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# REVIEW AND ANALYSIS OF CUMULATIVE-FATIGUE-DAMAGE THEORIES

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# REVIEW AND ANALYSIS OF CUMULATIVE-FATIGUE-DAMAGE THEORIES

Lloyd Kaechele

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Title

## PREFACE

As the stress changes in an aircraft (or any other) structure, changes also occur in the material of the structure. Some stress changes or cycles are large enough to cause some amount of fatigue damage, even though this damage is not ordinarily discernible.

There are three main problem areas in designing aircraft structures so that the accumulation of fatigue damage does not threaten safety--evaluation of cumulative damage; predicting and describing the fatigue-producing stress variations that will be experienced; and accounting for the wide scatter in fatigue life that is observed for seemingly identical test specimens and structures.

The first, cumulative damage, arises because stress cycles in aircraft structures vary so greatly in size, number, and order that it is a practical impossibility to make enough tests to cover all variations. Therefore, fatigue damage caused by the many different stress cycles must be evaluated by adding up the effects of individual cycles. If we understood well enough the physical processes going on in a material during each stress cycle, this addition, or determination of cumulative effects, would be no problem. Lacking this understanding, evaluation of cumulative damage rests on theoretical methods, related in varying degree to the physical process of fatigue and to experiment.

This Memorandum presents an investigation of the basic concepts of cumulative damage and of several current cumulative-fatigue-damage theories.

## SUMMARY AND CONCLUSIONS


This Memorandum contains the results of a study of cumulative fatigue damage. The first part of the study investigates the general, basic foundation of this subject. The main conclusions of this phase of the study are these:

> What fatigue damage is, how it can be measured, and how it grows as a result of the stresses experienced by aircraft are questions that can not be adequately answered today. (However, an opportunity for important advances may exist in utilization of microscopic observation techniques in the study of cumulative fatigue damage.)
>
> Until a sound physical basis is established, evaluation of cumulative fatigue damage must depend on theories.

This Memorandum shows that there are certain key assumptions which can be identified in current theories. These assumptions determine general trends in the structural weight required to provide a satisfactory fatigue life when a particular theory is used for fatigue-preventive design of a flight structure.

The key assumptions have to do with the way fatigue damage is assumed to occur at different stress amplitudes when they are applied alone and when they are mixed with other stress amplitudes (as is the case in aircraft). In terms of these assumptions:

> Cumulative-fatigue-damage theories can be stress-dependent or stress-independent. That is, the amount of fatigue damage produced by a specified fraction of the number of cycles that would produce failure can be the same for all stress amplitudes (stress-independence) or different (stress-dependence).
>
> There can be interaction or interaction-free theories. The course of damage at one stress amplitude may be changed by applying other stress amplitudes (interaction), or it may be unaffected (interaction-free).

As a basis of comparison of cumulative-damage theories, Miner's theory has the advantages of simplicity and familiarity. It simply states that when a mixture of stress amplitudes is applied, fatigue failure will occur when the fractions of life expended at each stress amplitude add up to one. The fraction of life expended at any one

stress amplitude is the ratio of the number of cycles applied at that amplitude to the number that would cause fatigue failure at that amplitude.

A study of the effect of the key assumptions mentioned leads to these conclusions:

> Miner's theory amounts to a stress-independent, interaction-free theory, and other theories that have these features are therefore equivalent to Miner's theory.
>
> Stress-dependent, interaction-free cumulative fatigue damage theories, when used to design aircraft structures, will lead to heavier structures than would result if Miner's theory were used, since prediction of shorter fatigue life is inherent in this type of theory.
>
> Interaction effects can lead to designs either heavier or lighter than a design based on Miner's theory. Generally, investigation of the way that interaction effects are postulated in a theory will disclose the direction of design effects.

The utility of the results of the first phase of this study lies in the evaluation of current cumulative-fatigue-damage theories. By identifying the key assumptions mentioned above, conclusions about practical effects on design can be drawn. To provide a basis for evaluation, use of Miner's theory in design is outlined. Several other current theories are then examined, in terms of the key assumptions and also other assumptions (or in some cases, limitations resulting from a restricted experimental basis) that would be important in aircraft design. The theories studied were selected as being representative of the breadth of possibilities, not as being necessarily superior to others that have been proposed. Those selected and the conclusions drawn about them are:

> Valluri's Theory
>
> This theory is found to be equivalent to Miner's theory if certain restrictions concerning definition of fatigue damage are observed. The restrictions essentially require the use of experimental fatigue data based on a uniform measure of damage or else definition of theoretical damage in terms of a specified structural-failure criterion.

Grover's Theory

This theory proposes that fatigue damage proceeds in two discrete stages. It is shown that use of this theory would lead to increased structural weight, compared with use of Miner's theory (unless the fractional life at the transition between stages is the same for all stress amplitudes). The two-stage process is conceptually compatible with a reasonable physical model of fatigue, but sound experimental evidence justifying this theory for evaluation of cumulative damage is lacking.

Corten-Dolan Theory

This is an interaction theory. As developed, it contains the attractive assumption that interaction effects can be determined from fairly simple fatigue tests at two stress levels. Experiments designed to check the theory agree quite well with predictions, although Miner's theory also agrees quite well with the experiments. The experimental basis has not yet been provided for general application of this theory to aircraft structural design.

Freudenthal-Heller Theory

This is also an interaction theory, but interaction effects are related to experiment in a more complex way. Again, an experimental basis for applying this theory to aircraft structural design is not available.

Shanley's Theory

As originally presented, this theory postulated interaction effects which can be considered to result from varying amounts of initial damage produced by the first cycle at any one stress amplitude. This theory, in design work, would also lead to heavier structures than would Miner's theory. Theoretically, the interaction effects can be determined from experiments, and fatigue curves can be modified for use with Miner's theory, but this has not yet been verified.

2-x Method

This is an interpretation of Shanley's original theory, in which interaction effects are included, but in a complicated way not amenable to simple physical interpretation. The result, in design use, would consistently be increased structural weight, compared with Miner's theory.

In conclusion, no radical "breakthroughs" appear in the area of cumulative-fatigue-damage evaluation in aircraft structures, either in physical understanding or in theoretical techniques. Spectrum testing thus remains a vital ingredient in the process of designing fatigue-resistant aircraft structures. However, every effort should be made to so prescribe test conditions that the results will serve to aid in discerning useful features of cumulative-damage theories as well as to furnish design data.

The key characteristics of cumulative-damage theories concern effects that are difficult or impossible to observe in spectrum fatigue tests. Further, they are far from being soundly established by physical behavior and, worse, are often contrary to expected behavior in a structure as complicated as an airframe. For example, plastic strain is generally agreed to be a necessary factor in fatigue damage, but it can also be a factor in promoting fatigue life in a complex structure. Nevertheless, cumulative-damage theories fill an important need today. The results given in this Memorandum should be an aid in relating, for design use, spectrum-test results and cumulative-damage theories.

# CONTENTS

## LIST OF SYMBOLS

$A$ = constant

$a$ = fraction of cycles to failure during which first stage of damage occurs

$\bar{a}$ = exponent in damage equation

$C$ = constant

$D$ = fatigue damage

$D_f$ = fatigue damage at failure

$D_{in}$ = initial fatigue damage caused by first stress cycle

$E$ = modulus of elasticity

$h$ = stress-spectrum-shape parameter

$K$ = constant

$L$ = crack length

$L_{cr}$ = critical crack length

$L^*_{cr}$ = critical crack length for highest stress in spectrum

$L_{in}$ = crack length produced by first stress cycle

$L_o$ = hypothetical initial crack length

$M$ = interaction factor for spectrum loading

$m$ = number of applied stress cycles during second stage of damage

$\bar{m}$ = number of damage nuclei

$N$ = number of stress cycles applied at constant amplitude that produces sufficient damage for failure to occur at specified higher amplitude

$N'$ = number of stress cycles applied at constant amplitude that produces conventional failure

$N_I$ = revised cycles to failure at one stress amplitude due to interaction of other amplitudes

$N_r$ = number of cycles of spectrum loading that produces failure

$N_>$ = number of applied stress cycles with greater than a specified amplitude

$\bar{N}$ = cycles to failure at constant stress amplitude after increased initial damage

$n$ = number of applied stress cycles

$n^*$ = number of low stress cycles that produces same damage as initial high stress cycle

$\bar{n}$ = number of applied stress cycles starting from $n^*$

$R$ = rate of increase of damage with cycle ratio

$\bar{R}$ = ratio of crack-propagation constants

$r$ = crack-propagation constant

$z$ = number of stress amplitudes in a spectrum

$\alpha$ = ratio of number of applied cycles of a specified amplitude to number at a reference amplitude

$\beta$ = ratio of number of cycles to failure at a specified amplitude to number at a reference amplitude

$\gamma$ = $\alpha/\beta$

$\sigma_a$ = amplitude of stress cycle

$\sigma_{ar}$ = constant stress amplitude resulting in same number of cycles to failure as spectrum loading

$\sigma_e$ = fatigue-endurance limit

$\sigma_m$ = mean stress of stress cycle

$\sigma_o$ = maximum tensile stress in stress cycle

$\sigma_{ult}$ = failure stress in conventional tension test

## I. INTRODUCTION

The problem of structural fatigue confronted European engineers over a hundred years ago. At that time, the main problem was fatigue failure of axles in railroad equipment. While considerable progress has been made since fatigue was considered to be due to "crystallization" of the material, the phenomena occurring at the atomic level during fatigue are still not understood very well.

This does not preclude engineering solutions to the problem, as analogy with stability of structures in compression will illustrate. Long before there was basic understanding of phenomena occurring during elastic and plastic deformation of a material, it was possible, by using stress-strain curves, to design efficient structures to carry compressive loads without failure by instability.

By making repeated load tests of railroad axles, designs were found that provided the desired lifetime without fatigue failure. Tests were made at different stress levels, counting the number of cycles of repeated stress that produced failure. Plotting the relationship between stress (S) and number of cycles to failure (N) led to the first S - N curves. Constant amplitude tests (repetition of identical cycles) adequately simulated the loading in service in this case. It was found that below some stress level, called the endurance limit, cycles could be applied indefinitely without fatigue failure. A typical S - N curve is shown in Fig. 1. (Usually, S - N curves are displayed on log-log or semilog plots, where they appear as nearly straight lines.)

The problem of designing for stress cycles at one constant level occurs very rarely in flight structures. The actual stress variation with time that must be accounted for in fatigue considerations is far more complicated. The importance of keeping weight low and the absence of endurance limits in airframe materials rule out the simple solution of the railroad engineers. It is not practical to make enough tests to provide a firm experimental basis for guaranteeing adequate fatigue life for aircraft structures--at least not at the stage of design when thicknesses and sizes of structural members are chosen. Moreover,

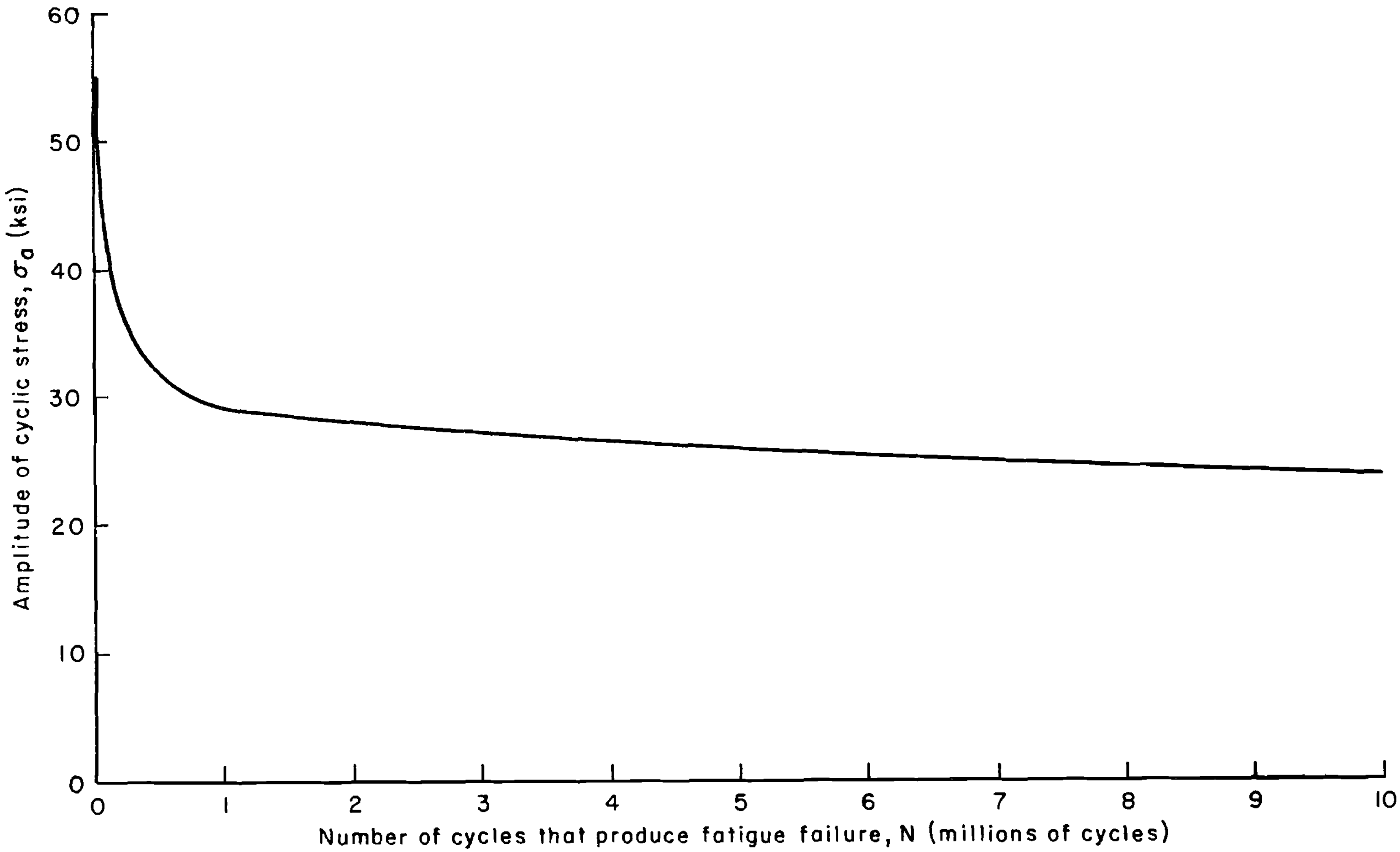



Fig. 1— Typical S-N curve (average values from ANC-5, for smooth-rotating-beam tests of 7075-T6 aluminum alloy)

there will always be some uncertainty in determining the important features of the stress variation experienced in service until the aircraft has been built and flown.

Thus it is necessary to evaluate fatigue damage caused by complex stress variations on the basis of "basic" data--the S - N relationship obtained from single-level tests. In order to use basic fatigue data in designing structures to withstand stress cycles of many different levels, it is necessary to decide (1) how damage progresses at one stress level and (2) how to properly add together the damage produced by each stress cycle when many different cycles are mixed together.

The first part of this Memorandum treats the subject of cumulative damage--the question of how fatigue damage accumulates when many different stress cycles are applied. It is shown that cumulative-damage theories can be categorized by determining the basic assumptions they contain regarding the two points above. General conclusions can be drawn about the relative fatigue life prediction of various theories when these basic assumptions are identified.

Next several current cumulative-damage theories are examined and compared in terms of prediction of fatigue life under spectrum loading. The basis of comparison is Miner's theory. However, life prediction is only an intermediate step in this study. The important consequence of differences in the theories considered is not different predicted lives, but rather differences in the amount of material that will provide a <u>specified</u> lifetime, as found by application of the theories in design.

This brings up fundamental differences between methods of <u>analysis</u> and methods of <u>design</u> (or synthesis). In fatigue, analysis attempts to answer the question, "Given a specified structural configuration and a specified variation of loads, when will the structure fail from fatigue?" The related design question is, "Given a specified length of time that the structure must perform its function and a specified variation of loads, what is the proper design configuration, i.e., the required amount and distribution of material in the structure?"

The two aspects are of course intimately interrelated, and design would not be possible in practice without the foundation of analysis. It is important to note, however, the difference of emphasis in the two approaches. The nature of experiments and the interpretation of results can differ markedly in fatigue work, depending on whether the analysis or the design orientation predominates.

Study of how these various theories would be applied in design, with realistic flight stress spectra, points out any limitations of these theories and also points out additional experimental work that is needed to establish their usefulness in design.

## II. DAMAGE CONCEPTS

Before considering cumulative damage, the whole concept of fatigue damage merits some remarks. Any attempt to evaluate cumulative damage in terms of basic fatigue behavior is hampered by a lack of understanding of what goes on, physically, in a material in the course of even simple fatigue tests. Unfortunately, in a large and important part of fatigue behavior, the term "damage" must be applied to a phenomenon that is neither directly observable nor measurable. In a simple, single-level fatigue test, damage to the test specimen occurs, in some undetermined fashion, as more and more stress cycles are applied. The amount of this damage keeps increasing until finally the specimen breaks. At this point damage may be designated as complete, or 100 per cent. The process is depicted schematically in Fig. 2.

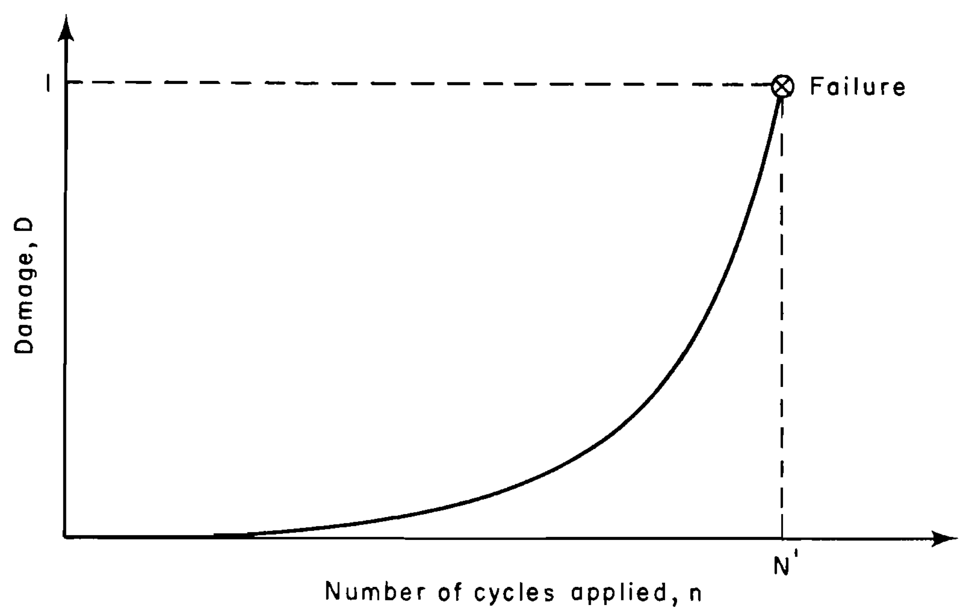



Fig. 2 — Representation of progress of fatigue damage with application of stress cycles (constant stress amplitude)

A useful measure of damage would be loss of strength. The strength of a specimen or structure decreases as stress cycles are applied. A specimen that would break at a stress of 75,000 psi in a static test will eventually break at a stress of only 20,000 psi if the stress is cycled between 0 and 20,000 psi often enough. By the time that fatigue failure occurs, there has been a loss in strength of 55,000 psi. Generally this loss occurs very gradually until shortly before failure. If the fatigue test were stopped before fatigue failure and a static test were then made, static failure would occur at a stress somewhere between 75,000 and 20,000 psi. As a measure of damage, the ordinate in Fig. 2 could be the ratio of the strength lost after any number of cycles to the strength lost when failure finally occurs in the fatigue test. There are two drawbacks to determination of this kind of relationship: First, measurement of static strength requires destruction of the test item and second, for a large part of the life, the strength decrease is so small that changes would be practically undetectable and could be obscured by scatter in initial static strength.

Another possible measure of damage is the size or length of fatigue cracks. Damage could be defined as the ratio of crack length after any number of cycles to the crack length that finally results in failure. The damage-cycle relationship would again be similar to that shown in Fig. 2. The same problems exist, in that for much of the range of interest cracks are so small that their depth is measurable only through sectioning and microscopic examination.

Physically meaningful damage concepts should include the microscopic phenomena--intrusions, extrusions, fissures, production of subgrains, fine structure of slip bands, and generation and motion of dislocations associated with plastic strain--that go on in the neighborhood of the damage. One of the most encouraging prospects for really understanding fatigue is the fairly recent emphasis on observation of these microscopic phenomena occurring within single grains. Reference 1 contains many interesting results of these observations. Recent development of techniques for obtaining transmission electron-microscope pictures[2,3] of "thick" metal specimens offers the possibility of observing dislocation movement within the material with resolution comparable to previous observations of a surface by replica methods. ("Thick" in this case may be only a few microns, but this is thick enough to contain something like 10,000 crystallographic planes.)

While these techniques have not yet been applied to rigorous study of cumulative damage, they give promise of helping to understand and evaluate cumulative damage. Equally important is the possibility that this understanding will lead to improvements in fatigue behavior of materials.

