}) + \alpha^2c_1(F_{22} + c_1H_{22})\right) + \mathcal{D}_x^2\left(\alpha c_1F_{12} + 2\alpha c_1F_{66} + \alpha c_1^2H_{12} + 2\alpha c_1^2H_{66}\right) \\
\mathcal{L}_{21} &= c_1\mathcal{D}_x^3(F_{11} + c_1H_{11}) + \mathcal{D}_x\left(-A_{55} - c_2(2D_{55} + c_2F_{55}) - \alpha^2c_1(F_{12} + 2F_{66} + c_1H_{12} + 2c_1H_{66})\right) \\
\mathcal{L}_{22} &= -A_{55} - c_2(2D_{55} + c_2F_{55}) + \mathcal{D}_x^2\left(D_{11} + 2c_1F_{11} + c_1^2H_{11}\right) - \alpha^2\left(D_{66} + 2c_1F_{66} + c_1^2H_{66}\right) \\
\mathcal{L}_{23} &= \mathcal{D}_x\left(-\alpha D_{12} - \alpha D_{66} - 2\alpha c_1F_{12} - 2\alpha c_1F_{66} - \alpha c_1^2H_{12} - \alpha c_1^2H_{66}\right) \\
\mathcal{L}_{31} &= -\alpha\left(A_{44} + c_2(2D_{44} + c_2F_{44}) + \alpha^2c_1(F_{22} + c_1H_{22})\right) + \alpha c_1\mathcal{D}_x^2(F_{12} + 2F_{66} + c_1H_{12} + 2c_1H_{66}) \\
\mathcal{L}_{32} &= \alpha\mathcal{D}_x(D_{12} + D_{66} + c_1(2F_{12} + 2F_{66} + c_1(H_{12} + H_{66}))) \\
\mathcal{L}_{33} &= -A_{44} - c_2(2D_{44} + c_2F_{44}) - \alpha^2(D_{22} + c_1(2F_{22} + c_1H_{22})) + \mathcal{D}_x^2(D_{66} + c_1(2F_{66} + c_1H_{66}))
\end{aligned} \tag{14}$$

where $\mathcal{D}_x = \frac{\mathrm{d}}{\mathrm{dx}}$ and A_{ij}, B_{ij}, C_{ij}, D_{ij}, E_{ij}, F_{ij}, H_{ij} have already been defined in Eq. (57). Expanding the determinant of the matrix in Eq. (13) the following differential equation is obtained:

$$\left(a_1\,\mathcal{D}_x^8 + a_2\,\mathcal{D}_x^6 + a_3\,\mathcal{D}_x^4 + a_4\,\mathcal{D}_x^2 + a_5\right)\Psi = 0 \tag{15}$$

where

$$\Psi = W_m,\ \Phi_{y_m},\ \Phi_{x_m} \tag{16}$$

Using a trial solution e^{λ} in Eq. (63) yields the following auxiliary equation:

$$a_1\,\lambda^8 + a_2\,\lambda^6 + a_3\,\lambda^4 + a_4\,\lambda^2 + a_5\ = 0 \tag{17}$$

Substituting $\mu = \lambda^2$, the 8^{th} order polynomial of Eq. (17) can be reduced to a quartic as:

$$a_1\,\mu^4 + a_2\,\mu^3 + a_3\,\mu^2 + a_4\,\mu + a_5 = 0 \tag{18}$$

the four roots for the quartic equation are given by:

$$
\begin{aligned}
\mu_1 &= -s_1 - \frac{1}{2}\sqrt{-s_5 + s_2 - \frac{s_8}{4\sqrt{s_9}} - \frac{s_6}{3\,a_1\,s_7}} - \frac{1}{2}\sqrt{s_9} \\
\mu_2 &= -s_1 + \frac{1}{2}\sqrt{-s_5 + s_2 - \frac{s_8}{4\sqrt{s_9}} - \frac{s_6}{3\,a_1\,s_7}} - \frac{1}{2}\sqrt{s_9} \\
\mu_3 &= -s_1 - \frac{1}{2}\sqrt{-s_5 + s_2 + \frac{s_8}{4\sqrt{s_9}} - \frac{s_6}{3\,a_1\,s_7}} + \frac{1}{2}\sqrt{s_9} \\
\mu_4 &= -s_1 + \frac{1}{2}\sqrt{-s_5 + s_2 + \frac{s_8}{4\sqrt{s_9}} - \frac{s_6}{3\,a_1\,s_7}} + \frac{1}{2}\sqrt{s_9}
\end{aligned}
\tag{19}
$$

where

$$
\begin{aligned}
&s_1 = \frac{a_1}{4\,a_2}, \quad s_2 = -\frac{4\,a_3}{3\,a_1^2} + \frac{a_2^2}{2\,a_1^2}, \quad s_3 = 2\,a_3 - 9\,a_2\,a_3\,a_4 - 72\,a_2^2\,a_5 + 27\,a_2^2\,a_5 + 27\,a_1\,a_4^2, \\
&s_4 = a_3^2 - 3\,a_2\,a_4 + 12\,a_1\,a_5, \quad s_5 = \frac{1}{a_1}\left(\frac{s_3 + \sqrt{s_3 - 4\,s_4}}{32}\right)^{\frac{1}{3}}, \\
&s_6 = \sqrt{2^2}^{\,3}\,s_4, \quad s_7 = \sqrt{32}^{\,3}\,s_5\,a_1, \quad s_8 = \left(\frac{a_2}{a_1}\right)^3 \frac{4\,a_2\,a_3}{a_1^2} - \frac{8\,a_4}{a_1}, \quad s_9 = s_5 + \frac{s_2}{2} + \frac{s_6}{3\,s_7\,a_1}
\end{aligned}
\tag{20}
$$

The explicit form of the polynomials coefficients $a_j \quad (j = 1, 2, 3, 4, 5)$ are given in Appendix B. Some pair or pairs of complex roots may occur when computing $\mu_j \quad (j = 1, 2, 3, 4)$, but the amplitude of the displacements $W_m(x)$, $\Phi_{x_m}(x)$, $\Phi_{y_m}(x)$ are all real, whilst the associated coefficients can be complex. As complex roots always occur in conjugate pairs, the associated coefficients will also occur as conjugates. The solution of the system of ordinary differential equations in Eq. (13) can thus be written as:

$$
\begin{aligned}
W_m(x) &= A_1\,e^{+\mu_1 x} + A_2\,e^{-\mu_1 x} + A_3\,e^{+\mu_2 x} + A_4\,e^{-\mu_2 x} \\
&\quad + A_5\,e^{+\mu_3 x} + A_6\,e^{-\mu_3 x} + A_7\,e^{+\mu_4 x} + A_8\,e^{-\mu_4 x} \\[1em]
\Phi_{x_m}(x) &= B_1\,e^{+\mu_1 x} + B_2\,e^{-\mu_1 x} + B_3\,e^{+\mu_2 x} + B_4\,e^{-\mu_2 x} \\
&\quad + B_5\,e^{+\mu_3 x} + B_6\,e^{-\mu_3 x} + B_7\,e^{+\mu_4 x} + B_8\,e^{-\mu_4 x} \\[1em]
\Phi_{y_m}(x) &= C_1\,e^{+\mu_1 x} + C_2\,e^{-\mu_1 x} + C_3\,e^{+\mu_2 x} + C_4\,e^{-\mu_2 x} \\
&\quad + C_5\,e^{+\mu_3 x} + C_6\,e^{-\mu_3 x} + C_7\,e^{+\mu_4 x} + C_8\,e^{-\mu_4 x}
\end{aligned}
\tag{21}
$$

where $A_1 - A_8$, $B_1 - B_8$, $C_1 - C_8$, are three sets of integration constants. The sets of constants are not all independent. Only one set of eight constants are needed to relate each set. Constants $B_1 - B_8$ are chosen to be the independent base. By substituting

Eqs. (66) into (13) the following relationships are obtained using symbolic computation:

$$\begin{array}{llll} A_1 = \delta_1\, B_1, & A_2 = -\delta_1\, B_2, & C_1 = \gamma_1\, B_1, & C_2 = -\gamma_1\, B_2 \\ A_3 = \delta_2\, B_3, & A_4 = -\delta_2\, B_4, & C_3 = \gamma_2\, B_3, & C_4 = -\gamma_2\, B_4 \\ A_5 = \delta_3\, B_5, & A_6 = -\delta_3\, B_6, & C_5 = \gamma_3\, B_5, & C_6 = -\gamma_3\, B_6 \\ A_7 = \delta_4\, B_7, & A_8 = -\delta_4\, B_8, & C_7 = \gamma_4\, B_7, & C_8 = -\gamma_4\, B_8 \end{array} \tag{22}$$

where

$$\begin{aligned}
\delta_i = -\Big[& - A_{55}\alpha^2 D_{22} - 2A_{55}c_2 D_{44} - 2\alpha^2 c_2 D_{22} D_{55} - 4c_2^2 D_{44} D_{55} - \alpha^4 D_{22} D_{66} - 2\alpha^2 c_2 D_{44} D_{66} - 2A_{55}\alpha^2 c_1 F_{22} \\
& - 4\alpha^2 c_1 c_2 D_{55} F_{22} - 2\alpha^4 c_1 D_{66} F_{22} - A_{55} c_2^2 F_{44} - 2c_2^3 D_{55} F_{44} - \alpha^2 c_2^2 D_{66} F_{44} - \alpha^2 c_2^2 D_{22} F_{55} - 2c_2^3 D_{44} F_{55} \\
& - 2\alpha^2 c_1 c_2^2 F_{22} F_{55} - c_2^4 F_{44} F_{55} - 2\alpha^4 c_1 D_{22} F_{66} - 4\alpha^2 c_1 c_2 D_{44} F_{66} - 4\alpha^4 c_1^2 F_{22} F_{66} - 2\alpha^2 c_1 c_2^2 F_{44} F_{66} \\
& - A_{55}\alpha^2 c_1^2 H_{22} - 2\alpha^2 c_1^2 c_2 D_{55} H_{22} - \alpha^4 c_1^2 D_{66} H_{22} - \alpha^2 c_1^2 c_2^2 F_{55} H_{22} - 2\alpha^4 c_1^3 F_{66} H_{22} - \alpha^4 c_1^2 D_{22} H_{66} - 2\alpha^2 c_1^2 c_2 D_{44} H_{66} \\
& - 2\alpha^4 c_1^3 F_{22} H_{66} - \alpha^2 c_1^2 c_2^2 F_{44} H_{66} - \alpha^4 c_1^4 H_{22} H_{66} - A_{44}\Big(A_{55} + 2c_2 D_{55} + c_2^2 F_{55} + \alpha^2 (D_{66} + 2c_1 F_{66} + c_1^2 H_{66})\Big) \\
& + A_{44}(D_{11} + c_1(2F_{11} + c_1 H_{11}))\mu_i^2 + \Big(A_{55}(D_{66} + c_1(2F_{66} + c_1 H_{66})) + 2c_2(D_{11} D_{44} + D_{55} D_{66} + c_1(2D_{44} F_{11} \\
& + 2D_{55} F_{66} + c_1 D_{44} H_{11} + c_1 D_{55} H_{66})) + c_2^2(D_{11} F_{44} + D_{66} F_{55} + c_1(2F_{11} F_{44} + 2F_{55} F_{66} + c_1 F_{44} H_{11} + c_1 F_{55} H_{66})) \\
& - \alpha^2(D_{12}^2 - D_{11}(D_{22} + c_1(2F_{22} + c_1 H_{22})) + 2D_{12}(D_{66} + c_1(2F_{12} + 2F_{66} + c_1(H_{12} + H_{66}))) + c_1(4F_{12}(D_{66} + c_1 F_{12}) \\
& - D_{22}(2F_{11} + c_1 H_{11}) + c_1(8F_{12} F_{66} + 2D_{66} H_{12} - 2F_{11}(2F_{22} + c_1 H_{22}) + c_1(-2F_{22} H_{11} + H_{12}(4(F_{12} + F_{66}) + c_1 H_{12}) \\
& - c_1 H_{11} H_{22} + 4F_{12} H_{66} + 2c_1 H_{12} H_{66}))))\Big)\mu_i^2 - (D_{11} + c_1(2F_{11} + c_1 H_{11}))(D_{66} + c_1(2F_{66} + c_1 H_{66}))\mu_i^4\Big] \\
/\Big[& \alpha^2(D_{12} + D_{66} + c_1(2F_{12} + 2F_{66} + c_1(H_{12} + H_{66})))\mu_i\Big(A_{44} + 2c_2 D_{44} + c_2^2 F_{44} + \alpha^2 c_1(F_{22} + c_1 H_{22}) \\
& - c_1(F_{12} + 2F_{66} + c_1 H_{12} + 2c_1 H_{66})\mu_i^2\Big) - \mu_i\Big(A_{55} + 2c_2 D_{55} + c_2^2 F_{55} + \alpha^2 c_1(F_{12} + 2F_{66} + c_1 H_{12} + 2c_1 H_{66}) \\
& - c_1(F_{11} + c_1 H_{11})\mu_i^2\Big)\Big(A_{44} + c_2(2D_{44} + c_2 F_{44}) + \alpha^2(D_{22} + 2c_1 F_{22} + c_1^2 H_{22}) - (D_{66} + 2c_1 F_{66} + c_1^2 H_{66})\mu_i^2\Big)\Big]
\end{aligned}$$

$$\begin{aligned}
\gamma_i = \Big[& \alpha\Big(A_{44} A_{55} + 2A_{55} c_2 D_{44} + 2A_{44} c_2 D_{55} + 4c_2^2 D_{44} D_{55} + A_{44}\alpha^2 D_{66} + 2\alpha^2 c_2 D_{44} D_{66} + A_{55}\alpha^2 c_1 F_{22} + 2\alpha^2 c_1 c_2 D_{55} F_{22} \\
& + \alpha^4 c_1 D_{66} F_{22} + A_{55} c_2^2 F_{44} + 2c_2^3 D_{55} F_{44} + \alpha^2 c_2^2 D_{66} F_{44} + A_{44} c_2^2 F_{55} + 2c_2^3 D_{44} F_{55} + \alpha^2 c_1 c_2^2 F_{22} F_{55} + c_2^4 F_{44} F_{55} \\
& + 2A_{44}\alpha^2 c_1 F_{66} + 4\alpha^2 c_1 c_2 D_{44} F_{66} + 2\alpha^4 c_1^2 F_{22} F_{66} + 2\alpha^2 c_1 c_2^2 F_{44} F_{66} + A_{55}\alpha^2 c_1^2 H_{22} + 2\alpha^2 c_1^2 c_2 D_{55} H_{22} + \alpha^4 c_1^2 D_{66} H_{22} \\
& + \alpha^2 c_1^2 c_2^2 F_{55} H_{22} + 2\alpha^4 c_1^3 F_{66} H_{22} + A_{44}\alpha^2 c_1^2 H_{66} + 2\alpha^2 c_1^2 c_2 D_{44} H_{66} + \alpha^4 c_1^3 F_{22} H_{66} + \alpha^2 c_1^2 c_2^2 F_{44} H_{66} + \alpha^4 c_1^4 H_{22} H_{66} \\
& + \Big(A_{55}(D_{12} + D_{66}) - A_{44}(D_{11} + c_1(2F_{11} + c_1 H_{11})) - 2c_2(D_{11} D_{44} - D_{55}(D_{12} + D_{66}) + c_1(2D_{44} F_{11} - D_{55} F_{12} + c_1 D_{44} H_{11} \\
& + c_1 D_{55} H_{66})) + c_2^2(-D_{11} F_{44} + (D_{12} + D_{66})F_{55} - c_1(2F_{11} F_{44} - F_{12} F_{55} + c_1 F_{44} H_{11} + c_1 F_{55} H_{66})) + c_1(A_{55}(F_{12} - c_1 H_{66}) \\
& + \alpha^2(-D_{11} F_{22} + D_{12}(F_{12} + 2F_{66} + c_1(H_{12} + 2H_{66})) + c_1(2F_{12}^2 - D_{11} H_{22} - 2F_{11}(F_{22} + c_1 H_{22}) + F_{12}(4F_{66} + 3c_1 H_{12} \\
& + 4c_1 H_{66}) + c_1(H_{12}(2F_{66} + c_1 H_{12}) - H_{11}(F_{22} + c_1 H_{22}) + 2c_1 H_{12} H_{66}))))\Big)\mu_i^2 + c_1(D_{11}(F_{12} + 2F_{66}) - D_{12}(F_{11} + c_1 H_{11}) \\
& - D_{66}(F_{11} + c_1 H_{11}) + c_1(D_{11}(H_{12} + 2H_{66}) + c_1 H_{11}(-F_{12} + c_1 H_{66}) + F_{11}(2F_{66} + c_1(H_{12} + 3H_{66}))))\mu_i^4\Big)\Big] \\
/\Big[& \mu_i\Big(A_{44} A_{55} - A_{44}\alpha^2 D_{12} + A_{55}\alpha^2 D_{22} + 2A_{55} c_2 D_{44} - 2\alpha^2 c_2 D_{12} D_{44} + 2A_{44} c_2 D_{55} + 2\alpha^2 c_2 D_{22} D_{55} + 4c_2^2 D_{44} D_{55} \\
& - A_{44}\alpha^2 D_{66} - 2\alpha^2 c_2 D_{44} D_{66} - A_{44}\alpha^2 c_1 F_{12} + \alpha^4 c_1 D_{22} F_{12} - 2\alpha^2 c_1 c_2 D_{44} F_{12} + 2A_{55}\alpha^2 c_1 F_{22} - \alpha^4 c_1 D_{12} F_{22} + 4\alpha^2 c_1 c_2 D_{55} F_{22} \\
& - \alpha^4 c_1 D_{66} F_{22} + A_{55} c_2^2 F_{44} - \alpha^2 c_2^2 D_{12} F_{44} + 2c_2^3 D_{55} F_{44} - \alpha^2 c_2^2 D_{66} F_{44} - \alpha^2 c_1 c_2^2 F_{12} F_{44} + A_{44} c_2^2 F_{55} + \alpha^2 c_2^2 D_{22} F_{55} \\
& + 2c_2^3 D_{44} F_{55} + 2\alpha^2 c_1 c_2^2 F_{22} F_{55} + c_2^4 F_{44} F_{55} + 2\alpha^4 c_1 D_{22} F_{66} + 2\alpha^4 c_1^2 F_{22} F_{66} + \alpha^4 c_1^2 D_{22} H_{12} + \alpha^4 c_1^3 F_{22} H_{12} + A_{55}\alpha^2 c_1^2 H_{22} \\
& - \alpha^4 c_1^2 D_{12} H_{22} + 2\alpha^2 c_1^2 c_2 D_{55} H_{22} - \alpha^4 c_1^2 D_{66} H_{22} - \alpha^4 c_1^3 F_{12} H_{22} + \alpha^2 c_1^2 c_2^2 F_{55} H_{22} + A_{44}\alpha^2 c_1^2 H_{66} + 2\alpha^4 c_1^2 D_{22} H_{66} \\
& + 2\alpha^2 c_1^2 c_2 D_{44} H_{66} + 3\alpha^4 c_1^3 F_{22} H_{66} + \alpha^2 c_1^2 c_2^2 F_{44} H_{66} + \alpha^4 c_1^4 H_{22} H_{66} - \Big(A_{55}(D_{66} + c_1(2F_{66} + c_1 H_{66})) + c_2^2(D_{66} F_{55} \\
& + c_1(F_{11} F_{44} + 2F_{55} F_{66} + c_1 F_{44} H_{11} + c_1 F_{55} H_{66})) + 2c_2(c_1 D_{44}(F_{11} + c_1 H_{11}) + D_{55}(D_{66} + c_1(2F_{66} + c_1 H_{66}))) \\
& + c_1(A_{44}(F_{11} + c_1 H_{11}) + \alpha^2(-F_{12}(D_{12} + 2c_1 F_{12}) - 2D_{12} F_{66} + D_{22}(F_{11} + c_1 H_{11}) + c_1(-4F_{12} F_{66} + F_{11}(2F_{22} + c_1 H_{22}) \\
& - D_{12}(H_{12} + 2H_{66}) + c_1(2F_{22} H_{11} - H_{12}(3F_{12} + 2F_{66} + c_1 H_{12}) + c_1 H_{11} H_{22} - 4F_{12} H_{66} - 2c_1 H_{12} H_{66}))))\Big)\mu_i^2 \\
& + c_1(F_{11} + c_1 H_{11})(D_{66} + c_1(2F_{66} + c_1 H_{66}))\mu_i^4\Big)\Big]
\end{aligned} \tag{23}$$

with $i = 1, 2, 3, 4$. The procedure leading to Eqs. (22) and (23) must be undertaken with sufficient care, because if wrong equations are chosen from Eq. (66) to obtain the relationship connecting different sets of constant, numerical instability can occur. When

Eqs. (22) are substituted into Eqs. (66) a solution in terms of only 8 constants can be formulated for $W_m(x)$, $\Phi_{x_m}(x)$ and $\Phi_{y_m}(x)$, respectively. Thus

$$\begin{aligned}
W_m(x) &= B_1\,\delta_1\,\mathrm{e}^{+\mu_1 x} - B_2\,\delta_1\,\mathrm{e}^{-\mu_1 x} + B_3\,\delta_2\,\mathrm{e}^{+\mu_2 x} - B_4\,\delta_2\,\mathrm{e}^{-\mu_2 x}\\
&\quad + B_5\,\delta_3\,\mathrm{e}^{+\mu_3 x} - B_6\,\delta_3\,\mathrm{e}^{-\mu_3 x} + B_7\,\delta_4\,\mathrm{e}^{+\mu_4 x} - B_8\,\delta_4\,\mathrm{e}^{-\mu_4 x}\\[2ex]
\Phi_{x_m}(x) &= B_1\;\mathrm{e}^{+\mu_1 x} + B_2\;\mathrm{e}^{-\mu_1 x} + B_3\;\mathrm{e}^{+\mu_2 x} + B_4\;\mathrm{e}^{-\mu_2 x}\\
&\quad + B_5\;\mathrm{e}^{+\mu_3 x} + B_6\;\mathrm{e}^{-\mu_3 x} + B_7\;\mathrm{e}^{+\mu_4 x} + B_8\;\mathrm{e}^{-\mu_4 x}\\[2ex]
\Phi_{y_m}(x) &= B_1\,\gamma_1\,\mathrm{e}^{+\mu_1 x} - B_2\,\gamma_1\,\mathrm{e}^{-\mu_1 x} + B_3\,\gamma_2\,\mathrm{e}^{+\mu_2 x} - B_4\,\gamma_2\,\mathrm{e}^{-\mu_2 x}\\
&\quad + B_5\,\gamma_3\,\mathrm{e}^{+\mu_3 x} - B_6\,\gamma_3\,\mathrm{e}^{-\mu_3 x} + B_7\,\gamma_4\,\mathrm{e}^{+\mu_4 x} - B_8\,\gamma_4\,\mathrm{e}^{-\mu_4 x}
\end{aligned} \tag{24}$$

The expressions for forces and moments can also be found in the same way by substituting Eqs. (69) into Eqs. (10) and using symbolic computation. In this way

$$\begin{aligned}
Q_x(x,y) = \Big(&\mathrm{e}^{\mu_1 x}\,(B_1 + B_2\,\mathrm{e}^{-2\,\mu_1 x})(A_{55} + A_{55}\,\delta_1\,\mu_1 + 2\,c_2\,(D_{55} + D_{55}\,\delta_1\,\mu_1) + c_2^2\,(F_{55} + \delta_1\,F_{55}\,\mu_1)\\
&+ c_1\,(\alpha\,\gamma_1\,(F_{12} + 2\,F_{66} + c_1\,H_{12} + 2\,c_1\,H_{66})\,\mu_1 - \mu_1^2\,(F_{11} + c_1\,H_{11} + c_1\,\delta_1\,H_{11}\,\mu_1) + \alpha^2\,(2\,F_{66}\\
&+ 2\,c_1\,H_{66} + c_1\,\delta_1\,H_{12}\,\mu_1 + 4\,c_1\,\delta_1\,H_{66}\,\mu_1)))+\\
&\mathrm{e}^{\mu_2 x}(B_3 + B_4\,\mathrm{e}^{-2\,\mu_2 x})(A_{55} + A_{55}\,\delta_2\,\mu_2 + 2\,c_2\,(D_{55} + D_{55}\,\delta_2\,\mu_2) + c_2^2\,(F_{55} + \delta_2\,F_{55}\,\mu_2)\\
&+ c_1\,(\alpha\,\gamma_2\,(F_{12} + 2\,F_{66} + c_1\,H_{12} + 2\,c_1\,H_{66})\,\mu_2 - \mu_2^2\,(F_{11} + c_1\,H_{11} + c_1\,\delta_2\,H_{11}\,\mu_2) + \alpha^2\,(2\,F_{66}\\
&+ 2\,c_1\,H_{66} + c_1\,\delta_2\,H_{12}\,\mu_2 + 4\,c_1\,\delta_1\,H_{66}\,\mu_2)))+\\
&\mathrm{e}^{\mu_3 x}(B_5 + B_6\mathrm{e}^{-2\,\mu_3 x})(A_{55} + A_{55}\,\delta_3\,\mu_3 + 2\,c_2\,(D_{55} + D_{55}\,\delta_3\,\mu_3) + c_2^2\,(F_{55} + \delta_3\,F_{55}\,\mu_3)\\
&+ c_1\,(\alpha\,\gamma_3\,(F_{12} + 2\,F_{66} + c_1\,H_{12} + 2\,c_1\,H_{66})\,\mu_3 - \mu_3^2\,(F_{11} + c_1\,H_{11} + c_1\,\delta_3\,H_{11}\,\mu_3) + \alpha^2\,(2\,F_{66}\\
&+ 2\,c_1\,H_{66} + c_1\,\delta_3\,H_{12}\,\mu_3 + 4\,c_1\,\delta_3\,H_{66}\,\mu_3)))+\\
&\mathrm{e}^{\mu_4 x}(B_3 + B_4\,\mathrm{e}^{-2\,\mu_4 x})(A_{55} + A_{55}\,\delta_4\,\mu_4 + 2\,c_2\,(D_{55} + D_{55}\,\delta_4\,\mu_4) + c_2^2\,(F_{55} + \delta_4\,F_{55}\,\mu_4)\\
&+ c_1\,(\alpha\,\gamma_4\,(F_{12} + 2\,F_{66} + c_1\,H_{12} + 2\,c_1\,H_{66})\,\mu_4 - \mu_4^2\,(F_{11} + c_1\,H_{11} + c_1\,\delta_4\,H_{11}\,\mu_4) + \alpha^2\,(2\,F_{66}\\
&+ 2\,c_1\,H_{66} + c_1\,\delta_4\,H_{12}\,\mu_4 + 4\,c_1\,\delta_4\,H_{66}\,\mu_4)))\Big)\;\sin(\alpha\,y) = \mathcal{Q}_x\;\sin(\alpha\,y)
\end{aligned}$$

$$\begin{aligned}
M_{xx}(x,y) = \Big(&\mathrm{e}^{\mu_1 x}(B_1 + B_2\,\mathrm{e}^{-2\,\mu_1 x})(\alpha^2\,c_1\,\delta_1\,(F_{12} + c_1\,H_{12}) + \alpha\,\gamma_1\,(D_{12} + c_1\,(2\,F_{12} + c_1\,H_{12})) - \mu_1\,(D_{11}\\
&+ c_1\,(2\,F_{11} + c_1\,H_{11} + \delta_1\,F_{11}\,\mu_1 + c_1\,\delta_1\,H_{11}\,\mu_1)))+\\
&\mathrm{e}^{\mu_2 x}(B_3 + B_4\,\mathrm{e}^{-2\,\mu_2 x})(\alpha^2\,c_1\,\delta_2\,(F_{12} + c_1\,H_{12}) + \alpha\,\gamma_2\,(D_{12} + c_1\,(2\,F_{12} + c_1\,H_{12})) - \mu_2\,(D_{11}\\
&+ c_1\,(2\,F_{11} + c_1\,H_{11} + \delta_2\,F_{11}\,\mu_2 + c_1\,\delta_2\,H_{11}\,\mu_2)))+\\
&\mathrm{e}^{\mu_3 x}(B_5 + B_5\,\mathrm{e}^{-2\,\mu_3 x})(\alpha^2\,c_1\,\delta_3\,(F_{12} + c_1\,H_{12}) + \alpha\,\gamma_3\,(D_{12} + c_1\,(2\,F_{12} + c_1\,H_{12})) - \mu_3\,(D_{11}\\
&+ c_1\,(2\,F_{11} + c_1\,H_{11} + \delta_3\,F_{11}\,\mu_3 + c_1\,\delta_3\,H_{11}\,\mu_3)))+\\
&\mathrm{e}^{\mu_4 x}\,(B_7 + B_8\,\mathrm{e}^{-2\,\mu_4 x})(\alpha^2\,c_1\,\delta_4\,(F_{12} + c_1\,H_{12}) + \alpha\,\gamma_4\,(D_{12} + c_1\,(2\,F_{12} + c_1\,H_{12})) - \mu_4\,(D_{11}\\
&+ c_1\,(2\,F_{11} + c_1\,H_{11} + \delta_4\,F_{11}\,\mu_4 + c_1\,\delta_4\,H_{11}\,\mu_4)))\Big)\;\sin(\alpha\,y) = \mathcal{M}_{xx}\;\sin(\alpha\,y)
\end{aligned}$$

