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COMPUTATION OF OPTIMAL ACTUATOR/SENSOR LOCATIONS

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Final Report

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14. ABSTRACT Many systems, such as acoustic noise and structural vibrations are distributed in space. The location of control actuators, and also the sensors, is a variable the design of a control system. It has been known for some time that controlled system performance depends on actuator location. However, selection of actuator/sensor locations has not previously been systematically addressed. A mechatronic approach where controller design is integrated with actuator location was used. This effort was complicated by the fact that approximations to the governing equation are required. Different objectives require different cost functions and lead to different algorithms as well as conditions for computation. Algorithms for calculating actuator locations that optimize various cost functions were developed. These algorithms were tested with computer simulations and also an experiment on a beam. The studies demonstrated dramatically improved performance at optimal over non-optimal actuator locations. Even for relatively simple systems, the optimal location is often not intuitively obvious. It was also investigated how the optimal actuator location is affected by the choice of cost function. Controllability is a very popular criterion. It was found to lead to locations that were considerably worse than those selected to optimize the controller design objective.					
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COMPUTATION OF OPTIMAL ACTUATOR/SENSOR LOCATIONS

AFOSR GRANT FA9550-10-1-0530

Final Report

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Overview

Many systems, such as acoustic noise and structural vibrations are distributed in space. (See Figure 1.) Other applications include control of diffusion (such as heating) and welding. The location of control actuators, and also the sensors, is a variable the design of a control system. The fact that controlled system performance depends on actuator location has been known for some time, but actuator/sensor location has not previously been systematically addressed. Analysis of the best locations for actuators and sensors is important for several reasons. Some actuators, such as piezo-electric actuators, commonly used in control of structures, are difficult to move once they have been attached to a structure. Furthermore, for problems with multiple actuators, there are a huge number of possible locations. Thus, a “trial and error” approach is either impossible, or else is unlikely to lead to locations that are close to optimal.

A mechatronic approach where controller design is integrated with actuator location was used. The overall aim of this project is the integration of controller design and actuator/sensor location. The models for these systems are partial differential equations (PDE's). Approximations to the governing PDE, often of very high order, are required and this complicates both controller design and optimization of the actuator locations. However, improvements in actuator/sensor location will lead to improved performance of a number of controlled systems.

Different objectives require different cost functions and lead to different algorithms as well as conditions for computation. If reducing the response to the initial condition is of concern, a linear quadratic (LQ) objective is appropriate. The presence of disturbances leads to different objectives. If the disturbance is known, then the objective is to minimize the $\mathbb{H}_2$ -norm of the output. A disturbance with known spatial distribution, but unknown frequency content leads to an H_∞ -optimal problem. The theory for optimal actuator location, for all these controller design objectives, along

with assumptions necessary for the use of approximations in calculations was completed. Algorithms for calculating linear-quadratic (including $\mathbb{H}_2$ -) and $\mathbb{H}_\infty$ optimal actuator locations were developed.

In the case of a linear-quadratic (and $\mathbb{H}_2$ -) cost, the problem was formulated as a convex optimization problem. Comparison of the algorithm against the popular genetic algorithm showed that the new algorithm is more accurate and considerably faster. The case of an $\mathbb{H}_\infty$ -cost was very challenging. Since derivative information is unavailable, a derivative-free method was used. It was combined with a game-theoretic algorithm for solving designing controllers for large systems developed in the previous year and tested with positive results.

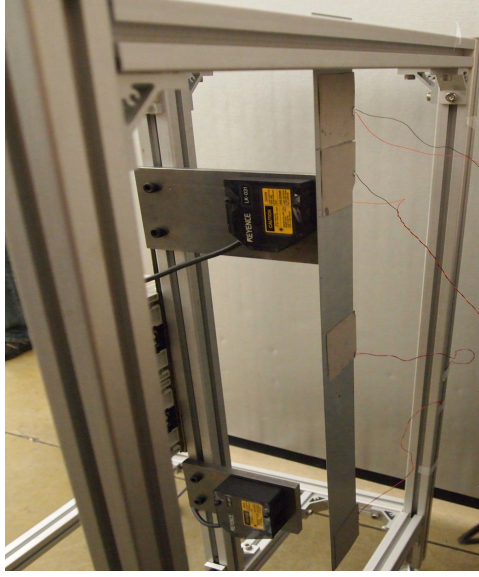
The LQ-optimization algorithm was implemented for a beam with piezo-electric patches, both numerically and in experiments. Since full state feedback is not possible in a lab, the optimal location of the laser sensors was obtained using duality and the LQ-optimization algorithm. The experiment demonstrated dramatically improved performance at optimal over non-optimal actuator locations.

A number of simulation studies were carried out to investigate how the optimal actuator location is affected by the choice of cost function. Controllability is a very popular criterion. It is numerically not reliable however. Furthermore, it was found to lead to locations that were considerably worse than those selected by LQ, $\mathbb{H}_2$ or H_∞ -costs. Even within these cost criteria, different criteria generally led to different predictions of the optimal location. The location and spatial distribution of the disturbance affects the optimal actuator location. Use of the optimal location for a given situation led to better performance, without a significant increase in overall control effort.

Controllability

Controllability is a very common actuator location criterion in the literature. Since a system is controllable if and only the controllability grammian $L_c(r)$ is positive definite, it seems reasonable to maximize controllability by maximizing the smallest eigenvalue.

However, there are serious issues with using controllability as a criterion for actuator location. Controllability $c(r) = \lambda_{\min}(L_c(r))$ versus actuator location is shown in Figure 3. The controllability of the system modelled using the first 10 modes is very different from that using the first 5 modes. Points of high controllability for 5 modes can be almost uncontrollable for the model with 10 modes. As the number of modes considered increases, there will be more points with zero controllability. The fundamental issue is that even though it is possible to select actuator locations and widths that yield a controllable system and so $L_c(r)$ is invertible and all the eigenvalues of $L_c(r)$ for the original partial differential equation model are positive, the lower bound on the eigenvalues is 0 [Curtain & Zwart, 1995]. Thus, although the controllability grammian L_c may be positive definite for each approximation



(a) Beam Vibrations



(b) Acoustic Noise in Duct

Figure 1: In control of systems that are distributed in space, there is generally a choice of where to place the actuators (and sensors). This selection can be regarded as part of the controller design. (Apparatus at University of Waterloo)

order, the smallest eigenvalue will tend to 0 as approximation order increases. This behaviour is typical of PDE models. To illustrate this point, a plot of controllability versus the number of modes for a fixed actuator location is shown in Figure 2.

In the next section, a criterion for actuator placement is explicitly considers the controlled system performance is analyzed. It is shown that this approach yields better controlled system performance than using controllability as a criterion.