In summary, there is no simple, unambiguous measure of damage in the early stages of fatigue, even in the simplest fatigue tests. Therefore, in order to evaluate damage caused by complex stress variations, an assumption for the way that damage proceeds is required. Reduced to essentials, cumulative-fatigue-damage theories postulate the form of damage growth and state a rule for adding the damage produced by stress cycles of different amplitudes.

## III. STRESS DEPENDENCE

A distinguishing feature of various cumulative-damage theories is the manner in which the course of damage depends on stress amplitude. Some effect of stress amplitude must be included, of course, because the number of cycles to failure does depend on stress amplitude. However, as described below, certain cumulative-damage theories can be termed stress-independent theories. Stress independence simply means that equal amounts of damage are produced at equal fractions of life for all stress amplitudes. Care is required in defining damage and life, however.

Figure 3 shows the effect of stress amplitude on the number of cycles to failure for several stresses. With the definitions of damage previously given, equal amounts of damage on the curves shown are not equivalent in an absolute sense, because damage at failure (either loss of strength or crack length) will be different for different stress amplitudes. Measured in terms of the amount of strength lost, there must be less damage at failure in tests at high stress levels than at low, simply because it is not possible to lose as much strength when applying the high-stress cycles.

The damage measure must be revised so that equal ordinates do represent the same amount of damage. Considering cycles of many different amplitudes (that will be mixed and applied randomly) the damage--strength loss or crack length--at failure for the highest stress in the spectrum can be used as a reference damage. Then, if damage for any stress amplitude is divided by this reference damage, the damage-cycle relationships that result (shown in Fig. 4) represent the same true damage at equal values of the ordinate.

At this point, the details of the high end of the damage curves must be examined. With damage measured as shown in Fig. 4, greater damage is possible before failure at low cyclic stresses than at high cyclic stresses. However, in most practical applications, failure must be assumed to occur when the amount of damage reaches the level that would result in failure upon application of the highest stress amplitude. In other words, the cycles to failure in a constant-

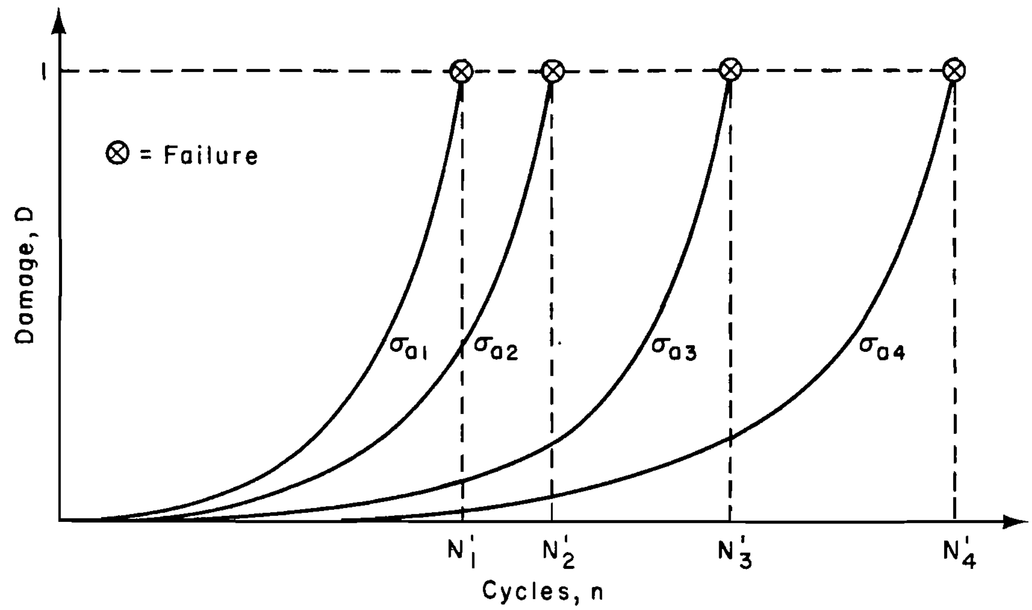



Fig. 3 — Damage-cycle relationships for several stress amplitudes, constant-stress-amplitude tests ($\sigma_{a1} > \sigma_{a2} > \sigma_{a3} > \sigma_{a4}$)

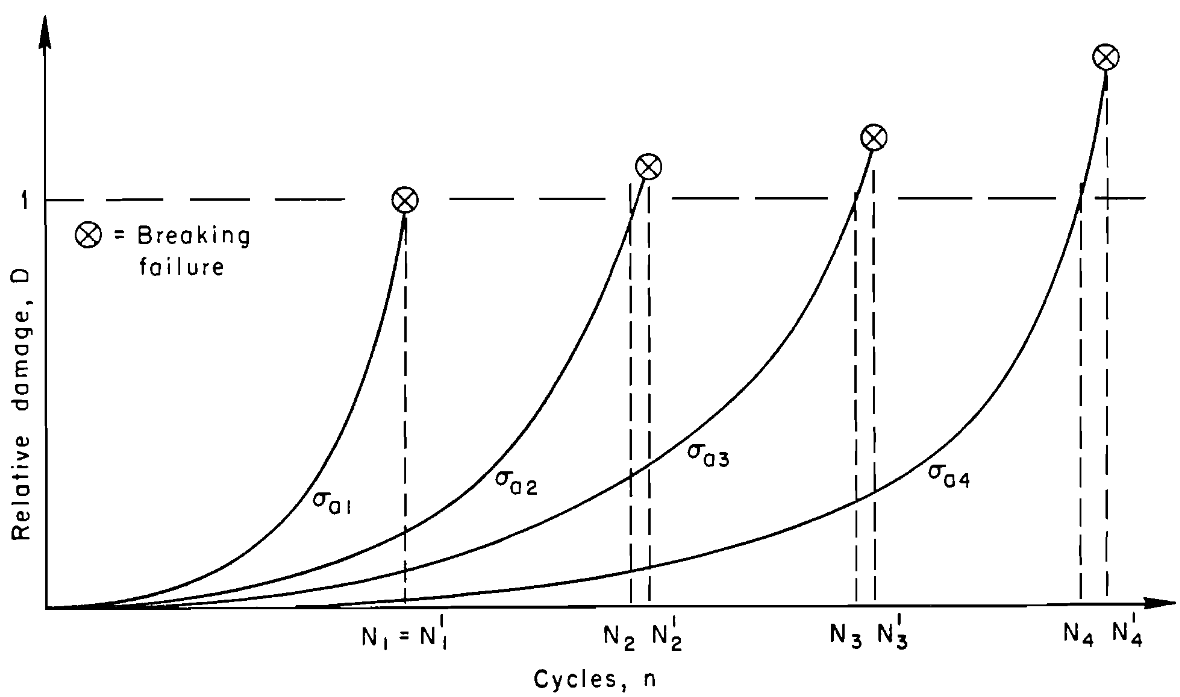



Fig. 4 — Damage-cycle relationships from Fig. 3 normalized to failure damage at highest stress amplitude in spectrum, $\sigma_{a1}$

amplitude test, $N_i'$, should be replaced by $N_i$, the number of cycles that produce an amount of damage equal to damage at failure at the highest stress amplitude.

There are several reasons for this approach:

1. A spectrum of stress amplitudes will be encountered during the lifetime of a flight structure, with the highest stress amplitudes occurring too frequently to be neglected. A designer could not count, as part of the design lifetime, any time which depends on not encountering the high-amplitude stress cycles.

2. In many cases, the rate of strength loss or crack growth near the end of lifetime becomes so rapid that the difference between N and N' becomes insignificant.

3. The experimental S - N curve may be obtained by determining N rather than N', i.e., counting cycles until some prescribed crack length or strength decrease is reached.

The second and third statements above are often implicit in cumulative-damage theories.

If the curves in Fig. 4 are normalized by dividing the number of cycles applied by the number of cycles that produces the selected damage level (damage at failure at the highest stress amplitude), $N_i$, either of two possibilities, Fig. 5 or Fig. 6, will result. Figure 5 results from cumulative-damage theories that are stress-independent, while cumulative-damage theories that reduce to the representation of Fig. 6 are stress-dependent. (The curves in Fig. 6 do not have to be nonintersecting between the origin and 1,1.)

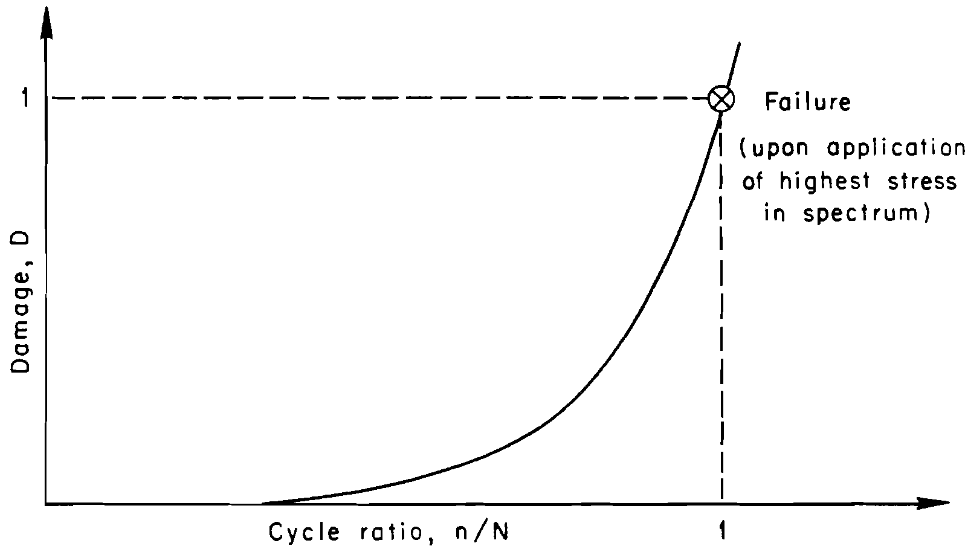



Fig. 5—Typical stress-independent damage-cycle relationship (Fig. 4 with cycles normalized; curves are identical for all stress amplitudes)

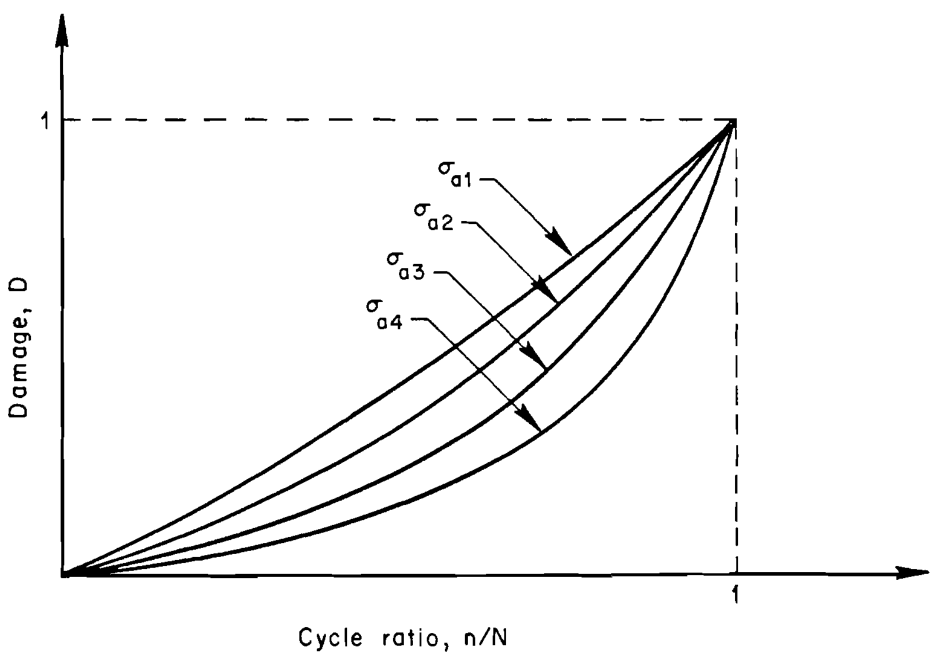



Fig. 6—Typical stress-dependent damage-cycle relationships

## IV. INTERACTION EFFECTS

The other important characteristic of cumulative-damage theories is interaction. If the relationship between damage and number of cycles for a specified stress amplitude is assumed to be valid whether or not other stress amplitudes are applied, the cumulative-damage theory is termed an interaction-free theory. A theory that assumes that application of other stress amplitudes changes the course of damage due to the specified stress amplitude is then an interaction theory.

Consider material in a structure (or laboratory specimen) that has had enough stress cycles of various amplitudes applied to produce some specified damage. Now one more cycle is applied, of amplitude $\sigma_{a3}$ for example. For an interaction-free cumulative-damage theory, the increment of damage produced by this cycle would be the slope of the damage-cycle curve for the given stress amplitude ($\sigma_{a3}$) at the specified amount of damage multiplied by the increment in cycle ratio. For one additional cycle, the increment in cycle ratio, $d(n/N)$, would be $1/N_3$. In equation form

$$\Delta D = \frac{dD}{d(\frac{n}{N})} \cdot \frac{1}{N_3} \tag{1}$$

where $dD/d(n/N)$ is determined at the specified damage level on the damage curve for the specified stress amplitude, $\sigma_{a3}$. Cumulative-damage theories in which the above expression is not valid for incremental damage are interaction theories.

The definition of interaction-free theories applies equally well to either the stress-independent theories of Fig. 5 or the stress-dependent theories of Fig. 6.

## V. THE ROLE OF DAMAGE EXPERIMENTS

Most of the experimental work exploring differences in damage rates is of limited usefulness to the designer faced with the spectrum of stress amplitudes that a flight structure experiences. This is mainly due to the fact that in the damage range of most interest (before obvious, measurable cracks appear) the problem of measuring damage appears to be intractable. As a result, experimental investigations utilize the number of stress cycles, which can be measured. Typical tests can be termed one-step tests, since a certain number of cycles of one stress amplitude is applied; the stress amplitude is then stepped up (or down), and cycles at the second amplitude are applied until failure occurs. Since the damage at the first amplitude cannot be measured, any curve from the origin to the point (1,1) on Fig. 5 or 6 will suffice for a damage curve, although a monotonically increasing curve is preferable. At the amount of damage on this curve corresponding to the value of $(n/N)_1$, which is the ratio of the number of applied stress cycles at the first amplitude to the number of cycles to failure at that amplitude, the remaining life at the second amplitude is plotted. (Actually the calculated equivalent fraction of cycles used $(N_2 - n_2)/N_2$ is plotted, or, equivalently, $n_2/N_2$ is plotted, measured to the left from the line $n/N = 1$.) When this is done for several values of $n_1/N_1$, curves such as those shown in Fig. 7 result. These are suitable for showing how many cycles at the second stress amplitude may be applied after a specified number are applied at the first amplitude, but there is no reason to assume that these curves are valid as damage curves when cycles of the two amplitudes are mixed in together. As a matter of fact, it seems likely that the one-step type of test engenders extreme emphasis on interaction effects that could be insignificant in spectrum tests. Considering well-mixed cycles of many amplitudes, there is no way by which this technique can lead to the proper damage curves.

Spectrum tests can be considered to be damage experiments only in a very limited sense. They can confirm or deny the validity of a cumulative-damage theory, but they cannot provide specific knowledge of how the early damage progresses at the various stress amplitudes in the spectrum.

At present, at least until more refined microscopic-damage measurements are utilized in study of early-stage cumulative damage, the damage-cycle relationships incorporated in cumulative-damage theories must remain primarily hypothetical.

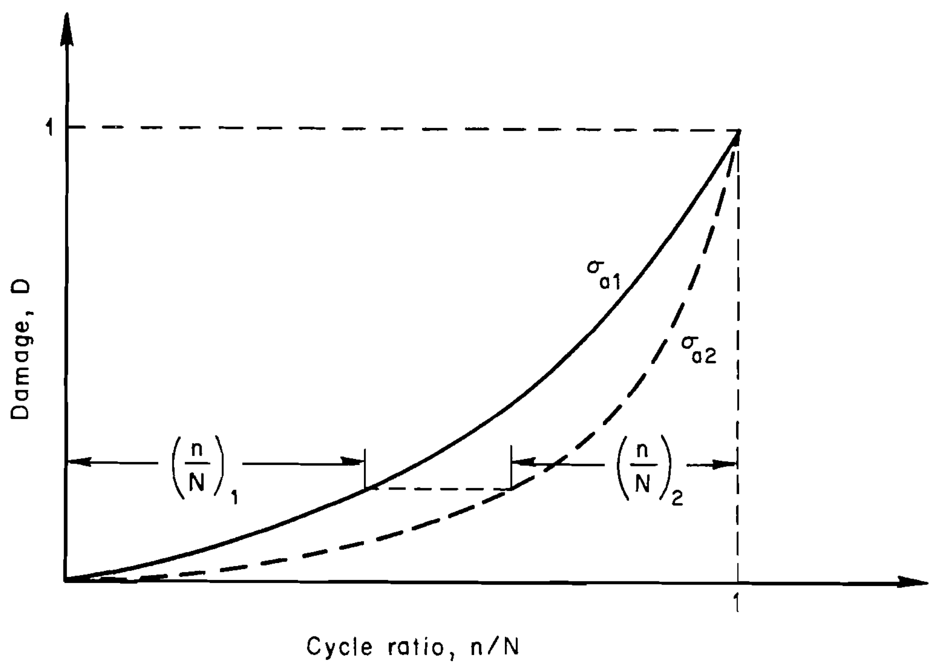



Fig. 7—Typical damage curves obtained from one-step fatigue tests

## VI. GENERAL COMPARISON OF CUMULATIVE-DAMAGE THEORIES

In the previous sections, some general characteristics of cumulative-damage theories were explored, namely, the nature of assumptions regarding dependence on stress amplitude and on interaction effects. It will be shown that these characteristics determine general properties of cumulative-damage theories. The most generally known cumulative-damage theory, that usually attributed to Miner[(4)] in this country, will be used as a basis of comparison. Miner's theory is important because of its relative simplicity, its close-enough approximation to reality to be of practical use, and the widespread knowledge and utilization of this theory.

Actually Palmgren[(5)] presented cumulative-damage equations identical to Miner's some 20 years earlier, apparently the result of a purely intuitive approach. Langer's excellent paper[(6)] presents a very perceptive inquiry into fatigue. His work includes Miner's theory in a more general framework but lacks the attractively simple cumulative-damage equations and the supporting experimental work presented by Miner.

Miner's theory simply states that the fraction of life used up by application of stress cycles of any amplitude is just the ratio of the number applied to the number that would cause failure at that amplitude. When different amplitude cycles are mixed together, failure occurs when the fractions of life expended at each amplitude add up to one.

The key features of cumulative-damage theories, concerning stress and interaction effects, lead (as shown in Appendix A) to these statements:

1. Cumulative-damage theories that reduce to a stress-independent damage-cycle relationship when normalized in the manner of Fig. 5 and are interaction-free are equivalent to Miner's theory.
2. Cumulative-damage theories that are stress-dependent, reducing to the form shown in Fig. 6 when normalized, and are

interaction-free will predict shorter life under random, spectrum loading than would be predicted by Miner's theory.

3. In order for a cumulative-damage theory to predict longer life than does Miner's theory, it must be an interaction type of cumulative-damage theory (and can be either stress-dependent or stress-independent). Interaction-type damage theories can be constructed so as to predict either shorter or longer life than does Miner's theory.

Classification and comparison of cumulative-damage theories can be made fairly simply. This allows general conclusions to be drawn about predictions and consequences of the use of a theory--whether more or less structural material than the amount determined with Miner's theory is required for a specified life. In following sections, several current cumulative-damage theories will be dissected in order to make these comparisons.

## VII. DESIGNING FOR STRESS SPECTRA WITH MINER'S THEORY

Procedures for predicting the spectra of stress amplitudes experienced in service contribute some of the major uncertainties in designing to prevent fatigue. For the moment, only some brief remarks are necessary, since comparisons of cumulative-damage theories are meaningful only insofar as they are based on specific stress spectra.

Current Air Force specifications on structural fatigue lead to results such as those shown in Fig. 8. In this figure, $N_>$ is the number of times that cyclic stresses of amplitude $\sigma_a$ or greater are expected to be experienced at some specified part of the structure in the course of a specified amount of flying time (e.g., 100 hr). The shape of the curve shown is representative of curves in which the cyclic stresses are produced by atmospheric turbulence.

For an existing airplane, this curve would change with changes in speed, altitude, weight, weight distribution, and location in the structure. The curve shown is for a constant mean stress, and this too could change during flight. However, these are not the changes of interest for the present. We are interested in designing for particular flight conditions. The required amount of material at the specified location is being determined, and changes in this amount will change both the mean stress level and the amplitude of stress cycles in Fig. 8. In other words, variations in stress levels in the following discussion reflect design choices of material thickness or cross-sectional area, not changes in stress levels in a specific design due to changes in flight conditions.

Application of Miner's cumulative-damage theory[4] to stress-amplitude spectra from atmospheric turbulence is illustrated in Figs. 9, 10, 11, and 12. Procedures are outlined here in a greatly simplified fashion. We begin by assuming that the information required for Fig. 8 is already available. The thickness of material at a specified point in the structure has been selected (as determined by static strength requirements, for example), and thus a mean stress is determined for the flight condition associated with Fig. 8.