$$
\begin{aligned}
M_{xy}\,(x,y) = \Big(&\mathrm{e}^{\mu_1 x}\,(B_1 + B_2\,\mathrm{e}^{-2\mu_1 x})(\gamma_1\,(D_{66} + c_1\,(2\,F_{66} + c_1\,H_{66}))\,\mu_1 + \alpha\,(D_{66} + c_1\,(2\,F_{66} + c_1\,H_{66} \\
&+ 2\,\delta_1\,F_{66}\,\mu_1 + 2\,c_1\,\delta_1\,H_{66}\,\mu_1)))+ \\
&\mathrm{e}^{\mu_2 x}\,(B_1 + B_2\,\mathrm{e}^{-2\mu_2 x})(\gamma_2\,(D_{66} + c_1\,(2\,F_{66} + c_1\,H_{66}))\,\mu_2 + \alpha\,(D_{66} + c_1\,(2\,F_{66} + c_1\,H_{66} \\
&+ 2\,\delta_2\,F_{66}\,\mu_1 + 2\,c_1\,\delta_2\,H_{66}\,\mu_2)))+ \\
&\mathrm{e}^{\mu_3 x}\,(B_1 + B_2\,\mathrm{e}^{-2\mu_3 x})(\gamma_3\,(D_{66} + c_1\,(2\,F_{66} + c_1\,H_{66}))\,\mu_3 + \alpha\,(D_{66} + c_1\,(2\,F_{66} + c_1\,H_{66} \\
&+ 2\,\delta_3\,F_{66}\,\mu_3 + 2\,c_1\,\delta_3\,H_{66}\,\mu_2)))+ \\
&\mathrm{e}^{\mu_4 x}\,(B_1 + B_2\,\mathrm{e}^{-2\mu_4 x})(\gamma_4\,(D_{66} + c_1\,(2\,F_{66} + c_1\,H_{66}))\,\mu_4 + \alpha\,(D_{66} + c_1\,(2\,F_{66} + c_1\,H_{66} \\
&+ 2\,\delta_4\,F_{66}\,\mu_1 + 2\,c_1\,\delta_4\,H_{66}\,\mu_4)))\Big)\,\cos(\alpha\,y) = \mathcal{M}_{xy}\,\cos(\alpha\,y)
\end{aligned}
$$

$$
\begin{aligned}
P_{xx}\,(x,y) = \Big(&\mathrm{e}^{\mu_1 x}\,(-B_1 + B_2\,\mathrm{e}^{-2\,\mu_1\,x})\,(\alpha^2\,c_1\,\delta_1\,H_{12} + \alpha\,\gamma_1\,(F_{12} + c_1\,H_{12}) - \mu_1\,(F_{11} + c_1\,H_{11}\,(1 + \delta_1\,\mu_1)))+ \\
&\mathrm{e}^{\mu_2 x}\,(-B_1 + B_2\,\mathrm{e}^{-2\,\mu_2\,x})\,(\alpha^2\,c_1\,\delta_2\,H_{12} + \alpha\,\gamma_2\,(F_{12} + c_1\,H_{12}) - \mu_2\,(F_{11} + c_1\,H_{11}\,(1 + \delta_2\,\mu_2)))+ \\
&\mathrm{e}^{\mu_3 x}\,(-B_1 + B_2\,\mathrm{e}^{-2\,\mu_3\,x})\,(\alpha^2\,c_1\,\delta_3\,H_{12} + \alpha\,\gamma_3\,(F_{12} + c_1\,H_{12}) - \mu_3\,(F_{11} + c_1\,H_{11}\,(1 + \delta_3\,\mu_3)))+ \\
&\mathrm{e}^{\mu_4 x}\,(-B_1 + B_2\,\mathrm{e}^{-2\,\mu_4\,x})\,(\alpha^2\,c_1\,\delta_4\,H_{12} + \alpha\,\gamma_4\,(F_{12} + c_1\,H_{12}) - \mu_4\,(F_{11} + c_1\,H_{11}\,(1 + \delta_4\,\mu_4)))\Big) \\
&\sin(\alpha\,y) = \mathcal{P}_{xx}\,\sin(\alpha\,y)
\end{aligned}
\tag{25}
$$

At this point, zero boundary conditions are generally used to eliminate the constants when using the classical method which establishes the stability equation for a single individual plate. By contrast, the development of the dynamic stiffness matrix entails imposition of general boundary conditions in algebraic form and widens the possibility of the analysis of multi-plate systems. In order to develop the dynamic stiffness matrix, the following boundary conditions are applied next.

$$
\begin{aligned}
x = 0:&\qquad W_m = W_{m_1}\,,\ \Phi_{x_m} = \Phi_{x_1}\,,\ \Phi_{y_m} = \Phi_{y_1}\,,\ W_{m,x} = W_{m1,x} \\
x = b:&\qquad W_m = W_{m_2}\,,\ \Phi_{x_m} = \Phi_{x_2}\,,\ \Phi_{y_m} = \Phi_{y_2}\,,\ W_{m,x} = W_{m2,x} \\
x = 0:&\qquad \mathcal{Q}_x = -\mathcal{Q}_{x_1}\,,\ \mathcal{M}_{xx} = -\mathcal{M}_{xx_1}\,,\ \mathcal{M}_{xy} = -\mathcal{M}_{xy_1}\,,\ \mathcal{P}_{xx} = -\mathcal{P}_{xx_1} \\
x = b:&\qquad \mathcal{Q}_x = \mathcal{Q}_{x_2}\,,\ \mathcal{P}_{xx} = \mathcal{P}_{xx_2}\,,\ \mathcal{M}_{xy} = \mathcal{M}_{xy_2}\,,\ \mathcal{P}_{xx} = \mathcal{P}_{xx_2}
\end{aligned}
\tag{26}
$$

By substituting Eq. (26) into Eq.(69), the following matrix relations for the displacements are obtained:

$$
\begin{bmatrix} W_1 \\ \Phi_{x_1} \\ \Phi_{y_1} \\ W_{1,x} \\ W_2 \\ \Phi_{x_2} \\ \Phi_{y_2} \\ W_{2,x} \end{bmatrix}
=
\begin{bmatrix}
\delta_1 & -\delta_1 & \delta_2 & -\delta_2 & \delta_3 & -\delta_3 & \delta_4 & -\delta_4 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\gamma_1 & -\gamma_1 & \gamma_2 & -\gamma_2 & \gamma_3 & -\gamma_3 & \gamma_4 & -\gamma_4 \\
f_1 & -f_1 & f_2 & -f_2 & f_3 & -f_3 & f_4 & -f_4 \\
\delta_1\,\mathrm{e}^{b\,\mu_{o1}} & -\delta_1\,\mathrm{e}^{-b\,\mu_{o1}} & \delta_2\,\mathrm{e}^{b\,\mu_{o2}} & -\delta_2\,\mathrm{e}^{-b\,\mu_{o2}} & \delta_3\,\mathrm{e}^{b\,\mu_{o3}} & -\delta_3\,\mathrm{e}^{-b\,\mu_{o3}} & \delta_4\,\mathrm{e}^{b\,\mu_{o4}} & -\delta_4\,\mathrm{e}^{-b\,\mu_{o4}} \\
\mathrm{e}^{b\,\mu_{o1}} & -\mathrm{e}^{-b\,\mu_{o1}} & \mathrm{e}^{b\,\mu_{o2}} & -\mathrm{e}^{-b\,\mu_{o2}} & \mathrm{e}^{b\,\mu_{o3}} & -\mathrm{e}^{-b\,\mu_{o3}} & \mathrm{e}^{b\,\mu_{o4}} & -\mathrm{e}^{-b\,\mu_{o4}} \\
\gamma_1\,\mathrm{e}^{b\,\mu_{o1}} & -\gamma_1\,\mathrm{e}^{-b\,\mu_{o1}} & \gamma_2\,\mathrm{e}^{b\,\mu_{o2}} & -\gamma_2\,\mathrm{e}^{-b\,\mu_{o2}} & \gamma_3\,\mathrm{e}^{b\,\mu_{o3}} & -\gamma_3\,\mathrm{e}^{-b\,\mu_{o3}} & \gamma_4\,\mathrm{e}^{b\,\mu_{o4}} & -\gamma_4\,\mathrm{e}^{-b\,\mu_{o4}} \\
f_1\,\mathrm{e}^{b\,\mu_{o1}} & -f_1\,\mathrm{e}^{-b\,\mu_{o1}} & f_2\,\mathrm{e}^{b\,\mu_{o2}} & -f_2\,\mathrm{e}^{-b\,\mu_{o2}} & f_3\,\mathrm{e}^{b\,\mu_{o3}} & -f_3\,\mathrm{e}^{-b\,\mu_{o3}} & f_4\,\mathrm{e}^{b\,\mu_{o4}} & -f_4\,\mathrm{e}^{-b\,\mu_{o4}}
\end{bmatrix}
\begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \\ B_6 \\ B_7 \\ B_8 \end{bmatrix}
\tag{27}
$$

where

$$f_i = \delta_i\,\mu_i; \quad \text{with}\, i = 1, 2, 3, 4$$

Equations (82) and (27) can be written as

$$\boldsymbol{\delta} = \boldsymbol{A}\,\boldsymbol{C} \tag{28}$$

By applying the same procedure for forces and moments, i.e. substituting Eq. (26) into Eq.(25) the following matrix relations are obtained:

$$\begin{bmatrix} \mathcal{Q}_{x_1} \\ \mathcal{M}_{xx_1} \\ \mathcal{M}_{xy_1} \\ \mathcal{P}_{xx_1} \\ \mathcal{Q}_{x_2} \\ \mathcal{M}_{xx_2} \\ \mathcal{M}_{xy_2} \\ \mathcal{M}_{xx_2} \end{bmatrix} = \begin{bmatrix} \mathcal{Q}_1 & \mathcal{Q}_1 & \mathcal{Q}_2 & \mathcal{Q}_2 & \mathcal{Q}_3 & \mathcal{Q}_3 & \mathcal{Q}_4 & \mathcal{Q}_4 \\ \mathcal{T}_1 & -\mathcal{T}_1 & \mathcal{T}_2 & -\mathcal{T}_2 & \mathcal{T}_3 & -\mathcal{T}_3 & \mathcal{T}_4 & -\mathcal{T}_4 \\ -\mathcal{I}_1 & -\mathcal{I}_1 & -\mathcal{I}_2 & -\mathcal{I}_2 & -\mathcal{I}_3 & -\mathcal{I}_3 & -\mathcal{I}_4 & -\mathcal{I}_4 \\ \mathcal{L}_1 & -\mathcal{L}_1 & \mathcal{L}_2 & -\mathcal{L}_2 & \mathcal{L}_3 & -\mathcal{L}_3 & \mathcal{Y}_L & -\mathcal{L}_4 \\ \mathcal{Q}_1\,\mathrm{e}^{b\,\mu_{o1}} & -\mathcal{Q}_1\,\mathrm{e}^{-b\,\mu_{o1}} & \mathcal{Q}_2\,\mathrm{e}^{b\,\mu_{o2}} & -\mathcal{Q}_2\,\mathrm{e}^{-b\,\mu_{o2}} & \mathcal{Q}_3\,\mathrm{e}^{b\,\mu_{o3}} & -\mathcal{Q}_3\,\mathrm{e}^{-b\,\mu_{o3}} & \mathcal{Q}_4\,\mathrm{e}^{b\,\mu_{o4}} & -\mathcal{Q}_4\,\mathrm{e}^{-b\,\mu_{o4}} \\ -\mathcal{T}_1\mathrm{e}^{b\,\mu_{o1}} & \mathcal{T}_1\mathrm{e}^{-b\,\mu_{o1}} & -\mathcal{T}_2\mathrm{e}^{b\,\mu_{o2}} & \mathcal{T}_2\mathrm{e}^{-b\,\mu_{o2}} & -\mathcal{T}_3\mathrm{e}^{b\,\mu_{o3}} & \mathcal{T}_3\mathrm{e}^{-b\,\mu_{o3}} & -\mathcal{T}_4\mathrm{e}^{b\,\mu_{o4}} & \mathcal{T}_4\mathrm{e}^{-b\,\mu_{o4}} \\ \mathcal{I}_1\mathrm{e}^{b\,\mu_{o1}} & \mathcal{I}_1\mathrm{e}^{-b\,\mu_{o1}} & \mathcal{I}_2\mathrm{e}^{b\,\mu_{o2}} & \mathcal{I}_2\mathrm{e}^{-b\,\mu_{o2}} & \mathcal{I}_3\mathrm{e}^{b\,\mu_{o3}} & \mathcal{I}_3\mathrm{e}^{-b\,\mu_{o3}} & \mathcal{I}_4\mathrm{e}^{b\,\mu_{o4}} & \mathcal{I}_4\mathrm{e}^{-b\,\mu_{o4}} \\ -\mathcal{L}_1\mathrm{e}^{b\,\mu_{o1}} & \mathcal{L}_1\mathrm{e}^{-b\,\mu_{o1}} & -\mathcal{L}_2\mathrm{e}^{b\,\mu_{o2}} & \mathcal{L}_2\mathrm{e}^{-b\,\mu_{o2}} & -\mathcal{L}_3\mathrm{e}^{b\,\mu_{o3}} & \mathcal{L}_3\mathrm{e}^{-b\,\mu_{o3}} & -\mathcal{L}_4\mathrm{e}^{b\,\mu_{o4}} & \mathcal{L}_4\mathrm{e}^{-b\,\mu_{o4}} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \\ B_6 \\ B_7 \\ B_8 \end{bmatrix} \tag{29}$$

where

$$\begin{aligned} \mathcal{Q}_i = &-A_{55}(1+\delta_i\mu_{oi}) - 2c_2(D_{55}+D_{55}\delta_i\mu_{oi}) - c_2^2(F_{55}+\delta_iF_{55}\mu_{oi}) - c_1(\alpha\gamma_i(F_{12}+2F_{66}+c_1H_{12} \\ &+2c_1H_{66})\mu_{oi} - \mu_{oi}^2(F_{11}+c_1H_{11}+c_1\delta_iH_{11}\mu_{oi}) + \alpha^2(2F_{66}+2c_1H_{66}+c_1\delta_iH_{12}\mu_{oi}+4c_1\delta_iH_{66}\mu_{oi})) \end{aligned}$$

$$\begin{aligned} \mathcal{T}_i = &\,\alpha^2c_1\delta_i(F_{12}+c_1H_{12}) + \alpha\gamma_i(D_{12}+c_1(2F_{12}+c_1H_{12})) - \mu_{oi}(D_{11}+c_1(2F_{11}+c_1H_{11} \\ &+\delta_iF_{11}\mu_{oi}+c_1\delta_iH_{11}\mu_{oi})) \end{aligned}$$

$$\mathcal{I}_i = \gamma_1(D_{66}+c_1(2F_{66}+c_1H_{66}))\mu_{oi} - \alpha(D_{66}+c_1(2F_{66}+c_1H_{66}+2\delta_iF_{66}\mu_{oi}+2c_1\delta_iH_{66}\mu_{oi}))$$

$$\mathcal{L}_i = c_1(\alpha^2c_1\delta_iH_{12}+\alpha\gamma_i(F_{12}+c_1H_{12}) - \mu_{oi}(F_{11}+c_1H_{11}(1+\delta_i\mu_{oi}))) \qquad \text{with} \quad i=1,2,3,4 \tag{30}$$

Equation (29) can be written as

$$\boldsymbol{F} = \boldsymbol{R}\boldsymbol{C} \tag{31}$$

By eliminating the constants vector $\boldsymbol{C}$ form Eqs. (28) and (31) the dynamic stiffness matrix is formulated as follows:

$$\boldsymbol{K} = \boldsymbol{R}\,\boldsymbol{A}^{-1} \tag{32}$$

or more explicitly

$$\boldsymbol{K} = \begin{bmatrix} s_{qq} & s_{qm} & s_{qt} & s_{qh} & f_{qq} & f_{qm} & f_{qt} & f_{qh} \\ & s_{mm} & s_{mt} & s_{mh} & -f_{qm} & f_{mm} & f_{mt} & f_{mh} \\ & & s_{tt} & s_{th} & f_{qt} & -f_{mt} & f_{tt} & f_{th} \\ & & & s_{hh} & -f_{qh} & f_{mh} & -f_{th} & f_{hh} \\ & & Sym & & s_{qq} & -s_{qm} & s_{qt} & -s_{qh} \\ & & & & & s_{mm} & -s_{mt} & s_{mh} \\ & & & & & & s_{tt} & -s_{th} \\ & & & & & & & s_{hh} \end{bmatrix} \tag{33}$$

Finally the dynamic stiffness matrix related to the force and displacement vectors can be written as follows:

$$\begin{bmatrix} \mathcal{Q}_{x_1} \\ \mathcal{M}_{xx_1} \\ \mathcal{M}_{xy_1} \\ \mathcal{P}_{xx_1} \\ \mathcal{Q}_{x_2} \\ \mathcal{M}_{xx_2} \\ \mathcal{M}_{xy_2} \\ \mathcal{P}_{xx_2} \end{bmatrix} = \begin{bmatrix} s_{qq} & s_{qm} & s_{qt} & s_{qh} & f_{qq} & f_{qm} & f_{qt} & f_{qh} \\ & s_{mm} & s_{mt} & s_{mh} & -f_{qm} & f_{mm} & f_{mt} & f_{mh} \\ & & s_{tt} & s_{th} & f_{qt} & -f_{mt} & f_{tt} & f_{th} \\ & & & s_{hh} & -f_{qh} & f_{mh} & -f_{th} & f_{hh} \\ & & \text{Sym} & & s_{qq} & -s_{qm} & s_{qt} & -s_{qh} \\ & & & & & s_{mm} & -s_{mt} & s_{mh} \\ & & & & & & s_{tt} & -s_{th} \\ & & & & & & & s_{hh} \end{bmatrix} \begin{bmatrix} W_1 \\ \Phi_{x_1} \\ \Phi_{y_1} \\ W_{1,x} \\ W_2 \\ \Phi_{x_2} \\ \Phi_{y_2} \\ W_{2,x} \end{bmatrix} \tag{34}$$

which in compact matrix form:

$$\boldsymbol{F} = \boldsymbol{K}\,\boldsymbol{D} \tag{35}$$

The above dynamic stiffness matrix will now be used in conjunction with the Wittrick-Williams algorithm [13] to analyze composite simple and stiffened plates for their buckling behavior based on HSDT. Explicit expressions for each element of the DS matrix were obtained via symbolic computation, but they are far too extensive and voluminous to report here. The correctness of these expressions was further checked by implementing them in a MATLAB program and carrying out a wide rage of numerical simulations.

2.4 Assembly procedure, boundary conditions and similarities with FEM

Once the DS matrix of a laminate element has been developed, it can be rotated and/or offset if required and thus can be assembled to form the global DS matrix of the final structure. The assembly procedure is schematically shown in Fig. 2 which is similar to that of FEM. Although like the FEM, a mesh is required in the DSM, it should be noted that unlike the former, the latter is not mesh dependent in the sense that additional elements are required only when there is a change in the geometry of the structure. A single DS laminate element is enough to compute any number of its buckling loads to any desired accuracy, which, of course, is impossible in the FEM. However, for the type of structures under consideration DS plate elements do not have point nodes, but have line nodes instead. Also no change in the geometry along longitudinal direction is admitted. This assumption is in addition to the assumed simple support boundary conditions on two opposite sides. The other two sides of the plate can have any boundary conditions. The application of the boundary conditions of the global dynamic stiffness matrix involves the use of the so-called penalty method. This consists of adding a large stiffness to the appropriate position on the leading diagonal term which corresponds to the degree of freedom of the node that needs to be suppressed. It is thus possible to apply free, simple support and clamped boundary conditions on the structure by penalizing the appropriate degrees of freedom. Note that in accordance with the notation and sign convention used in Fig. 2 for simple support boundary condition V, W and Φ_y have to be penalized whereas for clamped boundary condition U, V, W, Φ_y, Φ_x, W,x have to be penalized. Clearly for the free-edge boundary condition no penalty will be applied.

Because of the similarities between DSM and FEM, DS elements can be implemented in FEM codes and thus the accuracy of results can be increased substantially.

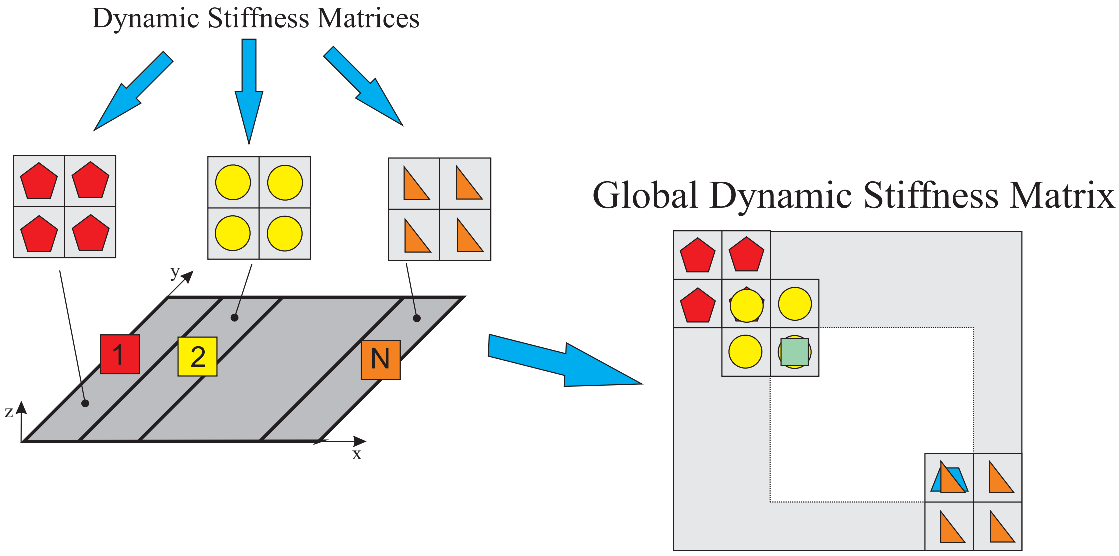


Figure 2: Direct assembly of dynamic stiffness elements.

2.5 Application of the Wittrick-Williams Algorithm

In order to compute the critical buckling loads of a structure by using the DSM, an efficient way to solve the eigen-problem is to apply the Wittrick and Williams algorithm [13] which has featured in literally hundreds of papers. For the sake of completeness the procedure is briefly summarized as follows.

First the global dynamic stiffness matrix of the final structure K^* is computed for an arbitrarily chosen trial critical buckling load λ^*. Next, by applying the usual form of Gauss elimination the global stiffness matrix, is transformed into its upper triangular $K^{*\triangle}$ form. The number of negative terms on the leading diagonal of $K^{*\triangle}$ is now defined as the sign count $s(K^*)$ which forms the fundamental basis of the algorithm. In its simplest form, the algorithm states that j, the number of critical buckling loads (λ) of a structures that lie below an arbitrarily chosen trial buckling load (λ^*) is given by:

$$j = j_0 + s(K^*) \tag{36}$$

where j_0 is the number of critical buckling loads of all single strip elements within the structure which are still lower than the trial buckling load (λ^*) when their opposite sides are fully clamped. It is necessary to account for this clamped-clamped critical buckling loads because exact buckling analysis using DSM allows an infinity number of critical buckling loads to be accounted for when all the nodes of the structures are fully clamped.(i.e. in the overall formulation $\boldsymbol{K}\,\boldsymbol{\delta} = 0$, these critical buckling loads correspond to $\boldsymbol{\delta} = 0$ modes.) Thus j_0 is an integral part of the algorithm and is not a peripheral issue. However, j_0 is usually zero and the dominant term of the algorithm is the sign-count $s(K^*)$ of Eq. (36). One way of avoiding the computation of troublesome j_0 is to split the structure into sufficient number of elements so that the clamped-clamped buckling loads of an individual element in the structure are never exceeded. Once $s(K^*)$ and j_0 of Eq. (36) are known, any suitable method, for example, bi-section technique can be devised to bracket any critical buckling load within any desired accuracy. The mode shapes are routinely computed by using the standard eigen-solution procedure in which the global dynamic stiffness matrix is computed at the critical buckling load and the force vector is set to zero whilst deleting one row of the DS matrix and giving one of the nodal displacement component an arbitrarily chosen value and determining the rest of the displacements in terms of the chosen one.

3 Results ans Discussion

Results are shown accounting for the in-plane load conditions in Fig. 3 and the boundary conditons in Fig. 4. A preliminary validation of the critical buckling load analysis

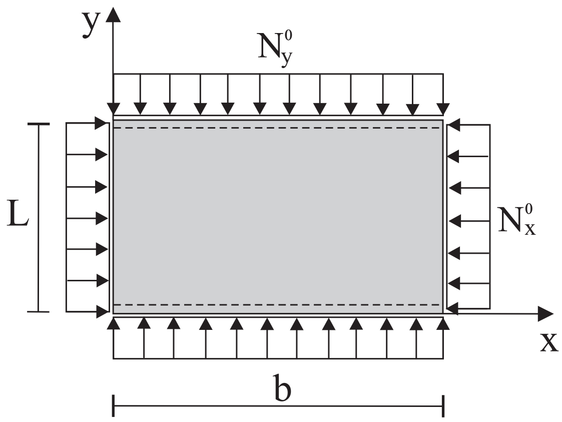


Figure 3: Laminated composite plate under in-plane loadings.

for moderately thick ($a/h = 10$) simply-supported cross-ply square plates uniaxially loaded in the x direction is carried out and the results are shown in Table 1 for different orthotropy ratos. The dimensionless critical buckling load, obtained using HSDT within the framework of the DSM are in excellent agreement when compared with the 3D elasticity solution and the results also lead to the same findings of the classical Lèvy-type closed form solution. Note that for all practical purpose, it is only the first buckling load that matters. Therefore only the first critical loads is presented in this paper. As expected the percentage error, with respect to the 3D elasticity solution, increases when increasing the orthotropic ratio. In Table 2 the dimensionless critical buckling load for the same case study of Table 1 is computed but taking into account the effects of the length-to-thickness and orthotropy ratios and boundary conditions (see Fig. 4). At a

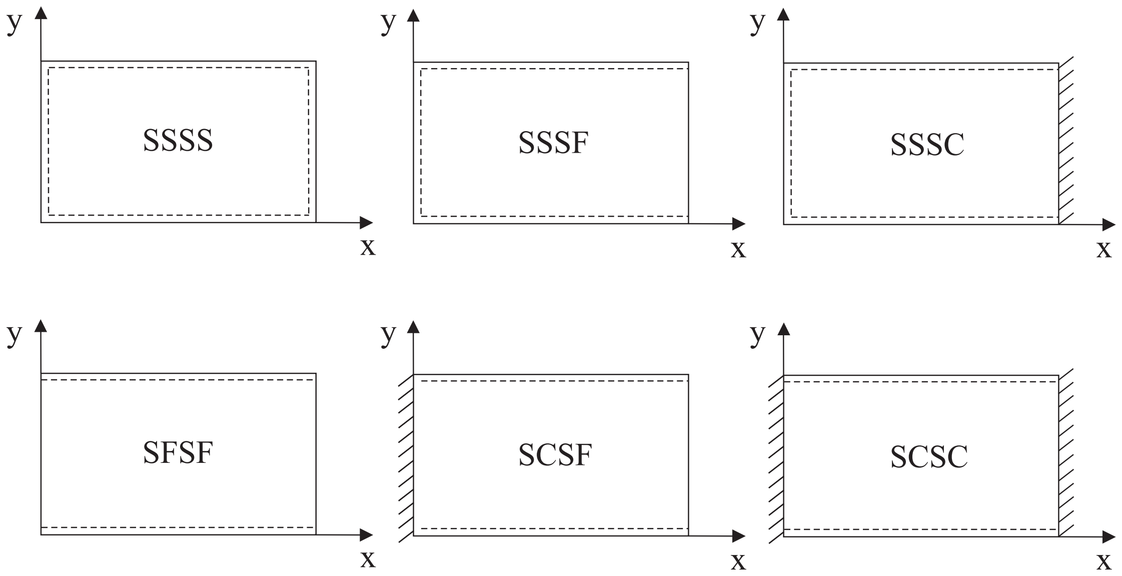


Figure 4: Boundary conditions.