Linear-Quadratic Control

If the aim of the controller is to reduce the system's response to initial conditions, while including a cost on the control, a very common and appropriate criterion for controller design is linear-quadratic (LQ) control. The design problem is to find a controller u to achieve

$$\min_{u \in L_2(0, \infty; \mathcal{U})} \underbrace{\int_0^\infty \langle Cz(t), Cz(t) \rangle + \langle u(t), u(t) \rangle dt}_{J_r(u, z_0)} \quad (1)$$

subject to

$$\dot{z}(t) = Az(t) + B(r)u(t), \quad z(0) = z_0. \quad (2)$$

where $C \in \mathcal{L}(\mathcal{Z}, Y)$, A with domain $D(A)$ generates a strongly continuous semigroup $S(t)$ on a Hilbert space $\mathcal{Z}$ (the state space), $B(r) \in \mathcal{L}(\mathcal{U}, \mathcal{Z})$ and U, Y are Hilbert

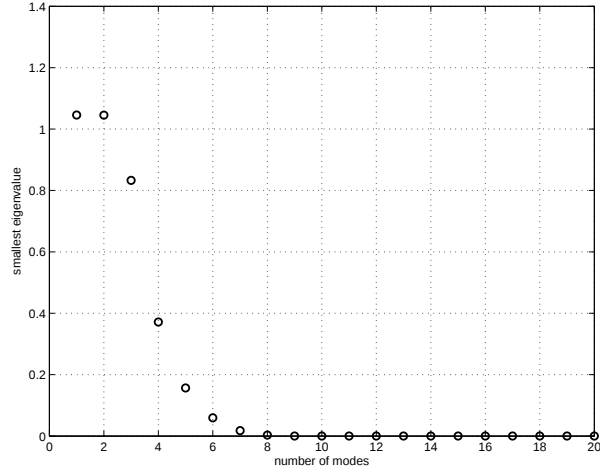


Figure 2: Controllability versus number of modes for a simply supported beam, with the actuator fixed at $x = 0.11$. Regardless of the choice of actuator location, the controllability measure $c(r)$ converges to 0 as the number of modes increases.

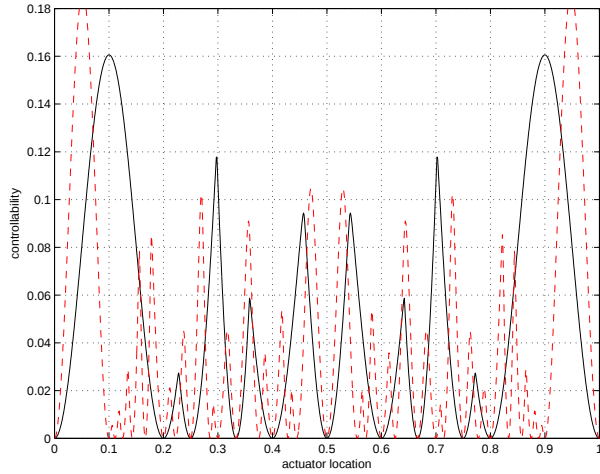


Figure 3: Controllability of the first 5 and 10 modes versus actuator location for a simply supported beam. Points of high controllability for 5 modes can be almost uncontrollable for the model with 10 modes.

spaces (usually finite-dimensional). The parameter r indicates the dependence of the control operator $B(r)$ on the actuator locations r . Suppose that there are M actuators, each of which may lie in some set Ω so that $r \in \Omega^M$. For each r , the optimal cost is $\langle \Pi(r)z_0, z_0 \rangle$ where $\Pi(r)$ solves an operator algebraic Riccati equation (ARE) [Curtain&Zwart, 1995]. We may choose r to minimize response to the worst initial condition:

$$\begin{aligned} \max_{\substack{z_0 \in \mathcal{H} \\ \|z_0\|=1}} \min_{u \in L_2(0, \infty; U)} J_r(u, z_0) &= \max_{\substack{z_0 \in \mathcal{H} \\ \|z_0\|=1}} \langle \Pi(r)z_0, z_0 \rangle \\ &= \|\Pi(r)\|. \end{aligned}$$

The cost or performance for each actuator location is $\mu(r) = \|\Pi(r)\|$ and the optimization problem is therefore

$$\hat{\mu} = \inf_{r \in \Omega^m} \|\Pi(r)\|.$$

If the initial condition is regarded as random, then the cost should be chosen differently. For example, if it is random with zero mean and variance V , the appropriate cost to minimize is $\text{Tr} V^{\frac{1}{2}} \Pi(r) V^{\frac{1}{2}}$ [Morris, 2011]. Studies conducted as part of the research effort showed that these different criteria in general lead to different optimal actuator locations [12].

The solution to the operator ARE on the state space cannot generally be calculated. It is approximated by the solution to a finite-dimensional ARE. Even for fixed actuator location, a set of assumptions in addition to those required for simulation are needed to ensure convergence of controller design. See, for example [Morris, 2009]. Rather than reproduce the list of assumptions here, we refer to them henceforth as the *standard assumptions* for approximations. However, examples show that even if Π_n (at a fixed actuator location) converges strongly to Π , the optimal cost and actuator location of the approximations may not converge to the optimal cost and location. In general, in addition to the conditions required for strong convergence of Π_n , the input operator B and cost C should be compact [Morris, 2011]. This assumption may be weakened if the underlying dynamics involve an analytic semigroup [Morris, 2011]. Conditions guaranteeing (1) well-posedness of the optimal actuator location problem and also (2) the validity of calculations using approximations to the PDE (2) were obtained [Morris, 2011].

These results, like most results on the use of approximations in controller design for systems with partial differential equation models, assume that the control and observation operators are bounded. In most models, these operators are indeed bounded, if actuator/sensor dynamics are included. The major exception is control of structures with piezo-electric patches. Modelling of these systems with magnetic effects has led to the conclusion that these structures do have bounded control operators, if magnetic effects are included and current control, rather than the usual voltage control, is used [9, 10, 11]. Experiments have previously indicated that current control leads to considerably reduced hysteresis over voltage control; see for

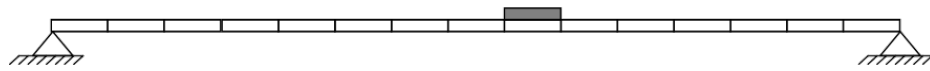
example, [Main&Garcia,1997]. This research further supports the use of current-controlled piezo-electrics. Also, although magnetic effects have a small effect on the dynamics, they affect the control properties.

The remaining problems in linear-quadratic control were computational: obtaining an algorithm for the calculation of optimal actuator locations and corresponding controller that can be used for systems that may require large-order approximations. The LQ- cost function is differentiable with respect to the actuator locations and so gradient-based methods can be considered. However, even though the optimal LQ control problem is strictly convex, the optimal actuator location problem in general has multiple local minima; see for example [Geromel,1989],[Morris,2011]. However, if the possible actuator locations are discretized by M possible locations, and whether there is an actuator is at a particular location is identified by a 0 or a 1, then the resulting problem is convex in the larger space $\mathbb{R}^M$ [Geromel,1989]. A secondary advantage to this formulation is that it explicitly prevents clustering of actuators at a single point. However, there are now a much larger number of variables than in the original problem.