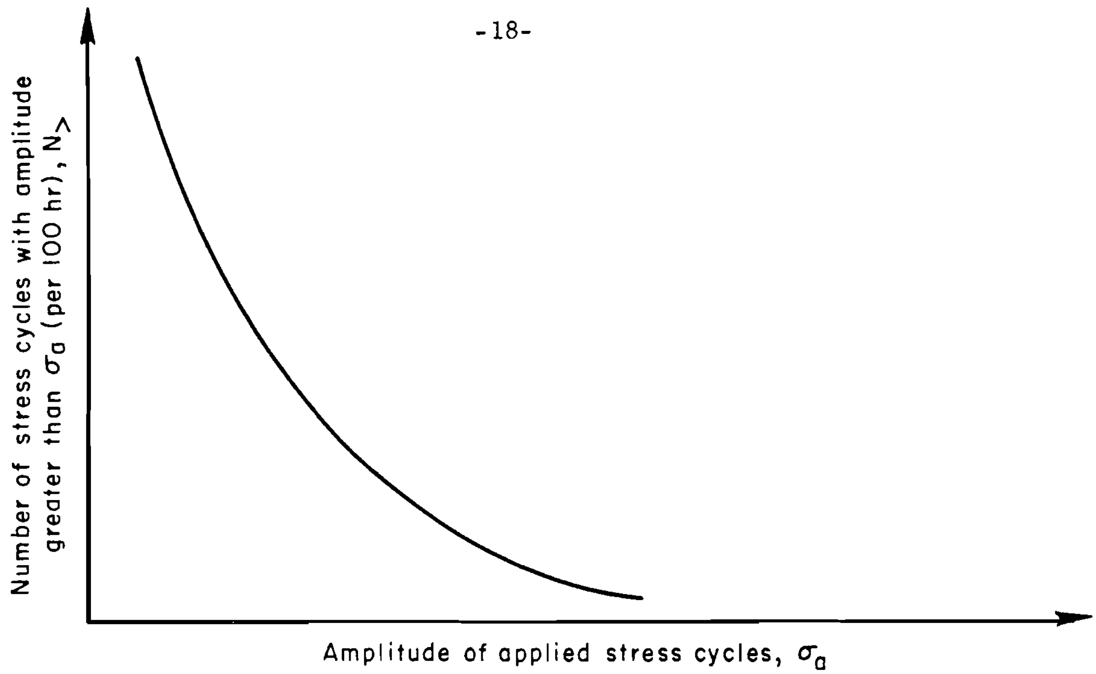



Fig. 8 — Typical stress spectrum resulting from atmospheric turbulence

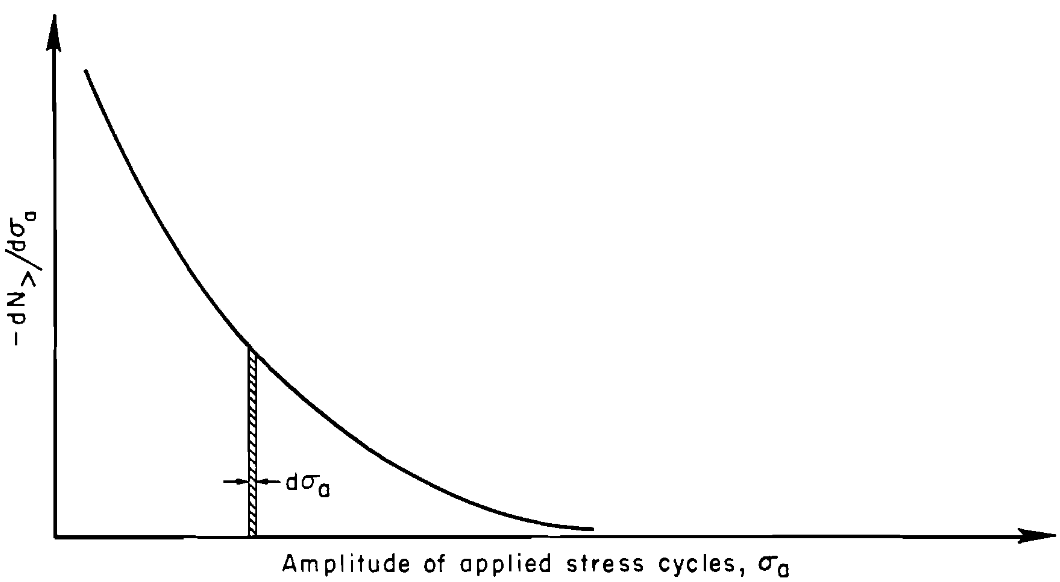



Fig. 9 — Incremental number of applied cycles per increment in stress amplitude

In Fig. 9, the negative slope of the curve in Fig. 8 is plotted against stress amplitude. (The negative slope of this curve is the same as the positive slope of the curve for the number of cycles of amplitude less than $\sigma_a$.) The increment of area shown in Fig. 9 is thus the number of cycles with amplitude between $\sigma_a$ and $\sigma_a + d\sigma_a$.

Figure 10 shows the basic fatigue curve where N is the number of cycles to failure as a function of stress amplitude, $\sigma_a$, for constant-amplitude tests at the same mean stress as in Fig. 8. (This is the familiar S - N curve with linear rather than logarithmic scales and with the coordinate axes reversed.) Dividing the ordinate of Fig. 9 by the ordinate of Fig. 10 at the same value of stress amplitude leads to the curve shown in Fig. 11. In this curve, the ordinate $-(dN_{>}/d\sigma_a)/N = -(dN_{>}/N)/d\sigma_a$ is termed the life expenditure rate, LER. Its magnitude shows how much fatigue damage is produced by the cycles of different stress amplitudes. The word "rate" in this context refers not to time, but to the increase in damage with increasing applied stress amplitude.

In Fig. 11, the incremental area shown is equal to $dN_{>}/N$, or the fraction of life expended due to cycles of stress amplitude between $\sigma_a$ and $\sigma_a + d\sigma_a$. The Miner summation, $\Sigma\ (n_i/N_i)$, is then the integral of this curve over the total range of stress amplitude. The value of this integral is the fraction of the total available fatigue life that is expended in the time period represented in Fig. 8. Dividing this quantity, the life fraction, into this time period would give the time to failure, or fatigue life if no stress cycles other than those represented in Fig. 8 were applied.

By selecting other values of material thickness and repeating the computations, curves such as those shown in Fig. 12 can be obtained. Changing the thickness also changes the mean stress, which in turn changes the "cycles-greater-than" curve in Fig. 8, its slope in Fig. 9, and also the basic S - N curve in Fig. 10. The remainder of this Memorandum will be devoted to other cumulative-damage theories--comparing their general characteristics with Miner's theory and determining methods for applying them to provide the relationship shown in Fig. 12.

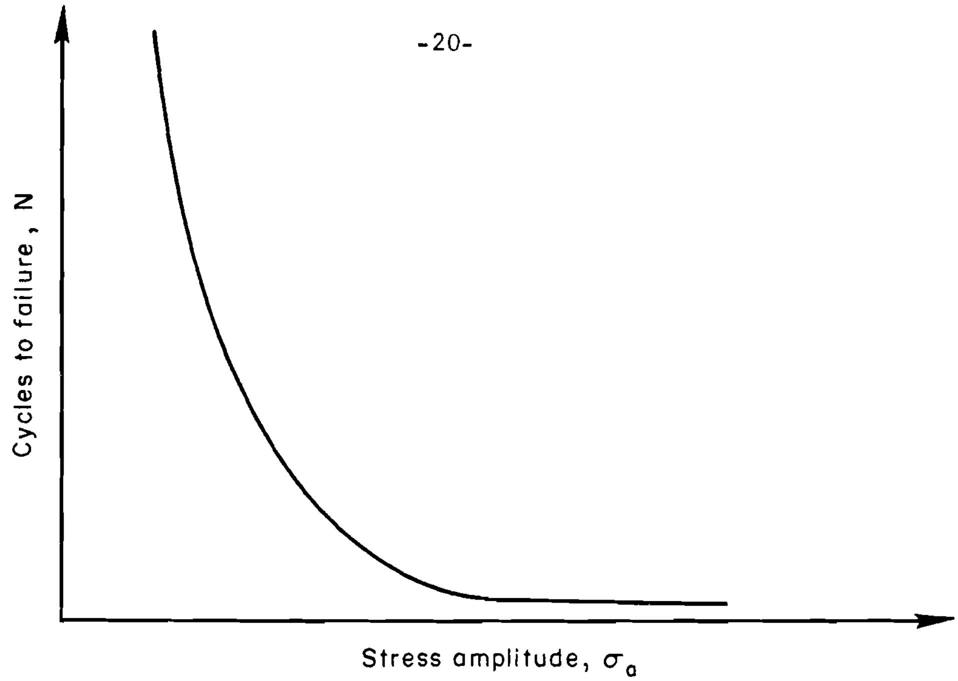



Fig. 10 — Basic fatigue data (S-N curve)

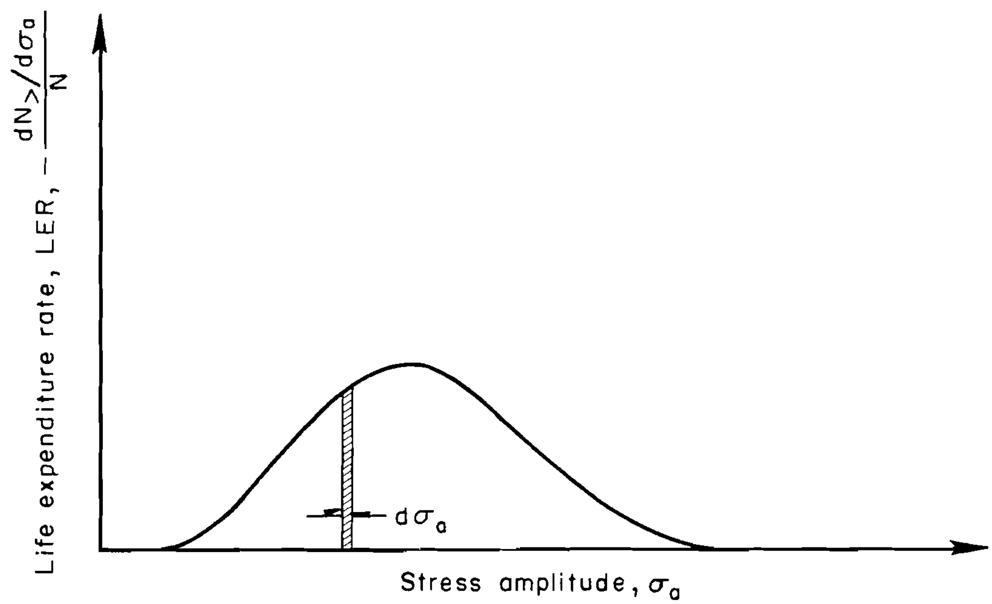



Fig. 11 — Life expenditure rate (area under curve is the fraction of fatigue life expended during time specified in Fig. 8)

The type of curve shown in Fig. 12 is useful as a basis for evaluating differences in cumulative-damage theories, but it is restricted to only a small portion of the total flight regime. Going back one step, the integral of the curve in Fig. 11 is the fatigue life fraction expended in the time period represented in Fig. 8. However, Fig. 8 represents only one flight condition (defined by speed, altitude, gross weight, etc.). The complete design process must include the stress cycles from the total flight experience, including maneuvers, take-off and landing, and the ground-air-ground cycle as well as turbulence for all altitudes, speeds, and weights involved in service.

With Miner's theory, it is simple (although it requires appreciable effort, in practice) to add the fatigue damage from all sources of cyclic stress and find the damage produced during the required service lifetime.

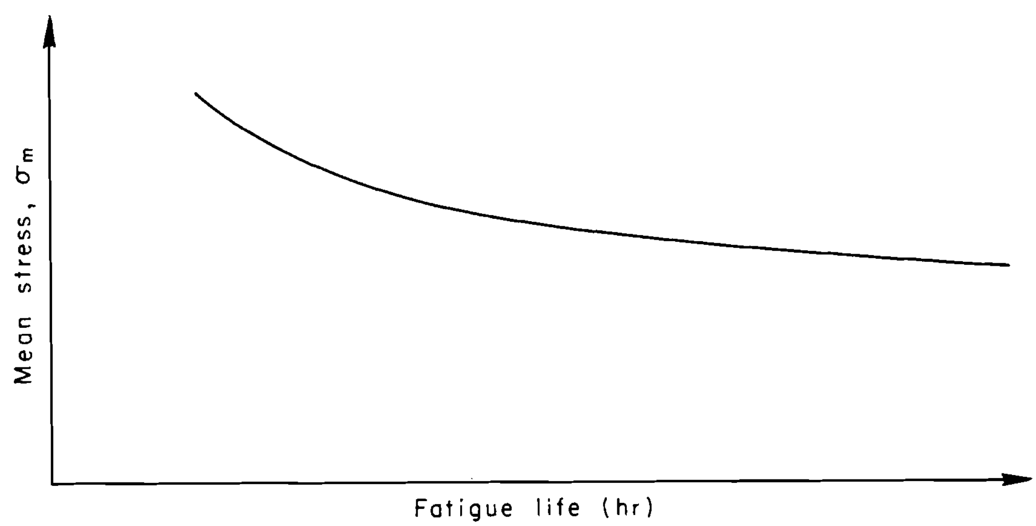



Fig. 12 — Design relationship for fatigue

# VIII. VALLURI'S THEORY

## ANALYSIS

Valluri has presented a cumulative-fatigue-damage theory based on dislocation theory and plastic deformation (and its reduction with cyclic stressing) at the tip of a crack.[7] He develops a relationship between crack length and the number of stress cycles applied which is fundamental to his theory of fatigue:

$$L = L_o \exp \left\{ C \frac{\left[\frac{\sigma_a + (\sigma_m - \sigma_e)}{E}\right]^2 \left(\frac{2\sigma_a}{\sigma_e}\right)^2}{\ell n \left[\frac{\sigma_a + (\sigma_m - \sigma_e)}{K}\right]} n \right\} \qquad (2)$$

Equation (2) is Eq. (19) of Ref. 7, rewritten in the terminology of this Memorandum, where L is crack length, $\sigma_a$ is the cyclic stress amplitude, $\sigma_m$ the mean stress, and n the number of applied stress cycles. The additional symbols are $L_o$, a crack length determined by applying Griffith crack theory to the ultimate tensile stress obtained from a static tension test; $\sigma_e$, fatigue endurance limit; and E, Young's modulus. In Eq. (2) C is a constant that contains (1) another constant expressing relationship between the number of cycles required to narrow a hysteresis loop and the logarithm of the ratio of the areas of the initial and final hysteresis loops, (2) a constant relating the size of a plastic enclave ahead of a crack to stress, Young's modulus, mean grain size, a Burgers vector, and length of crack, (3) mean grain size, and (4) a Burgers vector. The symbol K is also defined as a constant, the asymptotic width of the hysteresis loop (although it appears in the derivation that K also includes a parameter related to plastic strain in the first cycle, which would depend on stress for most materials). In application, the values of C and K and also the endurance limit, $\sigma_e$, are to be determined empirically from fatigue tests.

The form of the exponential growth of a crack with increasing number of stress cycles, in accordance with Eq. (2), is shown in

Fig. 13. (Experimental determination of K insures that the logarithm of the term in the denominator of Eq. (2) is positive and also that crack growth is faster for higher stress amplitudes.)

An analysis of the basic characteristics of Valluri's theory follows. Referring again to Fig. 13, a spectrum of stress amplitudes is considered, and the critical crack length for the highest stress in the spectrum is denoted by $L_{cr}^*$. Considering now the $i^{th}$ (lower) stress amplitude, $\sigma_{ai}$, the number of cycles to failure in a single-level fatigue test at this stress amplitude is $N_i'$. However, the number of cycles at this amplitude which would produce a crack long enough so that a single application of the highest stress in the spectrum would result in failure is, of course, a lesser number than $N_i'$, and is shown as $N_i$ in Fig. 13.

From Eq. (2), the crack-length - cycle relationship for the $i^{th}$ stress amplitude at constant mean stress can be written as

$$L_i = L_o \exp\left[f\ (\sigma_{ai})\ n_i\right] \qquad (3)$$

where $f(\sigma_{ai})$ is a function only of stress amplitude.

Since $n_i = N_i$ when $L_i = L_{cr}^*$

$$L_{cr}^* = L_o \exp\left[f\ (\sigma_{ai})\ N_i\right] \qquad (4)$$

$$\ln\left(\frac{L_{cr}^*}{L_o}\right) = f\ (\sigma_{ai})\ N_i$$

$$f\ (\sigma_{ai}) = \left[\ln\left(\frac{L_{cr}^*}{L_o}\right)\right]/N_i \qquad (5)$$

When this expression for $f\ (\sigma_{ai})$ is substituted in Eq. (3), the crack-growth equation becomes

$$L_i = L_o \exp\left\{\left[\ln\left(\frac{L_{cr}^*}{L_o}\right)\right]\left(\frac{n}{N}\right)_i\right\} \qquad (6)$$

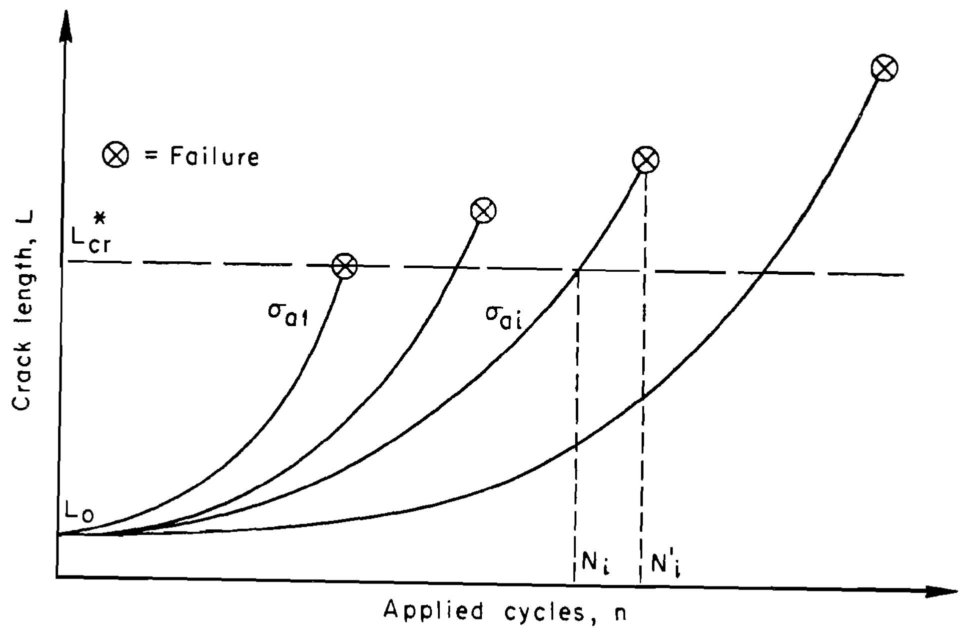



Fig. 13 —Crack-growth relationship (Valluri)

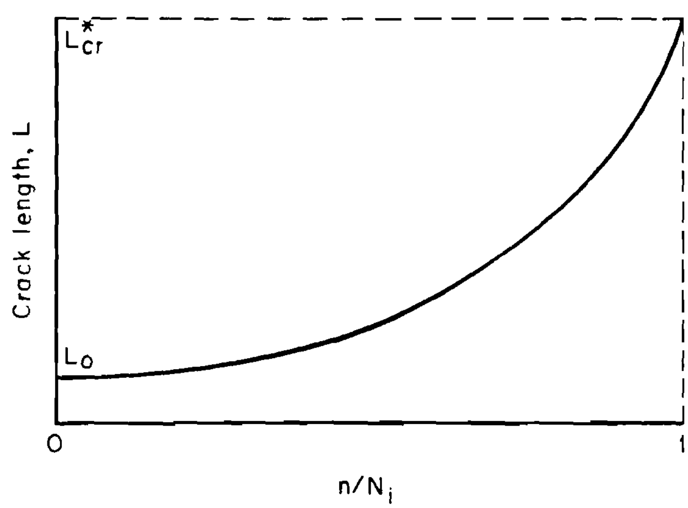



Fig. 14 —Crack-growth relationship for all stress amplitudes; cycles normalized by cycles to $L^*_{cr}$ (Valluri)

for all values of stress amplitude in the spectrum (that is, all stress amplitudes that are large enough to avoid endurance-limit effects).

Since Eq. (6) is independent of stress amplitude, a single curve as shown in Fig. 14 applies to all stress amplitudes. The crack length can be normalized by defining damage as

$$D = \frac{L_i - L_o}{L^*_{cr} - L_o} \tag{7}$$

The form of the relationship between damage thus defined and cycles normalized by the number of cycles that produce $L^*_{cr}$ is shown in Fig. 15.

