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MORE ON GAMES OF SURVIVAL

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SUMMARY. Let  $\Omega$  be the game of survival which is the repetition (until the bankruptcy of one of the players) of a normalized finite zero-sum two-person game,  $\Gamma = (\Gamma_{ij})$ , where each  $\Gamma_{ij}$  is a non-zero integer. It is shown that  $\Omega$  is inessential and has some easily described optimal strategies. It is also shown that if  $\max_{i,j} |\Gamma_{ij}|$  is small enough compared to the combined fortunes, then playing at the  $n$ -th play a  $\delta^n$ -optimal strategy for  $\Gamma$  is an  $\epsilon$ -optimal strategy for  $\Omega$ , if  $\delta$  is sufficiently small.

#### MORE ON GAMES OF SURVIVAL

M. P. Peisakoff

We are interested in the games  $\{\Omega(f_1, f_2)\}$  in which two players with finite fortunes,  $f_1$  and  $f_2$ , respectively, in chips repeat a normalized finite zero-sum two-person game,  $\Gamma = (\Gamma_{ij} | (1, 1) \leq (i, j) \leq (i_0, j_0))$ . At least one of  $f_1$  and  $f_2$  is positive and play is continued until the fortune of one of the players is non-positive, or ad infinitum if this never occurs. The payoff in money is  $(1, 0)$  if player 2 ends with a non-positive fortune and  $(0, 1)$  if player 1 ends with a non-positive fortune. If the game goes on indefinitely, then the payoff is  $(\alpha(C), \beta(C))$ , which can depend on the course of the game,  $C$ , but which satisfies  $(\alpha(C), \beta(C)) \leq (1, 1)$  and  $\alpha(C) + \beta(C) \leq 1$ . We shall show that if all the  $\Gamma_{ij}$ 's are non-zero integers, then  $\Omega$  is inessential\* and has some easily described optimal strategies. (In an inessential game, an optimal strategy for a player is one which secures for the player the maximum amount he can insure for himself. An  $\epsilon$ -optimal strategy secures for

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\* An inessential game is one in which the players together can secure only the sum of the (minorant) amounts insurable without cooperation.

him at least that amount less  $\varepsilon$ .) We shall also show that if  $\max_{i,j} |\Gamma_{ij}|$  is small enough compared to the combined fortunes, then playing at the  $n$ -th play a  $\delta^n$ -optimal strategy for  $\Gamma$  is an  $\varepsilon$ -optimal strategy for  $\Omega$ , if  $\delta$  is sufficiently small. ( $\delta^n$  is the  $n$ -th power of  $\delta$ .)

We assume that every column of  $\Gamma$  has a positive entry, and every row has a negative entry. Otherwise, there would be a negative column or a positive row. In the first case, player 2 can always force player 1's fortune to become non-positive, by playing the negative column repeatedly. In the second case, player 1 can force player 2's fortune to become non-positive by playing the positive row repeatedly.

Let  $\bar{\Omega}^{(n)}(f_1, f_2)$  be the game in which the two players repeat  $\Gamma$   $n$  times, or until one of the players has a non-positive fortune if this occurs first. The payoff in money is  $(0, 1)$  if player 1 ends with a non-positive fortune, and  $(1, 0)$  otherwise.  $\bar{\Omega}^{(n)}(f_1, f_2)$  is a constant-sum two-person game with value, say,  $(\bar{v}^{(n)}(f_1, f_2), 1 - \bar{v}^{(n)}(f_1, f_2))$ .