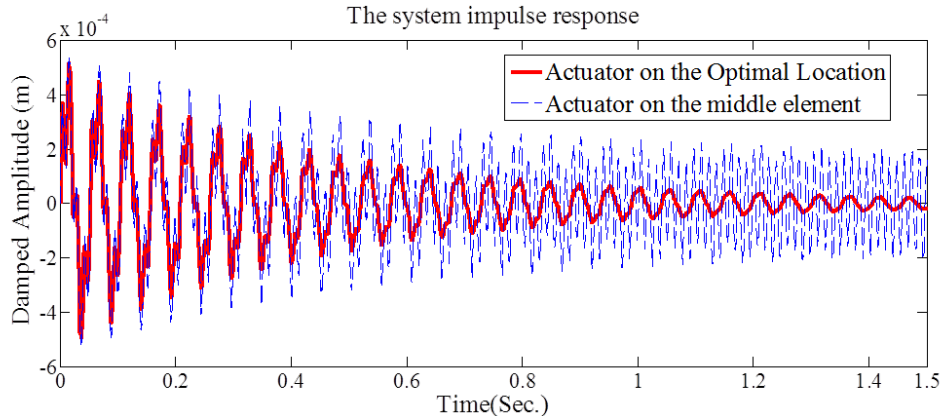
An algorithm to find LQ-optimal actuator location, using the convex reformulation of the problem, was developed [2]. This algorithm was successfully implemented on a number of examples [1, 2]. In comparisons of this algorithm with the popular genetic algorithm, the new algorithm found locations with better performance than the genetic algorithm, and was about 100 times faster. This improvement in speed is critical for problems requiring high-order approximations.

Results for a uniform simply supported beam with a single actuator are shown in Figure 4. We would expect the optimal location for the single actuator to be at the centre. However, the optimal location is slightly off-centre. Figure 4b compares the performance of the controlled system with the actuator at the optimal location to that with the actuator at the centre. Optimal location of the actuator yields considerably superior performance. This simple example illustrates not only the sensitivity of performance to actuator location; but also the fact that, even for a simple example, intuition does not always lead to the best actuator placement. (Modal analysis reveals that all the odd-numbered modes have a node at the centre. Thus, although the centre is best if only the first mode is considered, an off-centre location picks up the other modes while still using most of the energy in the first mode.)

In addition to the simulations, an experiment using the cantilevered beam shown in Figure 1a was conducted [2]. Two non-contact laser sensors were used to measure deformations and were used as inputs to a Luenberger observer. The optimal location of the sensors were chosen by solving a dual problem to the control problem. The optimal sensor locations are indicated in Figure 5. To study the optimal actuator location problem 4 patches were attached to the beam surface as shown in Figure 5. In each experiment only two actuators are activated to suppress the beam's vibration. The optimal location of two actuators on the beam are positions 1 and 2 as shown in Fig. 5. As illustrated by Fig. 6, placing actuators at the locations



(a) Optimal location



(b) Comparison of response with actuator at optimal location to that obtained with actuator at the centre.

Figure 4: LQ-optimal actuator location of a single actuator on a simply supported beam. The optimal actuator location is slightly off-centre, not at the centre as would be expected. Figure (b) shows that the optimal location yields considerably better system performance than if the actuator is placed at the centre.

selected by the algorithm leads to much shorter settling times than if other locations are used.

Comparison of different cost functions

Plots of the normalized LQ-cost, for a random initial condition, versus actuator location for various weighting matrices Q and R are shown in Figure 7. Optimal actuator location depends on the weighting matrices Q and R . As R decreases, that is the state cost is increasingly weighted more than the control, the optimal actuator location moves towards $x = 0.4$ (or $x = 0.6$). As R increases, that is the control cost is increasingly weighted more than the state, the optimal actuator location converges to the centre 0.5. For heavily weighted control cost, the best actuator location (the centre) is a point of minimum controllability. A similar pattern is seen if the objective is to minimize the response to the worst initial condition (Figure 8). If the state is heavily weighted, the centre is a poor spot for the actuator. This is likely reflecting the fact that although the most significant mode, the first mode, has a peak at the centre, and the centre is a good spot for controlling this mode, the odd-numbered modes have nodes at the centre and can't be controlled with an

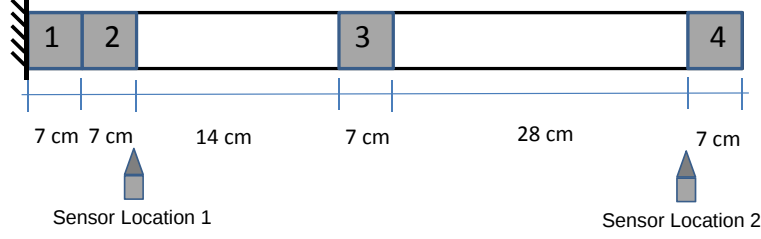


Figure 5: Cantilevered beam showing optimal actuator locations (1 & 2) as well as other actuator locations. The optimal sensor locations for 2 sensors are also indicated.

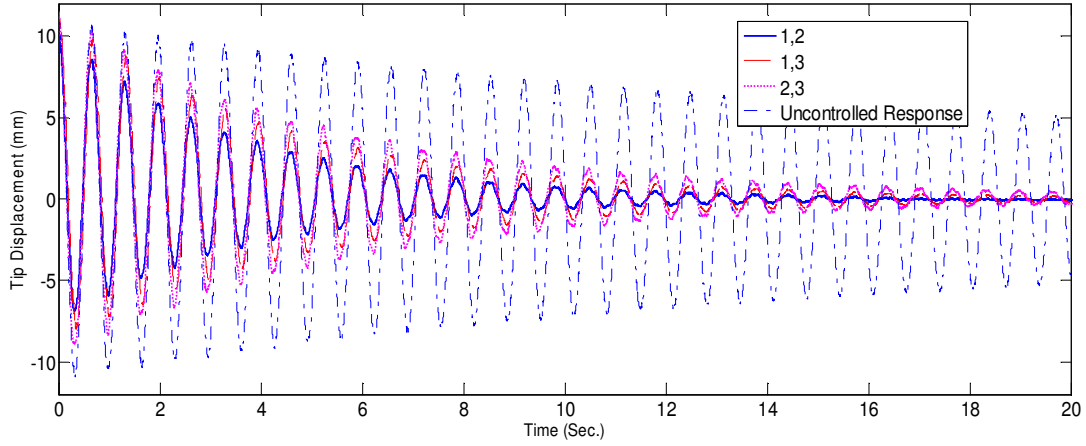


Figure 6: Experimental measurements of tip vibrations with controlled system with different actuator locations. Considerably smaller settling time is obtained with the actuators placed at the locations selected by the algorithm.

actuator placed there.

To determine whether the improvement in performance was accomplished by increased control cost, the control signal with the optimal LQ state feedback controller at the LQ-optimal location, and also at a non-optimal location was examined with different initial conditions. For weighting matrices $Q = I$ and $R = 0.01$, and a minimum variance LQ-cost (with $V = I$), a plot of the L_2 norm of the control signal versus actuator location for each initial condition is shown in Figure 9(a). Similarly, a plot of the L_∞ norms is shown in Figure 9(b). The initial conditions used are:

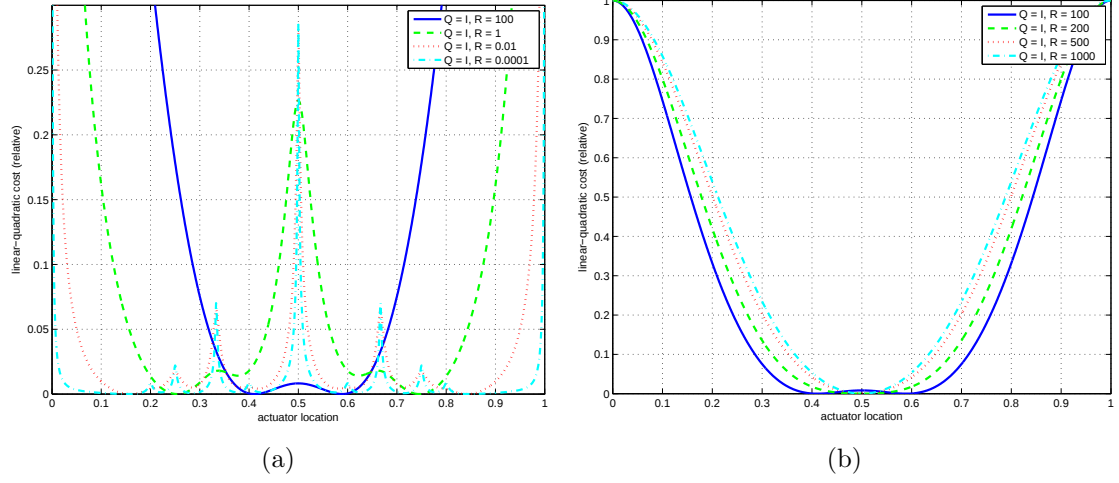


Figure 7: Normalized linear-quadratic cost $E[J_{min}(r)]$ for random initial condition with variance $V = I$, versus actuator location for different weighting matrices Q and R .

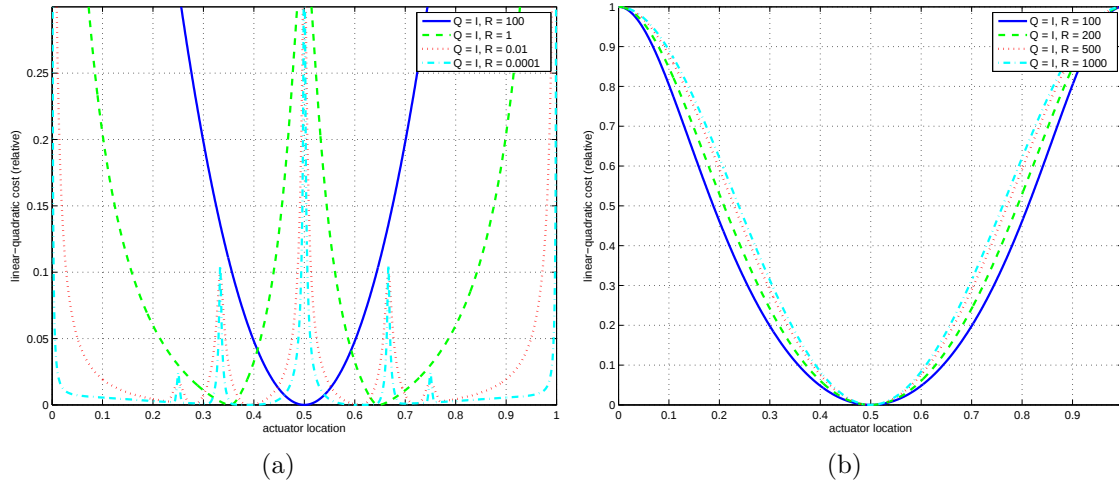


Figure 8: Normalized linear-quadratic cost $\|\Pi(r)\|$ versus actuator location for different weighting matrices Q and R .

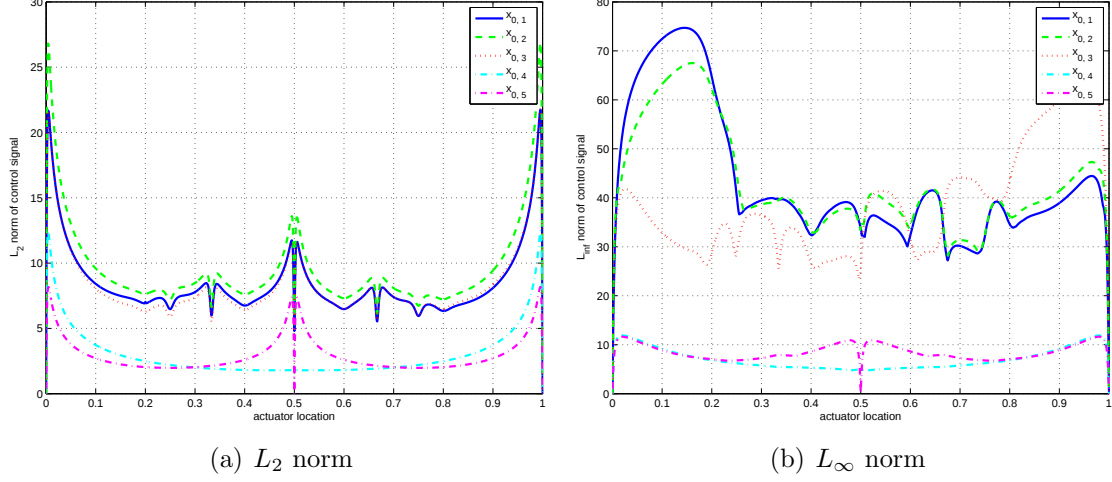


Figure 9: Norms of the control signal versus actuator location for different initial conditions. Cost is linear-quadratic with $Q = I$, $R = 0.01$.

$$x_{0,1} = [1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1]^T \quad (3)$$

$$x_{0,2} = [1.5 \ 1.4 \ 1.3 \ 1.2 \ 1.1 \ 1 \ 0.9 \ 0.8 \ 0.7 \ 0.6]^T \quad (4)$$

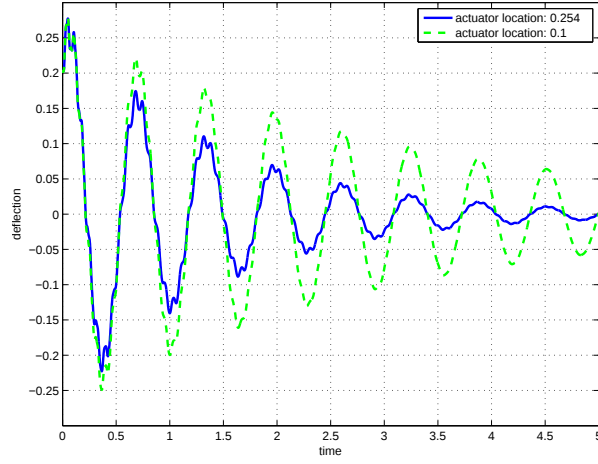
$$x_{0,3} = [-1 \ 1 \ -1 \ 1 \ -1 \ 1 \ -1 \ 1 \ -1 \ 1]^T \quad (5)$$

$$x_{0,4} = [1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]^T \quad (6)$$

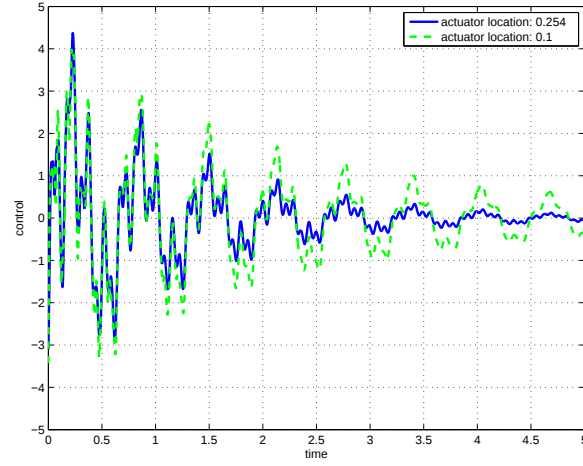
$$x_{0,5} = [0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]^T \quad (7)$$

From Figure 9(a), it is observed that poor actuator locations (which correspond to locations with zero controllability) result in control signals with L_2 norm at a local minimum for initial conditions $x_{0,1}$, $x_{0,2}$ and $x_{0,3}$. This is not always the case for $x_{0,4}$ and $x_{0,5}$. An explanation for this is that for $x_{0,4}$ and $x_{0,5}$, not all modes have been excited. A similar pattern can be observed in Figure 9(b) for the L_∞ norm, though not as strong.