Table 1: Dimensionless uniaxial buckling load (along x direction) $\mathrm{N}_{cr} = \bar{\mathrm{N}}_{cr} \frac{b^2}{E_2 h^3}$, for simply supported cross-ply square plates with $b/h = 10$, $E_1/E_2 =$ open, $G_{12}/E_2 = G_{13}/E_2 = 0.6$, $G_{23}/E_2 = 0.5$, $\nu_{12} = \nu_{13} = 0.25$.

Stacking Sequence	Models	E_1/E_2					
[0°/90°/90°/0°]		3		20		40	
	3D-Elasticity [24]	5.304	$\Delta^{\dagger}_{3\mathrm{D}}\%$	15.019	$\Delta_{3\mathrm{D}}\%$	22.881	$\Delta_{3\mathrm{D}}\%$
Classical Lèvy's solution	HSDT	5.393	(1.68)	15.298	(1.86)	23.340	(2.01)
	FSDT	5.399	(1.79)	15.351	(2.21)	23.453	(2.50)
	CLPT	5.754	(8.48)	19.712	(31.2)	36.160	(58.0)
DSM	HSDT	5.393	(1.68)	15.298	(1.86)	23.340	(2.01)

† $\Delta_{3\mathrm{D}}\% = \frac{\hat{\omega}-\hat{\omega}_{3\mathrm{D}}}{\hat{\omega}_{3\mathrm{D}}} \times 100$.

fixed length-to-thickness ratio, the dimensionless critical buckling load increases when increasing the orthotropic ratio for all the considered boundary conditions. A similar behavior can be observed when varying the length-to-thickness ratio but by fixing the orthotropic ratio. Understandably, the largest dimensionless critical buckling load is

Table 2: Dimensionless uniaxial buckling load (along x direction) $\text{N}_{cr} = \bar{\text{N}}_{cr} \frac{b^2}{E_2 h^3}$, for simply supported cross-ply square plates, stacking sequence [0°/90°/90°/0°] and $E_1/E_2 =$ open, $G_{12}/E_2 = G_{13}/E_2 = 0.6$, $G_{23}/E_2 = 0.5$, $\nu_{12} = \nu_{13} = 0.25$.

Uniaxial Compression (UA-C)							
		b/h					
E_1/E_2	BCs	2	5	10	25	50	100
3	S-S-S-S	1.9557	4.5458	5.3933	5.6928	5.7384	5.7499
	S-S-S-F	1.5912	3.3640	4.3496	5.4623	6.2621	6.8168
	S-S-S-C	1.9821	5.4309	7.1169	7.8176	7.9304	7.9592
	S-F-S-F	1.5839	3.2200	4.1087	5.1745	6.0137	6.7261
	S-C-S-F	1.5954	3.3805	4.3651	5.4686	6.2624	6.8599
	S-C-S-C	2.0555	6.8502	10.273	12.012	12.314	12.392
10	S-S-S-S	2.2810	7.1554	9.9406	11.209	11.420	11.473
	S-S-S-F	2.0267	5.1516	6.9540	8.6366	9.7745	10.536
	S-S-S-C	2.3426	8.3895	14.133	17.865	18.587	18.778
	S-F-S-F	2.0040	4.5982	6.0871	7.9721	9.8185	11.836
	S-C-S-F	2.0746	7.1554	7.0981	9.0614	10.724	12.305
	S-C-S-C	2.5134	9.9930	20.515	30.068	32.273	32.878
20	S-S-S-S	2.5689	9.4219	15.298	18.825	19.482	19.654
	S-S-S-F	2.3144	6.7586	9.8649	12.398	14.130	15.513
	S-S-S-C	2.6643	10.376	20.977	31.075	33.495	34.166
	S-F-S-F	2.2874	5.8285	8.1129	10.893	13.785	17.359
	S-C-S-F	2.4425	7.0390	10.416	13.613	16.334	19.282
	S-C-S-C	2.9945	11.723	28.670	52.336	59.682	61.870
40	S-S-S-S	3.0749	11.997	23.340	33.131	35.347	35.953
	S-S-S-F	2.6827	8.7789	14.723	19.374	22.292	24.982
	S-S-S-C	3.1136	12.301	29.414	54.096	62.190	64.645
	S-F-S-F	2.6558	7.5209	11.386	15.744	20.251	26.398
	S-C-S-F	2.9731	9.4057	15.987	21.955	26.517	31.952
	S-C-S-C	3.5992	13.357	37.045	87.668	110.86	118.82

given by the boundary condition S-C-S-C and the lowest by S-F-S-F. In Table 3 the results are given for composite plates that are uniaxially loaded in the y direction, instead of the x direction for different values of length-to-thickness and orthotropy ratios. The dimensionless critical buckling load is generally lower for all the boundary conditions but for for the case with one or two sides free, namely, S-S-S-F and S-F-S-F, it decreases significantly.

Table 3: Dimensionless uniaxial buckling load (along y direction) $\mathrm{N}_{cr} = \bar{\mathrm{N}}_{cr}\,\frac{b^2}{E_2\,h^3}$, for simply supported cross-ply square plates, stacking sequence [0°/90°/90°/0°] and E_1/E_2 = open, $G_{12}/E_2 = G_{13}/E_2 = 0.6$, $G_{23}/E_2 = 0.5$, $\nu_{12} = \nu_{13} = 0.25$.

Uniaxial Compression (UA-C)							
		b/h					
E_1/E_2	BCs	2	5	10	25	50	100
3	S-S-S-S	1.7611	4.5458	5.3933	5.6928	5.7384	5.7499
	S-S-S-F	0.9288	1.4363	1.5677	1.6136	1.6219	1.6247
	S-S-S-C	1.8052	5.0418	7.0967	8.0978	8.2713	8.3164
	S-F-S-F	0.6575	0.9484	1.0136	1.0338	1.0369	1.0377
	S-C-S-F	1.0287	1.7630	2.0138	2.1123	2.1309	2.1371
	S-C-S-C	1.8552	5.3906	8.0926	9.6793	9.9819	10.062
10	S-S-S-S	1.8421	6.0365	9.2387	10.877	11.161	11.234
	S-S-S-F	1.0769	1.9529	2.2269	2.3238	2.3402	2.3452
	S-S-S-C	1.9124	6.4505	10.676	13.497	14.069	14.222
	S-F-S-F	0.8416	1.4919	1.6799	1.7416	1.7508	1.7531
	S-C-S-F	1.2723	2.6410	3.2331	3.4805	3.5247	3.5380
	S-C-S-C	1.9998	6.9366	12.551	17.665	18.940	19.301
20	S-S-S-S	1.9042	7.2304	12.728	16.249	16.923	17.100
	S-S-S-F	1.1977	2.5793	3.1263	3.3311	3.3647	3.3741
	S-S-S-C	2.0019	7.6623	14.591	20.721	22.214	22.634
	S-F-S-F	0.9876	2.1487	2.5906	2.7494	2.7737	2.7798
	S-C-S-F	1.4691	3.6342	4.8200	5.3662	5.4629	5.4902
	S-C-S-C	2.1313	8.2008	16.897	27.735	31.291	32.391
40	S-S-S-S	1.9744	8.5588	18.034	26.395	28.285	28.801
	S-S-S-F	1.3172	3.5410	4.7817	5.3152	5.4037	5.4272
	S-S-S-C	2.1210	8.9972	20.099	33.690	37.996	39.317
	S-F-S-F	1.1294	3.1544	4.2674	4.7367	4.8124	4.8317
	S-C-S-F	1.5871	5.0774	7.6226	9.0324	9.2956	9.3669
	S-C-S-C	2.3293	9.5963	22.614	44.455	54.516	58.136

Part II

A Refined Dynamic Stiffness Element for Free Vibration Analysis of Cross-Ply Laminated Composite Cylindrical and Spherical Shallow Shells

Summary

An exact free vibration analysis of laminated composite doubly-curved shallow shells has been carried out by combining the dynamic stiffness method (DSM) and a higher order shear deformation theory (HSDT) for the first time. In essence, the HSDT has been exploited to develop first the dynamic stiffness (DS) element matrix and then the global DS matrix of composite cylindrical and spherical shallow shell structures by assembling the individual DS elements. As an essential prerequisite, Hamilton's principle is used to derive the governing differential equations and the related natural boundary conditions. The equations are solved symbolically in an exact sense and the DS matrix is formulated by imposing the natural boundary conditions in algebraic form. The Wittrick-Williams algorithm is used as a solution technique to compute the eigenvalues of the overall DS matrix. The effect of several parameters such as boundary conditions, orthotropic ratio, length-to-thickness ratio, radius-to-length ratio and stacking sequence on the natural frequencies and mode shapes is investigated in details. Results are compared with those available in the literature. Finally some concluding remarks are drawn.

4 Introduction

Aerospace structures are generally made up of thin-walled cylindrical or spherical shell components. The application of composite shell structures is justified because of the extraordinary load-carrying capability. Furthermore, their structural efficiency is characterized by a high stiffness-to-weight and strength-to-weight ratios. Due to the harsh environment conditions in which aerospace structures work a comprehensive and accurate investigation of their dynamic behavior is required. During the last decades many theories have been developed to this purpose. In particular, most of the classical theories are mainly based on Kirchhoff-Love's hypotheses [25,26]. It is well-known that these shell theories neglecting the transverse shear and normal stresses lead to accurate results when high values of length-to-thickness ratio are considered and when 3D local effects are not present. When moderately thick or thick shells are investigated the inclusion of transverse shear and normal stresses are mandatory. However, on the other hand, a refinement in results accuracy cannot be only achieved by virtue of an enhancement in the kinematics description of the displacement model, but combining it with an adequate description of the curvature terms h/R_i with $i = \alpha$, β. In this regards, some interesting observations have been provided by Qatu [27] and, Carrera [28, 29] and recently Fazzolari [5,30]. In order to provide solutions of the Governing Differential Equations (GDEs) of practical interest, considerable efforts were focused and devoted to their simplification. As a result, the GDEs of shallow shells were derived. Most notably, it worths highlighting the articles of Donnell [31, 32], Mushtari [33, 34] and Vlasov [35, 36]. The hypothesis of thinness was discarded in the shell theories proposed by Flügge [37, 38], Lur'e [39] and Byrne [40], where higher-order expansion of the reciprocal of the Lamé parameters was considered. Other refinements of the developed shell theories were proposed by Sanders [41]. Additional effects in the development of shell theories were taken into account by Whitney and Sun [42], Librescu [43], Gulati and Essemberg [44], Zucas and Vinson [45] and Ambartsumian [46–53] amongst others. Additional references can be found in Naghdi [54], Ambartsumian [55] and Bert [56–58]. Reddy [59] proposed a generalization of Sander's theory to anisotropic doubly-curved shells. The application of layer-wise theories for shell structures can be found in the papers presented by Hsu and Wang [60], Cheung and Wu [61], Barbero *et al.* [62] and Carrera [28, 29, 63]. Reviews on finite element shell formulations have been given by Denis and Palazzotto [64] and Di and Ramm [65]. An exhaustive review on classical theories can be found in Librescu [43]. As regards the use of approximation methods, Qatu and Asadi [66] addressed the vibration analysis of doubly-curved shallow shells with arbitrary boundary conditions by using the Ritz method with algebraic polynomial displacement functions. Asadi *et al.* [67] employed a 3D and several shear deformation theories in order to carry out static and vibration analysis of thick deep laminated cylindrical shells. Ferreira *et al.* [68] used a wavelet collocation method for the analysis of laminated shells. The same author [69] combined a sinusoidal shear deformation theory with the radial basis functions collocation method to deal with static and vibration analyses of laminated composite shells. Tornabene *et al.* [70, 71] studied the free vibration behavior of doubly-curved anisotropic laminated composite shells and revolution panels by means of the Generalized Differential Quadrature (GDQ). Fazzolari *et al.* [5, 30, 72–76] proposed two advanced hierarchical trigonometric Ritz formulations based on the Principle of Virtual Displacements (PVD) and Reissner's Mixed Variational Theorem (RMVT) to cope with the free vibrations of anisotropic laminated composite plates and cylindrical, spherical and hyperbolic paraboloidal shells. The dynamic stiffness method (DSM) has not been extensively researched in the analysis of laminated composite shells, only few works have been devoted to the application of DSM to shell structures and very few to laminated composite cylindrical and spherical shells. In particular, Casimir *et. al* [77] developed the dynamic stiffness matrix of tubular shells with free edge boundary conditions, including rotatory inertia, transverse shear deformation and non-axisymmetric loadings, but the analysis was restricted to homogeneous and isotropic material. Thinh and Nguyen [78] proposed the dynamic stiffness matrix of continuous element for vibration of thick cross-ply laminated composite cylindrical shells. Khadimallah *et. al* [79] dealt with the dynamic response to harmonic loads of an axisymmetric shell by using a dynamic stiffness formulations. The dynamic stiffness matrix of toroidal shells was derived by Leung and Kwok [80]. El-Kaabazi and Kennedy [81] proposed dynamic stiffness equations for variable thickness cylindrical shells under the assumptions of Donnell, Timoshenko and Flügge theories. Langley [82] developed a dynamic stiffness technique for the vibration analysis of stiffened shell structures. Chronopoulos *et. al* [83] by using a dynamic stiffness approach studied the response of layered shells. Tounsi *et. al* [84] derived the dynamic stiffness matrix of circular rings. In the present article a new dynamic stiffness element has been developed in order to cope with the free vibration analysis of laminated composite doubly-curved shallow shells. The proposed formulation is based on the close-form Lèvy-type solution and the application of the Wittrick-Williams algorithm [85]. The latter is used as a solution technique to compute the eigenvalues of the overall DS matrix. Several symmetric cross-ply lamination schemes have been investigated. The effect of different parameters such as stacking sequence, boundary conditions, orthotropic ratio, length-to-thickness and radius-to-length ratios on the dimensionless frequencies parameters and mode shapes have been studied. Finally some conclusions have been drawn from the findings of the research.

5 Theoretical Formulation

In the derivation that follows, the hypotheses of straightness and normality of a transverse normal after deformation are assumed to be no longer valid for the displacement field which is now considered to be a cubic function in the thickness coordinate and hence the use of higher order shear deformation theory (HSDT). The composite circular cylindrical shell is assumed to be composed of N_l layers so that the theory is sufficiently general. The geometry of the shell and the coordinate system are shown in Fig. 5. The integer k is used as a superscript denoting the layer number where the numbering starts from the bottom. After imposing the transverse shear stress homogeneous conditions [19, 86] at the top/bottom surface of the shell, the displacements field is in the usual form given below:

$$
\begin{aligned}
u(\alpha,\beta,z,t) &= \left(1+\frac{z}{R_\alpha}\right) u_0(\alpha,\beta,t) + z\,\phi_\alpha(\alpha,\beta,t) - z^3\,\frac{4}{3\,h^2}\left(\phi_\alpha(\alpha,\beta,t) + \frac{1}{A_\alpha}\frac{\partial w_0(\alpha,\beta,t)}{\partial\alpha}\right)\\
v(\alpha,\beta,z,t) &= \left(1+\frac{z}{R_\beta}\right) v_0(\alpha,\beta,t) + z\,\phi_\beta(\alpha,\beta,t) - z^3\,\frac{4}{3\,h^2}\left(\phi_\beta(\alpha,\beta,t) + \frac{1}{A_\beta}\frac{\partial w_0(\alpha,\beta,t)}{\partial\beta}\right)\\
w(\alpha,\beta,z,t) &= w_0(\alpha,\beta,t)
\end{aligned}
\tag{37}
$$

where u, v, w are general displacements within the shell in the α, β, and z directions, respectively, whereas u_0, v_0, w_0 are the corresponding displacements of the reference surface (mid-plane Ω), ϕ_α, ϕ_β are the rotations of the normals to the α, β, respectively, A_α, A_β are the surface metrics and R_α and R_β are the radii of curvature in the α and β directions. The geometry of the shell is completely described by the parameters given in Fig. 5. Most notably, $\boldsymbol{r}(\alpha,\beta)$ is the position vector of a point on the middle surface Ω of

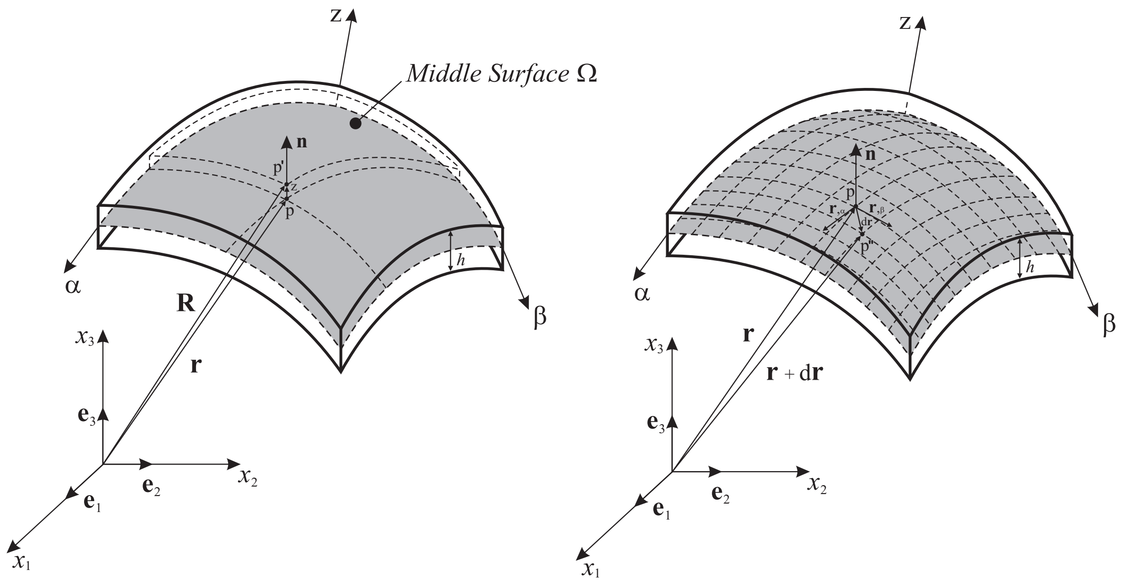


Figure 5: Shells geometry

the shell, $\boldsymbol{R}(\alpha,\beta,z)$ is the position vector of a generic point within the volume occupied by the shell. At each point P of the middle surface $\boldsymbol{n}(\alpha,\beta)$ indicates the unit normal vector. Subscripts and superscripts k refers that quantity at layer level. The square of line elements in orthogonal curvilinear coordinates is therefore defined as:

$$ds_k^2 = d\boldsymbol{r}_k \cdot d\boldsymbol{r}_k = \left(H_\alpha^k\right)^2 d\alpha_k^2 + \left(H_\beta^k\right)^2 d\beta_k^2 + \left(H_z^k\right)^2 dz_k^2 \tag{38}$$

the area of an infinitesimal rectangle on Ω_k as:

$$d\Omega_k = H_\alpha^k\, H_\beta^k\, d\alpha_k\, d\beta_k \tag{39}$$

and the infinitesimal volume as:

$$dV_k = H_\alpha^k \, H_\beta^k \, H_z^k \, d\alpha_k \, d\beta_k \, dz_k \tag{40}$$

where

$$H_\alpha^k = A_\alpha^k \left(1 + \frac{z_k}{R_\alpha^k}\right) \qquad H_\beta^k = A_\beta^k \left(1 + \frac{z_k}{R_\beta^k}\right) \qquad H_z^k = 1 \tag{41}$$

are referred to as Lamé parameters and A_α^k and A_β^k are the first fundamental magnitude of the first fundamental form $d\Omega_k$. Attention is herein focused on shells with a constant curvature, i.e., doubly-curved shells (cylindrical, spherical, toroidal geometries) for which $A_\alpha^k = A_\beta^k = 1$. In the case of a swallow shell the approximation $\left(1 + \frac{z}{R_\alpha}\right) \approx 1$ and $\left(1 + \frac{z}{R_\beta}\right) \approx 1$ is generally valid and leads to reasonably reliable results. The strain-displacement relationships referred to an orthogonal curvilinear coordinate system give rise to the following deformation field [19, 68]:

$$\begin{aligned}
&\varepsilon_{\alpha\alpha} = \varepsilon_{\alpha\alpha}^0 + z\,\left(k_{\alpha\alpha}^0 + z^2\,k_{\alpha\alpha}^2\right), \quad \varepsilon_{\beta\beta} = \varepsilon_{\beta\beta}^0 + z\,\left(k_{\beta\beta}^0 + z^2\,k_{\beta\beta}^2\right), \\
&\gamma_{\alpha\beta} = \gamma_{\alpha\beta}^0 + z\,\left(k_{\alpha\beta}^0 + z^2\,k_{\alpha\beta}^2\right), \quad \gamma_{\alpha z} = \gamma_{\alpha z}^0 + z^2\,k_{\alpha z}^2, \quad \gamma_{\beta z} = \gamma_{\beta z}^0 + z^2\,k_{\beta z}^2
\end{aligned} \tag{42}$$

where

$$\begin{aligned}
&\varepsilon_{\alpha\alpha}^0 = \frac{\partial u_0}{\partial \alpha} + \frac{w}{R_\alpha}, \quad \varepsilon_{\beta\beta}^0 = \frac{\partial u_0}{\partial \beta} + \frac{w}{R_\beta}, \quad \gamma_{\alpha\beta}^0 = \frac{\partial u_0}{\partial \beta} + \frac{\partial v_0}{\partial \alpha} \\
&\gamma_{\alpha z}^0 = \frac{\partial w_0}{\partial \alpha} + \phi_\alpha, \quad \gamma_{\beta z}^0 = \frac{\partial w_0}{\partial \beta} + \phi_\beta \\
&k_{\alpha\alpha}^0 = \frac{\partial \phi_\alpha}{\partial \alpha}, \quad k_{\beta\beta}^0 = \frac{\partial \phi_\beta}{\partial \beta}, \quad k_{\alpha\beta}^0 = \frac{\partial \phi_\alpha}{\partial \beta} + \frac{\partial \phi_\beta}{\partial \alpha} \\
&k_{\alpha\alpha}^2 = c_1\,\left(\frac{\partial \phi_\alpha}{\partial \alpha} + \frac{\partial^2 w_0}{\partial \alpha^2}\right), \quad k_{\beta\beta}^2 = c_1\,\left(\frac{\partial \phi_\beta}{\partial \beta} + \frac{\partial^2 w_0}{\partial \beta^2}\right) \\
&k_{\beta\alpha}^2 = c_1\,\left(\frac{\partial \phi_\beta}{\partial \alpha} + \frac{\partial \phi_\alpha}{\partial \beta} + 2\,\frac{\partial^2 w_0}{\partial \alpha \partial \beta}\right), \quad k_{\alpha z}^1 = c_2\left(\phi_\alpha + \frac{w_0}{\partial \alpha}\right), \quad k_{\beta z}^1 = c_2\left(\phi_\beta + \frac{w_0}{\partial \beta}\right)
\end{aligned} \tag{43}$$

The stress-stain relationship of the k–layer is

$$\boldsymbol{\sigma}^k = \tilde{\boldsymbol{C}}^k\,\boldsymbol{\varepsilon}^k \tag{44}$$

where $\tilde{\boldsymbol{C}}^k$ is the plane-stress constitutive matrix (see [19]) and $\boldsymbol{\sigma}^k$ and $\boldsymbol{\varepsilon}^k$ are the stress and strain vectors, respectively, and defined as

$$\begin{aligned}
\boldsymbol{\sigma}^k &= \left\{\ \sigma_{\alpha\alpha}^k \quad \sigma_{\beta\beta}^k \quad \sigma_{\alpha\beta}^k \quad \sigma_{\alpha z}^k \quad \sigma_{\beta z}^k\ \right\}^T \\
\boldsymbol{\varepsilon}^k &= \left\{\ \varepsilon_{\alpha\alpha}^k \quad \varepsilon_{\beta\beta}^k \quad \gamma_{\alpha\beta}^k \quad \gamma_{\alpha z}^k \quad \gamma_{\beta z}^k\ \right\}^T
\end{aligned} \tag{45}$$

where T denotes a transpose.

5.1 Governing differential equations

The governing differential equations (GDEs) are derived by using Hamilton's Principle. The variational statement can be written as

$$\sum_{k=1}^{N_l} \int_{t_1}^{t_2} \delta\,\mathcal{L}^k\,dt = 0 \tag{46}$$

where $\mathcal{L}^k$ is the Lagrangian for the kth layer of the composite shell. The first variation can be expressed as

$$\delta\mathcal{L}^k = \delta T^k - \delta U^k \tag{47}$$

where δU^k is the virtual strain energy, δT^k is the virtual kinetic energy, and assume the following form

$$\begin{aligned} \delta U^k &= \int_{\Omega^k} \int_{z^k}^{z^{k+1}} \left(\delta\boldsymbol{\varepsilon}^{k^T}\,\boldsymbol{\sigma}^k\right)\,d\Omega^k\,dz; \\ \delta T^k &= \int_{\Omega^k} \int_{z^k}^{z^{k+1}} \left(\rho^k\,\delta\dot{\boldsymbol{\eta}}^T\,\dot{\boldsymbol{\eta}}\right)\,d\Omega^k\,dz \end{aligned} \tag{48}$$

where stresses $\left(\boldsymbol{\sigma}^k\right)$ and strains $\left(\boldsymbol{\varepsilon}^k\right)$ are defined in Eq. (45) and $\boldsymbol{\eta}$ is the displacement vector given by

$$\boldsymbol{\eta} = \left\{\begin{array}{ccc} u & v & w \end{array}\right\}^T \tag{49}$$

ρ^k denotes mass density while an over dot denotes differentiation with respect to time. The symbol δ represents the variational operator. Imposing the condition in Eq. (46), the equations of motion are derived

$$\begin{aligned}
\delta u_0 :\quad & \frac{\partial N_{\alpha\alpha}}{\partial\alpha} + \frac{\partial N_{\alpha\beta}}{\partial\beta} = I_0\,\frac{\partial^2 u_0}{\partial t^2} + \bar{I}_1\,\frac{\partial^2\phi_\alpha}{\partial t^2} - I_3\,\frac{\partial}{\partial\alpha}\left(\frac{\partial^2 w_0}{\partial t^2}\right); \\
\delta v_0 :\quad & \frac{\partial N_{\alpha\beta}}{\partial\alpha} + \frac{\partial N_{\beta\beta}}{\partial\beta} = I_0\,\frac{\partial^2 v_0}{\partial t^2} + \bar{I}_1\,\frac{\partial^2\phi_\beta}{\partial t^2} - I_3\,\frac{\partial}{\partial\beta}\left(\frac{\partial^2 w_0}{\partial t^2}\right); \\
\delta w_0 :\quad & \frac{\partial Q_{\alpha\alpha}}{\partial\alpha} + \frac{\partial Q_{\alpha\beta}}{\partial\beta} + c_2\left(\frac{\partial K_{\alpha\alpha}}{\partial\alpha} + \frac{\partial K_{\beta\beta}}{\partial\beta}\right) - c_1\left(\frac{\partial^2 P_{\alpha\alpha}}{\partial\alpha^2} + \frac{\partial^2 P_{\beta\beta}}{\partial\beta^2} + 2\,\frac{\partial^2 P_{\alpha\beta}}{\partial\alpha\,\beta}\right) - \\
& \frac{N_{\alpha\alpha}}{R_\alpha} - \frac{N_{\beta\beta}}{R_\beta} = -c_1\,I_3\left[\frac{\partial}{\partial\alpha}\left(\frac{\partial^2 u_0}{\partial t^2}\right) + \frac{\partial}{\partial\beta}\left(\frac{\partial^2 v_0}{\partial t^2}\right)\right] - c_1\,\bar{I}_4\Big[\frac{\partial}{\partial\alpha}\left(\frac{\partial^2\phi_\alpha}{\partial t^2}\right) + \\
& \frac{\partial}{\partial\beta}\left(\frac{\partial^2\phi_\beta}{\partial t^2}\right)\Big] + I_0\,\frac{\partial^2 w_0}{\partial t^2} - c_1^2\,I_6\left[\frac{\partial^2}{\partial\alpha^2}\left(\frac{\partial^2 w_0}{\partial t^2}\right) + \frac{\partial^2}{\partial\beta^2}\left(\frac{\partial^2 w_0}{\partial t^2}\right)\right];
\end{aligned} \tag{50}$$

$$
\begin{aligned}
\delta\phi_\alpha: \quad & \frac{\partial M_{\alpha\,\alpha}}{\partial\alpha} + \frac{\partial M_{\alpha\,\beta}}{\partial\beta} - Q_{\alpha\,\alpha} - c_2\,K_{\alpha\,\alpha} + c_1\left(\frac{\partial P_{\alpha\,\alpha}}{\partial\alpha} + \frac{\partial P_{\alpha\,\beta}}{\partial\beta}\right) = \bar{I}_1\,\frac{\partial^2 u_0}{\partial t^2} + \\
& I_2^*\,\frac{\partial^2\phi_\alpha}{\partial t^2} + c_1\,\bar{I}_4\,\frac{\partial}{\partial\alpha}\left(\frac{\partial^2 w_0}{\partial t^2}\right); \\
\delta\phi_\beta: \quad & \frac{\partial M_{\beta\,\alpha}}{\partial\alpha} + \frac{\partial M_{\beta\,\beta}}{\partial\beta} - Q_{\beta\,\beta} - c_2\,K_{\beta\,\beta} + c_1\left(\frac{\partial P_{\alpha\,\beta}}{\partial\alpha} + \frac{\partial P_{\beta\,\beta}}{\partial\beta}\right) = \bar{I}_1\,\frac{\partial^2 v_0}{\partial t^2} + \\
& I_2^*\,\frac{\partial^2\phi_\beta}{\partial t^2} + c_1\,\bar{I}_4\,\frac{\partial}{\partial\beta}\left(\frac{\partial^2 w_0}{\partial t^2}\right)
\end{aligned}
$$