One additional relationship following from Valluri's theory is pertinent here. With his criterion for failure, the number of cycles to failure at the $i^{th}$ stress amplitude, $N'_i$ (which is not the same as $N_i$ in the above derivation), is given by Valluri as

$$N'_i = \left\{ \frac{\ell n \left[\frac{\sigma_{ai} + (\sigma_m - \sigma_e)}{K}\right]}{C\left[\frac{\sigma_{ai} + (\sigma_m - \sigma_e)}{E}\right]^2 \left[\frac{2\sigma_{ai}}{\sigma_e}\right]^2} \right\} \ell n \left(\frac{\sigma_{ult}}{\sigma_{oi}}\right)^2 \tag{8}$$

Equation (8) is Eq. (21) of Ref. 7 with the present terminology. The quantity $\sigma_{oi}$ is equal to the maximum applied tensile stress, or $\sigma_m$ plus $\sigma_{ai}$, and $\sigma_{ult}$ is the ultimate tensile stress. The terms within the braces in Eq. (8) can be substituted in Eq. (2) to give

$$L_i = L_o \exp\left\{\left[\ell n \left(\frac{\sigma_{ult}}{\sigma_{oi}}\right)^2\right] \left(\frac{n}{N'}\right)_i\right\} \tag{9}$$

$$(L/L_o)_i = (\sigma_{ult}/\sigma_{oi})^{2\,(n/N')_i} \tag{10}$$

This relationship is shown in Fig. 16.

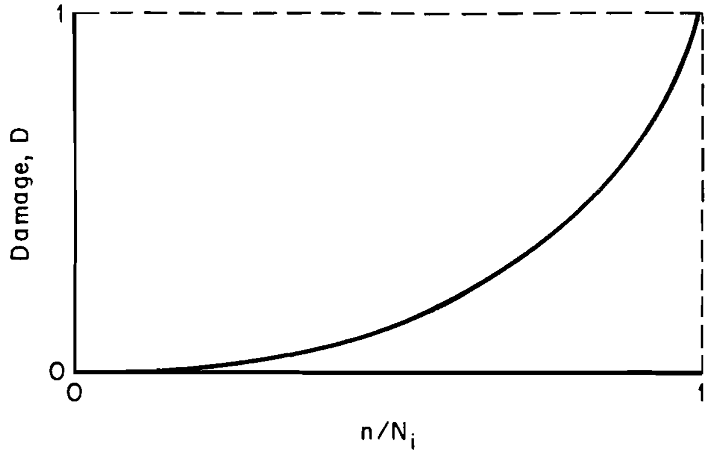



Fig. 15—Normalized damage-cycle relationship (Valluri)

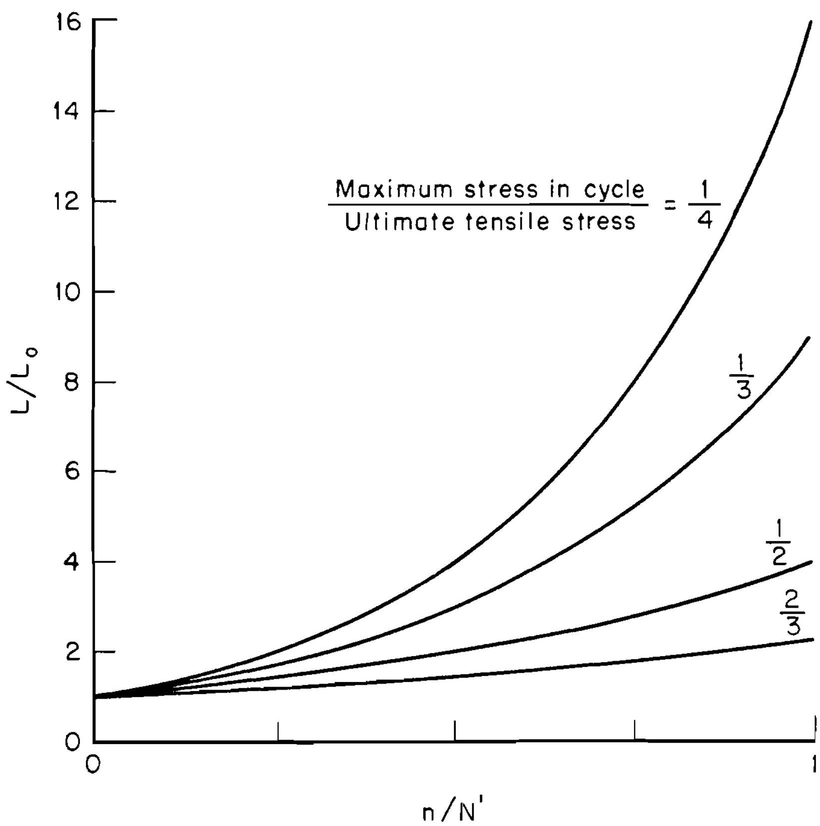



Fig. 16 — Crack growth for constant stress amplitude, cycles normalized by Valluri's cycles to failure

DISCUSSION

The preceding investigation, leading to Fig. 15, shows that this theory is stress-independent in terms of the general characteristics of damage theories presented previously. Since it is presented as an interaction-free theory in the original derivation, the conclusion follows that Valluri's theory is equivalent to Miner's for evaluating cumulative damage, and therefore

$$\Sigma \left(\frac{n}{N}\right)_i = 1$$

is the failure criterion. It must be noted that this conclusion follows from a specified definition; namely, that for well-mixed spectrum loading, failure is the state of damage such that application of the highest stress amplitude in the spectrum would break the structure. Consequently, the values of $N_i$ that must be used with lower stress amplitudes are those that produce this level of damage when the lower amplitudes are applied alone in a constant-amplitude test.

Any disagreement with Miner's theory stems from Valluri's assumption relating critical absolute damage to the magnitude of applied stress cycles. After damage has reached unity, with the above definition, many more cycles can be applied as long as their amplitude is below that which would produce failure at the existing crack length or damage level. Figure 13 illustrates this point.

When the number of applied cycles is normalized by $N'$, the summation $\Sigma\ (n/N')_i$ can be either much greater or much less than unity at failure, depending on the order of application of stress amplitudes. It will always be less than unity for the well-mixed spectrum considered previously. On the other hand, the summation $\Sigma\ (n/N)_i$ can be greater than unity at failure if, after critical damage is reached, the amplitude of succeeding cycles is appropriately controlled, to keep the stress below that which would cause failure. But it will never be less than unity and will be equal to unity for the well-mixed spectrum.

An important contribution of Valluri's theory is the emphasis placed on the possibility of damage at failure depending on stress level. From Eq. (10) (or from Eq. (4) of Ref. 7) an expression for failure stress in terms of crack length can be obtained:

$$\frac{\sigma_{fail}}{\sigma_{ult}} = \sqrt{\frac{L_o}{L}} \tag{11}$$

By substitution from Eq. (10), an expression can be obtained that gives the stress at which failure would occur if a static test were made on a specimen after any number of cycles of a lower stress amplitude had been applied:

$$\frac{\sigma_{fail}}{\sigma_{ult}} = \sqrt{\left(\frac{\sigma_o}{\sigma_{ult}}\right)^{2\left(\frac{n}{N'}\right)}} = \left(\frac{\sigma_o}{\sigma_{ult}}\right)^{\frac{n}{N'}} \tag{12}$$

Considering, for example, application of stress cycles in which the maximum stress is 25 per cent of the ultimate tensile stress, Fig. 17 shows the stress that would produce failure after application of a number of lower stress cycles, according to this theory. For this case, application of about 20 per cent of the cycles that would produce failure at the specified cyclic stress results in a decrease of the failure stress to about 75 per cent of the original ultimate tensile stress. There is a lack of experimental verification of this effect, and as a matter of fact, there is some evidence that the strength falls off much more slowly until late in the life.

However, Valluri's work emphasizes the importance of insuring that basic S - N curves are properly determined and understood in terms of what happens when the number of applied cycles approaches the value N. For application of spectrum loadings, it must be insured that N is the number of cycles that produces the same damage level at all stress levels in order for Miner's cumulative-damage theory to be valid. Some investigation is thus called for in particular cases to determine whether N and $N'$ are significantly different and what criterion was used in establishing the basic S - N curve.

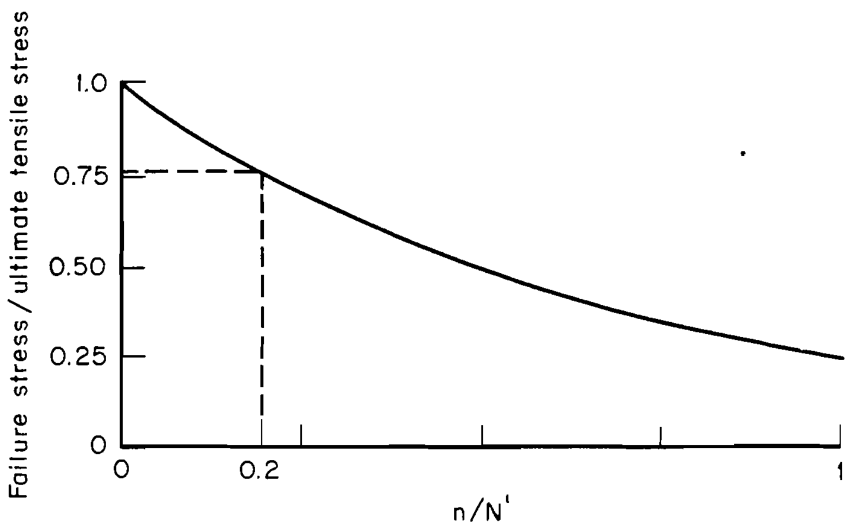



Fig. 17—Calculated decrease in strength with application of stress cycles (Valluri)

Many times the S - N curve is determined by cycles to appearance of first visible crack. In materials and configurations in which there is little or no strength loss at this point, Valluri's constants could not then be validly determined.

## IX. GROVER'S THEORY--A STRESS-DEPENDENT THEORY

Grover has proposed a theory that differs from Miner's theory in that it postulates a two-stage damage process.[8] The essential feature of this theory is the assumption that cracks are initiated during an initial stage of damage as stress cycles are applied. In a single-level test, this initial stage is completed at some fraction, denoted by "a," of the total cycles to failure. In the second stage, propagation of cracks to failure occurs in the course of application of the fraction (1 - a) of the total cycles to failure. It is assumed that the value of a for any stress amplitude is constant whether or not cycles of other stress amplitudes are applied. (It is interesting to note that the essential features of this theory are contained in the method proposed by Langer.[6])

For cumulative-damage calculations under spectrum loading, it is assumed that Miner's theory applies to each stage, which results in two equations:

$$\Sigma \frac{n_i}{a_i N_i} = 1 \tag{13}$$

$$\Sigma \frac{m_i}{(1-a_i)N_i} = 1 \tag{14}$$

where n denotes cycles applied during the initial stage, and m denotes cycles during the second.

As noted in Ref. 8, if $a_i$ is the same for all stress amplitudes in the spectrum, this theory is equivalent to Miner's theory.

The following discussion of this theory is directed toward comparing its general features with Miner's theory and investigating means of applying it to design problems.

For comparison of this theory with Miner's theory in terms of the general characteristics of cumulative-damage theories presented earlier, the normalized damage-cycle relationships postulated by this theory are shown in Fig. 18. The use of Miner's theory for the two stages of damage (Eqs. (13) and (14)) implies the curves shown in Figs. 19 and 20. The normalized damage-cycle relationships shown are identical for all stress amplitudes, in accordance with the stress-independent, interaction-free basis of Miner's theory as presented in the previous discussion of general cumulative-damage theory.

From that discussion it also follows that Grover's theory is stress-dependent, since the damage curves in Fig. 18 differ for different stress amplitudes. Thus, for the well-mixed stress spectrum (when values of the quantity a depend on stress amplitude) it can be concluded that Grover's theory will predict failure in fewer cycles than will Miner's theory. Consequently, for design requirements specifying a load spectrum and lifetime, more material would be required in a design based on Grover's theory than in one based on Miner's theory. (This conclusion is proved algebraically in Appendix B.)

Differences in life are illustrated in Fig. 21. This figure shows the ratio of life predicted by Grover's theory to that predicted by Miner's theory for a well-mixed two-level stress spectrum. Curves are shown for three combinations of $a_1$ and $a_2$, the fractional first-stage life for the two stress levels. The life ratio is seen to be a function of the relative amount of cycles of the two stress levels in the spectrum, $n_1/n_2$. (The life ratio approaches one for large values of the abscissa.)

While there is little point at present in proceeding to design considerations, since there are insufficient data to provide a meaningful determination of the value of the parameter a and how it varies with stress amplitude, this theory has interesting possibilities. These follow from the earlier discussion of the meaning of damage and possible measurements of damage. The current efforts to investigate fatigue damage at the microscopic level and at very early stages suggest that there are differences in the nature of damage at very early and later stages. For example, the delineation between the early and later stages might be the transition from primary

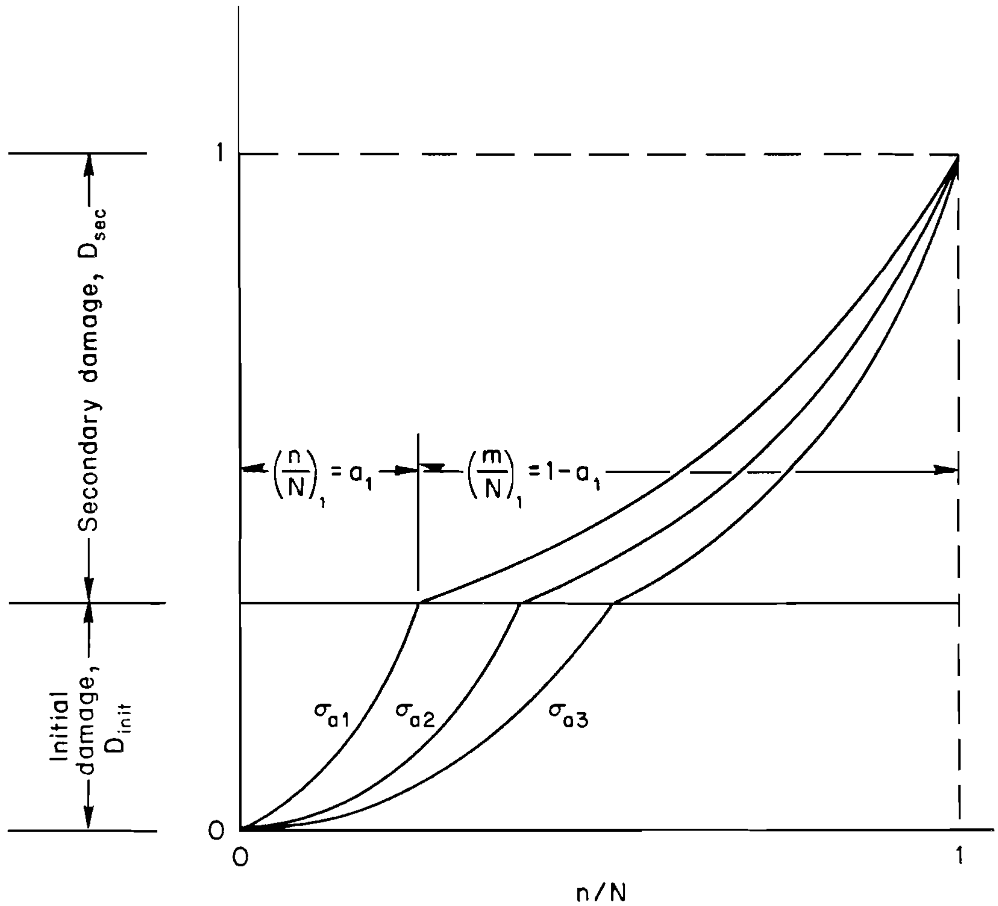



Fig. 18—Form of damage-cycle relationship (Grover)

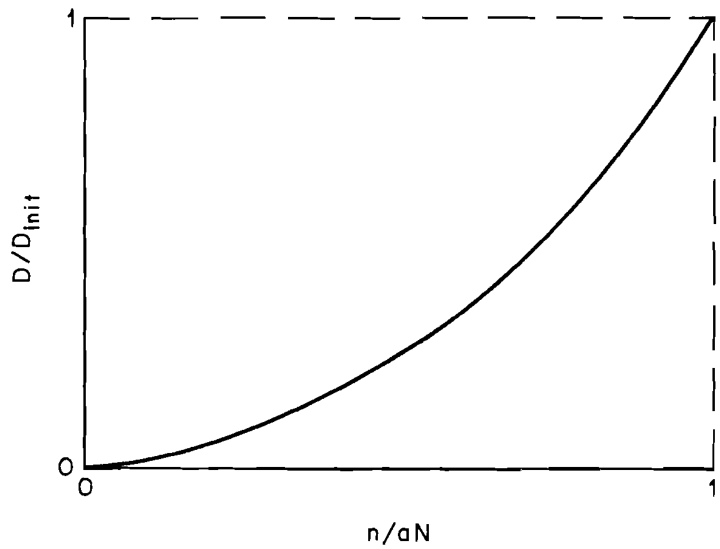



Fig. 19 — Damage-cycle relationship in initial-damage stage for all stress amplitudes (Grover)

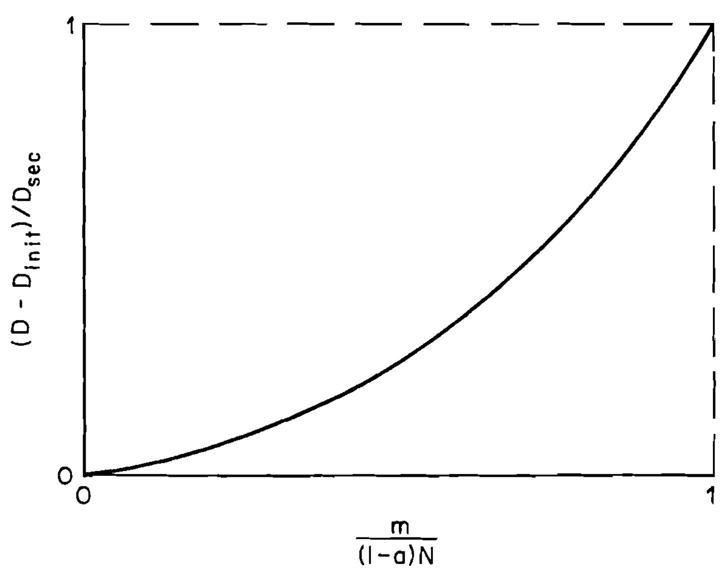



Fig. 20 — Damage-cycle relationship in secondary- or final-damage stage for all stress amplitudes (Grover)

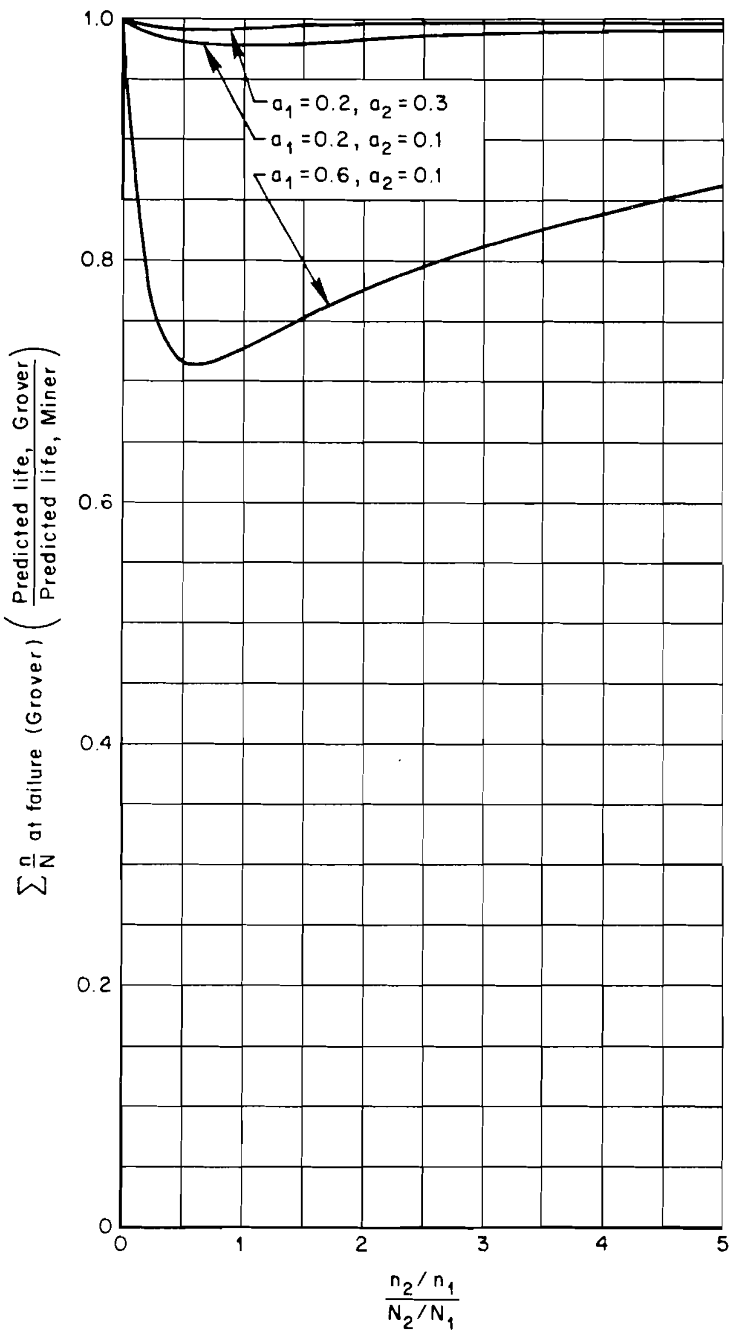



Fig. 21—Comparison of life-prediction data of Grover and Miner for selected initial-damage fractions

intragranular damage (characterized by progressive unbonding, intrusions, extrusions, fissures, etc., within grains) to intergranular crack formation and propagation on a grosser, more familiar scale--the joining together of the damaged portions that have occurred in individual grains to form cracks that traverse more and more grains as cycling proceeds. Work in this area is not yet adequate to discern relationships between stress amplitude and initial damage or transition to the second stage, but this certainly appears a possibility. It would be of extreme interest to investigate cumulative damage under spectrum loading at this initial microscopic-damage level.