Figures 10 and 11 display the deflection at the beam centre and the control signal for two different initial conditions. The LQ-optimal location yields better response than the location with maximum controllability, with no apparent increase in the control effort. These patterns can be observed using different weighting matrices Q and R and different examples. There is no apparent relationship between locations with high controllability and LQ-optimal actuator locations. Also, at the LQ-optimal actuator locations, the L_2 -norm of the control signal tends not to be large or small compared to other locations. However, it should be noted that the L_∞ norm is sometimes large. Therefore, as always, care should be taken to avoid possible saturation of control signals when placing actuators.

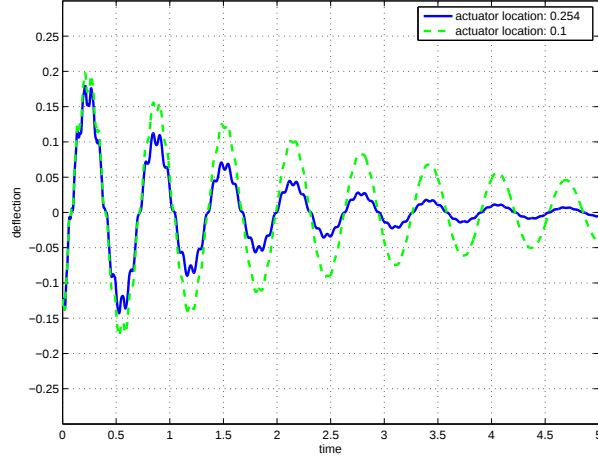


(a) deflection at centre of beam

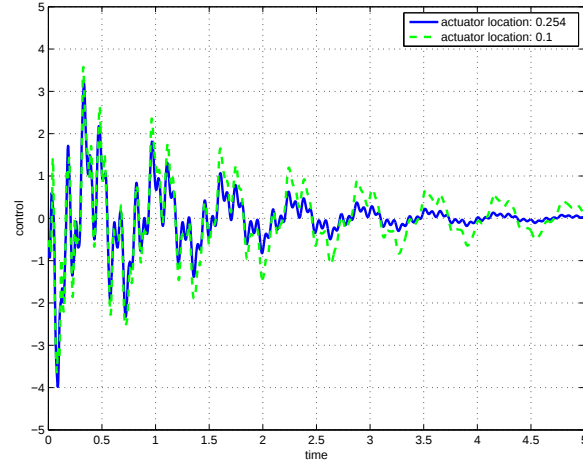


(b) control signal

Figure 10: Initial condition $x_{0,2}$. Deflection at centre of beam and control signal for actuator at 0.254 (LQ optimal actuator location) and 0.1 (optimal controllability). $Q = I$ and $R = 1$. Both the deflection and control signal are smaller for an actuator placed at the LQ-optimal actuator location than at the spot of optimal controllability.



(a) deflection at centre of beam



(b) control signal

Figure 11: Initial condition $x_{0,3}$. Deflection at centre of beam and control signal for actuator at 0.254 (LQ optimal actuator location) and 0.1 (optimal controllability). $Q = I$ and $R = 1$. Both the deflection and control signal are smaller for actuator at the LQ-optimal actuator location than at the spot of optimal controllability.

Disturbances

In many problems, the aim is to reduce the response to an exogenous disturbance. The model (2) becomes

$$\dot{z}(t) = Az(t) + B(r)u(t) + Dv(t). \quad (8)$$

where $D \in \mathcal{L}(W, \mathcal{Z})$ and W is a separable Hilbert space. Suppose that the disturbance $v(t) \in L_2(0, \infty; W)$ is white noise. (The more general situation where the disturbance is fixed, but not white noise, is handled by absorbing the description of the disturbance into the plant model and the subsequent treatment is identical.) For $C \in \mathcal{L}(\mathcal{Z}, Y)$, $R \in \mathcal{L}(U, U)$, where R is coercive, define the cost

$$y(t) = \begin{bmatrix} Cz(t) \\ Eu(t) \end{bmatrix}. \quad (9)$$

Since it is assumed that all states z are available to the controller, this is known as the *full information problem*. The standard assumption that E^*E is invertible will be made so that the control cost is non-singular. To simplify the formulae, it is also assumed that $E^*C = 0$ and $E^*E = I$. This cost is then identical to the linear quadratic cost (1) with $Q = C^*C$ and $R = E^*E = I$. The standard problem is to find the control law u so that $\|y\|_2$ is minimized. The difference between this problem and the LQ problem is that in LQ-control the aim is to reduce the response to the initial condition $z(0)$ with disturbance $v = 0$ while we now wish to reduce the response to the disturbance v and set $z(0) = 0$. Since the L_2 -norm of y equals the $\mathbb{H}_2$ -norm of the Laplace transform of y , this is known as an $\mathbb{H}_2$ -controller design problem. The relevance of the $\mathbb{H}_2$ -norm is also that, letting G indicate the transfer function, $\|G\|_2$ is the spectral power density, or expected power, of the output of a system subject to unit variance white noise (a signal with uniform spectral density). The following theorem is an extension of the analogous result for finite-dimensional systems.

Theorem 1 [4] *Consider the linear system (8) with cost (9) and assume that (A, B) is stabilizable. The $\mathbb{H}_2$ -optimal control is the state feedback*

$$u(t) = -B^*(r)\Pi(r)z(t)$$

where $\Pi(r)$ solves

$$A^*\Pi(r) + \Pi(r)A - \Pi(r)B(r)R^{-1}B^*(r)\Pi(r) + C^*C = 0. \quad (10)$$

The optimal cost is

$$\text{Tr}(D^*\Pi(r)D)$$

and the optimal norm of the closed loop transfer function is

$$\sqrt{\text{Tr}(D^*\Pi(r)D)}.$$

The $\mathbb{H}_2$ -optimal actuator location problem is thus to find the actuator location $\hat{r}$ that minimizes

$$Tr(D^*\Pi(r)D).$$

Since the optimal cost relies on the norm of the solution to a LQ-ARE (10), well-posedness of this problem follows using techniques and results from the LQ-case [4]. Mathematically, the problem is identical to that of minimizing the LQ-cost when the initial condition is random with variance D^*D . If $B(r)$ is a continuous function of r , both the optimal actuator location problem with a fixed disturbance location, and the problem where the disturbance is unknown, lead to well-posed optimization problems.

