

U. S. AIR FORCE

PROJECT RAND

RESEARCH MEMORANDUM

OPTIMAL STRATEGIES IN GAMES OF SURVIVAL

Melvin Hausner

RM-777

18 February 1952

Assigned to _____

This is a working paper. It may be expanded, modified, or withdrawn at any time. The views, conclusions, and recommendations expressed herein do not necessarily reflect the official views or policies of the United States Air Force.

The **RAND** *Corporation*

1700 MAIN ST. • SANTA MONICA • CALIFORNIA

Copyright 1952
The RAND Corporation

Report Documentation Page

Form Approved
OMB No. 0704-0188

Public reporting burden for the collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden, to Washington Headquarters Services, Directorate for Information Operations and Reports, 1215 Jefferson Davis Highway, Suite 1204, Arlington VA 22202-4302. Respondents should be aware that notwithstanding any other provision of law, no person shall be subject to a penalty for failing to comply with a collection of information if it does not display a currently valid OMB control number.

| | | | | | |
|---------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------|-------------------------------------|----------------------------|-----------------------------------------------------|---------------------------------|
| 1. REPORT DATE 18 FEB 1952 | | 2. REPORT TYPE | | 3. DATES COVERED 00-00-1952 to 00-00-1952 | |
| 4. TITLE AND SUBTITLE Optimal Strategies in Games of Survival | | | | 5a. CONTRACT NUMBER | |
| | | | | 5b. GRANT NUMBER | |
| | | | | 5c. PROGRAM ELEMENT NUMBER | |
| 6. AUTHOR(S) | | | | 5d. PROJECT NUMBER | |
| | | | | 5e. TASK NUMBER | |
| | | | | 5f. WORK UNIT NUMBER | |
| 7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) Rand Corporation, Project Air Force, 1776 Main Street, PO Box 2138, Santa Monica, CA, 90407-2138 | | | | 8. PERFORMING ORGANIZATION REPORT NUMBER | |
| 9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES) | | | | 10. SPONSOR/MONITOR'S ACRONYM(S) | |
| | | | | 11. SPONSOR/MONITOR'S REPORT NUMBER(S) | |
| 12. DISTRIBUTION/AVAILABILITY STATEMENT Approved for public release; distribution unlimited | | | | | |
| 13. SUPPLEMENTARY NOTES | | | | | |
| 14. ABSTRACT | | | | | |
| 15. SUBJECT TERMS | | | | | |
| 16. SECURITY CLASSIFICATION OF: | | | 17. LIMITATION OF ABSTRACT | 18. NUMBER OF PAGES | 19a. NAME OF RESPONSIBLE PERSON |
| a. REPORT unclassified | b. ABSTRACT unclassified | c. THIS PAGE unclassified | | | |

SUMMARY

A simple procedure is given to calculate the optimal strategies and the survival probability for a general class of games of survival. Theorem 1 shows that for games of this particular type, the optimal strategy for both players depends only on the amount of money held by the second player.

OPTIMAL STRATEGIES IN GAMES OF SURVIVAL

Melvin Hausner

In RM-776, the author considered games of the following type. Players I and II play a zero-sum, 2 x 2 game continually, play ceasing when one of the players runs out of money. If the payoff matrix to I is

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} ; \quad a_{11}, a_{22} > 0; \quad a_{12}, a_{21} < 0,$$

then the above game was shown to have a solution. The play for I's survival and II's ruin are equivalent for player I. Player I can assume himself a probability $v_n(k)$ of survival (n is the amount of money in the game, k is I's initial share of the money) if he plays optimally, while II can limit him to this probability of survival.

The difficulty, in practice, lies in the calculation of $v_n(k)$. (The optimal mixed strategies are simple functions of $v_n(k)$.) We give a case where some general statements may be

made about the solution and where the computation is greatly shortened. We shall always assume that n , k , and a_{ij} are integers.