We observe

- (1) Player 2 can always win as much money in  $\bar{\Gamma}^{(n+1)}$  as in  $\bar{\Gamma}^{(n)}$  by playing a  $\bar{\Gamma}^{(n)}$ -optimal strategy during the first  $n$  moves of  $\bar{\Gamma}^{(n+1)}$  and arbitrarily on the  $(n+1)$ th move. Hence

$$\bar{v}^{(n)}(f_1, f_2) \geq \bar{v}^{(n+1)}(f_1, f_2).$$

- (2) Since each column has a positive entry, by repeatedly playing the strategy which assigns each pure strategy probability  $1/i_0$ , player 1 insures that no matter what player 2 does, player 2's fortune will decrease each time with probability

at least  $1/i_0$ . Player 1 thereby insures that with probability at least  $i_0^{-[f_2]-1}$ , player 2 will be bankrupted in at most  $[f_2] + 1$  trials. ( $[f_2]$  is the largest integer not larger than  $f_2$ .) Hence if  $n \geq [f_2] + 1$  and  $(f_1, f_2) > (0, 0)$ , then

$$\bar{v}^{(n)}(f_1, f_2) \in [\delta, 1]$$

where  $\delta = i_0^{-[f_2]-1}$ . By definition we have also

$$\bar{v}^{(n)}(f_1, f_2) = 0 \quad \text{if } f_1 \leq 0$$

and

$$\bar{v}^{(n)}(f_1, f_2) = 1 \quad \text{if } f_2 \leq 0.$$

- (3) Let  $G(\Delta)$  be the game value of  $\Delta$ , for each game  $\Delta$ . If  $(f_1, f_2) > 0$ , after one move of  $\bar{F}^{(n+1)}(f_1, f_2)$ , the players are playing  $\bar{F}^{(n)}(f_1 + \Gamma_{ij}, f_2 - \Gamma_{ij})$ . Hence

$$\bar{v}^{(n+1)}(f_1, f_2) = G(\bar{v}^{(n)}(f_1 + \Gamma_{ij}, f_2 - \Gamma_{ij})).$$

- (4) Let  $\epsilon \geq 0$ . Player 1 can always win as much in  $\bar{F}^{(n)}(f_1 + \epsilon, f_2 - \epsilon)$  as in  $\bar{F}^{(n)}(f_1, f_2)$ . Hence

$$\bar{v}^{(n)}(f_1 + \epsilon, f_2 - \epsilon) \geq \bar{v}^{(n)}(f_1, f_2).$$

We can now conclude:

(A) From (1) and (2),

$$\begin{aligned} \bar{v}^{(n)}(f_1, f_2) &\longrightarrow \bar{v}(f_1, f_2) \in [\delta, 1] \text{ if } (f_1, f_2) > (0, 0) \\ &= 0 \quad \text{if } f_1 \leq 0 \\ &= 1 \quad \text{if } f_2 \leq 0. \end{aligned}$$

(B) From (3), if  $(f_1, f_2) > (0, 0)$ ,

$$\bar{v}(f_1, f_2) = G(\bar{v}(f_1 + \Gamma_{ij}, f_2 - \Gamma_{ij})).$$

(C) From (4), for  $\varepsilon \geq 0$ ,

$$\bar{v}(f_1 + \varepsilon, f_2 - \varepsilon) \geq \bar{v}(f_1, f_2).$$

Definition: A strategy for player 1 is called conditionally optimal if the conditional distribution of his strategy at any play of  $\Gamma$ , given the course of the game up to that play, is an optimal strategy for the game  $(v(\phi_1 + \Gamma_{ij}, \phi_2 - \Gamma_{ij}))$  where  $(\phi_1, \phi_2)$  is the fortune distribution immediately before the play in question.

Lemma 1: If player 1's strategy is conditionally optimal, and if with probability one the fortune of one of the players (not necessarily always the same one) eventually becomes non-positive, then player 1 can expect at least  $\bar{v}(f_1, f_2)$  in payoff.

Proof: It is sufficient to show that the probability that player 2's fortune becomes non-positive is at least  $\bar{v}(f_1, f_2)$ . Let  $((F_1^n, F_2^n) | n \geq 1)$  be the random variable of fortunes at play  $n$ , where if the game ends at play  $N$ ,  $(F_1^{N+j}, F_2^{N+j}) \equiv (F_1^N, F_2^N)$  for  $j \geq 1$ . Then since player 1's

strategy is conditionally optimal, if  $(F_1^n, F_2^n) > (0, 0)$ ,

$$\begin{aligned} E\bar{v}(F_1^{n+1}, F_2^{n+1}) &\geq EG(\bar{v}(F_1^n + r_{ij}, F_2^n - r_{ij})) \\ &= E\bar{v}(F_1^n, F_2^n), \end{aligned}$$