Define the optimal cost $\hat{\mu} = \inf_{r \in \Omega^M} \langle D, \Pi(r)D \rangle = \langle D, \Pi(\hat{r})D \rangle$ where $\hat{r}$ is an optimal actuator location. Provided that an approximation scheme is found so that $\Pi_n \rightarrow \Pi$ then $\mu_n \rightarrow \mu$. Convergence of the optimal actuator location is also implied.

Theorem 2 [4] *Consider the control system (8) where D is a compact operator and assume that for any $r \in \Omega^M$,*

$$\lim_{s \rightarrow r} \|B(s) - B(r)\| = 0,$$

and that the standard assumptions for approximations are satisfied by $(A_n, B_n(r), C_n)$. Then the approximating optimal costs converge to the exact optimal cost, that is

$$\hat{\mu} = \inf_{r \in \Omega^M} Tr(D^*\Pi(r)D) = \lim_{n \rightarrow \infty} \inf_{r \in \Omega^M} Tr(D_n^*\Pi_n(r)D_n),$$

Also there is a subsequence of approximating actuator locations $\hat{r}_m$ so

$$\hat{\mu} == \lim_{m \rightarrow \infty} Tr(D^*\Pi(\hat{r}_m)D); \tag{11}$$

that is, performance arbitrarily close to optimal can be achieved with the approximating actuator locations.

Consider a single disturbance so that $Dv = b_1v$ for some $b_1 \in \mathcal{Z}$. If the spatial distribution of the disturbance, b_1 , is not known then the objective is to find the actuator location that minimizes the $\mathbb{H}_2$ -cost over possible disturbance distributions: The problem now becomes that of choosing the actuator location r to minimize the closed loop response to the worst spatial disturbance distribution; that is

$$\inf_{r \in \Omega^M} \sup_{b_1 \in \mathcal{H}} \langle b_1, \Pi(r)b_1 \rangle = \inf_{r \in \Omega^M} \|\Pi(r)\|.$$

Thus, if the spatial distribution of the disturbance b_1 is unknown, the problem is to minimize $\|\Pi(r)\|$ over the actuator location r . This is identical to minimizing the LQ-cost with respect to the worst initial condition [4]. Letting $\hat{r}$ indicate the optimal actuator location, the worst disturbance is the eigenfunction corresponding to the largest eigenvalue of $\Pi(\hat{r})$.

A number of numerical tests were done on structures [3, 4]. These indicate that the best actuator location depends on the type of disturbance and its location.

If the disturbance v is not known, then the problem becomes one of minimizing the response to the worst disturbance. This can be shown to be equivalent to minimizing the $\mathbb{H}_\infty$ -norm of the closed-loop transfer function, and so this controller design approach is known as $\mathbb{H}_\infty$ -controller design. The basic $\mathbb{H}_\infty$ -controller design problem is to find a stabilizing controller so that (8) is stabilized with attenuation γ :

$$\int_0^\infty \|y(t)\|^2 dt < \gamma^2 \int_0^\infty \|v(t)\|^2 dt.$$

The norm of the map from v to y is equal to the $\mathbb{H}_\infty$ -norm of the transfer function which is why this type of control objective is generally known as $\mathbb{H}_\infty$ -controller design. This problem is solvable if and only if there exists a nonnegative, self-adjoint operator Π on $\mathcal{Z}$ solving the algebraic Riccati equation (ARE)

$$(A^*\Pi + \Pi A - \Pi \left(B(r)B(r)^* - \frac{1}{\gamma^2} DD^* \right) \Pi + C^*C)z = 0, \quad z \in D(A), \quad (12)$$

where $A - B(r)B(r)^*\Pi + \frac{1}{\gamma^2} DD^*\Pi$ generates an exponentially stable semigroup on $\mathcal{Z}$ [Bernhard & Bensoussan 1993],[Keulen,1993].

Often, the optimal disturbance attenuation is sought. The cost $\mu(r)$ is the smallest attenuation γ for which for which a stabilizing solution to (12) exists. In this case, unlike LQ- and $\mathbb{H}_2$ optimal control, an iterative procedure to find the optimal γ and hence $\mu(r)$ is required. The problem here is to calculate not only the optimal controller, but also the actuator location(s) that minimize the $\mathbb{H}_\infty$ -norm of the controlled system. Optimal disturbance attenuation as a function of actuator location is the cost function. The following theorem states conditions under which this problem is well-posed; and furthermore for when approximations can be used to find the optimal locations and corresponding controller.

Theorem 3 [6] *If*

- *for any r_0 , $\lim_{r \rightarrow r_0} \|B(r) - B(r_0)\| = 0$,*
- *$(A, B(r))$ are all stabilizable, (A, C) is detectable*
- *B and D are compact operators,*
- *$(A_n, [B_n(r) \ D_n], C_n)$ satisfies the standard assumptions on approximations for controller design*

then, letting $\mu(r) = \inf \gamma(r)$, $\hat{\mu} = \inf_{r \in \Omega^m} \mu(r)$,

- *there exists an optimal actuator location $\hat{r}$ so that $\hat{\mu} = \mu(\hat{r})$,*

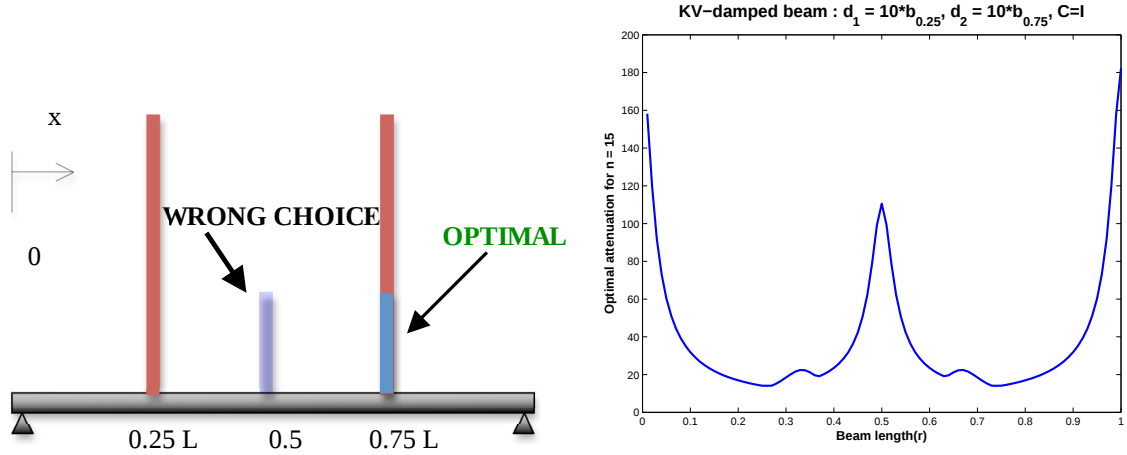
- $\hat{\mu} = \lim_{n \rightarrow \infty} \hat{\mu}_n$,
- and there exists a subsequence $\{\hat{r}_m\}$ of $\{\hat{r}_n\}$ such that $\hat{\mu} = \lim_{m \rightarrow \infty} \mu(\hat{r}_m)$.

Note that unlike linear-quadratic design, the penalty C on the state does not need to be compact.