Use of Grover's theory in design would of course require knowledge of the value of the parameter a as a function of stress amplitude. An example demonstrating the procedure, using a hypothetical relationship for the parameter a, is shown in Fig. 22. The assumed value of a as a function of stress amplitude is shown at the bottom of the figure. The upper curve is the life-expenditure-rate curve discussed in the previous section on Miner's theory. (The stress spectrum was determined for stress cycles from turbulence during 100 flying hours, and an experimental S - N curve for an aluminum alloy was used.) The Miner life for this particular case is

$$\text{Miner life} = \frac{100}{0.1744} = 573 \text{ hr}$$

The next curve shows the life-expenditure-rate for the initial-damage stage, using values for the parameter a as shown. The life at which the initial-damage stage is complete is thus

$$\text{Life during initial-damage stage} = \frac{100}{1.0652} = 94 \text{ hr}$$

The next curve shows the life-expenditure-rate for the second (final) stage of damage. The time in this case is

$$\text{Life during final-damage stage} = \frac{100}{0.2129} = 470 \text{ hr}$$

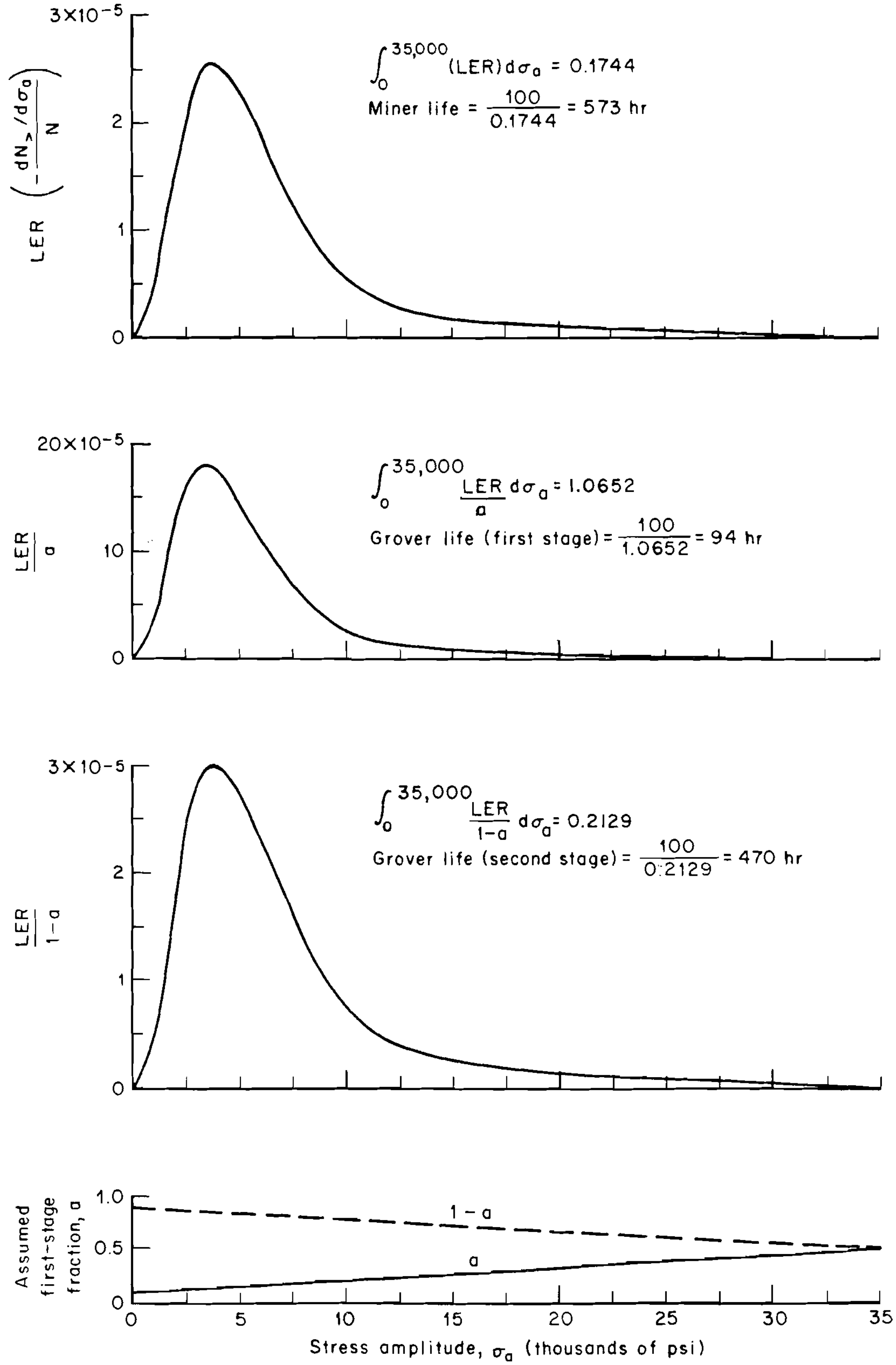



Fig. 22 — Use of Grover's theory in design calculations

Thus the total life from Grover's theory (with this purely arbitrary assumption for the value of a) is 564 hr, or only about 1-1/2 per cent less than the Miner life. The difference could be far more pronounced, of course, for different assumptions about a, but as noted above there is no way at present to establish the proper magnitude of this parameter. It can be seen, however, that use of this theory in design would be quite straightforward. As shown in this discussion, use of Grover's theory, for a specified load spectrum and life requirement will always result in a design requirement for more material than would be required if Miner's theory were used.

## X. CORTEN-DOLAN THEORY

As originally presented[9] the general form of this theory includes both stress-dependence and interaction effects. Subsequent experimental work[10,11] in connection with the theory led to formulation of a stress-independent, interaction theory.

In the original presentation, the expression for damage as a function of applied cycles (for stress amplitude $\sigma_{ai}$) was given by

$$D_i = \bar{m}_i \, r_i \, n_i^{\bar{a}_i} \tag{15}$$

In this expression, $\bar{m}_i$ is some number of "damage nuclei," $r_i$ is a crack-propagation constant, and $\bar{a}_i$ is a constant. These quantities are assumed constant for a specified stress amplitude (in a single-level test) but may be different for different stress amplitudes. Damage at failure is prescribed as unity (or 100 per cent), giving

$$D_{fi} = \bar{m}_i \, r_i \, N_i^{\bar{a}_i} = 1 \tag{16}$$

By dividing $\bar{m}_i \, r_i \, N_i^{\bar{a}_i} = 1$ into Eq. (15), the following expression for damage is obtained:

$$D_i = \left(\frac{n}{N}\right)_i^{\bar{a}_i} \tag{17}$$

Figure 23 shows the appearance of damage curves from this relationship. If the value of $\bar{a}_i$ depends on stress amplitude, a stress-dependent theory is proposed, while $\bar{a}_i$ independent of stress amplitude leads to a stress-independent theory and a single curve in Fig. 23 for all stress amplitudes. It was concluded that $\bar{a}_i$ was independent of stress for the materials and stress spectra of the experimental work mentioned above.

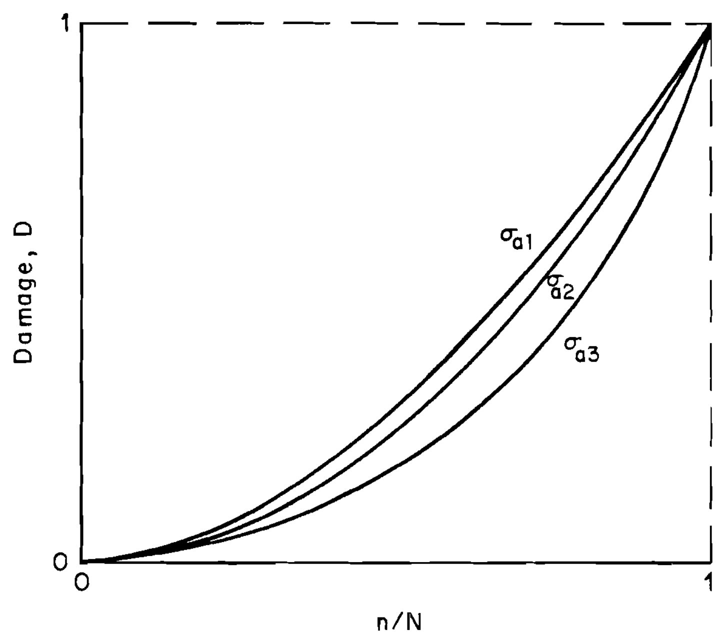



Fig. 23 —Theoretical damage-cycle relationship (exponent varying with stress amplitude; constant-amplitude tests) (Corten-Dolan theory)

Interaction effects are introduced into this theory, leading to the possibility of predictions that differ from Miner's theory. The possibility of interaction follows from the assumption that the number of damage nuclei produced by the highest stress in a spectrum will affect the growth of damage at other, lower stress amplitudes. The original assumption was that damage growth at lower stress amplitudes would be faster after a higher stress amplitude had been applied than when the lower amplitude was applied alone. In other words, $\bar{m}$ for the higher stress was considered larger than $\bar{m}$ for the lower stress. (Subsequent experiments indicated that, from the point of view of this theory, the damage growth rate at the lower stress was actually reduced in some cases when two stress levels were applied together.) With the interaction effect included, the damage relationship becomes

$$D_i = \frac{\bar{m}_1}{\bar{m}_i} \left(\frac{n}{N}\right)_i^{\bar{a}_i} \qquad (18)$$

In this equation, $\bar{m}_1$ is the damage-nuclei number for the highest stress in the spectrum (assumed to be applied early in the test), and $\bar{m}_i$ is the damage-nuclei number for the $i^{th}$ (lower) stress amplitude.

By defining a new parameter as

$$M_i = \left(\frac{\bar{m}_i}{\bar{m}_1}\right)^{1/\bar{a}_i} \qquad (19)$$

the damage relationship of Eq. (15) can be written as

$$D_i = \left(\frac{n_i}{M_i N_i}\right)^{\bar{a}_i} = \left(\frac{n_i}{N_{I_i}}\right)^{\bar{a}_i} \qquad (20)$$

In this equation, $N_{I_i}$, which is equal to $M_i N_i$, is a revised (because of interaction) number of cycles to failure at the $i^{th}$ amplitude which should be used in the damage equation when the $i^{th}$ amplitude is mixed together with a higher stress amplitude. The experimental work[10,11] in connection with this theory can be considered to be an investigation of the validity of the above concepts.

The experimental work is, in fact, directed to investigation of a different parameter, but the parameter M used in Eq. (20) can be derived from it. The original derivation of this theory[9] considers the possibility of an initial nucleation period, but this is later considered to be inconsequential. The original theory also considers stress-dependence (different values of the exponent in Eq. (15) for different stress amplitudes) but, as mentioned, in later work the exponent is considered to be constant, or a stress-independent theory is assumed.

(In a discussion of power-law damage rates in Ref. 10, it is shown that if exponents are identical for different stress levels but not equal to one, and there is no interaction, the cumulative damage resulting from spectrum loading will be different than if evaluated by Miner's theory. This disagrees with the conclusions of this Memorandum, where it is shown that use of identical exponents would lead to a stress-independent, interaction-free theory which is equivalent to Miner's theory. The disagreement results from the use of incorrect values of cycle ratio for determination of damage increments in Ref. 10. The derivation there is correct when all exponents equal one because the incorrect terms all become equal to unity and do not affect the result.)

The parameter investigated experimentally is denoted as $\bar{R}^{1/\bar{a}}$, where $\bar{R}$ is the ratio of crack-propagation constants for two different stress amplitudes. The initial experimental work consisted of spectrum loading with two stress levels in the spectrum. In the following, the higher stress amplitude will be called $\sigma_{a_1}$, and the cycles to failure at this amplitude alone, $N_1$. The lower stress amplitude and the cycles to failure when applied alone are $\sigma_{a_2}$ and $N_2$. The fictitious number of cycles to failure at the lower stress (when interaction effects are included) is $N_{I_2} = MN_2$, as in Eq. (20). Using the same exponent for the two stress levels and equating damage at failure from Eq. (16) for the two stress levels

$$\bar{m}_1 \; r_1 \; N_1^{\bar{a}} = \bar{m}_2 \; r_2 \; N_2^{\bar{a}}$$

$$\frac{N_1}{N_2} = \left(\frac{\bar{m}_2}{\bar{m}_1}\right)^{1/\bar{a}} \left(\frac{r_2}{r_1}\right)^{1/\bar{a}}$$

$$\frac{N_1}{N_2} = M \; \bar{R}^{1/\bar{a}}$$

$$\bar{R}^{1/\bar{a}} = \frac{N_1}{MN_2} = \frac{N_1}{N_{I_2}} \tag{21}$$

The experimental work was done to investigate the relationship of the parameter $\bar{R}^{1/\bar{a}}$ to the magnitude and the proportion of the two stress levels in a spectrum. The main conclusions were that $\bar{R}^{1/\bar{a}}$ was independent of the proportion of high and low stress cycles in the spectrum for specified stress levels and that the magnitude of $\bar{R}^{1/\bar{a}}$ could be adequately represented by a power function of the ratio of the high to the low stress amplitude. (Tests were made in reversed bending with zero mean stress. The above conclusions apply to the materials tested, 2024-T4 and 7075-T6 aluminum-alloy wire and two kinds of steel wire.)

As a consequence of the relationship between $\bar{R}^{1/\bar{a}}$ and the stress amplitudes (and the assumption of $\bar{R}^{1/\bar{a}}$ independent of the proportion of cycles), a fictitious S - N curve for use with spectrum loading can be constructed. On a plot of log $\sigma_a$ versus log N, this curve is a straight line passing through $N_1$ (the experimentally determined cycles to failure at the highest stress in the spectrum) and through a point $N_{I_2}$ at a lower stress amplitude. The value for $N_{I_2}$ is determined from two-level spectrum tests containing the two stress amplitudes $\sigma_{a_1}$ and $\sigma_{a_2}$.

With this theory, cumulative damage could then be assessed for any spectrum by using Miner's theory with the interaction S - N curve thus obtained. While this approach is attractive because of its relative simplicity and the generally good agreement obtained from its use with fairly complicated stress spectra, there are some reservations that must be disposed of by further experiments.

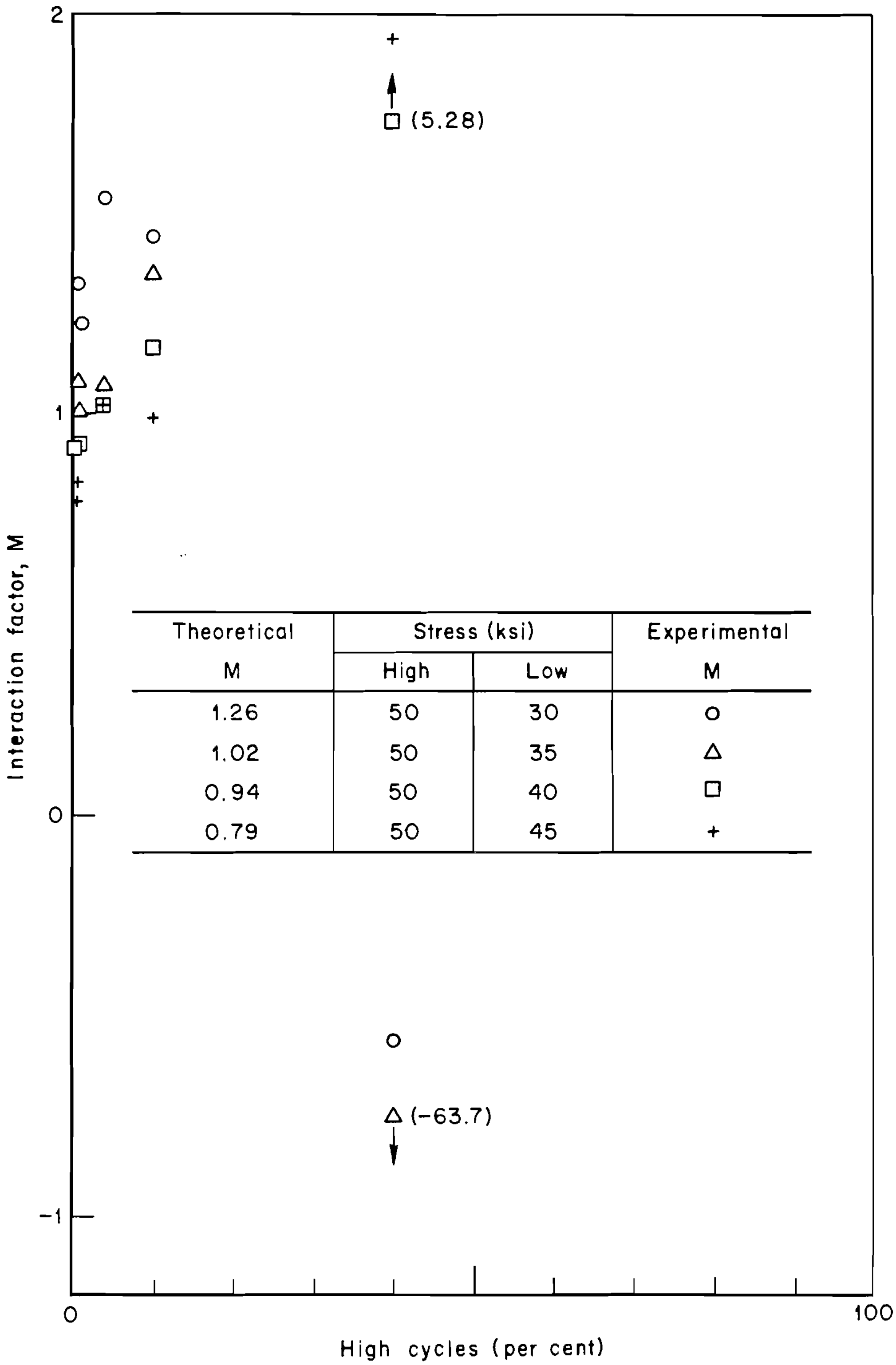
For one thing, similar experiments with specimens more closely related to aircraft structures and with a range of mean stress are necessary to evaluate effects of stress raisers and mean stress.

The effect of the proportion of cycles may also require closer study. An alternative way of looking at the results of experiments shows this. Values of $\bar{R}^{1/\bar{a}}$ determined from experiments are given for 2024 aluminum for several combinations of high and low stress and several different proportions of high stress cycles in the spectrum.[10] By dividing $\bar{R}^{1/\bar{a}}$ into $N_1/N_2$, from Eq. (21)

$$\frac{N_1/N_2}{\bar{R}^{1/\bar{a}}} = M \qquad (22)$$

where M is the number by which N is multiplied to give the value of $N_I$, the fictitious N, incorporating interaction effects, to be used for cumulative damage. In Fig. 24 experimental values of M are shown for the various stress levels and cycle proportions, together with the derived theoretical value. It is not clear from this figure that cycle proportions can be neglected, although it is stated that experimental values of 40 per cent of high cycles are uncertain. Negative values of M indicate that a negative value of $N_{I_2}$ should be used in the summation of cycle ratios. (This means that more high stress cycles can be applied if some lower stress cycles are mixed with them than can be applied at the high stress alone.)