while

$$E\bar{v}(F_1^{n+1}, F_2^{n+1}) = E\bar{v}(F_1^n, F_2^n)$$

otherwise. Hence, by induction,

$$E\bar{v}(F_1^n, F_2^n) \geq \bar{v}(f_1, f_2).$$

Let  $((P_1^n, P_2^n) | n \geq 1)$  be the random variable which is

- (0, 0) if neither player's fortune is non-positive by the end of the n-th play,
- (0, 1) if the first player's fortune is non-positive by the end of the n-th play,
- (1, 0) if the second player's fortune is non-positive by the end of the n-th play.

Then

$$E\bar{v}(F_1^n, F_2^n) \leq EF_1^n + E(1 - P_2^n - P_1^n).$$

But by assumption the second term on the right tends to zero. Hence, where  $\xi_n \rightarrow 0$ ,

$$EP_1^n + \epsilon_n \geq v(f_1, f_2),$$

which is the desired result.

Lemma 2: There is a conditionally optimal strategy for the first player which insures that the probability that the game ends by the n-th play tends to one as n tends to  $\infty$ , uniformly in the opponent's strategy.

Proof: First we show that for each  $(\phi_1, \phi_2) > (0, 0)$ , there is an optimal strategy I for the first player for the game  $(\bar{v}(\phi_1 + \Gamma_{ij}, \phi_2 - \Gamma_{ij}))$  such that for all J,  $\Pr \{ \Gamma_{IJ} > 0 \} > 0$ . Suppose, to the contrary, that for some  $(\phi_1, \phi_2) > (0, 0)$ , for all optimal I, there is a J such that  $\Pr \{ \Gamma_{IJ} > 0 \} = 0$ , or what is the same since  $\Gamma_{ij} \neq 0$ ,  $\Pr \{ \Gamma_{IJ} < 0 \} = 1$ . Then since player 1 is playing optimally,

$$\bar{v}(\phi_1, \phi_2) \leq E\bar{v}(\phi_1 + \Gamma_{IJ}, \phi_2 - \Gamma_{IJ}).$$

From the monotonicity of  $\bar{v}(\phi_1 + \epsilon, \phi_2 - \epsilon)$ ,

$$\bar{v}(\phi_1, \phi_2) \geq \bar{v}(\phi_1 + \Gamma_{IJ}, \phi_2 - \Gamma_{IJ}).$$

Combining,

$$\bar{v}(\phi_1, \phi_2) = \bar{v}(\phi_1 + \Gamma_{IJ}, \phi_2 - \Gamma_{IJ}),$$

or weaker, from monotonicity again,

$$\bar{v}(\phi_1, \phi_2) = \bar{v}(\phi_1 - 1, \phi_2 + 1).$$

If  $(\phi_1 - 1, \phi_2 + 1) > (0, 0)$ , this implies that an optimal strategy I for

the first player for the game  $(\bar{v}(\phi_1 + \Gamma_{ij} - 1, \phi_2 - \Gamma_{ij} + 1))$  is an optimal strategy for  $(\bar{v}(\phi_1 + \Gamma_{ij}, \phi_2 - \Gamma_{ij}))$ , since by using it against any  $J$ , the first player insures himself

$$\begin{aligned} E\bar{v}(\phi_1 + \Gamma_{IJ}, \phi_2 - \Gamma_{IJ}) &\geq E\bar{v}(\phi_1 + \Gamma_{IJ} - 1, \phi_2 - \Gamma_{IJ} + 1) \\ &\geq \bar{v}(\phi_1 - 1, \phi_2 + 1) \\ &= \bar{v}(\phi_1, \phi_2) . \end{aligned}$$