The next task was the construction of an algorithm to calculate the optimal actuator locations, and also the controller, to achieve the optimal attenuation. This was complicated by several factors. First, there is no generally accepted algorithm for solution of large $\mathbb{H}_\infty$ -AREs, particularly when the system is in descriptor form, as is the case for finite-element approximations. An efficient, stable algorithm that can be used for systems of very large order and also to calculate optimal attenuation was developed [7]. Furthermore, calculation of the cost function $\mu(r)$ requires the solution of many Riccati equations, and each such calculation is computationally intensive. Also, the cost $\mu(r)$ is a non-convex function of the locations r and it appears to be non-differentiable. In order to avoid the problem of lack of differentiability, directional direct search [Conn, 2009, chap. 7] was used. Directional direct-search is a derivative-free method that samples the objective function at finite number of points and searches for a better function value at each iteration. The decisions are made based on function values without any explicit or implicit derivative calculation. There are several advantages to this method for the calculation of $\mathbb{H}_\infty$ -optimal actuator location. One is that the function evaluations may be done in parallel, which is natural for optimal actuator location, and speeds up convergence in a multi-processor architecture.

Another advantage to a directional direct search method for calculation of optimal attenuation is that a simpler, so-called *surrogate function* may be used to replace most cost function evaluations. Calculation of the cost function, the optimal attenuation $\mu(r)$ with actuators at r , is very computationally intensive. However, the actual attenuation achieved by a given controller (for some attenuation γ) is generally quite close to the optimal attenuation. Also, the optimal attenuation is a continuous function of the actuator location [6]. Thus, the actual attenuation achieved with the controller from the previous iteration is used as a surrogate function $sm(\cdot)$. The surrogate function is used to order the candidate points. The cost at candidate points is then evaluated in the given order using the actual cost function μ . Since the actual optimal attenuation is close to the surrogate function value, usually the first one or two points evaluated reduces the current cost, allowing the optimization procedure to progress. The use of a surrogate function resulted in savings of computation time of 200-2000% [6].

Testing the algorithm with several common examples showed that even for simple examples, the best location does not always agree with intuition [5, 6]. Consider a simply supported beam with disturbances at 0.25 and 0.75 of the total length and a single actuator. The centre of the beam would seem to be the best location for the actuator. In fact, placing the actuator at either of the disturbance locations improves the attenuation nearly 8 times over placing it at the centre. (See Figure



(a) Location of disturbances and optimal actuator (b) Optimal attenuation as a function of actuator location

Figure 12: $\mathbb{H}_\infty$ -optimal placement of a single actuator for a simply supported beam with two disturbances. The optimal location yields nearly 8 times better attenuation of disturbances over placing the actuator at the centre. Placing the actuator at the same location as one of the disturbances yields a closed loop $\mathbb{H}_\infty$ -norm of 15 versus 118 if the actuator is placed at the centre.

12) For another problem, placement of 2 actuators when the disturbances are at $x = 0.4$ and $x = 0.9$, the optimal actuator locations were 0.23 and 0.58. Placing the actuators at these locations yielded 15% better attenuation than collocation of the actuators with the disturbances. Optimal actuator placement for control of diffusion on an irregular domain with spatially varying diffusivity, shown in Figure 13, was also investigated. The optimal actuator location is not towards the centre, but in a spot where the diffusivity is low. This suggests that diffusivity is a factor in actuator location for systems with variable diffusivity.

Conclusions

A number of theoretical and numerical results have been obtained. For optimal actuator location, compactness is important for well-posedness of the problem and for convergence of approximations. The original problem needs to be properly formulated, and a suitable approximation scheme chosen in order to obtain useful results from numerical calculations. Furthermore, the optimal actuator location depends on the choice of cost function so the cost function needs to be chosen appropriately. The popular choice of maximizing controllability is not numerically reliable, and also does not generally lead to controlled systems with the best performance.

Algorithms for optimal actuator placement using LQ, $\mathbb{H}_2$ and $\mathbb{H}_\infty$ -cost criteria were developed and successfully tested on a number of examples. The algorithm for $\mathbb{H}_\infty$ -optimal actuator is the first algorithm developed for problems of this type. The

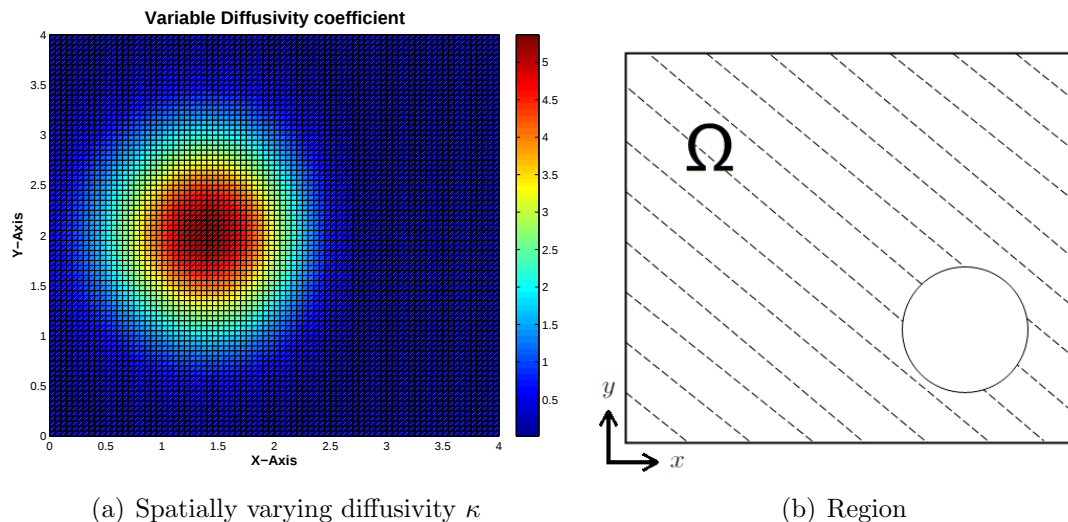


Figure 13: $\mathbb{H}_\infty$ -optimal actuator location for a uniformly distributed disturbance in a diffusion problem. The diffusivity is spatially dependent and the region is irregular. The optimal actuator location is not at the centroid of the region, but at (3.1, 3.35). This is a region of low diffusivity indicating that diffusivity is a factor in actuator efficacy for diffusion problems.

linear-quadratic algorithm was compared to the popular genetic algorithm and was 100 times faster on the examples tested. Experimental results on a beam corroborate the simulations.

The use of these algorithms, both in simulations and on an experiment have shown two important conclusions: (1) the best actuator locations do not always agree with physical intuition, even for simple examples, and (2) controlled system performance is strongly dependent on actuator location. Planned future work is to apply the algorithms to realistic applications in several space dimensions, such as space structures, aircraft fuselage, building dynamics and fluid problems. There is a natural extension of this research to sensor location in order to handle the usual situation where the state must be estimated. Estimation is also of interest in a number of situations where control may not be required. The results for actuator location indicate that significantly better estimation can be obtained with proper placement of sensors, but these questions need to be investigated.

Personnel Supported During Duration of Grant

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Name	Degree	Current Position
Neda Darivandi	PhD	Hatch Engineering Ltd.
Dhanaraja Kasinathan	PhD	research assistant, GM & UW
Arman Tavakoli	M. Math	in progress
Steven Yang	undergraduate research assistant	Master's student
Blake Martin	undergraduate research assistant	PhD student
Ozkan Ozer	post-doctoral fellow (8 months)	assistant professor, Univ. of Nevada

Student Theses

Dhanaraja Kasinathan, $\mathbb{H}_\infty$ -*optimal actuator location*
PhD (Applied Mathematics), 2012.