The natural boundary conditions are

$$
\begin{aligned}
\delta u_0: \quad & N_n = n_\alpha^2\,N_{\alpha\,\alpha} + n_\beta^2\,N_{\beta\,\beta} + 2\,n_\alpha\,n_\beta\,N_{\alpha\,\beta} \\
\delta v_0: \quad & N_{ns} = (N_{\beta\,\beta} - N_{\alpha\,\alpha})\,n_\alpha\,n_\beta + N_{\alpha\,\beta}\left(n_\alpha^2 - n_\beta^2\right) \\
\delta w_0: \quad & \tilde{Q}_n = N_n\,\frac{\partial w_0}{\partial n} + N_{ns}\,\frac{\partial w_0}{\partial s} - c_1\left(\frac{\partial P_n}{\partial n} + \frac{\partial P_{ns}}{\partial s}\right) + Q_n + c_2\,K_n \\
\delta\phi_\alpha: \quad & \tilde{M}_n = M_n + c_1\,P_n \\
\delta\phi_\beta: \quad & \tilde{M}_{ns} = M_{ns} + c_1\,P_{ns} \\
\delta w_{0,\alpha}: \quad & \tilde{P}_n = c_1\,P_n \\
\delta w_{0,\beta}: \quad & \tilde{P}_{ns} = c_1\,P_{ns}
\end{aligned}
\tag{51}
$$

where

$$
\begin{aligned}
P_n &= n_\alpha^2\,P_{\alpha\,\alpha} + n_\beta^2\,P_{\beta\,\beta} + 2\,n_\alpha\,n_\beta\,P_{\alpha\,\beta} \\
P_{ns} &= (P_{\beta\,\beta} - P_{\alpha\,\alpha})\,n_\alpha\,n_\beta + P_{\alpha\,\beta}\left(n_\alpha^2 - n_\beta^2\right) \\
M_n &= n_\alpha^2\,M_{\alpha\,\alpha} + n_\beta^2\,M_{\beta\,\beta} + 2\,n_\alpha\,n_\beta\,M_{\alpha\,\beta} \\
M_{ns} &= (M_{\beta\,\beta} - M_{\alpha\,\alpha})\,n_\alpha\,n_\beta + M_{\alpha\,\beta}\left(n_\alpha^2 - n_\beta^2\right) \\
Q_n &= Q_{\alpha\,\alpha}\,n_\alpha + Q_{\beta\,\beta}\,n_\beta \\
K_n &= K_{\alpha\,\alpha}\,n_\alpha + K_{\beta\,\beta}\,n_\beta
\end{aligned}
\tag{52}
$$

and $c_1 = -\frac{4}{3\,h^2}$, $c_2 = -\frac{4}{h^2}$. Moreover, n_α, n_β are the direction cosines of the unit normal on the boundary of the laminate. The force and moment resultants appearing in Eqs. (50) and (51) are defined as follows:

$$
\begin{aligned}
(N_i,\,M_i,\,P_i) &= \sum_{k=1}^{Nl}\int_{z^k}^{z^{k+1}}\left(\sigma_i^k\left(1, z, z^3\right)\right)\mathrm{d}z \quad \text{with } i = \alpha\alpha, \beta\beta, \beta\alpha \\
(Q_{\alpha\alpha},\,K_{\alpha\alpha}) &= \sum_{k=1}^{Nl}\int_{z^k}^{z^{k+1}}\left(\sigma_{\alpha z}^k\left(1, z^2\right)\right)\mathrm{d}z \\
(Q_{\beta\beta},\,K_{\beta\beta}) &= \sum_{k=1}^{Nl}\int_{z^k}^{z^{k+1}}\left(\sigma_{\beta z}^k\left(1, z^2\right)\right)\mathrm{d}z
\end{aligned}
\tag{53}
$$

The inertia terms in Eq. (50) are defined as

$$\bar{I}_i = I_i + c_1\, I_{i+2}; \qquad I_2^* = I_2 + 2\, c_1\, I_4 + c_1^2\, I_6 \tag{54}$$

and

$$(I_0, I_1, I_2, I_3, I_4, I_6) = \sum_{k=1}^{N_l} \int_{z^k}^{z^{k+1}} \rho^k \left(1\ z,\ z^2,\ z^3,\ z^4,\ z^6\right)\, dz \tag{55}$$

The resultants in Eq. (53) cab be expressed in terms of strain components

$$\begin{aligned} \{N\} &= [A]\,\{\varepsilon^0\} + [B]\,\{k^0\} + [E]\,\{k^2\} \\ \{M\} &= [B]\,\{\varepsilon^0\} + [D]\,\{k^0\} + [F]\,\{k^2\} \\ \{P\} &= [E]\,\{\varepsilon^0\} + [F]\,\{k^0\} + [H]\,\{k^2\} \\ \{Q\} &= [A^s]\,\{\gamma^0\} + [D^s]\,\{k^1\} \\ \{K\} &= [D^s]\,\{\gamma^0\} + [F^s]\,\{k^1\} \end{aligned} \tag{56}$$

Where laminate stiffnesses are defined as

$$\begin{aligned} (A_{ij}, B_{ij}, D_{ij}, E_{ij}, F_{ij}, H_{ij}) &= \sum_{k=1}^{N_l} \int_{z^k}^{z^{k+1}} \tilde{C}_{ij}^k \left(1\ z,\ z^2,\ z^3,\ z^4,\ z^6\right)\, dz \quad \text{with } i,j = 1,2,6 \\ \left(A_{ij}^s, D_{ij}^s, F_{ij}^s\right) &= \sum_{k=1}^{N_l} \int_{z^k}^{z^{k+1}} \tilde{C}_{ij}^k \left(1\ z^2,\ z^4\right)\, dz \quad \text{with } i,j = 4,5 \end{aligned} \tag{57}$$

The resultants in Eq. (56) are collected as follows

$$\begin{aligned} &\{N\} = \{N_{\alpha\alpha},\, N_{\beta\beta},\, N_{\alpha\beta}\}; \quad \{M\} = \{M_{\alpha\alpha},\, M_{\beta\beta},\, M_{\alpha\beta}\}; \quad \{P\} = \{P_{\alpha\alpha},\, P_{\beta\beta},\, P_{\alpha\beta}\}; \\ &\{Q\} = \{Q_{\alpha z},\, Q_{\beta z}\}; \quad \{K\} = \{K_{\alpha z},\, K_{\beta z}\} \end{aligned} \tag{58}$$

similarly for the strains which are given as

$$\begin{aligned} &\{\varepsilon^0\} = \{\varepsilon_{\alpha\alpha}^0,\, \varepsilon_{\beta\beta}^0,\, \gamma_{\alpha\beta}^0\}; \quad \{k^0\} = \{k_{\alpha\alpha}^0,\, k_{\beta\beta}^0,\, k_{\alpha\beta}^0\}; \quad \{k^2\} = \{k_{\alpha\alpha}^2,\, k_{\beta\beta}^2,\, k_{\alpha\beta}^2\}; \\ &\{\gamma^0\} = \{\gamma_{\alpha z}^0,\, \gamma_{\beta z}^0\}; \quad \{k^1\} = \{k_{\alpha z}^1,\, k_{\beta z}^1\} \end{aligned} \tag{59}$$

For cross-ply stacking sequences an exact solution can be sought in Navier's or Lèvy's form. For such laminates $\tilde{C}_{16}^k = \tilde{C}_{26}^k = \tilde{C}_{45}^k = 0$. Moreover, if the stacking sequence is symmetric the coupling elastic coefficients B_{ij} and E_{ij} (see Eq. (57)) are zero.

5.2 Dynamic Stiffness Method

Once the equations of motion Eqs.(50) and the natural boundary conditions Eqs.(51) are derived, the classical procedure to carry out exact free vibration analysis of the present type consists of (i) solving the system of differential equations in Navier or Lèvy type closed form in an exact sense, (ii) applying particular boundary conditions on the edges and finally (iii) obtaining the frequency equation by eliminating the integration constants [20–23]. This method, although extremely useful for analyzing an individual element, it lacks generality and cannot be easily applied to complex structures for which researchers usually resort to approximate methods such as the FEM. In this respect, the dynamic stiffness method has no such limitations as it always retains the exactness of the solution even thought when it is applied to complex structures. This is because once the dynamic stiffness matrix of a structural element is developed, it can be offset and/or rotated and assembled in a global DS matrix in the same way as the FEM. This global DS matrix contains implicitly all the exact natural frequencies of the structure which can be computed by using the well established Wittrick-Williams algorithm [85].
A general procedure to develop the dynamic stiffness matrix of a structural element can be summarized as follows:

(i) Seek a closed form analytical solution of the governing differential equations of motion of the structural element under consideration in free vibration.

(ii) Apply a number of general boundary conditions that are equal to twice the number of integration constants; these are usually nodal displacements and forces in algebraic forms.

(iii) Eliminate the integration constants by relating the amplitudes of the harmonically varying nodal forces to those of the corresponding displacements which essentially generates the frequency-dependent dynamic stiffness matrix, providing the force-displacement relationship of the nodal lines.

5.3 DS formulation and Lèvy-type closed form exact solution

The solution of the GDEs is sought in the following form:

$$\begin{aligned}
u^0(\alpha,\beta,t) &= \sum_{m=1}^{\infty} U_m(\alpha)\,\mathrm{e}^{i\omega t}\sin(\hat{\theta}\,\beta); \qquad v^0(\alpha,\beta,t) = \sum_{m=1}^{\infty} V_m(\alpha)\,\mathrm{e}^{i\omega t}\cos(\hat{\theta}\,\beta);\\
w^0(\alpha,\beta,t) &= \sum_{m=1}^{\infty} W_m(\alpha)\,\mathrm{e}^{i\omega t}\sin(\hat{\theta}\,\beta); \qquad \phi_\alpha(\alpha,\beta,t) = \sum_{m=1}^{\infty} \Phi_{\alpha_m}(\alpha)\,\mathrm{e}^{i\omega t}\sin(\hat{\theta}\,\beta);\\
\phi_\beta(\alpha,\beta,t) &= \sum_{m=1}^{\infty} \Phi_{\beta_m}(\alpha)\,\mathrm{e}^{i\omega t}\cos(\hat{\theta}\,\beta)
\end{aligned} \tag{60}$$

where ω is the unknown circular frequency, $\hat{\theta} = \frac{m\pi}{b}$ and $m = 1, 2, \ldots, \infty$. This is the so-called Lèvy's solution which assumes that two of the opposite sides of the shell are simply supported (S-S), i.e. $v_0 = w_0 = \phi_\alpha = 0$ at $\beta = 0$ and $\beta = b$. Substituting Eq.

(60) in the GDEs a set of ordinary differential equations (ODEs) is obtained which can be written in matrix forms as follows:

$$\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} & \mathcal{L}_{14} & \mathcal{L}_{15} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} & \mathcal{L}_{24} & \mathcal{L}_{25} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} & \mathcal{L}_{34} & \mathcal{L}_{35} \\ \mathcal{L}_{41} & \mathcal{L}_{42} & \mathcal{L}_{43} & \mathcal{L}_{44} & \mathcal{L}_{45} \\ \mathcal{L}_{51} & \mathcal{L}_{52} & \mathcal{L}_{53} & \mathcal{L}_{54} & \mathcal{L}_{55} \end{bmatrix} \begin{bmatrix} U_m \\ V_m \\ W_m \\ \Phi_\alpha \\ \Phi_\beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{61}$$

where $\mathcal{L}_{ij}$ with $i, j = 1, \cdots, 6$ are differential operators following defined:

$$\begin{aligned}
&\mathcal{L}_{11} = A_{66}\,\alpha^2 - A_{11}\,\mathcal{D}_\alpha^2 - I_0\,\omega^2; \quad \mathcal{L}_{12} = (A_{12} + A_{66})\,\alpha\,\mathcal{D}_\alpha;\\
&\mathcal{L}_{13} = \left(-A_{11}\,\frac{1}{R_\alpha} - A_{12}\,\frac{1}{R_\beta}\right)\mathcal{D}_\alpha;\\
&\mathcal{L}_{14} = \mathcal{L}_{15} = \mathcal{L}_{24} = \mathcal{L}_{25} = \mathcal{L}_{41} = \mathcal{L}_{42} = \mathcal{L}_{51} = \mathcal{L}_{52} = 0;\\
&\mathcal{L}_{21} = \mathcal{L}_{12}; \quad \mathcal{L}_{22} = -A_{22}\,\alpha^2 + A_{66}\,\mathcal{D}_\alpha^2 + I_0\,\omega^2; \quad \mathcal{L}_{23} = \alpha\,\left(A_{12}\,\frac{1}{R_\alpha} + A_{22}\,\frac{1}{R_\beta}\right);\\
&\mathcal{L}_{31} = \mathcal{L}_{13}; \quad \mathcal{L}_{32} = \mathcal{L}_{23};\\
&\mathcal{L}_{33} = -c_1^2\mathcal{D}_\alpha^4 H_{11} + I_0\,\omega^2 + \mathcal{D}_\alpha^2\Big(A_{55} + 2\,c_2\,D_{55} + c_2^2\,F_{55} + 2\,\alpha^2\,c_1^2\,H_{12} + 4\,\alpha^2\,c_1^2\,H_{66}\\
&\quad - c_1^2\,I_6\,\omega^2\Big) - \alpha^2\,\left(A_{44} + c_2\,(2\,D_{44} + c_2\,F_{44}) + c_1^2\,\left(\alpha^2\,H_{22} - I_6\,\omega^2\right)\right) - A_{11}\,\frac{1}{R_\alpha^2}\\
&\quad - 2\,A_{12}\,\frac{1}{R_\alpha\,R_\beta} - A_{22}\,\frac{1}{R_\beta^2};\\
&\mathcal{L}_{34} = \left(-c_1\,F_{11} - c_1^2\,H_{11}\right)\,\mathcal{D}_\alpha^3 + \Big(A_{55} + 2\,c_2\,D_{55} + \alpha^2\,c_1\,F_{12} + c_2^2\,F_{55} + 2\,\alpha^2\,c_1 F_{66}\\
&\quad + \alpha^2\,c_1^2\,H_{12} + 2\alpha^2\,c_1^2\,H_{66} - c_1\,I_4\,\omega^2 - c_1^2\,I_6\,\omega^2\Big)\,\mathcal{D}_\alpha;\\
&\mathcal{L}_{35} = \mathcal{D}_\alpha^2\left(\alpha\,c_1\,F_{12} + 2\,\alpha\,c_1\,F_{66} + \alpha\,c_1^2\,H_{12} + 2\,\alpha\,c_1^2\,H_{66}\right) - \alpha\,\Big(A_{44} + c_2\,(2\,D_{44} + c_2\,F_{44});\\
&\quad + \alpha^2\,c_1\,(F_{22} + c_1\,H_{22}) - c_1\,(I_4 + c_1\,I_6)\,\omega^2\Big);\\
&\mathcal{L}_{43} = -\mathcal{L}_{34}; \quad \mathcal{L}_{53} = -\mathcal{L}_{35}; \quad \mathcal{L}_{54} = -\mathcal{L}_{45};\\
&\mathcal{L}_{44} = -A_{55} - c_2\,(2\,D_{55} + c_2\,F_{55}) + \mathcal{D}_\alpha^2\,\left(D_{11} + 2\,c_1\,F_{11} + c_1^2\,H_{11}\right) - \alpha^2\,\Big(D_{66} + 2\,c_1\,F_{66}\\
&\quad + c_1^2\,H_{66}\Big) + \left(I_2 + 2\,c_1\,I_4 + c_1^2\,I_6\right)\,\omega^2;\\
&\mathcal{L}_{45} = \mathcal{D}_\alpha\,\left(-\alpha\,D_{12} - \alpha\,D_{66} - 2\,\alpha\,c_1\,F_{12} - 2\,\alpha\,c_1 F_{66} - \alpha\,c_1^2\,H_{12} - \alpha\,c_1^2\,H_{66}\right);\\
&\mathcal{L}_{55} = -A_{44} - c_2\,(2\,D_{44} + c_2\,F_{44}) - \alpha^2\,(D_{22} + c_1\,(2\,F_{22} + c_1\,H_{22})) + \mathcal{D}_\alpha^2\,(D_{66} + c_1\\
&\quad (2\,F_{66} + c_1\,H_{66})) + (I_2 + c_1\,(2\,I_4 + c_1\,I_6))\,\omega^2
\end{aligned} \tag{62}$$

where $\mathcal{D}_\alpha = \frac{\mathrm{d}}{\mathrm{d}\alpha}$, c_1 and c_2 have already been given previously, and A_{ij}, B_{ij}, C_{ij}, D_{ij}, E_{ij}, F_{ij}, H_{ij} have already been defined in Eq. (57). Expanding the determinant of the matrix in Eq. (61) the following ordinary differential equation is obtained:

$$\left(a_1\,\mathcal{D}_\alpha^{12} + a_2\,\mathcal{D}_\alpha^{10} + a_3\,\mathcal{D}_\alpha^{8} + a_4\,\mathcal{D}_\alpha^{6} + a_5\,\mathcal{D}_\alpha^{4} + a_6\,\mathcal{D}_\alpha^{2} + a_7\right)\,\Psi = 0 \tag{63}$$

where

$$\Psi = U_m \;\text{ or }\; V_m \;\text{ or }\; W_m \;\text{ or }\; \Phi_\alpha \;\text{ or }\; \Phi_\beta \tag{64}$$

Using a trial solution $\mathrm{e}^{\Psi\,\lambda}$ in Eq. (63) yields the following auxiliary equation:

$$a_1\,\lambda^{12} + a_2\,\lambda^{10} + a_3\,\lambda^{8} + a_4\,\lambda^{6} + a_4\,\lambda^{4} + a_6\,\lambda^{2} + a_7 = 0 \tag{65}$$

the polynomial coefficients a_j with $j = 1, \cdots, 7$ have been derived in symbolic form. The details are not reported here for brevity. Numerically some roots of Eq. (65) may turn out to be complex. As complex roots occur in conjugate pairs, the associated coefficients will also occur in conjugate pairs. The solution of the system of ordinary differential equations can thus be written as:

$$\begin{aligned} &U_m(\alpha) = \sum_{i=1}^{12} \mathscr{A}_i\,\mathrm{e}^{\lambda_i\,\alpha}; \quad V_m(\alpha) = \sum_{i=1}^{12} \mathscr{B}_i\,\mathrm{e}^{\lambda_i\,\alpha}; \quad W_m(\alpha) = \sum_{i=1}^{12} \mathscr{C}_i\,\mathrm{e}^{\lambda_i\,\alpha}; \\ &\Phi_\alpha(\alpha) = \sum_{i=1}^{12} \mathscr{D}_i\,\mathrm{e}^{\lambda_i\,\alpha}; \quad \Phi_\beta(\alpha) = \sum_{i=1}^{12} \mathscr{E}_i\,\mathrm{e}^{\lambda_i\,\alpha} \end{aligned} \tag{66}$$

where $\mathscr{A}_i$, $\mathscr{B}_i$, $\mathscr{C}_i$, $\mathscr{D}_i$, $\mathscr{E}_i$ with $i = 1, \cdots, 12$ are integration constants. Note that as a results of Eq. (61) the sets of the constants are not all independent. Thus a set of only twelve independent constants can be chosen and they can be related to the other sets. This choice is completely arbitrary. In the present case the set, $\mathscr{D}_i$, is chosen to be a base. By substituting Eqs. (66) into (61) the following relationships can be obtained:

$$\mathscr{A}_i = \gamma_i\,\mathscr{C}_i = \gamma_i\,\chi_i\,\mathscr{D}_i; \qquad \mathscr{B}_i = \delta_i\,\mathscr{C}_i = \delta_i\,\chi_i\,\mathscr{D}_i; \qquad \mathscr{C}_i = \chi_i\,\mathscr{D}_i; \qquad \mathscr{E}_i = \phi_i\,\mathscr{D}_i \tag{67}$$

where

$$\gamma_i = -\frac{\lambda_i\left(A_{12}^2\alpha^2 R_\alpha + A_{11}\left(-A_{22}\alpha^2 + I_0\omega^2 + A_{66}\lambda_i^2\right)R_\alpha + A_{22}A_{66}\alpha^2 R_\beta + A_{12}\left(A_{66}\alpha^2 R_\alpha + I_0\omega^2 R_\beta + A_{66}\lambda_i^2 R_\beta\right)\right)}{\left(A_{22}\alpha^2 - I_0\omega^2\right)\left(A_{66}\alpha^2 - I_0\omega^2\right) + \left(\left(A_{12}^2 - A_{11}A_{22} + 2A_{12}A_{66}\right)\alpha^2 + (A_{11} + A_{66})I_0\omega^2\right)\lambda_i^2 + A_{11}A_{66}\lambda_i^4}$$

$$\delta_i = -\frac{\alpha\left(-(A_{12} + A_{66})\lambda_i^2(A_{11}R_\alpha + A_{12}R_\beta) - \left(A_{66}\alpha^2 - I_0\omega^2 - A_{11}\lambda_i^2\right)\left(A_{12}R_\alpha + \mathrm{A22}R_\beta\right)\right)}{(A_{12}\alpha\lambda_i + A_{66}\alpha\lambda_i)^2 - \left(A_{66}\alpha^2 - I_0\omega^2 - A_{11}\lambda_i^2\right)\left(-\mathrm{A22}\alpha^2 + I_0\omega^2 + A_{66}\lambda_i^2\right)}$$

(68)

$$\chi_i = -\Big(- \alpha^2(D_{12} + D_{66} + c_1(2F_{12} + 2F_{66} + c_1(H_{12} + H_{66})))^2\lambda_i^2 + \Big(- A_{55} - c_2(2D_{55} + c_2F_{55}) - \alpha^2(D_{66} + c_1(2F_{66} + c_1H_{66})) + (I_2 + c_1(2I_4 + c_1I_6))\omega^2 + (D_{11} + c_1(2F_{11} + c_1H_{11}))\lambda_i^2\Big)\Big(A_{44} + c_2(2D_{44} + c_2F_{44}) + \alpha^2(D_{22} + c_1(2F_{22} + c_1H_{22})) - (I_2 + c_1(2I_4 + c_1I_6))\omega^2 - (D_{66} + c_1(2F_{66} + c_1H_{66}))\lambda_i^2\Big)\Big)/\Big(- \lambda_i\Big(A_{44} + c_2(2D_{44} + c_2F_{44}) + \alpha^2(D_{22} + c_1(2F_{22} + c_1H_{22})) - (I_2 + c_1(2I_4 + c_1I_6))\omega^2 - (D_{66} + c_1(2F_{66} + c_1H_{66}))\lambda_i^2\Big)\Big(A_{55} + 2c_2D_{55} + c_2^2F_{55} + c_1\left(\alpha^2(F_{12} + 2F_{66} + c_1(H_{12} + 2H_{66})) - (I_4 + c_1I_6)\omega^2 - (F_{11} + c_1H_{11})\lambda_i^2\right)\Big) + \alpha^2(D_{12} + D_{66} + c_1(2F_{12} + 2F_{66} + c_1(H_{12} + H_{66})))\lambda_i\Big(A_{44} + 2c_2D_{44} + c_2^2F_{44} + c_1\Big(\alpha^2(F_{22} + c_1H_{22}) - (I_4 + c_1I_6)\omega^2 - (F_{12} + 2F_{66} + c_1(H_{12} + 2H_{66}))\lambda_i^2\Big)\Big)\Big)$$

$$\phi_i = -\frac{1}{\alpha(D_{12} + D_{66} + c_1(2F_{12} + 2F_{66} + c_1(H_{12} + H_{66})))\lambda_i}\Big(A_{55} + 2c_2D_{55} + \alpha^2D_{66} + c_2^2F_{55} + 2\alpha^2c_1F_{66} + \alpha^2c_1^2H_{66} - I_2\omega^2 - 2c_1I_4\omega^2 - c_1^2I_6\omega^2 - D_{11}\lambda_i^2 - 2c_1F_{11}\lambda_i^2 - c_1^2H_{11}\lambda_i^2 - \Big(\alpha^2(D_{12} + D_{66} + c_1(2F_{12} + 2F_{66} + c_1(H_{12} + H_{66})))^2\lambda_i^3\Big(- A_{55} - c_2(2D_{55} + c_2F_{55}) + c_1\Big(- \alpha^2(F_{12} + 2F_{66} + c_1(H_{12} + 2H_{66})) + (I_4 + c_1I_6)\omega^2 + (F_{11} + c_1H_{11})\lambda_i^2\Big)\Big)\Big)/\Big(- \lambda_i\Big(A_{44} + c_2(2D_{44} + c_2F_{44}) + \alpha^2(D_{22} + c_1(2F_{22} + c_1H_{22})) - (I_2 + c_1(2I_4 + c_1I_6))\omega^2 - (D_{66} + c_1(2F_{66} + c_1H_{66}))\lambda_i^2\Big)\Big(A_{55} + 2c_2D_{55} + c_2^2F_{55} + c_1\Big(\alpha^2(F_{12} + 2F_{66} + c_1(H_{12} + 2H_{66})) - (I_4 + c_1I_6)\omega^2 - (F_{11} + c_1H_{11})\lambda_i^2\Big)\Big) + \alpha^2(D_{12} + D_{66} + c_1(2F_{12} + 2F_{66} + c_1(H_{12} + H_{66})))\lambda_i\Big(A_{44} + 2c_2D_{44} + c_2^2F_{44} + c_1\Big(\alpha^2(F_{22} + c_1H_{22}) - (I_4 + c_1I_6)\omega^2 - (F_{12} + 2F_{66} + c_1(H_{12} + 2H_{66}))\lambda_i^2\Big)\Big)\Big) - \Big(\lambda_i\Big(A_{55} + c_2(2D_{55} + c_2F_{55}) + \alpha^2(D_{66} + c_1(2F_{66} + c_1H_{66})) - (I_2 + c_1(2I_4 + c_1I_6))\omega^2 - (D_{11} + c_1(2F_{11} + c_1H_{11}))\lambda_i^2\Big)\Big(A_{44} + c_2(2D_{44} + c_2F_{44}) + \alpha^2(D_{22} + c_1(2F_{22} + c_1H_{22})) - (I_2 + c_1(2I_4 + c_1I_6))\omega^2 - (D_{66} + c_1(2F_{66} + c_1H_{66}))\lambda_i^2\Big)\Big(- A_{55} - c_2(2D_{55} + c_2F_{55}) + c_1\Big(- \alpha^2(F_{12} + 2F_{66} + c_1(H_{12} + 2H_{66})) + (I_4 + c_1I_6)\omega^2 + (F_{11} + c_1H_{11})\lambda_i^2\Big)\Big)\Big)/\Big(- \lambda_i\Big(A_{44} + c_2(2D_{44} + c_2F_{44}) + \alpha^2(D_{22} + c_1(2F_{22} + c_1H_{22})) - (I_2 + c_1(2I_4 + c_1I_6))\omega^2 - (D_{66} + c_1(2F_{66} + c_1H_{66}))\lambda_i^2\Big)\Big(A_{55} + 2c_2D_{55} + c_2^2F_{55} + c_1\Big(\alpha^2(F_{12} + 2F_{66} + c_1(H_{12} + 2H_{66})) - (I_4 + c_1I_6)\omega^2 - (F_{11} + c_1H_{11})\lambda_i^2\Big)\Big) + \alpha^2(D_{12} + D_{66} + c_1(2F_{12} + 2F_{66} + c_1(H_{12} + H_{66})))\lambda_i\Big(A_{44} + 2c_2D_{44} + c_2^2F_{44} + c_1\Big(\alpha^2(F_{22} + c_1H_{22}) - (I_4 + c_1I_6)\omega^2 - (F_{12} + 2F_{66} + c_1(H_{12} + 2H_{66}))\lambda_i^2\Big)\Big)\Big)\Big)$$

with $i = 1, \cdots, 12$. When Eqs. (67) are substituted into Eqs. (66) a solution in terms of only twelve integration constants is obtained. Thus

$$\begin{aligned}
U_m(\alpha) &= \sum_{i=1}^{12} \gamma_i\ \mathscr{C}_i\ \mathrm{e}^{\lambda_i \alpha} = \sum_{i=1}^{12} \gamma_i\ \chi_i\ \mathscr{D}_i\ \mathrm{e}^{\lambda_i \alpha} \\
V_m(\alpha) &= \sum_{i=1}^{12} \delta_i\ \mathscr{C}_i\ \mathrm{e}^{\lambda_i \alpha} = \sum_{i=1}^{12} \delta_i\ \chi_i\ \mathscr{D}_i\ \mathrm{e}^{\lambda_i \alpha} \\
W_m(\alpha) &= \sum_{i=1}^{12} \chi_i\ \mathscr{D}_i\ \mathrm{e}^{\lambda_i \alpha}, \qquad \Phi_\alpha(\alpha) = \sum_{i=1}^{12} \mathscr{D}_i\ \mathrm{e}^{\lambda_i \alpha}, \qquad \Phi_\beta(\alpha) = \sum_{i=1}^{12} \phi_i\ \mathscr{D}_i\ \mathrm{e}^{\lambda_i \alpha}
\end{aligned} \tag{69}$$