Another reason for further evaluation of this theory is the results of work to be discussed in the next section. The theory in this case also leads to a fictitious interaction S - N curve, but from the same general type of test on the same material (2024 aluminum) the interaction curve is found to lie considerably to the left of the basic S - N curve. In Figs. 25 and 26, basic S - N curves for 2024 and 7075 aluminum-alloy wire obtained during investigation[11] of the present theory are shown. The interaction curves determined by the




| Theoretical M | Stress (ksi) High | Stress (ksi) Low | Experimental M |
|---|---|---|---|
| 1.26 | 50 | 30 | ○ |
| 1.02 | 50 | 35 | △ |
| 0.94 | 50 | 40 | □ |
| 0.79 | 50 | 45 | + |

Fig. 24 — Theoretical and experimental interaction factors; two-level tests, 2024 aluminum (Liu and Corten[10])

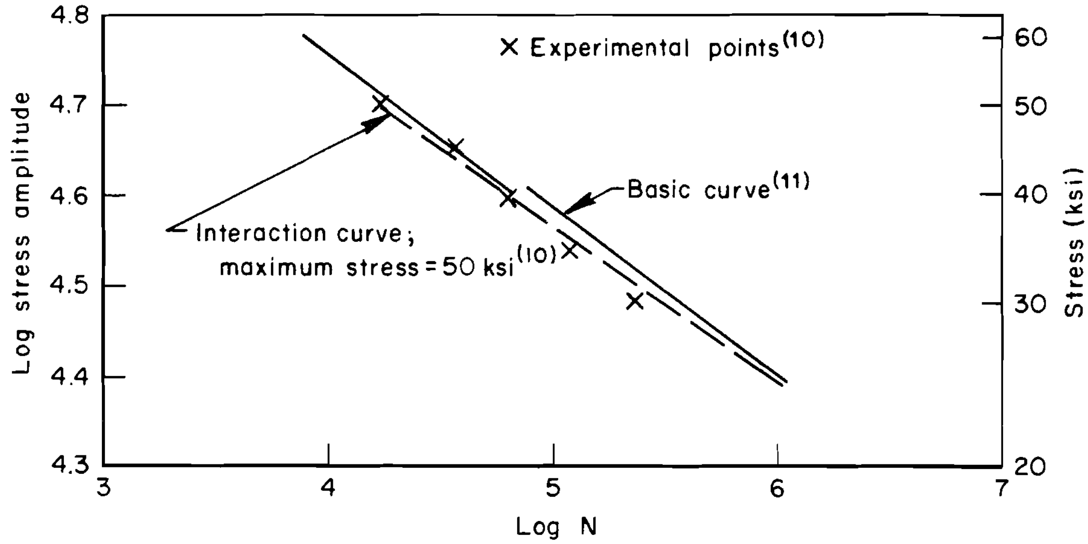



Fig. 25—Basic and interaction S-N curves, 2024 aluminum (mean stress = 0) (Liu and Corten)

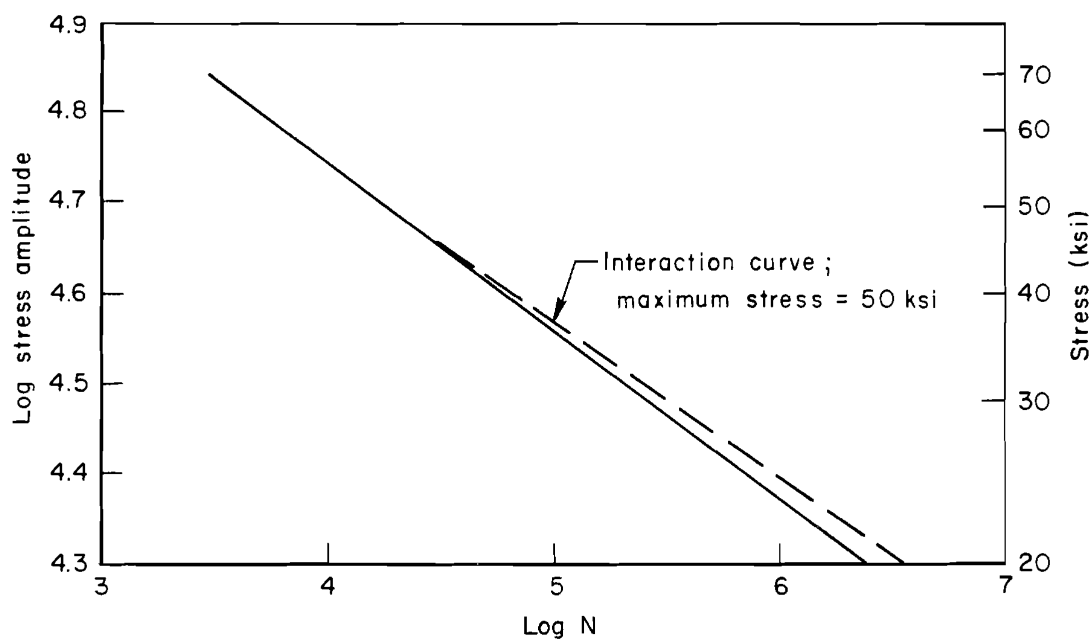



Fig. 26—Basic and interaction S-N curves, 7075 aluminum (mean stress = 0) (Liu and Corten[11])

investigation have been added. It can be seen that interaction effects are very slight. In terms of material required in designing for a specified life, it would make little difference whether Miner's theory were used with the basic S - N curves or with the interaction curves, for the materials and stress spectra considered in these experiments.

## XI. FREUDENTHAL-HELLER THEORY

In Ref. 12, Freudenthal and Heller present an interaction type of cumulative-damage theory. The first premise of their theory is that a fictitious S - N curve can be found such that Miner's theory, used with this curve, is valid for determining cumulative fatigue damage under spectrum loading. Secondly, interaction effects are such that the interaction S - N curve has an equation of the form

$$N_I = \frac{K}{\sigma_a^{\;x}} \qquad (23)$$

The interaction S - N curve is thus a straight line on a logarithmic plot of stress amplitude versus fictitious (with interaction) cycles to failure. It is also assumed that this interaction is effective only below a specified, fairly high stress level.

In connection with the above, it should be pointed out that if one is given the description of a stress spectrum and also the total number of cycles to failure when that spectrum is applied, there exists an infinite number of S - N curves, any one of which, used with Miner's theory, would predict failure exactly for the specified spectrum. Generation of these curves would, of course, be a useless pursuit. The merit of the work discussed in this section lies in the attempt to relate distinguishing characteristics of the applied stress spectrum to, in this case, the exponent x in Eq. (23). (Here K is a constant determined by the condition that the interaction curve passes through the point on the basic S - N curve corresponding to the upper stress limit on interaction.)

In a very good program, 220 spectrum tests were made on smooth 2024 aluminum-alloy test specimens in reversed bending. (Tests are also reported for 4340 steel, and previous tests were made to establish the basic S - N curves.) The stress spectra contained from four to six amplitudes, with the relative number of cycles at various amplitudes corresponding to distributions representative of gust-produced stresses in aircraft. The sequence of applications of the various stress levels was randomized by using random-number tables to prepare the load programming tape.

The results and analysis of these tests are quite interesting. Of the eleven test series, the results of all but one (with results noted as unreliable) were found to fit a fairly simple relationship for the value of the exponent in Eq. (23), involving a parameter related to the shape of the applied stress spectrum and also the basic S - N curve. The meaning of the spectrum shape parameter is illustrated in Fig. 27. This figure shows the relationship between the number of applied cycles and stress amplitude used in this program to establish the test stress spectra. The curves show the assumed number of stress cycles of amplitude greater than $\sigma_a$ that result from turbulence in flying a million miles (or from 2000 to 5000 flying hours for speeds of 500 mph down to 200 mph). The parameter h in these curves, for the tests on aluminum, is given by

$$h = \frac{1,032,000}{\sigma_a(1)} \tag{24}$$

where $\sigma_a(1)$ is the stress amplitude (in pounds per square inch) expected to occur on the average of once per million miles.

The values of stress amplitude selected for the test program are also shown in Fig. 27. The effect of different values of h on the test spectra is that for the steepest curve, the number of cycles applied at any of the test amplitudes is about 3 per cent of the number applied at the next lower amplitude. For the middle curve this figure is about 10 per cent, and it is about 18 per cent for the curve with h = 17.3. The other part of the parameter involved in the fictitious-exponent relationship is the predicted number of cycles to failure determined by Miner's theory and the basic S - N curve. The shape of the applied stress spectrum enters of course in this determination.

As stated in Ref. 12, the conclusions apply only to smooth specimens, and quite different results have been observed in speci-

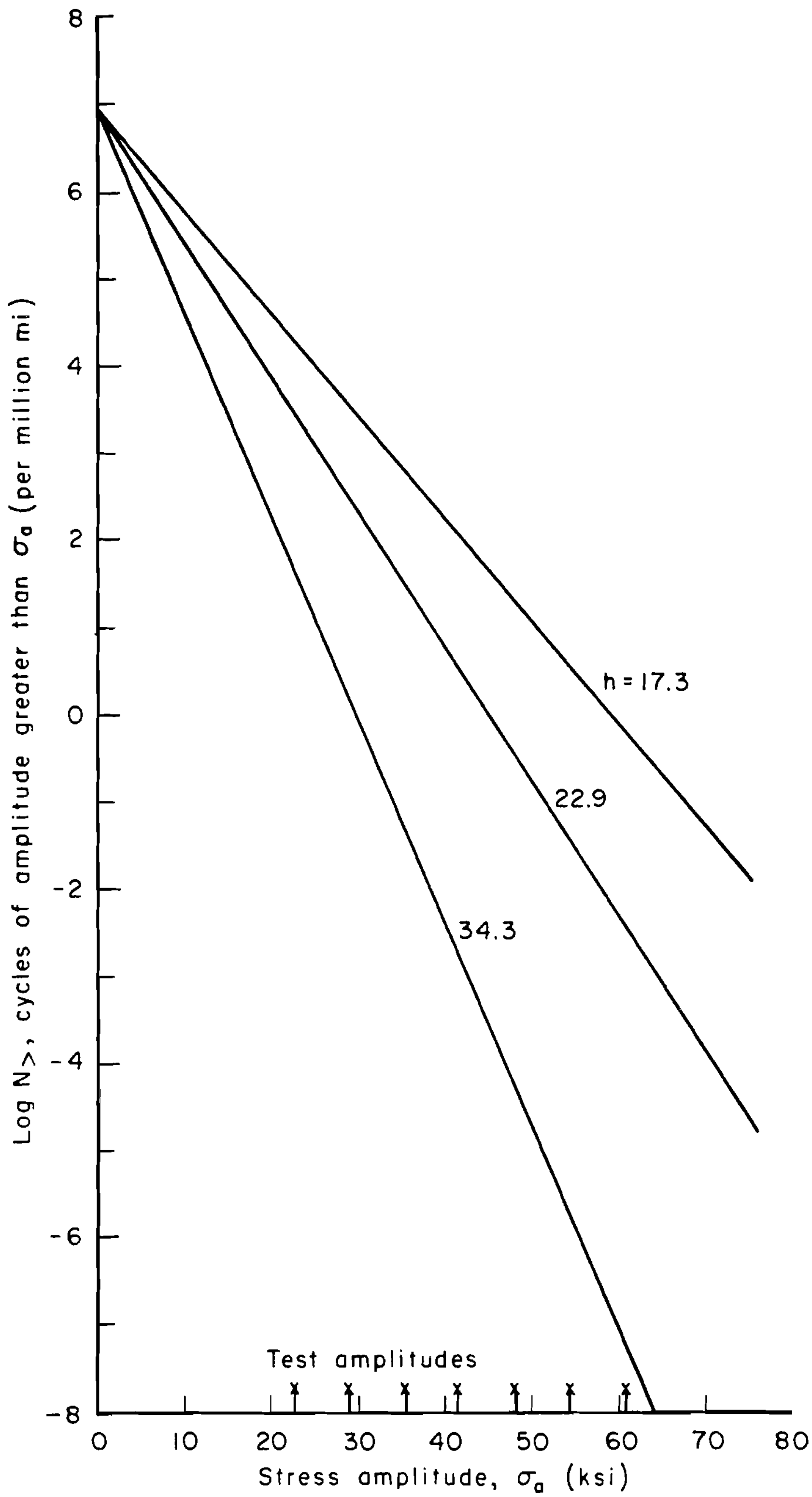



Fig. 27 — Stress-amplitude spectra used in Freudenthal-Heller tests;[12] 2024 aluminum, ultimate tensile stress = 64 ksi

mens more representative of aircraft structures. Further, the effect of nonzero mean stresses cannot be predicted from work done thus far. Finally, some observations can be made concerning disparities between this work and the work discussed in the preceding section. From Fig. 27 and test data it appears that the present experiments tend to emphasize the high end of the stress spectrum. Of the seven test amplitudes shown, those chosen for the tests were the lowest 6, the highest 6, the highest 5, and the highest 4. The values determined for the fictitious exponent are seen to decrease (i.e., the fictitious S - N curve becomes steeper) in the same order as this description of tests (aside from the series noted as unreliable). In other words, interaction effects become more pronounced when the higher stress amplitudes are selected for the test spectra. For the series that includes the lowest test stress amplitude (for each value of h) the values determined for the fictitious-exponent approach the value of the exponent of the first portion of the basic S - N curve. This can be illustrated by interpreting the test results discussed in this section in terms of the quantity M introduced in the previous section (M is the quantity by which the number of cycles to failure in a single-level test is multiplied to give the fictitious number of cycles to failure for including interaction effects in spectrum loading.) In the previous theory, interaction was assumed to be effective only for stress amplitudes below the highest in the spectrum, while in the present theory, interaction is assumed to occur at all stresses below a specified magnitude (95 per cent of ultimate tensile stress for the tests on aluminum).

The Freudenthal-Heller theory proposes that the value of M can be expressed as a power function of stress amplitude. Figure 28 shows the interaction factor, M, as a function of stress amplitude as determined from the experimental investigation of interaction. Curves are shown for the five test series with the lowest stress amplitudes. The curves are shown dashed below the lowest stress amplitude in the stress spectrum. (The discontinuity of slope in these curves results from the form of the basic S - N curve, which is composed of two linear segments on a log-log plot.) Also shown in Fig. 28 is the

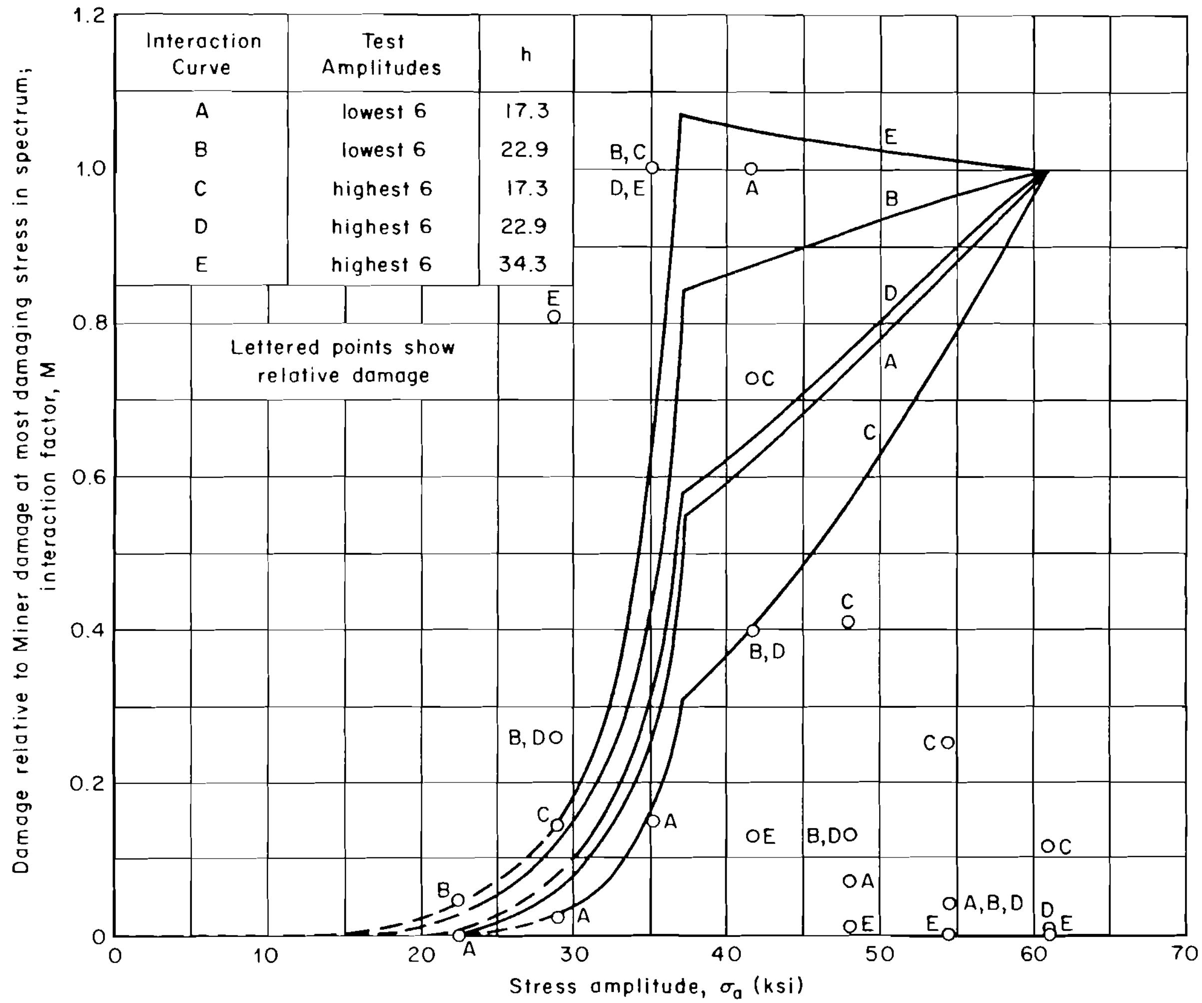



Fig. 28—Relative Miner damage at stress amplitudes in spectrum and theoretical interaction factor; lowest-stress-amplitude tests, 2024 aluminum (Freudenthal-Heller[12])

relative damage caused by the various stress amplitudes in the test spectra for the same five test series, as determined by Miner's theory and the basic S - N curve. Figure 28 thus shows a tendency for interaction effects to become more pronounced as the bulk of Miner damage shifts to the higher stress amplitudes.

In the work[11] discussed in the previous section, spectrum tests tended to include more of the lower stresses. Interaction curves that appear to be quite satisfactory for cumulative-damage calculations are, in this case, quite close to the basic S - N curves (Figs. 25 and 26). It may well be that the disparity results from the difference in test spectra and that this effect is adequately included in the relationship presented for determining the fictitious exponent, although this possibility warrants further investigation. At any rate, the work discussed in this and the preceding section appears to provide useful concepts for cumulative-damage evaluation under conditions approximating those of the tests.

A final comment concerns the design implications of these two theories. They are not applicable at present to fatigue design of an aircraft structural joint, particularly with non-zero mean stress. However, investigation of their possible use in design is instructive as a source of suggestions for further research and also for the possibility that similar concepts may later be found valid for more complex structures and loading.

In general terms, the problem facing the designer is that of designing for a spectrum of load cycles. He must design the structure (choose the proper amount and distribution of material) so that the stress spectrum that results from the load spectrum will not result in fatigue failure. (In some cases, the load spectrum may be changed slightly by changes in amount of material, if loads are affected by structural stiffness.) In practice, the amount of material is initially determined by a static strength requirement and the fatigue life of this design is determined. If the life is inadequate, reduction of stress levels by adding material (or changing material or design details) must be investigated and a configuration found that is satisfactory.

With the theory discussed here, the value of h (from Eq. (24)) and the basic fatigue life would be determined first (for the statically adequate design), and these in turn would permit determination of the fatigue life with interaction effects included. If this life were inadequate, additional material would be required. This would increase h and also the basic life, and both changes would tend to make interaction effects less severe. Repetition of this procedure would finally lead to a relationship between amount of material and life, from which the required amount of material for a specified life could be obtained. The important consideration in this process from the design viewpoint is the effect on fatigue of changes in the applied stress spectrum as changes in design are considered.

Further experimental investigation of these interesting theories is necessary to validate their application to design problems. In particular, tests with constant mean stress and also changing mean stress, stress raisers, and larger number of test amplitudes covering the spectrum more completely would be required.

## XII. SHANLEY'S THEORY

In a comprehensive discussion of many facets of fatigue, Shanley proposed a mechanism for the generation of fatigue cracks, based on the progressive unbonding of atoms as a result of reversed slip produced by cyclic stresses.[(13)] He also presents interesting possibilities for relating this mechanism to many fatigue phenomena. It is interesting to note that recent microscopic observations utilizing taper-sectioning techniques have disclosed "fissures" produced by cyclic stressing which are very similar to the physical picture of the result of progressive unbonding.* Also of interest are results of fatigue tests at extremely high vacuum. In a discussion of recent progress in ultrahigh vacuum[(14)] (in which so few gas molecules remain that the probability of one of them striking a freshly unbonded atom is very small), it is stated that ultrahigh vacuum results in tremendous increases in fatigue life, and it is suggested that cracks opened in tension may reweld in compression.

Shanley also presents two cumulative-damage theories. In the first, the expression proposed for crack growth is

$$L_i = L_{in} \exp \left(C \, \sigma_{ai}^{x} \, n_i\right) \qquad (25)$$

where $L_i$ is crack depth for the $i^{th}$ stress amplitude; $L_{in}$, an initial crack depth (produced by the first cycle) that is constant for all stress amplitudes; C, a constant; $\sigma_{ai}$, the magnitude of the $i^{th}$ stress amplitude; x, a constant; and $n_i$, the number of applied stress cycles.

Failure is defined as the attainment of a critical crack depth, $L_{cr}$, the same for all stress amplitudes, when the number of cycles reaches the number of cycles to failure, $N_i$, so that

$$L_{cr} = L_{in} \exp \left(C \, \sigma_{ai}^{x} \, N_i\right) \qquad (26)$$

---

*See "Some Basic Studies of Fatigue in Metals," by W. A. Wood in Ref. 1.

It can be seen from this equation that the equation of the S - N curve is established by this approach:

$$N = \frac{\text{constant}}{\sigma_a^x} = \frac{K}{\sigma_a^x} \qquad (27)$$

It is well known that many experimental S - N relationships can be adequately represented by this expression.

A general observation about cumulative-damage theories should be noted here concerning a point of difference in theoretical approaches. That is, a theory may or may not include as one of its features the prediction of the shape of the S - N curve. Theories that contain S - N relationship prediction as well as means for determining cumulative damage might be termed fundamental fatigue theories, while theories that are addressed only to cumulative damage, accepting the experimental S - N data, could be called simply cumulative-damage theories. Miner's theory falls in the latter category, as do several others, but it should also be noted that even the fundamental theories at present rely on experimental S - N curves for determination of constants (generally two or more) and in this respect reflect the present lack of truly fundamental understanding of fatigue.