Lemma 1: Suppose that I cannot lose more than 1 unit of money; i.e. the payoff matrix is of the form

$$(1) \quad A = \begin{pmatrix} a & -1 \\ -1 & b \end{pmatrix}; \quad a, b > 0.$$

then,

$$(2) \quad v_{n+1}(k+1) = v_n(k) + [1 - v_n(k)] v_{n+1}(1).$$

Proof: Let II play his optimal strategy against the following strategy of I's: I reserves one unit of money and plays his optimal strategy for the reduced game, where his

money and the total amount of money in the game is reduced by one. If I has only one unit, he plays arbitrarily. We

calculate I's probability of survival if both players play in the above fashion.

First, it is known that the payoff is independent of a player's strategy provided one of the players uses his optimal strategy. Assume that I has $k+1$ units and II has $n-k$. Since II plays his optimal strategy, the payoff (I's probability of survival) is $v_{n+1}(k+1)$. On the other hand, I can win in two mutually exclusive ways: he never falls down to one unit, or he does but comes back to win. The first case occurs with probability $v_n(k)$ since I plays his optimal strategy in the reduced game. The second case occurs with probability $1 - v_n(k)$ of falling down to one unit, and the probability $v_{n+1}(1)$ of

then surviving (since II plays optimally in the (n+1) game).

Hence, I's probability of survival is

$$v_n(k) + [1-v_n(k)] v_{n+1}(1).$$

thus,

$$(2) \quad v_{n+1}(k+1) = v_n(k) + [1-v_n(k)] v_{n+1}(1),$$

proving the lemma.

Theorem 1: For a game of the type (1), I's optimal strategy is to "reserve" one unit of money and play optimally in the reduced game.

By induction then, if I has k+1 unit of money and II has n, I's optimal strategy is to reserve k units of money and play optimally in the game in which he has 1 unit and II has n. Hence, I's optimal strategy depends only on the amount of money that II holds.

Proof: Let I have k+1 units and let there be n+1 units in the game. Suppose I plays his (unique) optimal strategy against any strategy of II's. I's strategy and II's strategy induce strategies for the reduced (n,k) game. Let p be the probability of I winning the reduced game (i.e. of his winning the original game without ever being reduced to one unit). Since I's strategy is optimal,

$$v_{n+1}(k+1) = p + (1-p) v_{n+1}(1).$$

By (2), $p = v_n(k)$. This is so for every one of II's strategies. Hence, I's reduced strategy is optimal in the reduced game, proving the theorems.

With the help of formula (4) of RM-776, we have

$$v_{n+1}(1) = \frac{v_{n+1}(1+a) v_{n+1}(1+b)}{v_{n+1}(1+a) + v_{n+1}(1+b)} .$$

Using formula (2), and substituting in this equation, we obtain a quadratic equation for $v_{n+1}(1)$, having only one positive root.

The solution is:

$$(3) \quad v_{n+1}(1) = \frac{\sqrt{v_n(a) v_n(b)}}{1 + \sqrt{v_n(a) v_n(b)}} .$$

Using the trivial

$$(4) \quad v_2(1) = \frac{1}{2} ,$$

it is possible to calculate $v_n(k)$ in general by induction on n with the help of (4), (3), and (2). The optimal strategies are then easily obtained. Letting $p_n(k)$ be the probability of I playing row 1 in his optimal strategy, we have by formula (3) of RM-776,

$$(5) \quad p_n(1) = \frac{v_n(1+b)}{v_n(1+a) + v_n(1+b)} = \frac{v_n(1)}{v_n(1+a)}$$

This is also II's probability of playing column 1 in his optimal strategy when I has k units and II has $n-k$. By theorem 1,

$$(6) \quad p_{n+1}(k+1) = p_n(k) = p_{n-k+1}(1),$$

so that the probabilities $p_n(1)$ give all the optimal strategies which may occur.

mhb