Abstract: There is often freedom in choosing the location of actuators on systems governed by partial differential equations. The actuator locations should be selected in order to optimize the performance criterion of interest. The main focus of this thesis is to consider $\mathbb{H}_\infty$ performance with state-feedback. That is, both the controller and the actuator locations are chosen to minimize the effect of disturbances on the output of a full-information plant.

Optimal $\mathbb{H}_\infty$ -disturbance attenuation as a function of actuator location is used as the cost function. It is shown that the corresponding actuator location problem is well-posed. In practice, approximations are used to determine the optimal actuator location. Conditions for the convergence of optimal performance and the corresponding actuator location to the exact performance and location are provided. Examples are provided to illustrate that convergence may fail when these conditions are not satisfied.

Systems of large model order arise in a number of situations; including approximation of partial differential equation models and power systems. The system descriptions are sparse when given in descriptor form but not when converted to standard first-order form. Numerical calculation of $\mathbb{H}_\infty$ -attenuation involves iteratively solving large $\mathbb{H}_\infty$ -algebraic Riccati equations ($\mathbb{H}_\infty$ -AREs) given in the descriptor form. An iterative algorithm that preserves the sparsity of the system description to calculate the solutions of large $\mathbb{H}_\infty$ -AREs is proposed. It is shown that the performance of our proposed algorithm is similar to a Schur method in many cases. However, on several examples, our algorithm is both faster and more accurate than other methods.

The calculation of $\mathbb{H}_\infty$ optimal actuator locations is an additional layer of optimization over the calculation of optimal attenuation. An optimization algorithm to calculate $\mathbb{H}_\infty$ optimal actuator locations using a derivative-free method is proposed. The results are illustrated using several examples motivated by partial differential equation models that arise in control of vibration and diffusion.

Thesis can be downloaded from:

https://uwspace.uwaterloo.ca/bitstream/handle/10012/6732/Kasinathan_Dhanaraja.pdf?sequence=1

Neda Darivandi, *Optimal Active Control of Flexible Structures Applying Piezoelectric Actuators*.

PhD (Mechanical and Mechatronics Engineering), 2013

Abstract: Piezoelectric actuators have proven to be useful in suppressing disturbances and shape control of flexible structures. Large space structures such as solar arrays are susceptible to large amplitude vibrations while in orbit. Moreover, Shape control of many high precision structures such as large membrane mirrors and space antenna is of great importance. Since most of these structures need to be ultra-light-weight, only a limited number of actuators can be used. Consequently, in order to obtain the most efficient control system, the locations of the piezoelectric elements as well as the feedback gain should be optimized. These optimization problems are generally non-convex. In addition, the models for these systems typically have a large number of degrees of freedom.

Researchers have used numerous optimization criteria and optimization techniques to find the optimal actuator locations in structural shape and vibration control. Due to the non-convex nature of the problem, evolutionary optimization techniques are extensively used. However, One drawback of these methods is that they do not use the gradient information and so convergence can be very slow. Classical gradient-based techniques, on the other hand, have the advantage of accurate computation; however, they may be computationally expensive, particularly since multiple initial conditions are typically needed to ensure that a global optimum is found. Consequently, a fast, yet global optimization method applicable to systems with a large number of degrees of freedom is needed.

In this study, the feedback control is chosen to be an optimal linear quadratic regulator. The optimal actuator location problem is reformulated as a convex optimization problem. A subgradient-based optimization scheme which leads to the global solution of the problem is introduced to optimize the actuator locations. The optimization algorithm is applied to optimize the placement of piezoelectric actuators in vibration control of flexible structures. This method is compared with a genetic algorithm, and is observed to be faster in finding the global optimum.

Moreover, by expanding the desired shape into the structures modes of vibration, a methodology for shape control of structures is presented. Applying this method, locations of piezoelectric actuators on flexible structures are optimized.

Very few experimental studies exist on shape and vibration control of structures. To the best knowledge of the author, optimal actuator placement in shape control has not been experimentally studied in the past. In this work, vibration control of a cantilever beam is investigated for various actuator locations and the effect of optimal actuator placement is studied on suppressing disturbances to the beam. Also using the proposed shape control method, the effect of optimal actuator placement

is studied on the same beam. The final shape of the beam and input voltages of actuators are compared for various actuator placements.

Thesis can be downloaded from:

https://uwspace.uwaterloo.ca/bitstream/handle/10012/7459/Darivandi_Shoushtari_Neda.pdf?sequence=1

Interactions/Transitions

The following talks on our results to date were presented.

- “Optimal Actuator Location”, University of Groningen, October, 2013.
- “Optimal Actuator Location”, IFAC Conference on Distributed Parameter Systems, Paris, September, 2013.
- “Comparison of Different Criteria for Actuator Location”, AFOSR Program reviews, Arlington, August, 2013.
- “Control of Piezo-electric Beams with Magnetic Effects,” SIAM Conference on Control Theory and Applications, San Diego, CA, July, 2013. (talk given by A. O. Ozer.)
- “Calculation of $\mathbb{H}_\infty$ -optimal actuator location for distributed parameter systems”, American Control Conference, Washington, DC, June 2013.
- “Optimal Actuator Location”, University of Alabama, Birmingham, USA, April, 2013.
- “Computational Issues in Optimal Actuator Location”, SIAM Conference on Computational Science and Engineering, Boston, USA, February, 2013.
- “ $\mathbb{H}_\infty$ -Optimal Actuator Location”, AFOSR Program Reviews, Arlington, August 2012.
- “Optimal Actuator Location in Structures” , 8th International Conference on Differential and Dynamical Systems, Waterloo, Canada, July 2012.
- “Linear-quadratic Optimal Actuator Location in Structures”, American Control Conference, Montreal, Canada, June 2012.
- “ $\mathbb{H}_\infty$ -Optimal Actuator Location”, AFOSR Program reviews, Arlington, August, 2013.
- “Numerical method for $\mathbb{H}_\infty$ control of large regular descriptor system”, Fifth Ontario Meeting on Systems and Control Theory, Toronto, Canada, May 2012. (talk given by D. Kasinathan)
- “Optimal Actuator Location”, Fields’ Institute Industrial Optimization Seminar, February, 2012.

- “Convergence of $\mathbb{H}_\infty$ -Optimal Actuator Locations”, IEEE Conference on Decision and Control, Orlando, December 2011. (talk given by D. Kasinathan)
- “Computation of Optimal Actuator Locations”, Workshop on Nonlinear Control, Monterey, November 2011.
- “LQ-optimal Actuator Location”, AFOSR Program reviews, Arlington, August, 2011.
- ‘ $\mathbb{H}_\infty$ -Optimal Actuator Locations”, Workshop on Distributed Parameter Systems, Wuppertal, July 2011.

Also, there have been discussions with several people about possible applications and extensions of this research. In particular, Ralph Smith at North Carolina State University is interested in whether proper actuator/sensor location, combined with uncertainty quantification, can reduce the effect of uncertain parameter values. There are also applications to improving the energy efficiency of buildings via better location of HVAC hardware and to locating sensors for estimation of the state of large lakes and the atmosphere.

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Publications arising from grant

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