The expressions for the generalized forces can then be found in the same way by substituting Eqs. (69) into Eqs. (51) written in terms of displacements. Thus, retaining the terms according to the symmetric cross-ply composite shallow shells the following expressions are derived:

$$N_{\alpha\alpha}(\alpha,\beta) = \sum_{m=1}^{\infty}\Big[A_{11}\sum_{i=1}^{12}\lambda_i\,\gamma_i\,\chi_i\,\mathscr{D}_i\,\mathrm{e}^{\lambda_i\,\alpha} - A_{12}\,\hat{\theta}\sum_{i=1}^{12}\delta_i\,\chi_i\,\mathscr{D}_i\,\mathrm{e}^{\lambda_i\,\alpha}$$
$$+\frac{A_{12}}{R_\beta}\sum_{i=1}^{12}\lambda_i\,\chi_i\,\mathscr{D}_i\,\mathrm{e}^{\lambda_i\,\alpha} + \frac{A_{11}}{R_\alpha}\sum_{i=1}^{12}\lambda_i\,\chi_i\,\mathscr{D}_i\,\mathrm{e}^{\lambda_i\,\alpha}\Big]\sin(\hat{\theta}\,\beta) = \mathcal{N}_{\alpha\alpha}\sin(\hat{\theta}\,\beta)$$

$$N_{\beta\beta}(\alpha,\beta) = \sum_{m=1}^{\infty}\left[A_{66}\sum_{i=1}^{12}\hat{\theta}\,\gamma_i\,\chi_i\,\mathscr{D}_i\,\mathrm{e}^{\lambda_i\,\alpha} + A_{66}\sum_{i=1}^{12}\lambda_i\,\delta_i\,\chi_i\,\mathscr{D}_i\,\mathrm{e}^{\lambda_i\,\alpha}\right]\cos(\hat{\theta}\,\beta)$$
$$= \mathcal{N}_{\beta\beta}\cos(\hat{\theta}\,\beta)$$

$$Q_{\alpha}(\alpha,\beta) = \sum_{m=1}^{\infty}\Big[H_{11}\sum_{i=1}^{12}c_2^2\,\lambda_i^2\,\mathscr{D}_i\,\mathrm{e}^{\lambda_i\,\alpha} + H_{11}\sum_{i=1}^{12}c_2^2\,\lambda_i^3\,\chi_i\,\mathscr{D}_i\,\mathrm{e}^{\lambda_i\,\alpha} + F_{11}\sum_{i=1}^{12}c_2\,\lambda_i^2\,\mathscr{D}_i\,\mathrm{e}^{\lambda_i\,\alpha}$$
$$- F_{12}\sum_{i=1}^{12}c_2\,\lambda_i\,\phi_i\,\mathscr{D}_i\,\mathrm{e}^{\lambda_i\,\alpha} - H_{12}\sum_{i=1}^{12}c_2^2\,\lambda_i\,\phi_i\,\mathscr{D}_i\,\mathrm{e}^{\lambda_i\,\alpha} - H_{12}\sum_{i=1}^{12}c_2^2\,\lambda_i\,\chi_i\,\mathscr{D}_i\,\mathrm{e}^{\lambda_i\,\alpha}$$
$$+ 4\,H_{66}\sum_{i=1}^{12}c_2^2\,\lambda_i\,\chi_i\,\mathscr{D}_i\,\mathrm{e}^{\lambda_i\,\alpha} - 2\,F_{66}\sum_{i=1}^{12}c_2\,\mathscr{D}_i\,\mathrm{e}^{\lambda_i\,\alpha} - 2\,F_{66}\sum_{i=1}^{12}c_2\,\lambda_i\,\phi_i\,\mathscr{D}_i\,\mathrm{e}^{\lambda_i\,\alpha}$$
$$- A_{55}\sum_{i=1}^{12}\mathscr{D}_i\,\mathrm{e}^{\lambda_i\,\alpha} - A_{55}\sum_{i=1}^{12}\lambda_i\,\chi_i\,\mathscr{D}_i\,\mathrm{e}^{\lambda_i\,\alpha} - 2\,D_{55}\sum_{i=1}^{12}c_2\,\mathscr{D}_i\,\mathrm{e}^{\lambda_i\,\alpha}$$
$$- 2\,D_{55}\sum_{i=1}^{12}c_2\,\lambda_i\,\chi_i\,\mathscr{D}_i\,\mathrm{e}^{\lambda_i\,\alpha} - F_{55}\sum_{i=1}^{12}c_2^2\,\mathscr{D}_i\,\mathrm{e}^{\lambda_i\,\alpha} - F_{55}\sum_{i=1}^{12}c_2^2\,\lambda_i\,\chi_i\,\mathscr{D}_i\,\mathrm{e}^{\lambda_i\,\alpha}\Big]$$
$$\times\sin(\hat{\theta}\,\beta) = \mathcal{Q}_{\alpha}\sin(\hat{\theta}\,\beta)$$

$$M_{\alpha\alpha}(\alpha,\beta) = \sum_{m=1}^{\infty}\Big[D_{11}\sum_{i=1}^{12}\lambda_i\,\mathscr{D}_i\,\mathrm{e}^{\lambda_i\,\alpha} + H_{11}\sum_{i=1}^{12}c_2^2\,\lambda_i\,\mathscr{D}_i\,\mathrm{e}^{\lambda_i\,\alpha} + H_{11}\sum_{i=1}^{12}c_2^2\,\lambda_i^2\,\chi_i\,\mathscr{D}_i\,\mathrm{e}^{\lambda_i\,\alpha}$$
$$+ 2\,F_{11}\sum_{i=1}^{12}c_2\,\lambda_i\,\mathscr{D}_i\,\mathrm{e}^{\lambda_i\,\alpha} + 2\,F_{11}\sum_{i=1}^{12}c_2\,\lambda_i^2\,\chi_i\,\mathscr{D}_i\,\mathrm{e}^{\lambda_i\,\alpha} - D_{12}\sum_{i=1}^{12}\phi_i\,\mathscr{D}_i\,\mathrm{e}^{\lambda_i\,\alpha}$$
$$- 2\,F_{12}\sum_{i=1}^{12}c_2\,\phi_i\,\mathscr{D}_i\,\mathrm{e}^{\lambda_i\,\alpha} - F_{12}\sum_{i=1}^{12}c_2\,\chi_i\,\mathscr{D}_i\,\mathrm{e}^{\lambda_i\,\alpha} - H_{12}\sum_{i=1}^{12}c_2^2\,\phi_i\,\mathscr{D}_i\,\mathrm{e}^{\lambda_i\,\alpha}$$
$$- H_{12}\sum_{i=1}^{12}c_2^2\,\chi_i\,\mathscr{D}_i\,\mathrm{e}^{\lambda_i\,\alpha}\Big]\sin(\hat{\theta}\,\beta) = \mathcal{M}_{\alpha\alpha}\sin(\hat{\theta}\,\beta)$$

$$\begin{aligned} M_{\beta\beta}(\alpha,\beta) = \sum_{m=1}^{\infty} \Big[& D_{12} \sum_{i=1}^{12} \lambda_i\, \phi_i\, \mathscr{D}_i\, \mathrm{e}^{\lambda_i\, \alpha} + D_{66} \sum_{i=1}^{12} \mathscr{D}_i\, \mathrm{e}^{\lambda_i\, \alpha} + D_{66} \sum_{i=1}^{12} \phi_i\, \mathscr{D}_i\, \mathrm{e}^{\lambda_i\, \alpha} \\ & + D_{12} \sum_{i=1}^{12} \lambda_i\, \phi_i\, \mathscr{D}_i\, \mathrm{e}^{\lambda_i\, \alpha} + 2\, F_{66} \sum_{i=1}^{12} c_2\, \mathscr{D}_i\, \mathrm{e}^{\lambda_i\, \alpha} + 2\, F_{66} \sum_{i=1}^{12} c_2\, \phi_i\, \mathscr{D}_i\, \mathrm{e}^{\lambda_i\, \alpha} \\ & + 2\, F_{66} \sum_{i=1}^{12} c_2\, \lambda_i\, \chi_i\, \mathscr{D}_i\, \mathrm{e}^{\lambda_i\, \alpha} + H_{66} \sum_{i=1}^{12} c_2^2\, \mathscr{D}_i\, \mathrm{e}^{\lambda_i\, \alpha} + H_{66} \sum_{i=1}^{12} c_2^2\, \phi_i\, \mathscr{D}_i\, \mathrm{e}^{\lambda_i\, \alpha} \\ & + 2\, H_{66} \sum_{i=1}^{12} c_2^2\, \lambda_i\, \chi_i\, \mathscr{D}_i\, \mathrm{e}^{\lambda_i\, \alpha} \Big] \cos(\hat{\theta}\, \beta) = \mathcal{M}_{\beta\beta} \cos(\hat{\theta}\, \beta) \end{aligned} \tag{70}$$

$$\begin{aligned} P_{\alpha\alpha}(\alpha,\beta) = \sum_{m=1}^{\infty} \Big[& H_{11} \sum_{i=1}^{12} c_2^2\, \lambda_i\, \mathscr{D}_i\, \mathrm{e}^{\lambda_i\, \alpha} + H_{11} \sum_{i=1}^{12} c_2^2\, \lambda_i^2\, \chi_i\, \mathscr{D}_i\, \mathrm{e}^{\lambda_i\, \alpha} F_{11} \sum_{i=1}^{12} c_2\, \lambda_i\, \mathscr{D}_i\, \mathrm{e}^{\lambda_i\, \alpha} \\ & - F_{12} \sum_{i=1}^{12} c_2\, \phi_i\, \mathscr{D}_i\, \mathrm{e}^{\lambda_i\, \alpha} - H_{12} \sum_{i=1}^{12} c_2^2\, \phi_i\, \mathscr{D}_i\, \mathrm{e}^{\lambda_i\, \alpha} - H_{12} \sum_{i=1}^{12} c_2^2\, \chi_i\, \mathscr{D}_i\, \mathrm{e}^{\lambda_i\, \alpha} \Big] \\ & \times \sin(\hat{\theta}\, \beta) = \mathcal{P}_{\alpha\alpha} \sin(\hat{\theta}\, \beta) \end{aligned}$$

Zero boundary conditions are then generally imposed to eliminate the constants in the classical method in order to establish the frequency equation for a single plate element. By contrast, the development of the dynamic stiffness matrix entails imposition of general boundary conditions in algebraic form which has much wider implications. Thus in order to develop the dynamic stiffness matrix the following boundary conditions are applied:

Generalized displacements

$$\begin{aligned} \alpha = 0: \quad & U_m = U_{m_1}; \quad V_m = V_{m_1}; \quad W_m = W_{m_1}; \quad \Phi_{\alpha_m} = \Phi_{\alpha_1}; \quad \Phi_{\beta_m} = \Phi_{\beta_1}; \\ & W_{m_{,\alpha}} = W_{m1,\alpha} \\ \alpha = L: \quad & U_m = U_{m_2}; \quad V_m = V_{m_2}; \quad W_m = W_{m_2}; \quad \Phi_{\alpha_m} = \Phi_{\alpha_2}; \quad \Phi_{\beta_m} = \Phi_{\beta_2}; \\ & W_{m_{,\alpha}} = W_{m2,\alpha} \end{aligned} \tag{71}$$

Generalized forces

$$\begin{aligned} \alpha = 0: \quad & \mathcal{N}_{\alpha\alpha} = -\mathcal{N}_{\alpha\alpha_1}; \quad \mathcal{N}_{\beta\beta} = -\mathcal{N}_{\beta\beta_1}; \quad \mathcal{Q}_{\alpha} = -\mathcal{Q}_{\alpha_1}; \quad \mathcal{M}_{\alpha\alpha} = -\mathcal{M}_{\alpha\alpha_1}; \\ & \mathcal{M}_{\alpha\beta} = -\mathcal{M}_{\alpha\beta_1}; \quad \mathcal{P}_{\alpha\alpha} = -\mathcal{P}_{\alpha\alpha_1}; \\ \alpha = L: \quad & \mathcal{N}_{\alpha\alpha} = \mathcal{N}_{\alpha\alpha_2}; \quad \mathcal{N}_{\beta\beta} = \mathcal{N}_{\beta\beta_2}; \quad \mathcal{Q}_{\alpha} = \mathcal{Q}_{\alpha_2}, \quad \mathcal{M}_{\alpha\alpha} = \mathcal{M}_{\alpha\alpha_2}; \\ & \mathcal{M}_{\alpha\beta} = \mathcal{M}_{\alpha\beta_2}; \quad \mathcal{P}_{\alpha\alpha} = \mathcal{P}_{\alpha\alpha_2} \end{aligned} \tag{72}$$

The imposition of Eq. (71) leads to the following relationship:

$$\boldsymbol{\delta} = \boldsymbol{A}\,\boldsymbol{D} \tag{73}$$

where

$$\boldsymbol{\delta} = \left\{\; U_{m_1} \quad V_{m_1} \quad W_{m_1} \quad \Phi_{\alpha_1} \quad \Phi_{\beta_1} \quad W_{m1,\alpha} \quad U_{m_2} \quad V_{m_2} \quad W_{m_2} \quad \Phi_{\alpha_2} \quad \Phi_{\beta_2} \quad W_{m2,\alpha} \;\right\}^T$$

$$\boldsymbol{D} = \left\{\; \mathscr{D}_1 \quad \mathscr{D}_2 \quad \mathscr{D}_3 \quad \mathscr{D}_4 \quad \mathscr{D}_5 \quad \mathscr{D}_6 \quad \mathscr{D}_7 \quad \mathscr{D}_8 \quad \mathscr{D}_9 \quad \mathscr{D}_{10} \quad \mathscr{D}_{11} \quad \mathscr{D}_{12} \;\right\}^T \tag{74}$$

and

$$\boldsymbol{A} = \begin{bmatrix}
\mathcal{A}_1^1 & \mathcal{A}_1^2 & \mathcal{A}_1^3 & \mathcal{A}_1^4 & \mathcal{A}_1^5 & \mathcal{A}_1^6 & \mathcal{A}_1^7 & \mathcal{A}_1^8 & \mathcal{A}_1^9 & \mathcal{A}_1^{10} & \mathcal{A}_1^{11} & \mathcal{A}_1^{12} \\
\mathcal{A}_2^1 & \mathcal{A}_2^2 & \mathcal{A}_2^3 & \mathcal{A}_2^4 & \mathcal{A}_2^5 & \mathcal{A}_2^6 & \mathcal{A}_2^7 & \mathcal{A}_2^8 & \mathcal{A}_2^9 & \mathcal{A}_2^{10} & \mathcal{A}_2^{11} & \mathcal{A}_2^{12} \\
\mathcal{A}_3^1 & \mathcal{A}_3^2 & \mathcal{A}_3^3 & \mathcal{A}_3^4 & \mathcal{A}_3^5 & \mathcal{A}_3^6 & \mathcal{A}_3^7 & \mathcal{A}_3^8 & \mathcal{A}_3^9 & \mathcal{A}_3^{10} & \mathcal{A}_3^{11} & \mathcal{A}_3^{12} \\
\mathcal{A}_4^1 & \mathcal{A}_4^2 & \mathcal{A}_4^3 & \mathcal{A}_4^4 & \mathcal{A}_4^5 & \mathcal{A}_4^6 & \mathcal{A}_4^7 & \mathcal{A}_4^8 & \mathcal{A}_4^9 & \mathcal{A}_4^{10} & \mathcal{A}_4^{11} & \mathcal{A}_4^{12} \\
\mathcal{A}_5^1 & \mathcal{A}_5^2 & \mathcal{A}_5^3 & \mathcal{A}_5^4 & \mathcal{A}_5^5 & \mathcal{A}_5^6 & \mathcal{A}_5^7 & \mathcal{A}_5^8 & \mathcal{A}_5^9 & \mathcal{A}_5^{10} & \mathcal{A}_5^{11} & \mathcal{A}_5^{12} \\
\mathcal{A}_6^1 & \mathcal{A}_6^2 & \mathcal{A}_6^3 & \mathcal{A}_6^4 & \mathcal{A}_6^5 & \mathcal{A}_6^6 & \mathcal{A}_6^7 & \mathcal{A}_6^8 & \mathcal{A}_6^9 & \mathcal{A}_6^{10} & \mathcal{A}_6^{11} & \mathcal{A}_6^{12} \\
\mathcal{A}_7^1 & \mathcal{A}_7^2 & \mathcal{A}_7^3 & \mathcal{A}_7^4 & \mathcal{A}_7^5 & \mathcal{A}_7^6 & \mathcal{A}_7^7 & \mathcal{A}_7^8 & \mathcal{A}_7^9 & \mathcal{A}_7^{10} & \mathcal{A}_7^{11} & \mathcal{A}_7^{12} \\
\mathcal{A}_8^1 & \mathcal{A}_8^2 & \mathcal{A}_8^3 & \mathcal{A}_8^4 & \mathcal{A}_8^5 & \mathcal{A}_8^6 & \mathcal{A}_8^7 & \mathcal{A}_8^8 & \mathcal{A}_8^9 & \mathcal{A}_8^{10} & \mathcal{A}_8^{11} & \mathcal{A}_8^{12} \\
\mathcal{A}_9^1 & \mathcal{A}_9^2 & \mathcal{A}_9^3 & \mathcal{A}_9^4 & \mathcal{A}_9^5 & \mathcal{A}_9^6 & \mathcal{A}_9^7 & \mathcal{A}_9^8 & \mathcal{A}_9^9 & \mathcal{A}_9^{10} & \mathcal{A}_9^{11} & \mathcal{A}_9^{12} \\
\mathcal{A}_{10}^1 & \mathcal{A}_{10}^2 & \mathcal{A}_{10}^3 & \mathcal{A}_{10}^4 & \mathcal{A}_{10}^5 & \mathcal{A}_{10}^6 & \mathcal{A}_{10}^7 & \mathcal{A}_{10}^8 & \mathcal{A}_{10}^9 & \mathcal{A}_{10}^{10} & \mathcal{A}_{10}^{11} & \mathcal{A}_{10}^{12} \\
\mathcal{A}_{11}^1 & \mathcal{A}_{11}^2 & \mathcal{A}_{11}^3 & \mathcal{A}_{11}^4 & \mathcal{A}_{11}^5 & \mathcal{A}_{11}^6 & \mathcal{A}_{11}^7 & \mathcal{A}_{11}^8 & \mathcal{A}_{11}^9 & \mathcal{A}_{11}^{10} & \mathcal{A}_{11}^{11} & \mathcal{A}_{11}^{12} \\
\mathcal{A}_{12}^1 & \mathcal{A}_{12}^2 & \mathcal{A}_{12}^3 & \mathcal{A}_{12}^4 & \mathcal{A}_{12}^5 & \mathcal{A}_{12}^6 & \mathcal{A}_{12}^7 & \mathcal{A}_{12}^8 & \mathcal{A}_{12}^9 & \mathcal{A}_{12}^{10} & \mathcal{A}_{12}^{11} & \mathcal{A}_{12}^{12}
\end{bmatrix} \tag{75}$$

where:

$$\begin{aligned}
&\mathcal{A}_1^i = (-1)^{i+1}\ \chi_i\ \gamma_i, && \mathcal{A}_2^i = (-1)^{i+1}\ \chi_i\ \delta_i, && \mathcal{A}_3^i = (-1)^{i+1}\ \chi_i \\
&\mathcal{A}_4^i = 1, && \mathcal{A}_5^i = (-1)^{i+1}\ \phi_i, && \mathcal{A}_6^i = \chi_i\ \lambda_i, \\
&\mathcal{A}_7^i = (-1)^{i+1}\ \chi_i\ \gamma_i\ \mathrm{e}^{(a\ \lambda_i)}, && \mathcal{A}_8^i = (-1)^{i+1}\ \chi_i\ \delta_i\ \mathrm{e}^{(a\ \lambda_i)}, && \mathcal{A}_9^i = (-1)^{i+1}\ \chi_i\ \mathrm{e}^{(a\ \lambda_i)} \\
&\mathcal{A}_{10}^i = \mathrm{e}^{(a\ \lambda_i)}, && \mathcal{A}_{11}^i = (-1)^{i+1}\ \phi_i\ \mathrm{e}^{(a\ \lambda_i)}, && \mathcal{A}_{12}^i = \chi_i\ \lambda_i\ \mathrm{e}^{(a\ \lambda_i)}
\end{aligned} \tag{76}$$

with $i = 1, \cdots, 12$. By applying the same procedure for boundary conditions of the generalized forces and thus exploiting Eq. (72), the following relationship is obtained:

$$\boldsymbol{F} = \boldsymbol{R}\,\boldsymbol{D} \tag{77}$$

where

$$\boldsymbol{F} = \left\{\; \mathcal{N}_{\alpha\alpha_1} \quad \mathcal{N}_{\beta\beta_1} \quad \mathcal{Q}_{\alpha_1} \quad \mathcal{M}_{\alpha\alpha_1} \quad \mathcal{M}_{\alpha\beta_1} \quad \mathcal{P}_{\alpha\alpha_1} \quad \mathcal{N}_{\alpha\alpha_2} \quad \mathcal{N}_{\beta\beta_2} \quad \mathcal{Q}_{\alpha_2} \quad \mathcal{M}_{\alpha\alpha_2} \quad \mathcal{M}_{\alpha\beta_2} \quad \mathcal{P}_{\alpha\alpha_2} \;\right\}^T$$

$$\boldsymbol{D} = \left\{\; \mathscr{D}_1 \quad \mathscr{D}_2 \quad \mathscr{D}_3 \quad \mathscr{D}_4 \quad \mathscr{D}_5 \quad \mathscr{D}_6 \quad \mathscr{D}_7 \quad \mathscr{D}_8 \quad \mathscr{D}_9 \quad \mathscr{D}_{10} \quad \mathscr{D}_{11} \quad \mathscr{D}_{12} \;\right\}^T \tag{78}$$

and

$$\boldsymbol{R} = \begin{bmatrix} \mathcal{R}_1^1 & \mathcal{R}_1^2 & \mathcal{R}_1^3 & \mathcal{R}_1^4 & \mathcal{R}_1^5 & \mathcal{R}_1^6 & \mathcal{R}_1^7 & \mathcal{R}_1^8 & \mathcal{R}_1^9 & \mathcal{R}_1^{10} & \mathcal{R}_1^{11} & \mathcal{R}_1^{12} \\ \mathcal{R}_2^1 & \mathcal{R}_2^2 & \mathcal{R}_2^3 & \mathcal{R}_2^4 & \mathcal{R}_2^5 & \mathcal{R}_2^6 & \mathcal{R}_2^7 & \mathcal{R}_2^8 & \mathcal{R}_2^9 & \mathcal{R}_2^{10} & \mathcal{R}_2^{11} & \mathcal{R}_2^{12} \\ \mathcal{R}_3^1 & \mathcal{R}_3^2 & \mathcal{R}_3^3 & \mathcal{R}_3^4 & \mathcal{R}_3^5 & \mathcal{R}_3^6 & \mathcal{R}_3^7 & \mathcal{R}_3^8 & \mathcal{R}_3^9 & \mathcal{R}_3^{10} & \mathcal{R}_3^{11} & \mathcal{R}_3^{12} \\ \mathcal{R}_4^1 & \mathcal{R}_4^2 & \mathcal{R}_4^3 & \mathcal{R}_4^4 & \mathcal{R}_4^5 & \mathcal{R}_4^6 & \mathcal{R}_4^7 & \mathcal{R}_4^8 & \mathcal{R}_4^9 & \mathcal{R}_4^{10} & \mathcal{R}_4^{11} & \mathcal{R}_4^{12} \\ \mathcal{R}_5^1 & \mathcal{R}_5^2 & \mathcal{R}_5^3 & \mathcal{R}_5^4 & \mathcal{R}_5^5 & \mathcal{R}_5^6 & \mathcal{R}_5^7 & \mathcal{R}_5^8 & \mathcal{R}_5^9 & \mathcal{R}_5^{10} & \mathcal{R}_5^{11} & \mathcal{R}_5^{12} \\ \mathcal{R}_6^1 & \mathcal{R}_6^2 & \mathcal{R}_6^3 & \mathcal{R}_6^4 & \mathcal{R}_6^5 & \mathcal{R}_6^6 & \mathcal{R}_6^7 & \mathcal{R}_6^8 & \mathcal{R}_6^9 & \mathcal{R}_6^{10} & \mathcal{R}_6^{11} & \mathcal{R}_6^{12} \\ \mathcal{R}_7^1 & \mathcal{R}_7^2 & \mathcal{R}_7^3 & \mathcal{R}_7^4 & \mathcal{R}_7^5 & \mathcal{R}_7^6 & \mathcal{R}_7^7 & \mathcal{R}_7^8 & \mathcal{R}_7^9 & \mathcal{R}_7^{10} & \mathcal{R}_7^{11} & \mathcal{R}_7^{12} \\ \mathcal{R}_8^1 & \mathcal{R}_8^2 & \mathcal{R}_8^3 & \mathcal{R}_8^4 & \mathcal{R}_8^5 & \mathcal{R}_8^6 & \mathcal{R}_8^7 & \mathcal{R}_8^8 & \mathcal{R}_8^9 & \mathcal{R}_8^{10} & \mathcal{R}_8^{11} & \mathcal{R}_8^{12} \\ \mathcal{R}_9^1 & \mathcal{R}_9^2 & \mathcal{R}_9^3 & \mathcal{R}_9^4 & \mathcal{R}_9^5 & \mathcal{R}_9^6 & \mathcal{R}_9^7 & \mathcal{R}_9^8 & \mathcal{R}_9^9 & \mathcal{R}_9^{10} & \mathcal{R}_9^{11} & \mathcal{R}_9^{12} \\ \mathcal{R}_{10}^1 & \mathcal{R}_{10}^2 & \mathcal{R}_{10}^3 & \mathcal{R}_{10}^4 & \mathcal{R}_{10}^5 & \mathcal{R}_{10}^6 & \mathcal{R}_{10}^7 & \mathcal{R}_{10}^8 & \mathcal{R}_{10}^9 & \mathcal{R}_{10}^{10} & \mathcal{R}_{10}^{11} & \mathcal{R}_{10}^{12} \\ \mathcal{R}_{11}^1 & \mathcal{R}_{11}^2 & \mathcal{R}_{11}^3 & \mathcal{R}_{11}^4 & \mathcal{R}_{11}^5 & \mathcal{R}_{11}^6 & \mathcal{R}_{11}^7 & \mathcal{R}_{11}^8 & \mathcal{R}_{11}^9 & \mathcal{R}_{11}^{10} & \mathcal{R}_{11}^{11} & \mathcal{R}_{11}^{12} \\ \mathcal{R}_{12}^1 & \mathcal{R}_{12}^2 & \mathcal{R}_{12}^3 & \mathcal{R}_{12}^4 & \mathcal{R}_{12}^5 & \mathcal{R}_{12}^6 & \mathcal{R}_{12}^7 & \mathcal{R}_{12}^8 & \mathcal{R}_{12}^9 & \mathcal{R}_{12}^{10} & \mathcal{R}_{12}^{11} & \mathcal{R}_{12}^{12} \end{bmatrix} \tag{79}$$

where

$$\mathcal{R}_1^i = (-1)^{i+1}\ \Big(A_{12}\,\alpha\,\chi_i\,\delta_i - A_{12}\,\chi_i\,\gamma_i\,\lambda_i - A_{12}\,\chi_i\,\frac{1}{R_\beta} - A_{11}\,\chi_i\,\frac{1}{R_\alpha}\Big);$$

$$\mathcal{R}_2^i = (-1)^{i+1}\ \Big(-A_{66}\,\alpha\,\chi_i\,\gamma_i - A_{66}\,\chi_i\,\delta_i\,\lambda_i\Big);$$