Returning to the damage relationships, if damage is defined as $L/L_{cr}$, then from Eqs. (25) and (26)

$$D = \exp\,(C\,\sigma_{ai}^{x}\,n_i - C\,\sigma_{ai}^{x}\,N_i)$$

$$= \exp\,[C\,\sigma_{ai}^{x}\,N_i\,(\frac{n_i}{N_i} - 1)]$$

$$= \exp\,[C\,K\,(\frac{N_i}{n_i} - 1)] \qquad (28)$$

As Shanley pointed out, this first theory is equivalent to Miner's theory. This is also shown by the form of Eq. (28), which shows this theory to be stress-independent. It is stated to be interaction-free and is therefore equivalent to Miner's theory.

Shanley also proposed a modification of the first theory by assuming that the initial crack length is not a constant but depends on the magnitude of the stress amplitude. (The initial crack is presumed to be caused by the first cycle, and n is the number of cycles thereafter.) The crack-growth equation in this case is

$$L_i = A \sigma_{ai}^x \exp (C \sigma_{ai}^x n_i) \tag{29}$$

The initial crack depth is

$$L_{in\ i} = A \sigma_{ai}^x \tag{30}$$

and the critical crack depth is

$$L_{cr} = L_{in\ i} \exp (C \sigma_{ai}^x N_i) \tag{31}$$

Defining damage again as

$$D_i = L_i/L_{cr}$$

$$D_i = \exp (C \sigma_{ai}^x n_i - C \sigma_{ai}^x N_i)$$

$$= \exp [C \sigma_{ai}^x N_i (\frac{n_i}{N_i} - 1)] \tag{32}$$

With this theory the S - N relationship differs from that given by Eq. (27), and $\sigma_{ai}^x N_i$ is not a constant. Equation (32) thus indicates that the damage-cycle relationship for this theory is stress-dependent. The S - N relationship for this theory can be written as

$$N = \frac{K_1 (1 - K_2 \log \sigma_a)}{\sigma_a^x} \tag{33}$$

In this form it can be seen that this relationship can approach that given in Eq. (27), depending on the determination of constants. (The additional constant in Eq. (33) permits better fit of S - N data into the high stress region.)

If initial damage is defined as

$$D_{in\ i} = \frac{L_{in\ i}}{L_{cr}} = \exp\ (- C\ \sigma_{ai}^{x}\ N_i) \tag{34}$$

the damage relationship from Eq. (32) can be written as

$$D_i = \left(\frac{1}{D_{in\ i}}\right)^{\left(\frac{n_i}{N_i} - 1\right)}$$

$$= (D_{in\ i})^{\left[1 - \left(\frac{n}{N}\right)_i\right]} \tag{35}$$

The normalized damage-cycle relationship for this theory would then appear as shown in Fig. 29, with the higher stress amplitudes starting at higher initial damage.

When cumulative damage from a well-mixed spectrum is considered, it can be assumed that the first application of the highest stress in the spectrum (denoted here by $\sigma_{a1}$) will immediately produce damage equal to at least $D_{in\ 1}$. Lower stress cycles applied thereafter will result in damage progressing from that level. Thus, for the well-mixed spectrum the damage curves should be revised to show this effect. This can be done by adjusting the damage curves for the lower stress amplitudes to begin at $D_{in\ 1}$. The revised number of cycles to failure for the lower stresses is given by

$$\overline{N}_i = N_i - n_i^* \tag{36}$$

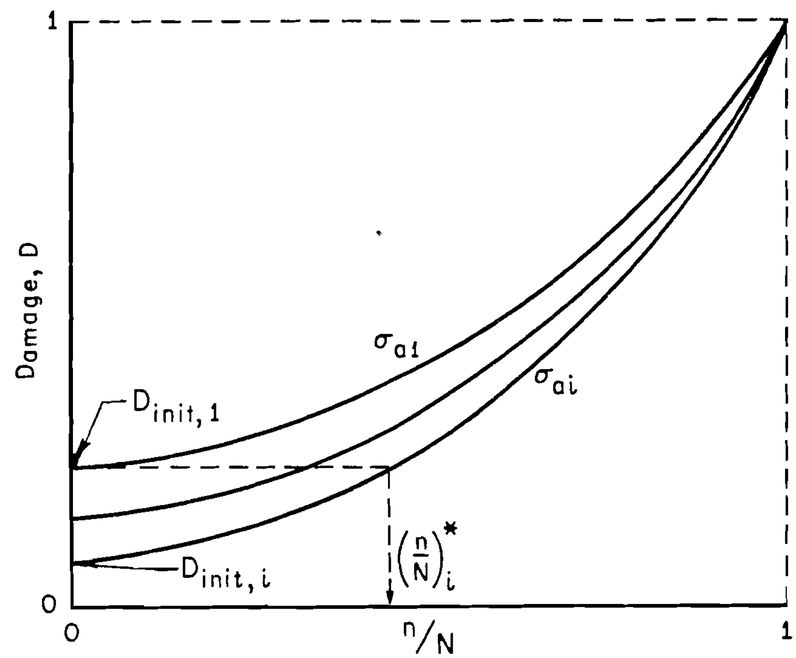



Fig. 29 — Damage-cycle relationship (Shanley's stress-dependent initial-damage theory[13])

where $n_i^*$ is the number of cycles required at the $i^{th}$ stress amplitude (when applied alone) to produce an amount of damage equal to the initial damage produced at the highest stress amplitude. This is given by

$$n_i^* = N_i \left(\frac{n}{N}\right)_i^* \qquad (37)$$

To find the value of $(n/N)_i^*$, damage at the $i^{th}$ amplitude (from Eq. (35)) is set equal to initial damage at the highest amplitude:

$$D_{in\ 1} = (D_{in\ i})^{\left[1 - \left(\frac{n}{N}\right)_i^*\right]} \qquad (38)$$

In the original damage curve, applied cycles were denoted by $n_i$. Now, for stress amplitudes other than the highest in the spectrum, cycles should be counted starting from $n_i^*$, giving a revised cycle count with $\bar{n}_i$ denoting the number of applied cycles

$$\bar{n}_i = n_i - n_i^* \tag{39}$$

The revised cycle ratio is then

$$\frac{\bar{n}_i}{\bar{N}_i} = \left(\frac{\bar{n}}{N}\right)_i = \frac{n_i - n_i^*}{N_i - n_i^*} = \frac{\left(\frac{n}{N}\right)_i - \left(\frac{n}{N}\right)_i^*}{1 - \left(\frac{n}{N}\right)_i^*} \tag{40}$$

The original cycle ratio in terms of the revised is thus

$$\left(\frac{n}{N}\right)_i = \left[1 - \left(\frac{n}{N}\right)_i^*\right] \left(\frac{\bar{n}}{N}\right)_i + \left(\frac{n}{N}\right)_i^* \tag{41}$$

Then, to transform the damage relationship to a function of the revised cycle ratio

$$D_i = (D_{in\ i})^{[1 - \left(\frac{n}{N}\right)_i]} = (D_{in\ i})^{\{1 - \left(\frac{n}{N}\right)_i^* - [1 - \left(\frac{n}{N}\right)_i^*] \left(\frac{\bar{n}}{N}\right)_i\}}$$

$$= (D_{in\ i})^{[1 - \left(\frac{n}{N}\right)_i^*][1 - \left(\frac{\bar{n}}{N}\right)_i]} \tag{42}$$

Using Eq. (38), this equation becomes

$$D_i = (D_{in\ 1})^{[1 - \left(\frac{\bar{n}}{N}\right)_i]} \tag{43}$$

Thus, if the damage curves are adjusted to begin at the initial damage corresponding to the highest stress in the spectrum, and the cycles to failure are adjusted accordingly, this theory can be viewed as equivalent to Miner's theory, but with a revised S - N curve. The quantity $1 - (\frac{n}{N})^*$ can be considered as an interaction factor which multiplies values of $N_i$ to give $\overline{N}_i$, the cycles to failure to be used with Miner's theory for cumulative damage under spectrum loading.

The value of this factor can be determined from Eq. (38)

$$1 - (\frac{n}{N})^*_i = \frac{\ell n\ (D_{in\ 1})}{\ell n\ (D_{in\ i})} \qquad (44)$$

From Eqs. (32) and (34)

$$D_{in\ i} = D_{in\ 1}\ (\frac{\sigma_{ai}}{\sigma_{a1}})^x \qquad (45)$$

The interaction factor can therefore be expressed as

$$1 - (\frac{n}{N})^*_1 = \frac{\ell n\ (D_{in\ 1})}{\ell n \left[ D_{in\ 1} \left( \frac{\sigma_{ai}}{\sigma_{a1}} \right)^x \right]} \qquad (46)$$

The value of initial damage at the highest stress amplitude can be determined from Eq. (34). The value of C is determined by constants in the basic S - N relationship, Eq. (33), together with Eq. (31).

This theory clearly predicts shorter lifetimes than does Miner's theory, but it is limited, strictly speaking, to the particular S - N relationship used in Eq. (33). As yet, there has not been experimental verification of variation of initial damage (caused by the first stress cycle) with stress amplitude, which is basic to this theory. (Note that the cumulative-damage relationships discussed above are not those of Shanley's original "2-x" method, which is discussed in the next section.)

# XIII. 2-x METHOD

In the original derivation of Shanley's 2-x method,[13] one of the steps implies the presence of interaction. In effect, it is assumed that at a specified damage level, under spectrum loading, high stress amplitudes cause greater damage and low stresses less than they would cause when applied alone at the same damage level, and initial-damage differences are neglected. Results are theoretically limited to the specific S - N relationship of Eq. (33).

In the following, a modified 2-x method will be considered, applicable to any S - N relationship. In other words, it will be reinterpreted as simply a cumulative-damage theory.

The original derivation leads to definition of a reduced stress amplitude:

$$\sigma_{ar} = \left[\frac{\Sigma\, dn_i\, \sigma_{ai}^{2x}}{\Sigma\, dn_i}\right]^{1/2x} \qquad (47)$$

where $\sigma_{ar}$ is the stress amplitude at which failure would result after the same number of cycles as the number of cycles in the spectrum that produces failure. The quantities $dn_i$ are the number of cycles of the $i^{th}$ amplitude at failure or, as they will be used later, the number of cycles of the $i^{th}$ amplitude applied in a specified period of time.

Now assuming that the S - N relationship of Eq. (27) can be used with Eq. (47),

$$\sigma_{ar}^{2x} = \frac{\Sigma\, dn_i\, \sigma_{ai}^{2x}}{\Sigma\, dn_i}$$

$$\frac{K^2}{N_r^2} = \frac{\Sigma \dfrac{K^2\, dn_i}{N_i^2}}{\Sigma\, dn_i}$$

$$\frac{\Sigma\, dn_i}{N_r^{\,2}} = \Sigma \frac{dn_i}{N_i^{\,2}}$$

$$\left(\frac{\Sigma\, dn_i}{N_r}\right)^2 = \Sigma\left(\frac{dn_i}{N_i} \cdot \frac{\Sigma\, dn_i}{N_i}\right)$$

$$\frac{\Sigma\, dn_i}{N_r} = \left[\Sigma\left(\frac{dn_i}{N_i} \cdot \frac{\Sigma\, dn_i}{N_i}\right)\right]^{1/2} \tag{48}$$

where $N_r$ is the total number of spectrum cycles that produces failure. If the quantities $dn_i$ are defined as the number of cycles of the $i^{th}$ amplitude applied in a specified length of time, the left-hand quantity in Eq. (48) is the fraction of fatigue life expended in that specified length of time. Denoting this as $LF_{2x}$, it can be compared with the comparable quantity from Miner's theory, $LF_M$:

$$LF_{2x} = \left[\Sigma\left(\frac{dn_i}{N_i} \cdot \frac{\Sigma\, dn_i}{N_i}\right)\right]^{1/2} \tag{49}$$

$$LF_M = \Sigma \frac{dn_i}{N_i} \tag{50}$$

In this form it can be seen that in the 2-x method, Miner fractions are multiplied by a weighting quantity, and the square root of the summation provides the life fraction.

It is fairly easy to show that the 2-x method in this form will always predict shorter life under spectrum loading than the Miner life, by proving that the life fraction expended during identical spectrum loading is larger as computed by the 2-x method, or

$$LF_{2x} > LF_M$$

Then

$$(LF_{2x})^2 > LF_M^{\,2}$$

$$\Sigma \left(\frac{dn_i}{N_i} \cdot \frac{\Sigma\, dn_i}{N_i}\right) > \left(\Sigma \frac{dn_i}{N_i}\right)^2$$

$$\Sigma \left(\frac{dn_i}{N_i}\right)^2 + \sum_{i=1}^{z-1} \sum_{j=i+1}^{z} dn_i\, dn_j \left(\frac{1}{N_i^{\,2}} + \frac{1}{N_j^{\,2}}\right) > \Sigma \left(\frac{dn_i}{N_i}\right)^2 + \sum_{i=1}^{z-1} \sum_{j=i+1}^{z} 2\, \frac{dn_i\, dn_j}{N_i N_j}$$

(z is the number of stress amplitudes in the spectrum.)

$$\frac{1}{N_i^{\,2}} + \frac{1}{N_j^{\,2}} > \frac{2}{N_i\, N_j}$$

$$\frac{N_j^{\,2} + N_i^{\,2}}{N_i N_j} > 2$$

$$N_j^{\,2} + N_i^{\,2} > 2\, N_i N_j$$

$$(N_j - N_i)^2 > 0$$

Therefore the 2-x method, as interpreted here, results in prediction of more rapid expenditure of fatigue life than does Miner's theory for the same spectrum, or the fatigue life predicted by the 2-x method is shorter. Consequently, use of the 2-x method in design will result in a requirement for a greater amount of material than would be required if Miner's theory were used.

There appears to be no simple way of interpreting this approach in terms of interaction effects, so it remains to be shown how this method can be applied to design. Equation (49) can be written as

$$LF_{2x} = \left[\Sigma\left(dn_i \frac{\Sigma\, dn_i}{N_i^2}\right)\right]^{1/2}$$

In the previous discussion of design computations with Miner's theory, the life-expenditure-rate curve in Fig. 11 was integrated to obtain the life fraction for the specified time, S - N curve, and spectrum. If the ordinates of this curve are divided by N from the S - N curve at all values of stress amplitude, the integral of the resulting curve between any two values of stress amplitude gives the quantity

$$\frac{\Sigma\, dn_i}{N_i^2} = \int_{\sigma_{a1}}^{\sigma_{a2}} - \frac{dN_{>}/d\sigma_a}{N^2}\, d\sigma_a \qquad (52)$$

for all of the applied cycles between the two stress amplitudes. The summation of applied cycles between the two stress amplitudes during the specified time is

$$\Sigma\, dn_i = \int_{\sigma_{a1}}^{\sigma_{a2}} - \frac{dN_{>}}{d\sigma_a}\, d\sigma_a \qquad (53)$$

The life fraction expended due to cycles between the two stress amplitudes is therefore

$$LF_{2x} = \left[ \int_{\sigma_{a1}}^{\sigma_{a2}} - \frac{dN_{>}}{d\sigma_a} \, d\sigma_a \int_{\sigma_{a1}}^{\sigma_{a2}} - \frac{dN_{>}/d\sigma_a}{N^2} \, d\sigma_a \right]^{1/2} \tag{54}$$

$$LF_{2x} = \left[ (N_{>\sigma_{a1}} - N_{>\sigma_{a2}}) \int_{\sigma_{a1}}^{\sigma_{a2}} - \frac{dN_{>}/d\sigma_a}{N^2} \, d\sigma_a \right]^{1/2} \tag{55}$$

The view taken here is that Eq. (49) can be applied regardless of the form of the S - N relationship, i.e., N can be determined either from a theoretical relationship or from experimental S - N data if available.

For use in design, procedure with this method would follow that outlined for Miner's theory.

## Appendix A

## EFFECT OF STRESS DEPENDENCE ON FATIGUE-LIFE PREDICTION

Several statements were made in Section VI about the relationship between key assumptions contained in a theory and predicted fatigue lives based on that theory.

The key assumptions concern stress-dependence--whether or not, at different stress amplitudes, equal fractions of life produce equal amounts of damage; and interaction--whether or not the course of damage at one stress amplitude is changed by mixing in other stress amplitudes.

It should be evident that interaction effects can be proposed that would result in prediction of either longer or shorter fatigue lives than those predicted by Miner's theory.

In the case of stress dependence, the outcome is not as evident. Following is a proof that stress-dependent theories predict shorter fatigue life than does Miner's theory. The proof applies to well-mixed application of the stress amplitudes in a spectrum, the condition of most interest in flight structures.

The first step is to plot the slope of the curves in Fig. 6 as a function of damage, D. (These curves are assumed to increase monotonically so that the slope is always positive.) This slope, $dD/d(n/N)$, denoted by R and called the damage rate, is shown in Fig. 30. The reciprocal of R is plotted, again as a function of damage, in Fig. 31. Considering any curve in Fig. 31, for one stress amplitude, the area under that curve must be equal to unity, or

$$\int_0^1 (1/R)_i \, dD = \int_0^1 \frac{d(\frac{n}{N})}{dD} \, dD = 1 \qquad \text{(A1)}$$

The next step is consideration of the effect of application of a mixture of cycles of various stress amplitudes. Consider a group of stress cycles that is large enough to contain representative proportions of the various stress-amplitude cycles applied, but small compared

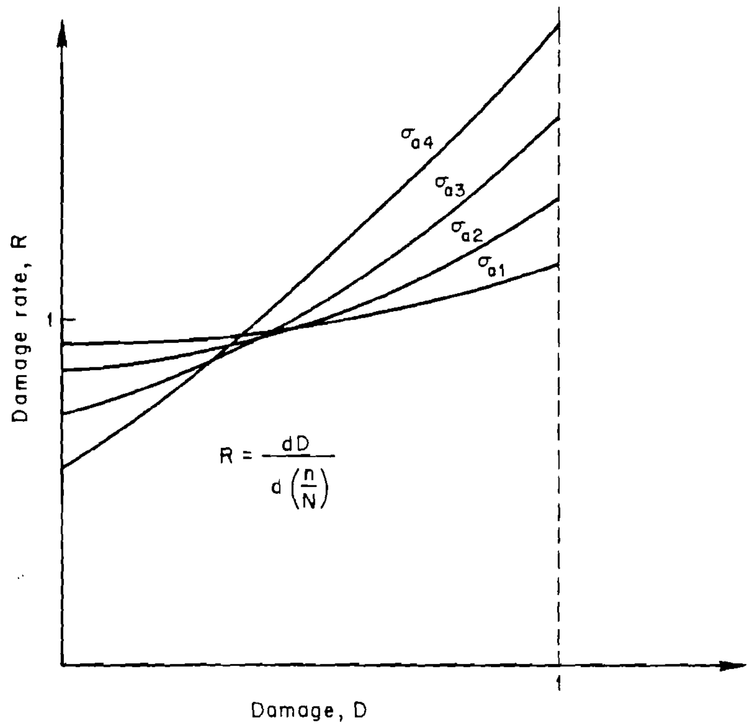



Fig. 30 — Damage rate versus damage curves (from damage–cycle curves of Fig. 6)

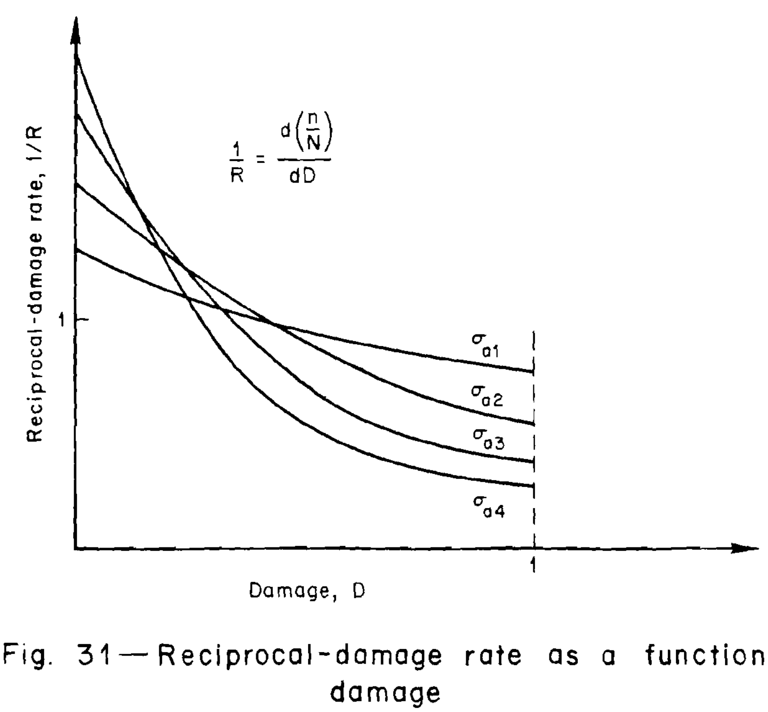



Fig. 31 — Reciprocal-damage rate as a function of damage

with the total number of cycles to be applied. The total (as yet unknown) number of cycles of the $i^{th}$ stress amplitude that can be applied before failure will be designated by $n_i$; $dn_i$ is the number of cycles of the $i^{th}$ amplitude in the representative group; $N_i$ is the number of cycles to failure when the $i^{th}$ amplitude is applied alone.