$$\mathcal{R}_3^i = -A_{55} - c_2\,(2\,D_{55} + c_2 F_{55}) - 2\,\alpha^2\,c_1\,F_{66} - 2\,\alpha^2\,c_1^2\,H_{66} - \chi_i\,(A_{55} + c_2\\ (2\,D_{55} + c_2\,F_{55}))\,\lambda_i - \alpha^2\,c_1^2\,\chi_i\,H_{12}\,\lambda_i - 4\,\alpha^2\,c_1^2\,\chi_i H_{66}\,\lambda_i - \alpha\,c_1\,F_{12}\,\phi_i\,\lambda_i\\ -2\,\alpha\,c_1\,F_{66}\phi_i\,\lambda_i - \alpha\,c_1^2\,H_{12}\,\phi_i\,\lambda_i - 2\,\alpha\,c_1^2\,H_{66}\,\phi_i\,\lambda_i + c_1\,F_{11}\,\lambda_i^2 + c_1^2\,H_{11}\,\lambda_i^2\\ + c_1^2\,\chi_i\,H_{11}\,\lambda_i^3;$$

$$\mathcal{R}_4^i = (-1)^{i+1}\ \Big(\alpha^2\,c_1\,\chi_i\,F_{12} + \alpha^2\,c_1^2\,\chi_i\,H_{12} + \alpha\,D_{12}\,\phi_i + 2\,\alpha\,c_1\,F_{12}\,\phi_i + \alpha\,c_1^2\,H_{12}\,\phi_i\\ - D_{11}\,\lambda_i - 2\,c_1\,F_{11}\,\lambda_i - c_1^2\,H_{11}\,\lambda_i - c_1\,\chi_i\,(F_{11} + c_1\,H_{11})\,\lambda_i^2\Big);$$

$$\mathcal{R}_5^i = -\alpha\,D_{66} - 2\,\alpha\,c_1\,F_{66} - \alpha\,c_1^2\,H_{66} - 2\,\alpha\,c_1\,\chi_i\,F_{66}\,\lambda_i - 2\,\alpha\,c_1^2\,\chi_i\,H_{66}\,\lambda_i - (D_{66}\\ + c_1\,(2\,F_{66} + c_1\,H_{66}))\,\phi_i\,\lambda_i;$$

$$\mathcal{R}_6^i = -c_1\,(-\alpha^2\,c_1\,\chi_i\,H_{12} - \alpha\,F_{12}\,\phi_i - \alpha\,c_1\,H_{12}\,\phi_i + F_{11}\,\lambda_i + c_1\,H_{11}\,\lambda_i + c_1\,\chi_i\,H_{11}\,\lambda_i^2);$$

$$\mathcal{R}_7^i = (-1)^{i+1}\ \Big(-A_{12}\,\alpha\,\chi_i\,\delta_i\,\mathrm{e}^{(a\,\lambda_i)} + A_{11}\,\chi_i\,\mathrm{e}^{(a\,\lambda_i)}\,\gamma_i\lambda_i + A_{12}\,\chi_i\,\mathrm{e}^{(a\,\lambda_i)}\,\frac{1}{R_\beta}\\ + A_{11}\,\chi_i\,\mathrm{e}^{(a\,\lambda_i)}\,\frac{1}{R_\alpha}\Big);$$

$$\mathcal{R}_8^i = (-1)^{i+1}\ \Big(A_{66}\,\alpha\,\chi_i\,\mathrm{e}^{(a\,\lambda_i)}\,\gamma_i + A_{66}\,\chi_i\,\delta_i\,\mathrm{e}^{(a\,\lambda_i)}\,\lambda_i\Big);$$

$$\mathcal{R}_9^i = \mathrm{e}^{(a\,\lambda_i)}\,(A_{55} + c_2\,(2\,D_{55} + c_2\,F_{55})) + 2\,\alpha^2\,c_1\,\mathrm{e}^{(a\,\lambda_i)}\,F_{66} + 2\,\alpha^2\,c_1^2\,\mathrm{e}^{(a\,\lambda_i)}\,H_{66}\\ + \chi_i\,\mathrm{e}^{(a\,\lambda_i)}\,(A_{55} + c_2\,(2\,D_{55} + c_2\,F_{55}))\,\lambda_i + \alpha^2\,c_1^2\,\chi_i\,\mathrm{e}^{(a\,\lambda_i)}\,H_{12}\,\lambda_i + 4\,\alpha^2\,c_1^2\,\chi_i\,\mathrm{e}^{(a\,\lambda_i)}\\ \times H_{66}\,\lambda_i + \alpha\,c_1\,\mathrm{e}^{(a\,\lambda_i)}\,F_{12}\,\phi_i\,\lambda_i + 2\,\alpha\,c_1\,\mathrm{e}^{\,L\,\lambda_i}\,F_{66}\,\phi_i\,\lambda_i + \alpha\,c_1^2\,\mathrm{e}^{(a\,\lambda_i)}\,H_{12}\,\phi_i\,\lambda_i\\ + 2\,\alpha\,c_1^2\,\mathrm{e}^{(a\,\lambda_i)}\,H_{66}\,\phi_i\,\lambda_i - c_1\,\mathrm{e}^{(a\,\lambda_i)}\,F_{11}\,\lambda_i^2 - c_1^2\,\mathrm{e}^{(a\,\lambda_i)}\,H_{11}\,\lambda_i^2 - c_1^2\,\chi_i\,\mathrm{e}^{(a\,\lambda_i)}\,H_{11}\,\lambda_i^3;$$

$$
\begin{aligned}
\mathcal{R}_{10}^{i} &= (-1)^{i+1} \Big(-\alpha^2\, c_1\, \chi_i\, \mathrm{e}^{(a\,\lambda_i)}\, F_{12} - \alpha^2\, c_1^2\, \chi_i\, \mathrm{e}^{(a\,\lambda_i)}\, H_{12} - \alpha\, D_{12}\, \mathrm{e}^{(a\,\lambda_i)}\, \phi_i \\
&\quad - 2\,\alpha\, c_1 \mathrm{e}^{(a\,\lambda_i)}\, F_{12}\, \phi_i - \alpha\, c_1^2\, \mathrm{e}^{(a\,\lambda_i)}\, H_{12}\, \phi_i + D_{11}\, \mathrm{e}^{(a\,\lambda_i)} + 2\, c_1\, \mathrm{e}^{(a\,\lambda_i)}\, F_{11}\, \lambda_i \\
&\quad + c_1^2\, \mathrm{e}^{(a\,\lambda_i)}\, H_{11}\, \lambda_i + c_1\, \chi_i\, \mathrm{e}^{(a\,\lambda_i)}\, (F_{11} + c_1\, H_{11})\, \lambda_i^2 \Big); \\
\mathcal{R}_{11}^{i} &= \alpha\, D_{66}\, \mathrm{e}^{(a\,\lambda_i)} + 2\,\alpha\, c_1 \mathrm{e}^{(a\,\lambda_i)}\, F_{66} + \alpha\, c_1^2\, \mathrm{e}^{(a\,\lambda_i)} H_{66} + 2\,\alpha\, c_1\, \chi_i\, \mathrm{e}^{(a\,\lambda_i)}\, F_{66}\, \lambda_i \\
&\quad + 2\,\alpha\, c_1^2\, \chi_i\, \mathrm{e}^{(a\,\lambda_i)}\, H_{66}\, \lambda_i + \mathrm{e}^{(a\,\lambda_i)}\, (D_{66} + c_1\, (2\, F_{66} + c_1\, H_{66}))\, \phi_i\, \lambda_i; \\
\mathcal{R}_{12}^{i} &= (-1)^{i+1} \Big(c_1\, (-\alpha^2\, c_1\, \chi_i\, \mathrm{e}^{(a\,\lambda_i)}\, H_{12} - \alpha\, \mathrm{e}^{(a\,\lambda_i)}\, F_{12}\, \phi_i - \alpha\, c_1\, \mathrm{e}^{(a\,\lambda_i)}\, H_{12}\, \phi_i \\
&\quad + \mathrm{e}^{(a\,\lambda_i)}\, F_{11}\, \lambda_i + c_1\, \mathrm{e}^{(a\,\lambda_i)} H_{11} \lambda_i + c_1\, \chi_i\, \mathrm{e}^{(L\,\lambda_i)}\, H_{11}\, \lambda_i^2) \Big)
\end{aligned}
\tag{80}
$$

with $i = 1, \cdots, 12$. Now by eliminating the constants vectors $\boldsymbol{D}$ in the Eqs. (73) and (77) the dynamic stiffness matrix which links the forces and moments vector $\boldsymbol{F}$ with the generalized displacements vector $\boldsymbol{\delta}$ is derived:

$$
\boldsymbol{F} = \boldsymbol{K}\,\boldsymbol{\delta}, \qquad \boldsymbol{K} = \boldsymbol{R}\,\boldsymbol{A}^{-1} \tag{81}
$$

i.e.

$$
\boldsymbol{K} = \begin{bmatrix}
\mathcal{K}_1^1 & \mathcal{K}_1^2 & \mathcal{K}_1^3 & \mathcal{K}_1^4 & \mathcal{K}_1^5 & \mathcal{K}_1^6 & \mathcal{K}_1^7 & \mathcal{K}_1^8 & \mathcal{K}_1^9 & \mathcal{K}_1^{10} & \mathcal{K}_1^{11} & \mathcal{K}_1^{12} \\
\mathcal{K}_2^1 & \mathcal{K}_2^2 & \mathcal{K}_2^3 & \mathcal{K}_2^4 & \mathcal{K}_2^5 & \mathcal{K}_2^6 & \mathcal{K}_2^7 & \mathcal{K}_2^8 & \mathcal{K}_2^9 & \mathcal{K}_2^{10} & \mathcal{K}_2^{11} & \mathcal{K}_2^{12} \\
\mathcal{K}_3^1 & \mathcal{K}_3^2 & \mathcal{K}_3^3 & \mathcal{K}_3^4 & \mathcal{K}_3^5 & \mathcal{K}_3^6 & \mathcal{K}_3^7 & \mathcal{K}_3^8 & \mathcal{K}_3^9 & \mathcal{K}_3^{10} & \mathcal{K}_3^{11} & \mathcal{K}_3^{12} \\
\mathcal{K}_4^1 & \mathcal{K}_4^2 & \mathcal{K}_4^3 & \mathcal{K}_4^4 & \mathcal{K}_4^5 & \mathcal{K}_4^6 & \mathcal{K}_4^7 & \mathcal{K}_4^8 & \mathcal{K}_4^9 & \mathcal{K}_4^{10} & \mathcal{K}_4^{11} & \mathcal{K}_4^{12} \\
\mathcal{K}_5^1 & \mathcal{K}_5^2 & \mathcal{K}_5^3 & \mathcal{K}_5^4 & \mathcal{K}_5^5 & \mathcal{K}_5^6 & \mathcal{K}_5^7 & \mathcal{K}_5^8 & \mathcal{K}_5^9 & \mathcal{K}_5^{10} & \mathcal{K}_5^{11} & \mathcal{K}_5^{12} \\
\mathcal{K}_6^1 & \mathcal{K}_6^2 & \mathcal{K}_6^3 & \mathcal{K}_6^4 & \mathcal{K}_6^5 & \mathcal{K}_6^6 & \mathcal{K}_6^7 & \mathcal{K}_6^8 & \mathcal{K}_6^9 & \mathcal{K}_6^{10} & \mathcal{K}_6^{11} & \mathcal{K}_6^{12} \\
\mathcal{K}_7^1 & \mathcal{K}_7^2 & \mathcal{K}_7^3 & \mathcal{K}_7^4 & \mathcal{K}_7^5 & \mathcal{K}_7^6 & \mathcal{K}_7^7 & \mathcal{K}_7^8 & \mathcal{K}_7^9 & \mathcal{K}_7^{10} & \mathcal{K}_7^{11} & \mathcal{K}_7^{12} \\
\mathcal{K}_8^1 & \mathcal{K}_8^2 & \mathcal{K}_8^3 & \mathcal{K}_8^4 & \mathcal{K}_8^5 & \mathcal{K}_8^6 & \mathcal{K}_8^7 & \mathcal{K}_8^8 & \mathcal{K}_8^9 & \mathcal{K}_8^{10} & \mathcal{K}_8^{11} & \mathcal{K}_8^{12} \\
\mathcal{K}_9^1 & \mathcal{K}_9^2 & \mathcal{K}_9^3 & \mathcal{K}_9^4 & \mathcal{K}_9^5 & \mathcal{K}_9^6 & \mathcal{K}_9^7 & \mathcal{K}_9^8 & \mathcal{K}_9^9 & \mathcal{K}_9^{10} & \mathcal{K}_9^{11} & \mathcal{K}_9^{12} \\
\mathcal{K}_{10}^1 & \mathcal{K}_{10}^2 & \mathcal{K}_{10}^3 & \mathcal{K}_{10}^4 & \mathcal{K}_{10}^5 & \mathcal{K}_{10}^6 & \mathcal{K}_{10}^7 & \mathcal{K}_{10}^8 & \mathcal{K}_{10}^9 & \mathcal{K}_{10}^{10} & \mathcal{K}_{10}^{11} & \mathcal{K}_{10}^{12} \\
\mathcal{K}_{11}^1 & \mathcal{K}_{11}^2 & \mathcal{K}_{11}^3 & \mathcal{K}_{11}^4 & \mathcal{K}_{11}^5 & \mathcal{K}_{11}^6 & \mathcal{K}_{11}^7 & \mathcal{K}_{11}^8 & \mathcal{K}_{11}^9 & \mathcal{K}_{11}^{10} & \mathcal{K}_{11}^{11} & \mathcal{K}_{11}^{12} \\
\mathcal{K}_{12}^1 & \mathcal{K}_{12}^2 & \mathcal{K}_{12}^3 & \mathcal{K}_{12}^4 & \mathcal{K}_{12}^5 & \mathcal{K}_{12}^6 & \mathcal{K}_{12}^7 & \mathcal{K}_{12}^8 & \mathcal{K}_{12}^9 & \mathcal{K}_{12}^{10} & \mathcal{K}_{12}^{11} & \mathcal{K}_{12}^{12}
\end{bmatrix} \tag{82}
$$

The above dynamic stiffness matrix will now be used in conjunction with the Wittrick-Williams algorithm [85] to analyze assemblies of laminated composite cylindrical and spherical shallow shells to investigate their free vibration characteristics based on HSDT. Explicit expressions for each shell element of the DS matrix were obtained via symbolic computation using Mathematica®. They are far too extensive and voluminous to report in this paper. The correctness of these expressions was further checked by implementing them in a MATLAB® program and then carrying out a wide range of numerical simulations.

5.4 Assembly procedure and similarities with FEM

Once the DS matrix of a laminated composite element has been developed, it can be assembled to form the global DS matrix of the final structure (see Fig 6). Although

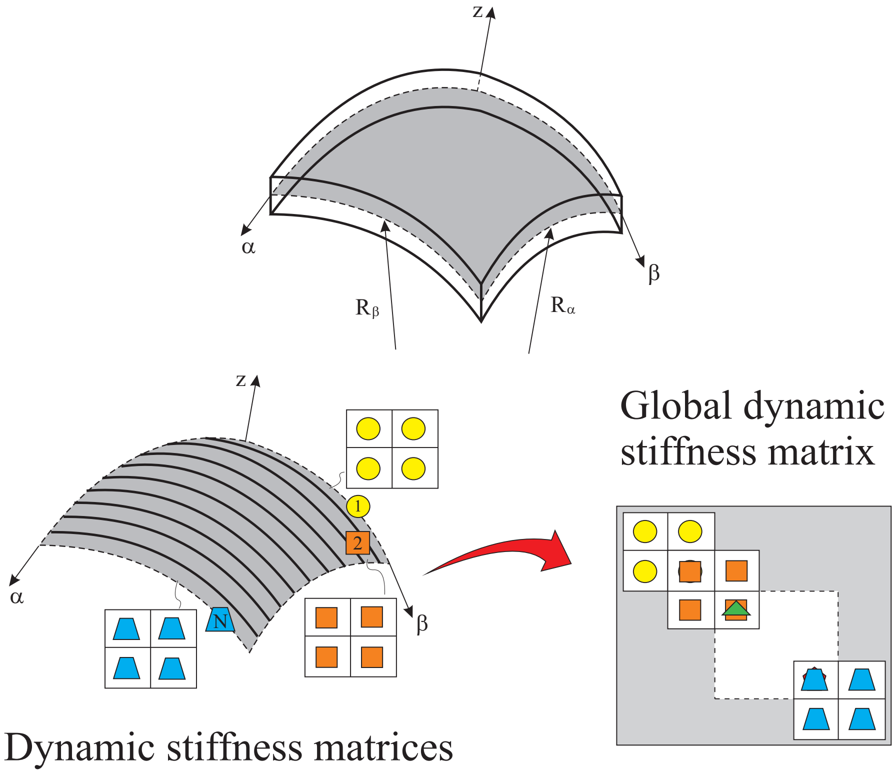


Figure 6: Direct assembly of dynamic stiffness element

like the FEM, a mesh is required in the DSM, but it should be noted that the latter is mesh independent in the sense that additional elements are required only when there is a change in the geometry of the structure. A single DS laminate element is enough to compute any number of its natural frequencies to any desired accuracy, which, of course, is impossible in the FEM. However, for the type of structures under consideration DS shell elements do not have point nodes, but have line nodes instead. In this particular case, no change in geometry along the longitudinal direction is admitted. This is in addition to the assumed simple support boundary conditions on two opposite sides, inherent in DSM for shell elements at present. The other two sides of the shell can have any boundary conditions. The application of the boundary conditions of the global dynamic stiffness matrix involves the use of the so-called penalty method. This consists of adding a large stiffness to the appropriate leading diagonal term which corresponds to the degree of freedom of the node that needs to be suppressed. It is thus possible to apply free (F), simple support (S) and clamped (C) boundary conditions on the structure by penalizing the appropriate degrees of freedom. Clearly for simple support boundary condition, the generalized displacement amplitudes V_i, W_i and Φ_{y_i} are assigned zero values. On the other hand, for clamped boundary condition U_i, V_i, W_i, Φ_{x_i}, Φ_{y_i} and W_{x_i} on the boundary are assigned zero values. Of course for the free-edge boundary condition stress resultants are assigned zero values and then no penalty will be applied at the generalized displacement amplitudes. Because of the similarities between DSM and FEM, DS elements can be implemented in FEM codes to enhance the accuracy of results in FEM very considerably.

6 Results and discussion

Free vibration characteristics of laminated composite cylindrical and spherical shallow shells with symmetric cross-ply lamination schemes are investigated. Results have been obtained in exact sense by using the Lèvy-type closed form solution within the framework of the DSM. The material used in all the analysis carried out is

$$\frac{E_1}{E_2} = 25; \quad G_{12} = G_{13} = 0.5 \;\; E_2; \quad G_{23} = 0.2 \;\; E_2; \quad \nu_{12} = 0.25;$$

if not differently stated. A first validation is undertaken in Table 4, the fundamental circular frequency parameter of three-layer symmetric cross-ply and moderately thick spherical and cylindrical shallow shells has been computed. The results obtained by using the present DSM formulation have been compared with those proposed by Khdeir and Reddy [87] by using the classical Lèvy-type solution. As can be seen in Table 4 the

Table 4: Fundamental circular frequency parameter $\hat{\omega} = \omega \, \frac{a^2}{h} \sqrt{\frac{\rho}{E_2}}$, of square cylindrical and spherical shells with staking sequence [0°/90°/0°] length-to-thickness ratio $a/h = 10$ and radius-to-length ratio $R_\beta/a = 20$.

	SSSS	SCSS	SFSF	SCSC	SFSS	SFSC
Spherical shell						
Khdeir and Reddy [87]	11.807	13.481	3.797	16.100	4.328	6.088
Present HSDT DSM	11.807	13.480	3.796	16.100	4.328	6.088
Cylindrical shell						
Khdeir and Reddy [87]	11.793	13.825	3.789	15.999	4.322	6.089
Present HSDT DSM	11.793	13.825	3.789	15.999	4.322	6.089

results perfectly match each other for all the considered boundary conditions. In Tables 5 and 6 further assessments have been carried out. Most notably, the fundamental circular frequency parameter has been computed for three and four-layer symmetric cross-ply and moderately thick spherical and cylindrical shallow shells, respectively. Results are compare with several meshless formulations proposed by Ferreira *el. all* [68, 88, 89], in particular a radial basis formulation (RBF) based on a HSDT, a wavelet collocation (WLC) formulation based on a FSDT and a radial basis collocation (RBFC) solution

Table 5: Dimensionless fundamental circular frequency parameter $\hat{\omega} = \omega \frac{a^2}{h} \sqrt{\frac{\rho}{E_2}}$, of square spherical and cylindrical shells with staking sequence [0°/90°/0°], length-to-thickness ratio $a/h = 10$ and varying the radius-to-length ratio R_β/a and R/a.

Shell Configuration	Theory	R/a					
		5	10	20	50	100	Plate
Spherical							
	HSDT RBF [68]	12.063	11.861	11.810	11.796	11.794	11.793
	FSDT WLC [88]	12.417	12.227	12.179	12.165	12.164	12.163
	SSDT RBFC [89]	12.125	11.966	11.925	11.914	11.912	11.911
	HSDT DSM	12.054	11.857	11.807	11.793	11.790	11.790
		R_β/a					
		5	10	20	50	100	Plate
Cylindrical							
	HSDT RBF [68]	11.851	11.808	11.797	11.794	11.793	11.793
	FSDT WLC [88]	12.214	12.176	12.166	12.163	12.163	12.163
	SSDT RBFC [89]	11.923	11.915	11.913	11.912	11.912	11.912
	HSDT DSM	11.846	11.804	11.793	11.790	11.790	11.790

Table 6: Dimensionless fundamental circular frequency parameter $\hat{\omega} = \omega \frac{a^2}{h} \sqrt{\frac{\rho}{E_2}}$, of square spherical and cylindrical shells with staking sequence [0°/90°/90°/0°], length-to-thickness ratio $a/h = 10$ and varying the radius-to-length ratio R_β/a and R/a.

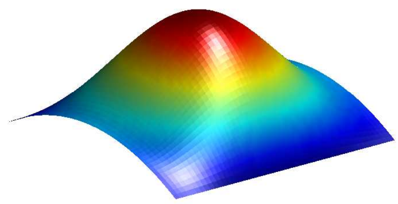
Shell Configuration	Theory	R/a					
		5	10	20	50	100	Plate
Spherical							
	HSDT RBF [68]	12.051	11.848	11.796	11.782	11.780	11.779
	FSDT WLC [88]	12.493	12.299	12.250	12.236	12.234	12.233
	SSDT RBFC [89]	12.099	11.938	11.896	11.885	11.883	11.883
	HSDT DSM	12.043	11.844	11.793	11.779	11.777	11.776
		R_β/a					
		5	10	20	50	100	Plate
Cylindrical							
	HSDT RBF [68]	11.838	11.794	11.783	11.780	11.779	11.779
	FSDT WLC [88]	12.279	12.240	12.230	12.228	12.227	12.227
	SSDT RBFC [89]	11.901	11.887	11.884	11.883	11.883	11.883
	HSDT DSM	11.832	11.790	11.780	11.777	11.777	11.776

procedure based on a sinusoidal shear deformation theory (SSDT). As can be seen in Tables 5 and 6 the present formulation leads to results which are in an excellent agreement with those obtained by using the aforementioned theories, for different values of the radius-to-length ratio (R/a). From Tables 7 to 10 the investigation is focused on symmetric cross-ply cylindrical shells, two different stacking sequences are investigate, and both moderately thick and thin shallow shells are taken into account. The first five circular frequency parameters are computed for several boundary conditions. From Fig. 7 to Fig. 9 the first six mode shapes of SCSC, SFSC and SFSF square cylindrical shallow shells are depicted. Most notably, the mode shapes are relative to a symmetric cross-ply $[0°/90°/0°]$ stacking sequence, $a/h = 100$ and $R/a = 30$. An assessment of the present formulation is carried out in Table 6 for isotropic spherical panels. The

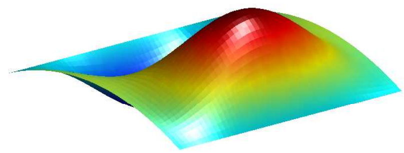
Table 7: First five circular frequency parameters $\hat{\omega} = \omega \frac{a^2}{h}\sqrt{\frac{\rho}{E_2}}$, of square cylindrical shells with staking sequence $[0°/90°/0°]$ length-to-thickness ratio $a/h = 10$ and varying the radius-to-length ratio R_β/a.

Mode		SSSS			SFSS			SFSF		
	R_β/a	5	50	100	5	50	100	5	50	100
1		11.846	11.790	11.790	4.376	4.326	4.326	3.783	3.790	3.790
2		18.489	18.487	18.487	14.398	14.401	14.401	5.726	5.576	5.574
3		29.399	29.333	29.332	17.307	17.219	17.218	13.898	13.905	13.905
4		30.856	30.860	30.860	22.971	22.959	22.959	15.741	15.734	15.734
5		32.961	32.949	32.948	28.330	28.336	28.336	23.019	23.020	23.020
Mode		SFSC			SCSS			SCSC		
	R_β/a	5	50	100	5	50	100	5	50	100
1		6.129	6.090	6.089	13.865	13.823	13.823	16.026	15.997	15.997
2		14.978	14.981	14.981	19.722	19.720	19.720	21.131	21.130	21.130
3		18.869	18.793	18.793	30.980	30.920	30.920	32.306	32.310	32.310
4		24.118	24.107	24.107	31.524	31.528	31.528	32.467	32.414	32.413
5		28.594	28.600	28.600	34.318	34.308	34.308	35.629	35.618	35.618

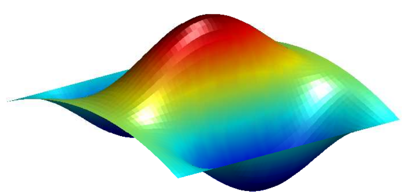
first six natural frequencies are computed and compared with the 3D elasticity solution and many other theories, which are refined in the displacement field or in the curvature description. As can be seen from the Table the proposed DSM formulation provide the best accuracy for the fundamental natural frequency with respect to the 3D elasticity solution, but for higher frequencies there is an increase in the error percentage. This slight loss of accuracy is due to the fact that approximated curvature descriptions have been employed. In Table 12 the first three circular frequency parameters of three-layer symmetric cross-ply shallow spherical shells are calculated. The boundary conditions SSSS, SCSS and SCSC are taken into account. The investigation is carried out for different values of the radius-to-length ratio and for moderately thick and thin spherical panels. In Table 13 the effect of the orthotropic ratio on the dimensionless fundamental frequency parameter is studied. The boundary conditions SSSS, SCSS and SCSC



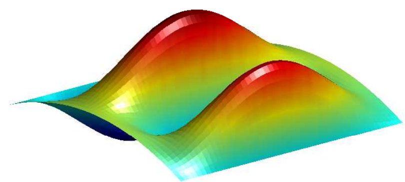
(a) Mode 1, $\hat{\omega} = 31.916$



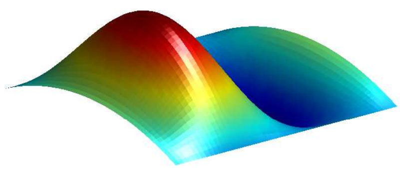
(b) Mode 2, $\hat{\omega} = 36.277$



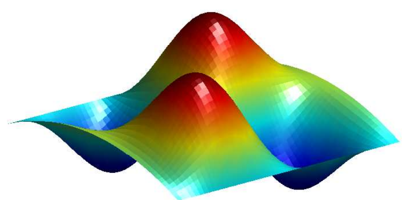
(c) Mode 3, $\hat{\omega} = 49.256$



(d) Mode 4, $\hat{\omega} = 72.406$

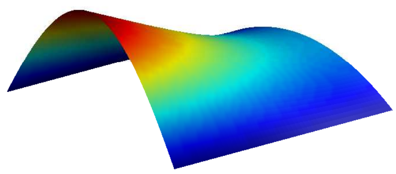


(e) Mode 5, $\hat{\omega} = 84.982$

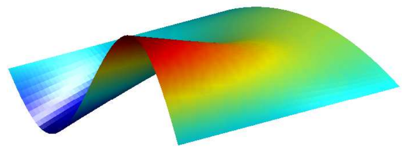


(f) Mode 6, $\hat{\omega} = 87.738$

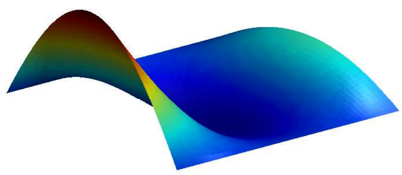
Figure 7: First six mode shapes of a symmetric cross-ply cylindrical shell with SCSC boundary condition.



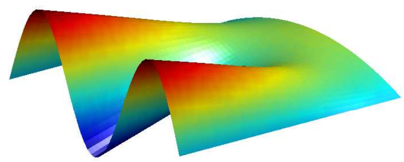
(a) Mode 1, $\hat{\omega} = 7.031$



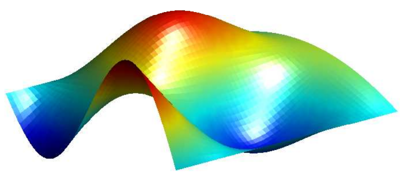
(b) Mode 2, $\hat{\omega} = 17.257$



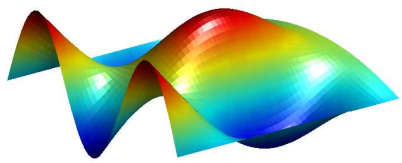
(c) Mode 3, $\hat{\omega} = 32.260$



(d) Mode 4, $\hat{\omega} = 36.340$

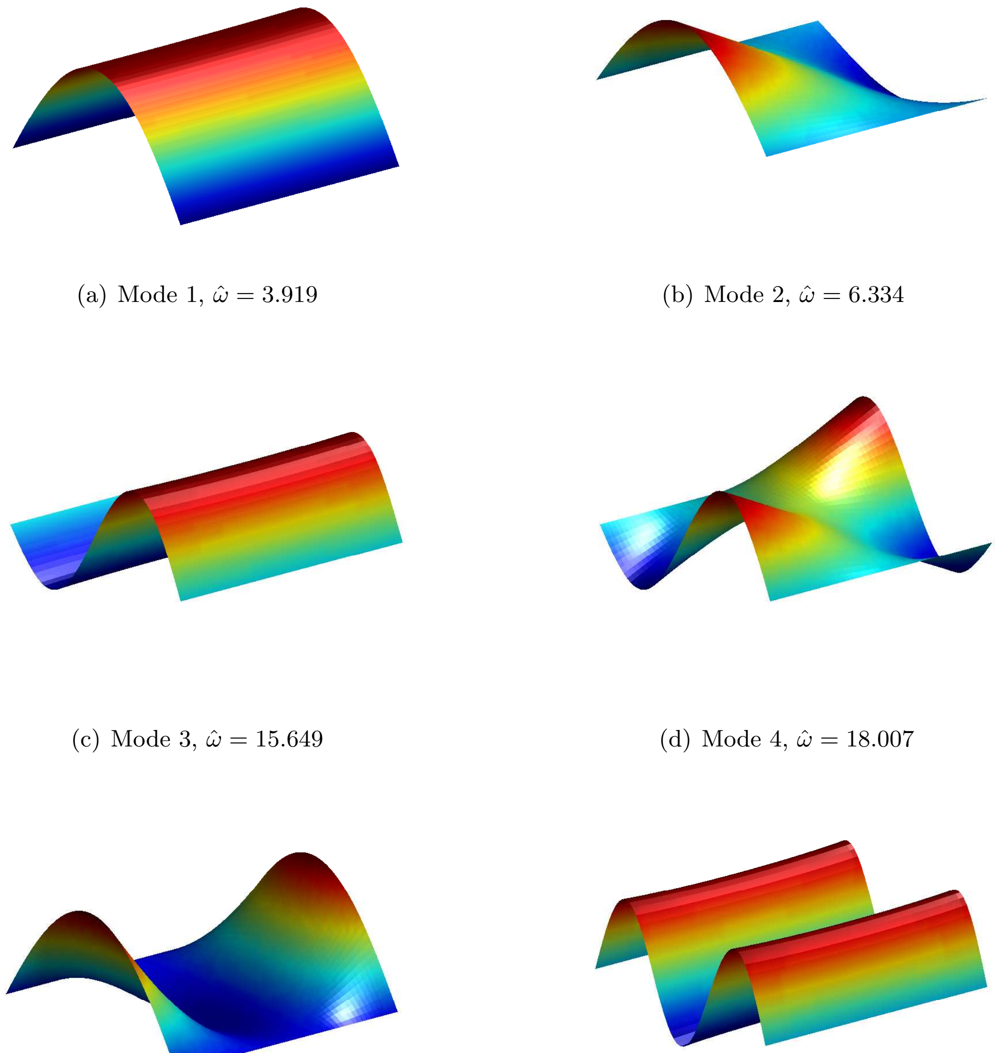


(e) Mode 5, $\hat{\omega} = 37.804$



(f) Mode 6, $\hat{\omega} = 51.898$

Figure 8: First six mode shapes of a symmetric cross-ply cylindrical shell with SFSC boundary condition.



(a) Mode 1, $\hat{\omega} = 3.919$

(b) Mode 2, $\hat{\omega} = 6.334$

(c) Mode 3, $\hat{\omega} = 15.649$

(d) Mode 4, $\hat{\omega} = 18.007$

(e) Mode 5, $\hat{\omega} = 33.435$

(f) Mode 6, $\hat{\omega} = 35.162$

Figure 9: First six mode shapes of a symmetric cross-ply cylindrical shell with SFSF boundary condition.

Table 8: First five circular frequency parameters $\hat{\omega} = \omega \frac{a^2}{h} \sqrt{\frac{\rho}{E_2}}$, of square cylindrical shells with staking sequence [0°/90°/0°] length-to-thickness ratio $a/h = 100$ and varying the radius-to-length ratio R_β/a.

Mode		SSSS			SFSS			SFSF		
	R_β/a	5	50	100	5	50	100	5	50	100
1		20.329	15.238	15.193	8.271	4.556	4.511	3.913	3.921	3.913
2		23.745	22.818	22.811	16.599	16.270	16.262	10.742	6.061	5.950
3		40.355	56.135	40.153	31.030	23.301	23.231	15.657	15.649	15.649
4		61.528	60.093	56.089	32.099	30.232	30.218	19.269	17.984	17.976
5		61.560	66.369	60.084	35.833	35.773	35.773	35.160	33.247	33.173
Mode		SFSC			SCSS			SCSC		
	R_β/a	5	50	100	5	50	100	5	50	100
1		10.746	6.933	6.888	26.683	22.586	22.549	34.921	31.856	31.833
2		17.765	17.249	17.249	29.277	28.410	28.402	36.984	36.266	36.259
3		36.438	32.124	32.071	43.951	43.726	43.726	49.460	49.251	49.251
4		38.696	36.341	36.341	68.784	68.716	68.716	72.476	72.409	72.409
5		39.498	37.779	37.762	74.430	69.904	69.866	88.755	84.909	84.880

Table 9: First five circular frequency parameters $\hat{\omega} = \omega \frac{a^2}{h} \sqrt{\frac{\rho}{E_2}}$, of square cylindrical shells with staking sequence [0°/90°/90°/0°] length-to-thickness ratio $a/h = 10$ and varying the radius-to-length ratio R_β/a.

Mode		SSSS			SFSS			SFSF		
	R_β/a	5	50	100	5	50	100	5	50	100
1		11.832	11.777	11.777	5.782	5.750	5.749	5.367	5.377	5.378
2		21.836	21.837	21.837	16.726	16.631	16.631	6.826	6.700	6.699
3		27.461	27.381	27.381	18.984	18.991	18.991	18.659	18.668	18.668
4		33.244	33.232	33.232	25.249	25.240	25.240	19.895	19.893	19.893
5		37.445	37.452	37.452	33.637	33.538	33.538	22.875	22.870	22.869
Mode		SFSC			SCSS			SCSC		
	R_β/a	5	50	100	5	50	100	5	50	100
1		7.049	7.018	7.018	13.491	13.446	13.446	15.304	15.271	15.270
2		17.991	17.906	17.905	22.691	22.693	22.693	23.694	23.695	23.695
3		19.371	19.377	19.377	28.772	28.699	28.698	30.001	29.934	29.933
4		26.068	26.060	26.060	34.292	34.281	34.281	35.304	35.294	35.294
5		35.144	35.053	35.053	37.899	37.905	37.905	38.435	38.442	38.442

Table 10: First five circular frequency parameters $\hat{\omega} = \omega \frac{a^2}{h} \sqrt{\frac{\rho}{E_2}}$, of square cylindrical shells with staking sequence [0°/90°/90°/0°] length-to-thickness ratio $a/h = 100$ and varying the radius-to-length ratio R_β/a.

Mode		SSSS			SFSS			SFSF		
	R_β/a	5	50	100	5	50	100	5	50	100
1		20.359	15.231	15.186	9.198	6.156	6.126	5.692	5.700	5.700
2		28.604	27.864	27.857	23.409	22.766	22.691	15.571	7.353	7.253
3		54.573	53.795	53.748	30.923	23.184	23.184	22.758	22.758	22.758
4		59.700	54.436	54.439	35.519	33.831	33.820	25.242	24.420	24.409
5		61.548	60.065	60.056	51.501	51.471	51.471	41.995	32.166	32.082
Mode		SFSC			SCSS			SCSC		
	R_β/a	5	50	100	5	50	100	5	50	100
1		11.411	7.942	7.898	26.272	22.026	21.996	33.964	30.720	30.697
2		24.209	23.835	23.835	32.962	32.215	32.207	39.376	38.726	38.718
3		38.139	31.045	30.985	57.062	56.905	56.905	61.024	60.867	60.867
4		41.815	40.169	40.157	71.834	66.814	66.777	85.287	80.995	80.958
5		51.904	51.837	51.837	73.372	72.054	72.045	86.582	85.426	85.419

are accounted for, a symmetric cross-ply [0°/90°/90°/0°] square spherical shallow shell ($R/a = 50$) is analyzed. As expected the fundamental frequency parameter increases when increasing both the orthotropic ratio and the length-to-thickness ratio, for all the considered boundary conditions. Finally, in Fig. 10 the first six mode shapes of SCSC square spherical shallow shells are represented. The geometrical characteristics and the lamination scheme are equal to the ones used for the cylindrical shallow shells shown in Figs. 7, 8 and 9.

7 Conclusions

A detailed literature reviews on the dynamic stiffness method for both buckling of laminated composite plates and free vibration of doubly curved shallow shells have been presented. In particular, in the first part of the report the stability equations have been obtained using the principle of minimum potential energy and the dynamic stiffness matrix has been derived for laminated composite plate elements based on the HSDT. The element stiffnesses have been implemented in a computer program and results for composite plate assemblies have been obtained and validated.

In the second part, an exact free vibration analysis of laminated composite shallow shells has been carried out by combining for the first time the dynamic stiffness method and a higher order shear deformation theory. The effect of several parameters such as length-to-thickness ratio and radius-to-length ratio, orthotropic ratio, stacking sequence and number of layers on the dimensionless circular frequency parameters has been investigated in details. Results have been compared with those available in the literature

Table 11: First six natural frequencies ω_{mn} rad/s, of square isotropic spherical shells.

Theory	Natural frequencies											
	ω_1 $(1,1)$		ω_2 $(2,1)$		ω_3 $(1,2)$		ω_4 $(2,2)$		ω_5 $(3,1)$		ω_6 $(1,3)$	
3D [90]	0.52543		0.58420		0.58487		0.67676		0.75219		0.75220	
CST [91]	0.53263	$(-1.370)^{\dagger}$	0.59041	(-1.063)	0.59080	(-1.014)	0.68486	(-1.197)	0.76020	(-1.065)	0.76260	(-1.383)
FSTD [92]	0.50211	$(+4.438)$	0.56247	$(+3.720)$	0.56248	$(+3.828)$	0.65706	$(+2.911)$	0.73915	$(+1.734)$	0.74035	$(+1.575)$
HSTD [92]	0.50223	$(+4.415)$	0.56276	$(+3.670)$	0.56277	$(+3.779)$	0.65788	$(+2.790)$	0.73966	$(+1.666)$	0.74081	$(+1.514)$
FSTD (DT‡) [93]	0.52864	(-0.611)	0.58954	(-0.914)	0.58954	(-0.798)	0.68370	(-1.025)	0.75974	(-1.004)	0.75974	(-1.002)
FSTD (ST*) [93]	0.52830	(-0.546)	0.58853	(-0.741)	0.58853	(-0.626)	0.68232	(-0.822)	0.75818	(-0.796)	0.75818	(-0.795)
HSDT DSM	0.52795	(-0.480)	0.58899	(-0.820)	0.58982	(-0.846)	0.68562	(-1.309)	0.75989	(-1.023)	0.76032	(-1.080)

Material and geometric properties: $a = b = 1.0118$, $h = 0.0191$, $R = 1.91$, $E = 1$, $\rho = 1$, $\nu = 0.3$

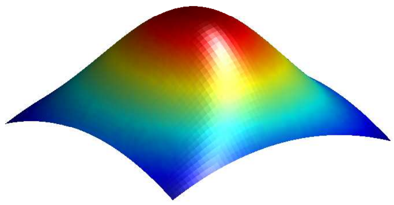
† $Error\% = \frac{\omega_{3D} - \omega}{\omega_{3D}} \times 100$; ‡ Donnell approximation; ∗ Sanders approximation

Table 12: First three circular frequency parameters $\hat{\omega} = \omega \frac{a^2}{h} \sqrt{\frac{\rho}{E_2}}$, of square spherical shells with staking sequence [0°/90°/0°] and varying the radius-to-length R/a and the length-to-thickness a/h ratios.

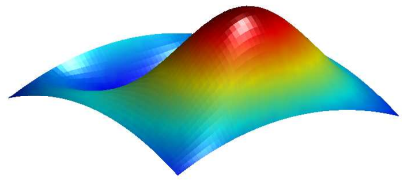
Mode	a/h		SSSS			SCSS			SCSC		
	10	R_β/a	20	50	100	20	50	100	20	50	100
1			8.611	8.601	8.600	9.999	9.990	9.988	11.494	11.485	11.484
2			14.572	14.563	14.562	15.345	15.335	15.335	16.237	16.227	16.226
3			20.876	20.868	20.868	21.989	21.982	21.981	23.038	23.031	23.030
Mode			SSSS			SCSS			SCSC		
	50	R_β/a	20	50	100	20	50	100	20	50	100
1			11.171	10.951	10.923	15.972	15.766	15.735	21.830	21.649	21.624
2			18.277	18.078	18.050	21.628	21.419	21.390	26.255	26.049	26.017
3			33.506	33.336	33.311	35.499	35.329	35.304	38.481	38.297	38.272
Mode			SSSS			SCSS			SCSC		
	100	R_β/a	20	50	100	20	50	100	20	50	100
1			10.279	11.187	11.067	14.130	16.314	16.201	23.562	22.824	22.718
2			16.662	18.357	18.243	22.832	21.981	21.860	28.044	27.207	27.087
3			34.561	33.895	33.795	36.767	36.086	35.987	40.249	39.525	39.419

Table 13: Fundamental circular frequency parameter $\hat{\omega} = \omega \frac{a^2}{h} \sqrt{\frac{\rho}{E_2}}$, of square spherical shells with staking sequence [0°/90°/90°/0°], $R/a = 50$ and varying orthotropic E_1/E_2 and the length-to-thickness a/h ratios.

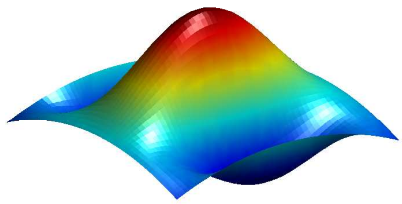
BCs	a/h	E_1/E_2				
		5	10	20	30	40
SSSS	5	6.4534	7.212	8.038	8.530	8.879
	10	7.697	9.201	11.110	12.336	13.223
	50	8.410	10.511	13.738	16.288	18.445
	100	8.702	10.803	14.050	16.660	18.904
SSSC	5	7.134	7.812	8.563	9.047	9.414
	10	9.427	11.102	12.877	13.915	14.653
	50	11.207	14.585	19.467	23.140	26.136
	100	11.495	14.957	20.098	24.101	27.467
SCSC	5	7.881	8.525	9.254	9.752	10.148
	10	11.360	13.112	14.764	15.686	16.341
	50	14.903	19.817	26.590	31.460	35.296
	100	15.259	20.432	27.881	33.563	38.282



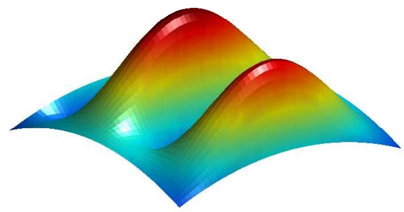
(a) Mode 1, $\hat{\omega} = 32.361$



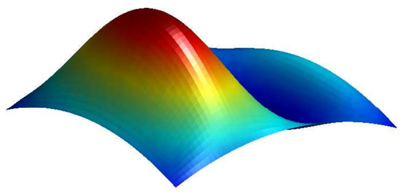
(b) Mode 2, $\hat{\omega} = 36.905$



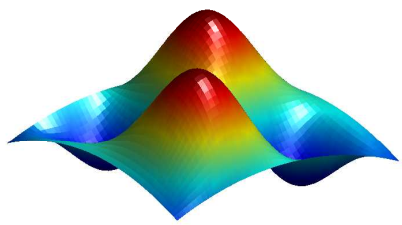
(c) Mode 3, $\hat{\omega} = 49.861$



(d) Mode 4, $\hat{\omega} = 72.907$



(e) Mode 5, $\hat{\omega} = 85.063$



(f) Mode 6, $\hat{\omega} = 87.862$

Figure 10: First six mode shapes of a symmetric cross-ply spherical shell with SCSC boundary condition.

including 3D elasticity solution. Different boundary conditions have been considered, according to the Lèvy-type closed-form solution. Representative mode shapes have been presented and discusses. The investigation has shown that the DSM allows computing all the natural frequencies of thin and thick laminated composite cylindrical and spherical shallow shells with high accuracy and low computational cost. The implementation of the HSDT within the framework of the DSM permits considerable improvements in results accuracy when compared to other analytical, FEM or meshless formulations based on FSDT.
All the Work packages have been successfully completed (see Gantt chart in Fig. 11 and the proposal [94]). The project is completed as scheduled.

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Appendix A: Laminate geometric and constitutive equations

The geometric relation for a lamina in the local or lamina reference system can be written as:

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{bmatrix} = \begin{bmatrix} \mathcal{D}_x & 0 & 0 \\ 0 & \mathcal{D}_y & 0 \\ \mathcal{D}_y & \mathcal{D}_x & 0 \\ 0 & \mathcal{D}_z & \mathcal{D}_y \\ \mathcal{D}_z & 0 & \mathcal{D}_x \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \tag{A.1}$$

and in terms of the functional degrees of freedom:

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{bmatrix} = \begin{bmatrix} \mathcal{D}_x & 0 & \left(c_1 z^3\right) \mathcal{D}_{xx} & \left(z + c_1 z^3\right) \mathcal{D}_x & 0 \\ 0 & \mathcal{D}_y & \left(c_1 z^3\right) \mathcal{D}_{yy} & 0 & \left(z + c_1 z^3\right) \mathcal{D}_y \\ \mathcal{D}_y & \mathcal{D}_x & \left(c_1 z^3 \mathcal{D}_{xy} + c_1 z^3 \mathcal{D}_{yx}\right) & \left(z + c_1 z^3\right) \mathcal{D}_y & \left(z + c_1 z^3\right) \mathcal{D}_x \\ 0 & 0 & \left(1 + 3 c_1 z^2\right) \mathcal{D}_y & 0 & \left(1 + 3 c_1 z^2\right) \\ 0 & 0 & \left(1 + 3 c_1 z^2\right) \mathcal{D}_x & \left(1 + 3 c_1 z^2\right) & 0 \end{bmatrix} \begin{bmatrix} u^0 \\ v^0 \\ w^0 \\ \phi_x \\ \phi_y \end{bmatrix} \tag{A.2}$$

where $\mathcal{D}_x$ and $\mathcal{D}_y$ are the derivatives in x and y respectively and $c_1 = -\frac{4}{3h^2}$. The constitutive equations in the lamina reference system can be written, in terms of reduced stiffness coefficients, as:

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \\ \tau_{23} \\ \tau_{13} \end{bmatrix} = \begin{bmatrix} \tilde{C}_{11} & \tilde{C}_{12} & 0 & 0 & 0 \\ \tilde{C}_{12} & \tilde{C}_{22} & 0 & 0 & 0 \\ 0 & 0 & \tilde{C}_{66} & 0 & 0 \\ 0 & 0 & 0 & \tilde{C}_{44} & 0 \\ 0 & 0 & 0 & 0 & \tilde{C}_{55} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{13} \end{bmatrix} \tag{A.3}$$

where the $\tilde{C}_{ij}$ are expressed in terms of stiffness coefficients C_{ij}, as:

$$\begin{aligned} &\tilde{C}_{11} = C_{11} - \frac{C_{13}^2}{C_{33}} = \frac{E_1}{1 - \nu_{12}\nu_{21}}, \qquad \tilde{C}_{12} = C_{12} - \frac{C_{13}\,C_{23}}{C_{33}} = \frac{\nu_{12}E_2}{1 - \nu_{12}\nu_{21}}, \qquad \tilde{C}_{22} = C_{22} - \frac{C_{23}^2}{C_{33}} = \frac{E_2}{1 - \nu_{12}\nu_{21}} \\ &\tilde{C}_{44} = C_{44} = G_{23}, \qquad \tilde{C}_{55} = C_{55} = G_{13} \qquad \tilde{C}_{66} = C_{66} = G_{12} \end{aligned} \tag{A.4}$$

where E_1 is the elastic modulus in the fibre direction, E_2 the elastic modulus in perpendicular to the fibre, ν_{12} and $\nu_{21} = \nu_{12}\,E_2/E_1$ the Poisson's ratios, $G_{12} = G_{13}$ and G_{23} the shear modulus of each single orthotropic lamina. If the lamina is placed at an angle θ in the laminate or global reference system, the equation need to be transformed as follows:

$$\begin{aligned} \overline{C}_{11} &= \tilde{C}_{11}\mathcal{C}^4 + 2(\tilde{C}_{12} + 2\tilde{C}_{66})\mathcal{S}^2\mathcal{C}^2 + \tilde{C}_{22}\mathcal{S}^4 \\ \overline{C}_{12} &= (\tilde{C}_{11} + \tilde{C}_{22} - 4\tilde{C}_{66})\mathcal{S}^2\mathcal{C}^2 + \tilde{C}_{12}(\mathcal{S}^4 + \mathcal{C}^4) \\ \overline{C}_{16} &= (\tilde{C}_{11} - \tilde{C}_{12} - 2\tilde{C}_{66})\mathcal{S}\mathcal{C}^3 + (\tilde{C}_{12} - \tilde{C}_{22} + 2\tilde{C}_{66})\mathcal{S}^3 \\ \overline{C}_{22} &= \tilde{C}_{11}\mathcal{S}^4 + 2(\tilde{C}_{12} + 2\tilde{C}_{66})\mathcal{S}^2\mathcal{C}^2 + \tilde{C}_{22}\mathcal{C}^4 \\ \overline{C}_{26} &= (\tilde{C}_{11} - \tilde{C}_{12} - 2\tilde{C}_{66})\mathcal{S}^3\mathcal{C} + (\tilde{C}_{12} - \tilde{C}_{22} + 2\tilde{C}_{66})\mathcal{S}\mathcal{C}^3 \\ \overline{C}_{66} &= (\tilde{C}_{11} + \tilde{C}_{22} - 2\tilde{C}_{12} - 2\tilde{C}_{66})\mathcal{S}^2\mathcal{C}^2 + \tilde{C}_{66}(\mathcal{S}^4 + \mathcal{C}^4) \\ \overline{C}_{44} &= \tilde{C}_{44}\mathcal{C}^2 + \tilde{C}_{55}\mathcal{S}^2 \\ \overline{C}_{55} &= \tilde{C}_{44}\mathcal{S}^2 + \tilde{C}_{55}\mathcal{C}^2 \\ \overline{C}_{45} &= (\tilde{C}_{55} - \tilde{C}_{44})\mathcal{C}\mathcal{S} \end{aligned} \tag{A.5}$$

where $\mathcal{C} = \cos(\theta)$ and $\mathcal{S} = \sin(\theta)$. This leads to the constitutive equation for the k-th lamina in the laminate or global reference system:

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{bmatrix} = \begin{bmatrix} \overline{C}_{11} & \overline{C}_{12} & \overline{C}_{16} & 0 & 0 \\ \overline{C}_{12} & \overline{C}_{22} & \overline{C}_{26} & 0 & 0 \\ \overline{C}_{16} & \overline{C}_{26} & \overline{C}_{66} & 0 & 0 \\ 0 & 0 & 0 & \overline{C}_{44} & \overline{C}_{45} \\ 0 & 0 & 0 & \overline{C}_{45} & \overline{C}_{55} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{bmatrix} \tag{A.6}$$

that in compact form can be written for each k-th lamina as:

$$\boldsymbol{\sigma}^k = \overline{\boldsymbol{C}}^k \boldsymbol{\varepsilon}^k \tag{A.7}$$