Two further definitions are required: $\alpha_i$ is equal to $dn_i/dn_1$, the ratio of the number of cycles of the $i^{th}$ amplitude in the representative group to the number of cycles of an arbitrarily selected amplitude (called the first amplitude) in the representative group. The assumption that the group is representative means that $\alpha_i$ is also equal to $n_i/n_1$. And $\beta_i$ is equal to $N_i/N_1$, the ratio of cycles to failure in a single-level test at the $i^{th}$ amplitude to cycles to failure in a single-level test at the designated first amplitude. The $\alpha_i$ are determined by the shape of the applied stress spectrum, and the $\beta_i$ by the S - N curve.

Now consider the situation after stress cycles have been applied until damage has reached some specified level. The aforesaid representative group of cycles is then applied. The damage produced by this group will be called dD. If the number of cycles in the group is small enough, dD will be independent of the order of application of the various amplitude cycles within the group. The situation is depicted graphically in Fig. 32, which shows a section of Fig. 31 with the horizontal scale greatly expanded. The total damage increment, dD, is made up of the sum of the damage increments due to each amplitude, or

$$dD = \Sigma dD_i \qquad (A2)$$

The incremental damage from the $i^{th}$ amplitude is given by

$$dD_i = \frac{dn_i}{N_i} \left( \frac{dD}{d(\frac{n}{N})} \right)_i = R_i \, d(\frac{n}{N})_i \qquad (A3)$$

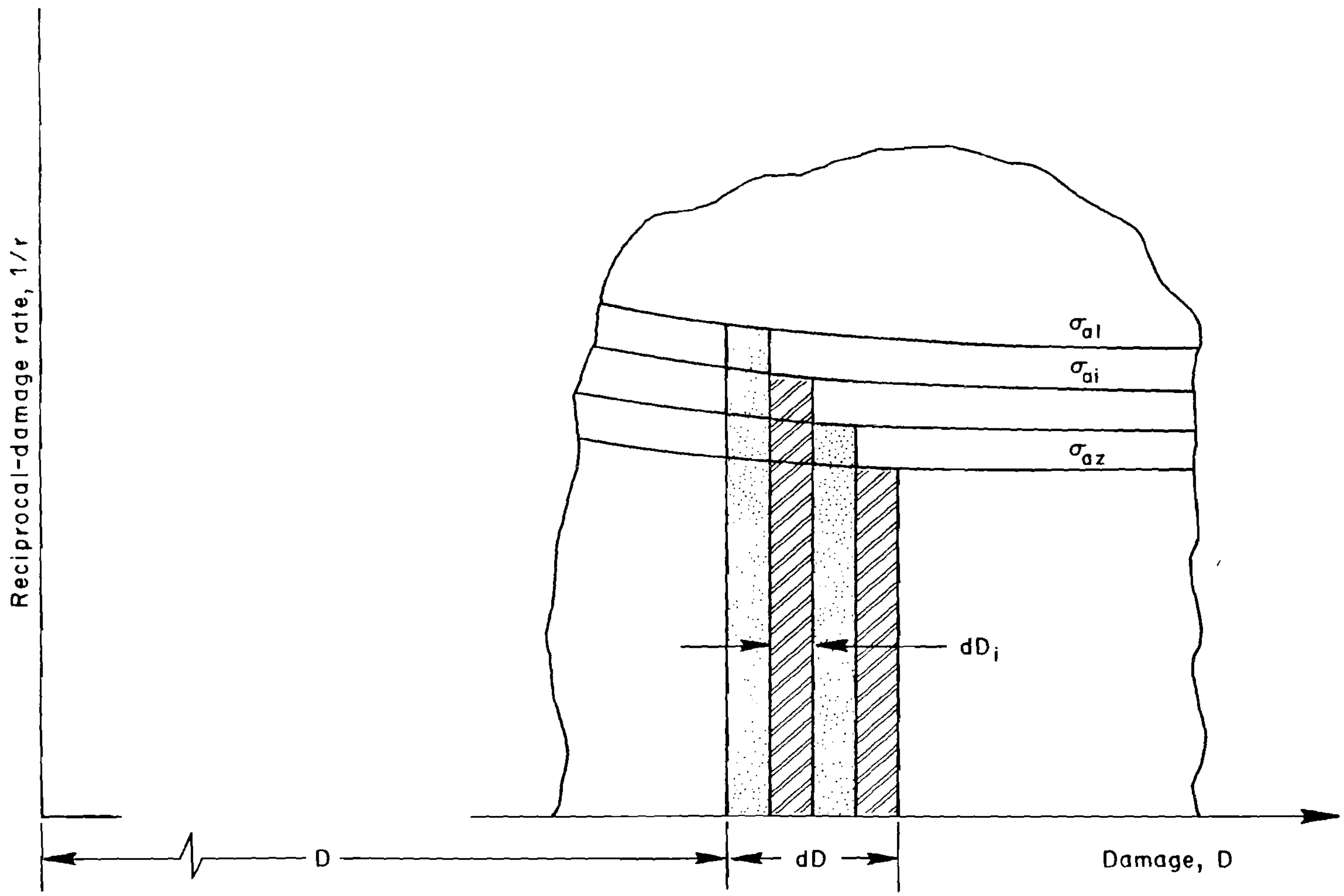



Fig. 32—Expanded section of Fig. 31, showing incremental damage produced by representative group of cycles within a spectrum of stress amplitudes

Then

$$d\left(\frac{n}{N}\right)_i = (1/R_i)\ dD_i \tag{A4}$$

and the hatched area associated with each 1/R curve in Fig. 32 is then equal to $d(\frac{n}{N})_i$. The value of $(\frac{n}{N})_i$ when damage becomes equal to 1 is then given by

$$\left(\frac{n}{N}\right)_i = \int_o^1 \frac{d(\frac{n}{N})_i}{dD}\ dD = \int_o^1 (1/R_i)\ \frac{dD_i}{dD}\ dD$$

$$= \int_o^1 (1/R_i)\ \frac{dD_i}{\Sigma dD_i}\ dD = \int_o^1 \frac{(1/R_i)\ dD}{(\Sigma dD_i)/dD_i}$$

$$= \int_o^1 \frac{(1/R_i)\ dD}{\frac{dD_1}{dD_i}\ \Sigma\ \frac{dD_i}{dD_1}} \tag{A5}$$

From Eq. (A3)

$$\frac{dD_i}{dD_1} = \frac{R_i(dn_i/N_i)}{R_1(dn_1/N_1)} = \frac{R_i}{R_1} \cdot \frac{dn_i}{dn_1} \cdot \frac{N_1}{N_i} = \frac{R_i}{R_1}\frac{\alpha_i}{\beta_i} \tag{A6}$$

Equation (A5) can then be written as

$$\left(\frac{n}{N}\right)_i = \int_o^1 \frac{(1/R_i)dD}{\frac{R_1}{R_i}\frac{\beta_i}{\alpha_i}(\Sigma\frac{\alpha_i}{\beta_i}\frac{R_i}{R_1})} = \int_o^1 \frac{\frac{\alpha_i}{\beta_i}(1/R_1)dD}{\Sigma\ \frac{\alpha_i}{\beta_i}\frac{R_i}{R_1}} \tag{A7}$$

The sum of $(n/N)_i$ for all of the stress amplitudes when failure occurs is

$$\Sigma\left(\frac{n}{N}\right)_i = \Sigma\int_o^1 \frac{\frac{\alpha_i}{\beta_i}(1/R_1)dD}{\Sigma\frac{\alpha_i}{\beta_i}\frac{R_i}{R_1}} = \int_o^1 \frac{\Sigma\frac{\alpha_i}{\beta_i}(1/R_1)dD}{\Sigma\frac{\alpha_i}{\beta_i}\frac{R_i}{R_1}} \quad \text{(A8)}$$

If R (which, it must be remembered, is a function of D) is the same for all stress amplitudes (as would be the case for the stress-independent damage-cycle relationship shown in Fig. 5), Eq. (A8) becomes

$$\Sigma\left(\frac{n}{N}\right)_i = \frac{\Sigma\frac{\alpha_i}{\beta_i}\int_o^1 (1/R_1)dD}{\Sigma\frac{\alpha_i}{\beta_i}} = 1 \quad \text{(A9)}$$

This result shows that the assumptions of stress-independent, interaction-free, monotonically increasing damage (not necessarily linear with cycles) in a cumulative fatigue damage theory make that theory equivalent to Miner's theory.

For the case of stress-dependence, evaluation of the summation in Eq. (A8) is more difficult. However, it can be shown that this summation is less than one. This will be done by proving the following inequality:

$$\frac{\Sigma\frac{\alpha_i}{\beta_i}(1/R_1)}{\Sigma\frac{\alpha_i}{\beta_i}\frac{R_i}{R_1}} < \frac{\Sigma\frac{\alpha_i}{\beta_i}(1/R_i)}{\Sigma\frac{\alpha_i}{\beta_i}} \quad \text{(A10)}$$

The left-hand side of this inequality is the integrand of Eq. (A8). The right-hand side is a quantity such that the value of its integral over the range of damage from zero to one is unity, i.e.

$$\int_o^1 \frac{\Sigma\frac{\alpha_i}{\beta_i}(1/R_i)}{\Sigma\frac{\alpha_i}{\beta_i}}dD = \frac{\Sigma\frac{\alpha_i}{\beta_i}\int_o^1(1/R_i)dD}{\Sigma\frac{\alpha_i}{\beta_i}} = 1 \quad \text{(A11)}$$

Thus if the inequality is true, it follows that the summation in Eq. (A8) must be less than one.

For simplification, $\alpha_i/\beta_i$ is replaced by $\gamma_i$ in the following steps. Inequality (A10) then becomes

$$\frac{\Sigma\ \gamma_i/R_1}{\Sigma\ \gamma_i\ \frac{R_i}{R_1}} < \frac{\Sigma\ \gamma_i/R_i}{\Sigma\ \gamma_i} \tag{A12}$$

$$\frac{\Sigma\ \gamma_i}{\Sigma\ \gamma_i R_i} < \frac{\Sigma\ \gamma_i/R_i}{\Sigma\ \gamma_i} \tag{A13}$$

$$(\Sigma\ \gamma_i)^2 < (\Sigma\ \frac{\gamma_i}{R_i})(\Sigma\ \gamma_i R_i) \tag{A14}$$

$$\Sigma\ \gamma_i^{\,2} + 2 \sum_{i=1}^{z-1} \sum_{j=i+1}^{z} \gamma_i\gamma_j < \Sigma\ \gamma_i^{\,2} + \sum_{i=1}^{z-1} \sum_{j=i+1}^{z} \gamma_i\gamma_j \left(\frac{R_i}{R_j} + \frac{R_j}{R_i}\right) \tag{A15}$$

$$\sum_{i=1}^{z-1} \sum_{j=i+1}^{z} 2\ \gamma_i\gamma_j < \sum_{i=1}^{z-1} \sum_{j=i+1}^{z} \left(\frac{R_i}{R_j} + \frac{R_j}{R_i}\right) \gamma_i\gamma_j \tag{A16}$$

In the above steps, z has been introduced as the number of different stress amplitudes. To prove the inequality, it only remains to be shown that the coefficient of $\gamma_i\gamma_j$ on the right-hand side is always greater than two, or

$$2 < \frac{R_i}{R_j} + \frac{R_j}{R_i} \tag{A17}$$

$$2 < \frac{R_i^{\,2} + R_j^{\,2}}{R_i R_j} \tag{A18}$$

$$2 R_i R_j < R_i^2 + R_j^2 \qquad (A19)$$

$$0 < (R_i - R_j)^2 \qquad (A20)$$

The inequality is thereby proved if R is not identically equal (at equal values of D) for all stress amplitudes. Values of R are not equal at equal values of D in a stress-dependent theory, and therefore these theories must predict shorter life than would Miner's theory for the same stress spectrum, barring assumption of interaction.

Appendix B

ALGEBRAIC ANALYSIS OF GROVER'S THEORY

In the discussion of Grover's theory it was stated that it could be shown algebraically that this theory predicts shorter lives than does Miner's theory (except for the special case when the ratio of first-stage cycles to total cycles is constant for all stress levels). In the following proof, the notation is the same as that in the original reference.[8]

The basic equations of the theory, for failure under spectrum loading, are

$$\sum_i \frac{n_i}{a_i N_i} = 1 \tag{B1}$$

$$\sum_i \frac{m_i}{(1-a_i)N_i} = 1 \tag{B2}$$

In these equations $n_i$ is the number of cycles applied at the $i^{th}$ stress amplitude during the first stage of fatigue damage, and $m_i$ is the corresponding number of cycles during the second stage; $N_i$ is the number of cycles to failure at the $i^{th}$ amplitude in a single-level test, and $a_i$ is the fraction of $N_i$ at which first-stage damage is complete and second-stage damage begins.

Considering a specified spectrum of stress amplitudes, one amplitude is selected as a reference amplitude, and the quantities defined above are denoted $n_1$, $m_1$, $N_1$, and $a_1$ for this amplitude. With a known spectrum and basic S - N curve, the quantities below are also known:

$$\alpha_i = \frac{n_i}{n_1} = \frac{m_i}{m_1} \tag{B3}$$

$$\beta_i = \frac{N_i}{N_1} \tag{B4}$$

$$\gamma_i = \alpha_i/\beta_i = \frac{n_i/N_i}{n_1/N_1} = \frac{m_i/N_i}{m_1/N_1} \tag{B5}$$

From Eq. (B1), by factoring out $\frac{n_1}{a_1 N_1}$ from all the terms in the summation

$$\frac{n_1}{a_1 N_1} \sum_i \frac{a_1}{a_i} \gamma_i = 1 \tag{B6}$$

$$\frac{n_1}{N_1} = \frac{a_1}{\sum_i \frac{a_1}{a_i} \gamma_i} \tag{B7}$$

Likewise, from Eq. (B2)

$$\frac{m_1}{(1-a_1)N_1} \sum_i \frac{(1-a_1)}{(1-a_i)} \gamma_i = 1 \tag{B8}$$

$$\frac{m_1}{N_1} = \frac{1-a_1}{\sum \frac{(1-a_1)}{(1-a_i)} \gamma_i} \tag{B9}$$

With the notation used here, the Miner failure criterion is

$$\sum_i \frac{n_i + m_i}{N_i} = \sum \frac{n_i}{N_i} + \sum \frac{m_i}{N_i} = 1 \tag{B10}$$

The question of interest here is how failure prediction from Grover's theory compares with Miner's. From Eqs. (B7) and (B5),

$$\frac{n_i}{N_i} = \frac{a_1 \gamma_i}{\sum_i \frac{a_1}{a_i} \gamma_i} \tag{B11}$$

From Eqs. (B9) and (B5)

$$\frac{m_i}{N_i} = \frac{(1-a_1)\ \gamma_i}{\sum_i \frac{(1-a_1)}{(1-a_i)} \gamma_i} \tag{B12}$$

Thus the Miner summation can be written

$$\sum \frac{n_i + m_i}{N_i} = \frac{a_1 \sum_i \gamma_i}{\sum_i \frac{a_1}{a_i} \gamma_i} + \frac{(1-a_1) \sum_i \gamma_i}{\sum_i \frac{(1-a_1)}{(1-a_i)} \gamma_i} \tag{B13}$$

$$= \frac{\sum_i \gamma_i}{\sum_i \frac{\gamma_i}{a_i}} + \frac{\sum_i \gamma_i}{\sum_i \frac{\gamma_i}{(1-a_i)}} \tag{B14}$$

$$= \frac{\sum_i \gamma_i \sum_i \frac{\gamma_i}{(1-a_i)} + \sum_i \gamma_i \sum_i \frac{\gamma_i}{a_i}}{\sum_i \frac{\gamma_i}{a_i} \sum_i \frac{\gamma_i}{1-a_i}} \tag{B15}$$

$$= \frac{\sum_i \gamma_i \left[\sum_i \frac{\gamma_i}{(1-a_i)} + \sum_i \frac{\gamma_i}{a_i}\right]}{\sum_i \frac{\gamma_i}{a_i} \sum_i \frac{\gamma_i}{(1-a_i)}} \tag{B16}$$

$$= \frac{\sum_i \gamma_i \sum_i \frac{\gamma_i a_i + \gamma_i - a_i \gamma_i}{a_i(1-a_i)}}{\sum \frac{\gamma_i}{a_i} \sum \frac{\gamma_i}{(1-a_i)}} \tag{B17}$$

$$= \frac{\sum_i \gamma_i \sum_i \frac{\gamma_i}{a_i(1-a_i)}}{\sum_i \frac{\gamma_i}{a_i} \sum_i \frac{\gamma_i}{(1-a_i)}} \tag{B18}$$

$$= \frac{\sum_i \frac{\gamma_i^2}{a_i(1-a_i)} + \sum_{i=1}^{z-1} \sum_{j=i+1}^{z} \gamma_i \gamma_j \left(\frac{1}{a_i(1-a_i)} + \frac{1}{a_j(1-a_j)}\right)}{\sum_i \frac{\gamma_i^2}{a_i(1-a_i)} + \sum_{i=1}^{z-1} \sum_{j=i+1}^{z} \gamma_i \gamma_j \left(\frac{1}{a_j(1-a_i)} + \frac{1}{a_i(1-a_j)}\right)} \tag{B19}$$

$$= \frac{\sum_i \frac{\gamma_i^2}{a_i(1-a_i)} + \sum_{i=1}^{z-1} \sum_{j=i+1}^{z} \gamma_i \gamma_j \frac{a_i+a_j-a_i^2-a_j^2}{a_i a_j (1-a_i)(1-a_j)}}{\sum_i \frac{\gamma_i^2}{a_i(1-a_i)} + \sum_{i=1}^{z-1} \sum_{j=i+1}^{z} \gamma_i \gamma_j \frac{a_i+a_j-2a_i a_j}{a_i a_j (1-a_i)(1-a_j)}} \tag{B20}$$

For each value of i and j, the numerator and denominator of Eq. (B20) are identical except for the numerator of the coefficient of $\gamma_i \gamma_j$. (The symbol z in the summations denotes the total number of amplitudes in the stress spectrum.) If the values of a are not equal for the $i^{th}$ and $j^{th}$ stress amplitudes

$$(a_i - a_j)^2 > 0 \tag{B21}$$

$$a_i^2 + a_j^2 > 2\, a_i\, a_j \tag{B22}$$

In view of the inequality (B22), it can be seen that the coefficient of $\gamma_i \gamma_j$ in the numerator of Eq. (B20) is less for each i and j than the corresponding coefficient in the denominator. Therefore it follows that the Miner summation in Eq. (B13) will be less than one unless the value of a is identical for all stress amplitudes. Thus, for any nonequal values of a, Grover's theory will predict failure in fewer cycles than will Miner's theory. In terms of design, for a specified load spectrum and a specified life requirement, use of Grover's theory would lead to a design requiring a greater amount of material than would be required by use of Miner's theory.

## REFERENCES


1. Averbach, B. L., et al., (eds.), Fracture, Proceedings of an International Conference on the Atomic Mechanisms of Fracture, The Technology Press of Massachusetts Institute of Technology and John Wiley and Sons, Inc., New York, 1959.

2. Grosskreutz, J. C., and P. Waldow, Research on the Mechanisms of Fatigue in Ultra Small Specimens, Midwest Research Institute, Kansas City, Missouri, Quarterly Progress Report No. 3, December 1961.

3. Dupouy, G., and F. Perrier, "Study of the Structure of Metals and Alloys with a Million Volt Electron Microscope," Comptes Rendus (Proceedings of the French Academy of Science), No. 22, November 27, 1961.

4. Miner. M. A., "Cumulative Damage in Fatigue," J. Appl. Mech., Vol. 12, No. 1, September 1945.

5. Palmgren, A., "Die Lebensdauer von Kugellagern," Zeitschrift des Vereines Deutscher Ingenieure, Vol. 68, No. 14, 1924.

6. Langer, B. F., "Fatigue Failure from Stress Cycles of Varying Amplitudes," J. Appl. Mech., Vol. 4, No. 4, December 1937.

7. Valluri, S. A., "A Unified Engineering Theory of High Stress Level Fatigue," Aerospace Eng., Vol. 20, No. 10, October 1961.

8. Grover, H. J., "An Observation Concerning the Cycle Ratio in Cumulative Damage," Symposium on Fatigue of Aircraft Structures, American Society for Testing Materials, Special Technical Publication No. 274, 1960.

9. Corten, H. T., and T. J. Dolan," Cumulative Fatigue Damage," International Conference on Fatigue of Metals, Institution of Mechanical Engineers and American Society of Mechanical Engineers, 1956.

10. Liu, H. W., and H. T. Corten, Fatigue Damage During Complex Stress Histories, NASA TN D-256, November 1959.

11. Liu, H. W., and H. T. Corten, Fatigue Damage Under Varying Stress Amplitudes, NASA TN D-647, November 1960.

12. Freudenthal, A. M., and R. A. Heller, "On Stress Interaction in Fatigue and a Cumulative Damage Rule," J. Aerospace Sci., Vol. 26, No. 7, July 1959.

13. Shanley, F. R., A Theory of Fatigue Based on Unbonding During Reversed Slip, The RAND Corporation, P-350, November 11, 1952.

14. Steinherz, H. A., and P. A. Redhead, "Ultrahigh Vacuum," *Sci. Amer.*, Vol. 206, No. 3, March 1962.