Appendix B: Polynomials Coefficients

The polynomial coefficients are following defined:

$$a_1 = c_1^2\,(F_{11}^2 - D_{11}\,H_{11})(D_{66} + c_1\,(2\,F_{66} + c_1\,H_{66}))$$

$$\begin{aligned}
a_2 = {} & -c_1^2 \left(A_{44} + \alpha^2 D_{22}\right) F_{11}^2 + 2\alpha^2 c_1^2 D_{12} F_{11} F_{12} - \alpha^2 c_1^2 D_{11} F_{12}^2 + 2\alpha^2 c_1^3 F_{11} F_{12}^2 - 2\alpha^2 c_1^3 F_{11}^2 F_{22} + 4\alpha^2 c_1^2 D_{12} \\
& F_{11} F_{66} - 4\alpha^2 c_1^2 D_{11} F_{12} F_{66} - 4\alpha^2 c_1^2 D_{11} F_{66}^2 - 8\alpha^2 c_1^3 F_{11} F_{66}^2 + A_{44} c_1^2 D_{11} H_{11} - \alpha^2 c_1^2 D_{12}^2 H_{11} + \alpha^2 c_1^2 D_{11} D_{22} H_{11} \\
& - 2\alpha^2 c_1^2 D_{12} D_{66} H_{11} - 2\alpha^2 c_1^3 D_{12} F_{12} H_{11} - 4\alpha^2 c_1^3 D_{66} F_{12} H_{11} - \alpha^2 c_1^4 F_{12}^2 H_{11} + 2\alpha^2 c_1^3 D_{11} F_{22} H_{11} - 4\alpha^2 c_1^4 F_{12} F_{66} \\
& H_{11} - 4\alpha^2 c_1^4 F_{66}^2 H_{11} + 2\alpha^2 c_1^2 D_{11} D_{66} H_{12} + 2\alpha^2 c_1^3 D_{12} F_{11} H_{12} + 4\alpha^2 c_1^3 D_{66} F_{11} H_{12} - 2\alpha^2 c_1^3 D_{11} F_{12} H_{12} + 2\alpha^2 c_1^4 F_{11} \\
& F_{12} H_{12} + 4\alpha^2 c_1^4 F_{11} F_{66} H_{12} - \alpha^2 c_1^4 D_{11} H_{12}^2 - \alpha^2 c_1^4 F_{11}^2 H_{22} + \alpha^2 c_1^4 D_{11} H_{11} H_{22} + 4\alpha^2 c_1^2 D_{11} D_{66} H_{66} + 4\alpha^2 c_1^3 D_{12} F_{11} \\
& H_{66} + 8\alpha^2 c_1^3 D_{66} F_{11} H_{66} - 4\alpha^2 c_1^3 D_{11} F_{12} H_{66} + 2\alpha^2 c_1^4 D_{12} H_{11} H_{66} + 4\alpha^2 c_1^4 D_{66} H_{11} H_{66} - 2\alpha^2 c_1^4 D_{11} H_{12} H_{66} + A_{55} D_{11} \\
& (D_{66} + c_1 (2F_{66} + c_1 H_{66})) + c_2^2 \left(D_{11} D_{66} F_{55} + 2c_1 D_{11} F_{55} F_{66} + c_1^2 \left(-F_{11}^2 F_{44} + D_{11} F_{44} H_{11} + D_{11} F_{55} H_{66}\right)\right) + 2c_2 \\
& \left(-c_1^2 D_{44} F_{11}^2 + D_{11} \left(c_1^2 D_{44} H_{11} + D_{55} \left(D_{66} + 2c_1 F_{66} + c_1^2 H_{66}\right)\right)\right) + (D_{11} + c_1 (2F_{11} + c_1 H_{11}))(D_{66} + c_1 (2F_{66} + c_1 H_{66}))\lambda N_{x0}
\end{aligned}$$

$$\begin{aligned}
a_3 = {} & 2\alpha^2 c_2 D_{12}^2 D_{55} - 2\alpha^2 c_2 D_{11} D_{22} D_{55} - 4c_2^2 D_{11} D_{44} D_{55} - 2\alpha^2 c_2 D_{11} D_{44} D_{66} + 4\alpha^2 c_2 D_{12} D_{55} D_{66} - 4\alpha^2 c_1 c_2 D_{12} D_{44} F_{11} \\
& - 8\alpha^2 c_1 c_2 D_{44} D_{66} F_{11} + 4\alpha^2 c_1 c_2 D_{11} D_{44} F_{12} + 4\alpha^2 c_1 c_2 D_{12} D_{55} F_{12} + 8\alpha^2 c_1 c_2 D_{55} D_{66} F_{12} + 2\alpha^4 c_1^2 D_{22} F_{11} F_{12} + 4\alpha^2 \\
& c_1^2 c_2 D_{44} F_{11} F_{12} - 2\alpha^4 c_1^2 D_{12} F_{12}^2 + 2\alpha^2 c_1^2 c_2 D_{55} F_{12}^2 - 4\alpha^4 c_1^3 F_{12}^3 - 4\alpha^2 c_1 c_2 D_{11} D_{55} F_{22} - 2\alpha^4 c_1^2 D_{12} F_{11} F_{22} - 2 \\
& \alpha^4 c_1^2 D_{66} F_{11} F_{22} + 2\alpha^4 c_1^2 D_{11} F_{12} F_{22} + 4\alpha^4 c_1^3 F_{11} F_{12} F_{22} - 2c_2^3 D_{11} D_{55} F_{44} - \alpha^2 c_2^2 D_{11} D_{66} F_{44} - 2\alpha^2 c_1 c_2^2 D_{12} F_{11} \\
& F_{44} - 4\alpha^2 c_1 c_2^2 D_{66} F_{11} F_{44} + 2\alpha^2 c_1 c_2^2 D_{11} F_{12} F_{44} + 2\alpha^2 c_1^2 c_2^2 F_{11} F_{12} F_{44} + \alpha^2 c_2^2 D_{12}^2 F_{55} - \alpha^2 c_2^2 D_{11} D_{22} F_{55} - 2c_2^3 \\
& D_{11} D_{44} F_{55} + 2\alpha^2 c_2^2 D_{12} D_{66} F_{55} + 2\alpha^2 c_1 c_2^2 D_{12} F_{12} F_{55} + 4\alpha^2 c_1 c_2^2 D_{66} F_{12} F_{55} + \alpha^2 c_1^2 c_2^2 F_{12}^2 F_{55} - 2\alpha^2 c_1 c_2^2 D_{11} \\
& F_{22} F_{55} - c_2^4 D_{11} F_{44} F_{55} + 4\alpha^2 c_1 c_2 D_{11} D_{44} F_{66} + 4\alpha^4 c_1^2 D_{22} F_{11} F_{66} + 8\alpha^2 c_1^2 c_2 D_{44} F_{11} F_{66} - 8\alpha^4 c_1^2 D_{12} F_{12} F_{66} + 8\alpha^2 c_1^2 \\
& c_2 D_{55} F_{12} F_{66} - 16\alpha^4 c_1^3 F_{12}^2 F_{66} + 4\alpha^4 c_1^2 D_{11} F_{22} F_{66} + 12\alpha^4 c_1^3 F_{11} F_{22} F_{66} + 2\alpha^2 c_1 c_2^2 D_{11} F_{44} F_{66} + 4\alpha^2 c_1^2 c_2^2 F_{11} F_{44} \\
& F_{66} + 4\alpha^2 c_1^2 c_2^2 F_{12} F_{55} F_{66} - 8\alpha^4 c_1^2 D_{12} F_{66}^2 + 8\alpha^2 c_1^2 c_2 D_{55} F_{66}^2 - 16\alpha^4 c_1^3 F_{12} F_{66}^2 + 4\alpha^2 c_1^2 c_2^2 F_{55} F_{66}^2 - 4\alpha^2 c_1^2 \\
& c_2 D_{12} D_{44} H_{11} - \alpha^4 c_1^2 D_{22} D_{66} H_{11} - 8\alpha^2 c_1^2 c_2 D_{44} D_{66} H_{11} + 2\alpha^4 c_1^3 D_{22} F_{12} H_{11} - 2\alpha^4 c_1^3 D_{12} F_{22} H_{11} - 4\alpha^4 c_1^3 D_{66} F_{22} \\
& H_{11} + 2\alpha^4 c_1^4 F_{12} F_{22} H_{11} - 2\alpha^2 c_1^2 c_2^2 D_{12} F_{44} H_{11} - 4\alpha^2 c_1^2 c_2^2 D_{66} F_{44} H_{11} + 2\alpha^4 c_1^3 D_{22} F_{66} H_{11} + 4\alpha^4 c_1^4 F_{22} F_{66} H_{11} + 2 \\
& \alpha^4 c_1^2 D_{12}^2 H_{12} - 2\alpha^4 c_1^2 D_{11} D_{22} H_{12} + 4\alpha^4 c_1^2 D_{12} D_{66} H_{12} - 2\alpha^4 c_1^3 D_{22} F_{11} H_{12} + 4\alpha^4 c_1^3 D_{12} F_{12} H_{12} + 8\alpha^4 c_1^3 D_{66} F_{12} \\
& H_{12} - 2\alpha^4 c_1^4 F_{12}^2 H_{12} - 2\alpha^4 c_1^3 D_{11} F_{22} H_{12} - 2\alpha^4 c_1^4 F_{11} F_{22} H_{12} - 8\alpha^4 c_1^4 F_{12} F_{66} H_{12} - 8\alpha^4 c_1^4 F_{66}^2 H_{12} + 2\alpha^4 c_1^4 D_{12} \\
& H_{12}^2 + 4\alpha^4 c_1^4 D_{66} H_{12}^2 - 2\alpha^2 c_1^2 c_2 D_{11} D_{55} H_{22} - \alpha^4 c_1^2 D_{11} D_{66} H_{22} - 2\alpha^4 c_1^3 D_{12} F_{11} H_{22} - 4\alpha^4 c_1^3 D_{66} F_{11} H_{22} + 2\alpha^4 c_1^3 \\
& D_{11} F_{12} H_{22} + 2\alpha^4 c_1^4 F_{11} F_{12} H_{22} - \alpha^2 c_1^2 c_2^2 D_{11} F_{55} H_{22} + 2\alpha^4 c_1^3 D_{11} F_{66} H_{22} + 4\alpha^4 c_1^4 F_{11} F_{66} H_{22} - 2\alpha^4 c_1^4 D_{12} H_{11} H_{22} \\
& - 4\alpha^4 c_1^4 D_{66} H_{11} H_{22} + 4\alpha^4 c_1^2 D_{12}^2 H_{66} - 4\alpha^4 c_1^2 D_{11} D_{22} H_{66} - 2\alpha^2 c_1^2 c_2 D_{11} D_{44} H_{66} - 4\alpha^2 c_1^2 c_2 D_{12} D_{55} H_{66} + 8\alpha^4 c_1^2 D_{12} \\
& D_{66} H_{66} - 8\alpha^2 c_1^2 c_2 D_{55} D_{66} H_{66} - 4\alpha^4 c_1^3 D_{22} F_{11} H_{66} + 8\alpha^4 c_1^3 D_{12} F_{12} H_{66} + 16\alpha^4 c_1^3 D_{66} F_{12} H_{66} - 4\alpha^4 c_1^3 D_{11} F_{22} H_{66} \\
& - 2\alpha^4 c_1^4 F_{11} F_{22} H_{66} - \alpha^2 c_1^2 c_2^2 D_{11} F_{44} H_{66} - 2\alpha^2 c_1^2 c_2^2 D_{12} F_{55} H_{66} - 4\alpha^2 c_1^2 c_2^2 D_{66} F_{55} H_{66} - \alpha^4 c_1^4 D_{22} H_{11} H_{66} + 4\alpha^4 c_1^4 \\
& D_{12} H_{12} H_{66} + 8\alpha^4 c_1^4 D_{66} H_{12} H_{66} - \alpha^4 c_1^4 D_{11} H_{22} H_{66} + \alpha^2 D_{12}^2 \lambda N_{x0} - \alpha^2 D_{11} D_{22} \lambda N_{x0} - 2c_2 D_{11} D_{44} \lambda N_{x0} + 2\alpha^2 D_{12} D_{66} \\
& \lambda N_{x0} - 2c_2 D_{55} D_{66} \lambda N_{x0} - 2\alpha^2 c_1 D_{22} F_{11} \lambda N_{x0} - 4c_1 c_2 D_{44} F_{11} \lambda N_{x0} + 4\alpha^2 c_1 D_{12} F_{12} \lambda N_{x0} + 4\alpha^2 c_1 D_{66} F_{12} \lambda \\
& N_{x0} + 4\alpha^2 c_1^2 F_{12}^2 \lambda N_{x0} - 2\alpha^2 c_1 D_{11} F_{22} \lambda N_{x0} - 4\alpha^2 c_1^2 F_{11} F_{22} \lambda N_{x0} - c_2^2 D_{11} F_{44} \lambda N_{x0} - 2c_1 c_2^2 F_{11} F_{44} \lambda \\
& N_{x0} - c_2^2 D_{66} F_{55} \lambda N_{x0} + 4\alpha^2 c_1 D_{12} F_{66} \lambda N_{x0} - 4c_1 c_2 D_{55} F_{66} \lambda N_{x0} + 8\alpha^2 c_1^2 F_{12} F_{66} \lambda N_{x0} - 2c_1 c_2^2 F_{55} F_{66} \lambda \\
& N_{x0} - \alpha^2 c_1^2 D_{22} H_{11} \lambda N_{x0} - 2c_1^2 c_2 D_{44} H_{11} \lambda N_{x0} - 2\alpha^2 c_1^3 F_{22} H_{11} \lambda N_{x0} - c_1^2 c_2^2 F_{44} H_{11} \lambda N_{x0} + 2\alpha^2 c_1^2 D_{12} H_{12} \lambda \\
& N_{x0} + 2\alpha^2 c_1^2 D_{66} H_{12} \lambda N_{x0} + 4\alpha^2 c_1^3 F_{12} H_{12} \lambda N_{x0} + 4\alpha^2 c_1^3 F_{66} H_{12} \lambda N_{x0} + \alpha^2 c_1^4 H_{12}^2 \lambda N_{x0} - \alpha^2 c_1^2 D_{11} H_{22} \lambda \\
& N_{x0} - 2\alpha^2 c_1^3 F_{11} H_{22} \lambda N_{x0} - \alpha^2 c_1^4 H_{11} H_{22} \lambda N_{x0} + 2\alpha^2 c_1^2 D_{12} H_{66} \lambda N_{x0} - 2c_1^2 c_2 D_{55} H_{66} \lambda N_{x0} + 4\alpha^2 c_1^3 F_{12} H_{66} \\
& \lambda N_{x0} - c_1^2 c_2^2 F_{55} H_{66} \lambda N_{x0} + 2\alpha^2 c_1^4 H_{12} H_{66} \lambda N_{x0} - A_{44} \Big(A_{55} D_{11} + 2c_2 D_{11} D_{55} + c_2^2 D_{11} F_{55} + \alpha^2 (2c_1 (F_{11} (D_{12} + 2D_{66}
\end{aligned}$$

$$
\begin{aligned}
&- c_1(F_{12} + 2F_{66})) + c_1(D_{12} + 2D_{66})H_{11}) + D_{11}(D_{66} + c_1(-2(F_{12} + F_{66}) + c_1H_{66}))) + (D_{11} + c_1(2F_{11} + c_1H_{11}))\lambda N_{x0}\Big) \\
&+ A_{55}\Big(- c_2D_{11}(2D_{44} + c_2F_{44}) + \alpha^2\Big(D_{12}^2 - D_{11}(D_{22} + c_1(2F_{22} + c_1H_{22})) + 2D_{12}(D_{66} + c_1(F_{12} - c_1H_{66})) \\
&+ c_1\Big(c_1(F_{12} + 2F_{66})^2 + 4D_{66}(F_{12} - c_1H_{66})\Big)\Big) - (D_{66} + c_1(2F_{66} + c_1H_{66}))\lambda N_{x0}\Big) - \alpha^2(D_{11} + c_1(2F_{11} + c_1H_{11})) \\
&(D_{66} + c_1(2F_{66} + c_1H_{66}))\lambda N_{y0}
\end{aligned}
$$

$$
\begin{aligned}
a_4 = {}& 4A_{55}\alpha^2c_2D_{12}D_{44} - 2\alpha^4c_2D_{12}^2D_{44} + 2\alpha^4c_2D_{11}D_{22}D_{44} + 8\alpha^2c_2^2D_{12}D_{44}D_{55} + A_{55}\alpha^4D_{22}D_{66} + 8A_{55}\alpha^2c_2D_{44}D_{66} \\
&- 4\alpha^4c_2D_{12}D_{44}D_{66} + 2\alpha^4c_2D_{22}D_{55}D_{66} + 16\alpha^2c_2^2D_{44}D_{55}D_{66} + 4\alpha^4c_1c_2D_{22}D_{44}F_{11} - 2A_{55}\alpha^4c_1D_{22}F_{12} - 4\alpha^4c_1c_2D_{12} \\
&D_{44}F_{12} - 4\alpha^4c_1c_2D_{22}D_{55}F_{12} - 8\alpha^4c_1c_2D_{44}D_{66}F_{12} - \alpha^6c_1^2D_{22}F_{12}^2 - 2\alpha^4c_1^2c_2D_{44}F_{12}^2 + 2A_{55}\alpha^4c_1D_{12}F_{22} + 4\alpha^4c_1 \\
&c_2D_{12}D_{55}F_{22} + 4A_{55}\alpha^4c_1D_{66}F_{22} + 8\alpha^4c_1c_2D_{55}D_{66}F_{22} - 2A_{55}\alpha^4c_1^2F_{12}F_{22} + 2\alpha^6c_1^2D_{12}F_{12}F_{22} - 4\alpha^4c_1^2c_2D_{55}F_{12} \\
&F_{22} + 2\alpha^6c_1^3F_{12}^2F_{22} - \alpha^6c_1^2D_{11}F_{22}^2 - 2\alpha^6c_1^3F_{11}F_{22}^2 + 2A_{55}\alpha^2c_2^2D_{12}F_{44} - \alpha^4c_2^2D_{12}^2F_{44} + \alpha^4c_2^2D_{11}D_{22}F_{44} + 4\alpha^2 \\
&c_2^3D_{12}D_{55}F_{44} + 4A_{55}\alpha^2c_2^2D_{66}F_{44} - 2\alpha^4c_2^2D_{12}D_{66}F_{44} + 8\alpha^2c_2^3D_{55}D_{66}F_{44} + 2\alpha^4c_1c_2^2D_{22}F_{11}F_{44} - 2\alpha^4c_1c_2^2D_{12} \\
&F_{12}F_{44} - 4\alpha^4c_1c_2^2D_{66}F_{12}F_{44} - \alpha^4c_1^2c_2^2F_{12}^2F_{44} + 4\alpha^2c_2^3D_{12}D_{44}F_{55} + \alpha^4c_2^2D_{22}D_{66}F_{55} + 8\alpha^2c_2^3D_{44}D_{66}F_{55} - 2\alpha^4 \\
&c_1c_2^2D_{22}F_{12}F_{55} + 2\alpha^4c_1c_2^2D_{12}F_{22}F_{55} + 4\alpha^4c_1c_2^2D_{66}F_{22}F_{55} - 2\alpha^4c_1^2c_2^2F_{12}F_{22}F_{55} + 2\alpha^2c_2^4D_{12}F_{44}F_{55} + 4\alpha^2 \\
&c_2^4D_{66}F_{44}F_{55} - 2A_{55}\alpha^4c_1D_{22}F_{66} - 4\alpha^4c_1c_2D_{22}D_{55}F_{66} - 4\alpha^6c_1^2D_{22}F_{12}F_{66} - 8\alpha^4c_1^2c_2D_{44}F_{12}F_{66} - 4A_{55}\alpha^4c_1^2F_{22} \\
&F_{66} + 4\alpha^6c_1^2D_{12}F_{22}F_{66} - 8\alpha^4c_1^2c_2D_{55}F_{22}F_{66} - 4\alpha^4c_1^2c_2^2F_{12}F_{44}F_{66} - 2\alpha^4c_1c_2^2D_{22}F_{55}F_{66} - 4\alpha^4c_1^2c_2^2F_{22}F_{55} \\
&F_{66} - 4\alpha^6c_1^2D_{22}F_{66}^2 - 8\alpha^4c_1^2c_2D_{44}F_{66}^2 - 8\alpha^6c_1^3F_{22}F_{66}^2 - 4\alpha^4c_1^2c_2^2F_{44}F_{66}^2 + 2\alpha^4c_1^2c_2D_{22}D_{44}H_{11} - \alpha^6c_1^4 \\
&F_{22}^2H_{11} + \alpha^4c_1^2c_2^2D_{22}F_{44}H_{11} + 2\alpha^6c_1^2D_{22}D_{66}H_{12} - 2\alpha^6c_1^3D_{22}F_{12}H_{12} + 2\alpha^6c_1^3D_{12}F_{22}H_{12} + 4\alpha^6c_1^3D_{66}F_{22}H_{12} \\
&+ 2\alpha^6c_1^4F_{12}F_{22}H_{12} + 4\alpha^6c_1^4F_{22}F_{66}H_{12} - \alpha^6c_1^4D_{22}H_{12}^2 + 2A_{55}\alpha^4c_1^2D_{12}H_{22} - \alpha^6c_1^2D_{12}^2H_{22} + \alpha^6c_1^2D_{11}D_{22}H_{22} \\
&+ 4\alpha^4c_1^2c_2D_{12}D_{55}H_{22} + 4A_{55}\alpha^4c_1^2D_{66}H_{22} - 2\alpha^6c_1^2D_{12}D_{66}H_{22} + 8\alpha^4c_1^2c_2D_{55}D_{66}H_{22} + 2\alpha^6c_1^3D_{22}F_{11}H_{22} - 2\alpha^6c_1^3 \\
&D_{12}F_{12}H_{22} - 4\alpha^6c_1^3D_{66}F_{12}H_{22} - \alpha^6c_1^4F_{12}^2H_{22} + 2\alpha^4c_1^2c_2^2D_{12}F_{55}H_{22} + 4\alpha^4c_1^2c_2^2D_{66}F_{55}H_{22} - 4\alpha^6c_1^4F_{12}F_{66}H_{22} \\
&- 4\alpha^6c_1^4F_{66}^2H_{22} + \alpha^6c_1^4D_{22}H_{11}H_{22} + A_{55}\alpha^4c_1^2D_{22}H_{66} + 4\alpha^4c_1^2c_2D_{12}D_{44}H_{66} + 2\alpha^4c_1^2c_2D_{22}D_{55}H_{66} + 4\alpha^6c_1^2D_{22} \\
&D_{66}H_{66} + 8\alpha^4c_1^2c_2D_{44}D_{66}H_{66} - 4\alpha^6c_1^3D_{22}F_{12}H_{66} + 4\alpha^6c_1^3D_{12}F_{22}H_{66} + 8\alpha^6c_1^3D_{66}F_{22}H_{66} + 2\alpha^4c_1^2c_2^2D_{12}F_{44}H_{66} \\
&+ 4\alpha^4c_1^2c_2^2D_{66}F_{44}H_{66} + \alpha^4c_1^2c_2^2D_{22}F_{55}H_{66} - 2\alpha^6c_1^4D_{22}H_{12}H_{66} + 2\alpha^6c_1^4D_{12}H_{22}H_{66} + 4\alpha^6c_1^4D_{66}H_{22}H_{66} + A_{55}\alpha^2 \\
&D_{22}\lambda N_{x0} + 2A_{55}c_2D_{44}\lambda N_{x0} + 2\alpha^2c_2D_{22}D_{55}\lambda N_{x0} + 4c_2^2D_{44}D_{55}\lambda N_{x0} + \alpha^4D_{22}D_{66}\lambda N_{x0} + 2\alpha^2c_2D_{44}D_{66}\lambda N_{x0} \\
&+ 2A_{55}\alpha^2c_1F_{22}\lambda N_{x0} + 4\alpha^2c_1c_2D_{55}F_{22}\lambda N_{x0} + 2\alpha^4c_1D_{66}F_{22}\lambda N_{x0} + A_{55}c_2^2F_{44}\lambda N_{x0} + 2c_2^3D_{55}F_{44}\lambda N_{x0} + \alpha^2 \\
&c_2^2D_{66}F_{44}\lambda N_{x0} + \alpha^2c_2^2D_{22}F_{55}\lambda N_{x0} + 2c_2^3D_{44}F_{55}\lambda N_{x0} + 2\alpha^2c_1c_2^2F_{22}F_{55}\lambda N_{x0} + c_2^4F_{44}F_{55}\lambda N_{x0} + 2\alpha^4c_1 \\
&D_{22}F_{66}\lambda N_{x0} + 4\alpha^2c_1c_2D_{44}F_{66}\lambda N_{x0} + 4\alpha^4c_1^2F_{22}F_{66}\lambda N_{x0} + 2\alpha^2c_1c_2^2F_{44}F_{66}\lambda N_{x0} + A_{55}\alpha^2c_1^2H_{22}\lambda N_{x0} \\
&+ 2\alpha^2c_1^2c_2D_{55}H_{22}\lambda N_{x0} + \alpha^4c_1^2D_{66}H_{22}\lambda N_{x0} + \alpha^2c_1^2c_2^2F_{55}H_{22}\lambda N_{x0} + 2\alpha^4c_1^3F_{66}H_{22}\lambda N_{x0} + \alpha^4c_1^2D_{22} \\
&H_{66}\lambda N_{x0} + 2\alpha^2c_1^2c_2D_{44}H_{66}\lambda N_{x0} + 2\alpha^4c_1^3F_{22}H_{66}\lambda N_{x0} + \alpha^2c_1^2c_2^2F_{44}H_{66}\lambda N_{x0} + \alpha^4c_1^4H_{22}H_{66}\lambda N_{x0} + \alpha^2 \\
&\Big(A_{55}(D_{66} + c_1(2F_{66} + c_1H_{66})) + 2c_2(D_{11}D_{44} + D_{55}D_{66} + c_1(2D_{44}F_{11} + 2D_{55}F_{66} + c_1D_{44}H_{11} + c_1D_{55}H_{66})) \\
&+ c_2^2(D_{11}F_{44} + D_{66}F_{55} + c_1(2F_{11}F_{44} + 2F_{55}F_{66} + c_1F_{44}H_{11} + c_1F_{55}H_{66})) - \alpha^2\Big(D_{12}^2 - D_{11}(D_{22} + c_1(2F_{22} \\
&+ c_1H_{22})) + 2D_{12}(D_{66} + c_1(2F_{12} + 2F_{66} + c_1(H_{12} + H_{66}))) + c_1(4F_{12}(D_{66} + c_1F_{12}) - D_{22}(2F_{11} + c_1H_{11}) \\
&+ c_1(8F_{12}F_{66} + 2D_{66}H_{12} - 2F_{11}(2F_{22} + c_1H_{22}) + c_1(-2F_{22}H_{11} + H_{12}(4(F_{12} + F_{66}) + c_1H_{12}) - c_1H_{11}H_{22} \\
&+ 4F_{12}H_{66} + 2c_1H_{12}H_{66}))))\Big)\Big)\lambda N_{y0} + A_{44}\Big(2A_{55}\alpha^2(D_{12} + 2D_{66}) + \alpha^4\Big(- D_{12}^2 + D_{11}D_{22} + c_1\Big(2D_{22}F_{11} - 4D_{66} \\
&F_{12} - c_1(F_{12} + 2F_{66})^2 + c_1D_{22}H_{11} + 4c_1D_{66}H_{66}\Big) - 2D_{12}(D_{66} + c_1(F_{12} - c_1H_{66}))\Big) + A_{55}\lambda N_{x0} + c_2(2D_{55} + c_2 \\
&F_{55})\lambda N_{x0} + \alpha^2\Big(4c_2D_{55}(D_{12} + 2D_{66}) + 2c_2^2(D_{12} + 2D_{66})F_{55} + (D_{66} + c_1(2F_{66} + c_1H_{66}))\lambda N_{x0} + (D_{11} + c_1 \\
&(2F_{11} + c_1H_{11}))\lambda N_{y0}\Big)\Big)
\end{aligned}
$$

$$
\begin{aligned}
a_5 = {}& -\alpha^2\left(A_{55} + 2c_2D_{55} + c_2^2F_{55} + \alpha^2\left(D_{66} + 2c_1F_{66} + c_1^2H_{66}\right)\right)\Big(\alpha^2\Big(A_{44}D_{22} + 2c_2D_{22}D_{44} + c_2^2D_{22}F_{44} + \alpha^2c_1^2 \\
&\left(-F_{22}^2 + D_{22}H_{22}\right)\Big) + \left(A_{44} + c_2(2D_{44} + c_2F_{44}) + \alpha^2\left(D_{22} + 2c_1F_{22} + c_1^2H_{22}\right)\right)\lambda N_{y0}\Big)
\end{aligned}
\tag{B.1}
$$

Appendix C: Gantt Chart

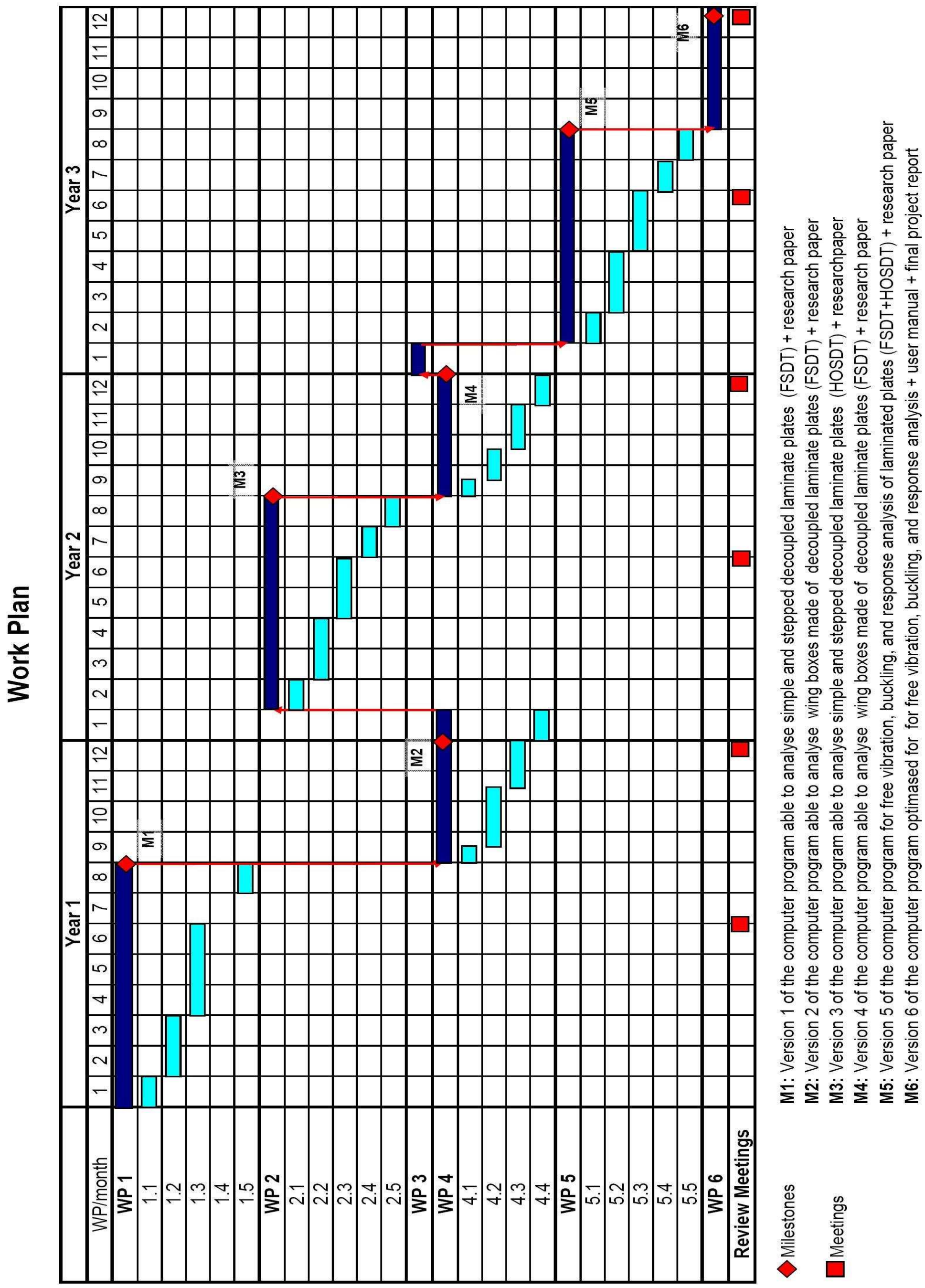

Figure 11: Gantt Chart