Thus for a fortune division  $(\phi_1 - 1, \phi_2 + 1) > (0, 0)$ , and by induction for a fortune division,  $(\phi_1 - n, \phi_2 + n) > (0, 0)$ , for all optimal strategies,  $I$ , there is a  $J$  such that  $\Gamma_{IJ} < 0$ . But eventually, perhaps for  $n = 0$ ,

$$(\phi_1 - n, \phi_2 + n) > (0, 0)$$

while  $\phi_1 \leq n + 1$ . Therefore, for an optimal  $I$  and some  $J$ ,

$$\begin{aligned} 0 < \delta &\leq v(\phi_1 - n, \phi_2 + n) \leq E\bar{v}(\phi_1 - n + \Gamma_{IJ}, \phi_2 + n - \Gamma_{IJ}) \\ &\leq \bar{v}(\phi_1 - n - 1, \phi_2 + n + 1) \\ &= 0 , \end{aligned}$$

which is the contradiction we were looking for.

We have now proved that for  $(\phi_1, \phi_2) > (0, 0)$ , there is an optimal  $I$  such that for all  $J$ ,  $\Pr\{\Gamma_{IJ} > 0\} > 0$ . For each  $(\phi_1, \phi_2) > (0, 0)$ , fix such an  $I$ . Call it  $I(\phi_1, \phi_2)$ . From the compactness of the second player's set of strategies and the fact that  $\Pr\{\Gamma_{IJ} > 0\}$  is a continuous

function of his strategy,  $\Pr \{ \Gamma_{IJ} > 0 \} \geq \rho(\phi_1, \phi_2) > 0$ . Define  $\sigma(\phi_1, \phi_2) = \min_k \rho(\phi_1 + k, \phi_2 - k) > 0$ , where  $k$  is an arbitrary positive, zero, or negative integer such that  $(\phi_1 + k, \phi_2 - k) > (0, 0)$ .

Let now player 1 use the conditionally optimal strategy which consists of playing  $I(\phi_1, \phi_2)$  when the fortune distribution is  $(\phi_1, \phi_2)$ . Let  $Q^{(n)}$  be the probability that one or the other player's fortune is exhausted on or before the  $n$ -th play. Then, where  $\sigma = \sigma(f_1, f_2)$ ,

$$Q^{([f_1+f_2]+1)} \geq 0$$

$$Q^{(n+[f_1+f_2]+1)} \geq Q^{(n)} + (1 - Q^{(n)})\sigma^{[f_1+f_2]+1}.$$

By induction,

$$Q^{(N\{[f_1+f_2]+1\})} \geq 1 - (1 - \sigma^{[f_1+f_2]+1})^{N-1}.$$

Hence  $Q^{(N)} \rightarrow 1$  as  $N \rightarrow \infty$ , which is the lemma.

Let  $\underline{\Omega}^{(n)}(f_1, f_2)$  be the game in which the two players repeat  $\Gamma$   $n$ -times, or until one of the players has a non-positive fortune if this occurs first, and the money payoff is  $(1, 0)$  if player 2 ends with a non-positive fortune,  $(0, 1)$  otherwise.  $\underline{\Omega}^{(n)}(f_1, f_2)$  is a constant-sum two-person game with value  $(\underline{v}^{(n)}(f_1, f_2), 1 - \underline{v}^{(n)}(f_1, f_2))$ . Obviously

$$(5) \quad \underline{v}^{(n)}(f_1, f_2) \leq \bar{v}^{(n)}(f_1, f_2),$$

since any strategy for player 1 in  $\underline{\Omega}^{(n)}(f_1, f_2)$ , will insure him as much money in  $\bar{\Omega}^{(n)}(f_1, f_2)$ . We therefore conclude, by the same reasoning as

earlier,

$$\begin{aligned}
 \text{(A)} \quad \underline{v}^{(n)}(f_1, f_2) &\rightarrow \underline{v}(f_1, f_2) \in [0, 1 - \delta'] \quad \text{if } (f_1, f_2) > (0, 0) \\
 &= 0 \quad \text{if } f_1 \leq 0 \\
 &= -1 \quad \text{if } f_2 \leq 0
 \end{aligned}$$

where  $\delta' > 0$ .

$$\text{(B)} \quad \text{If } (f_1, f_2) > (0, 0),$$

$$\underline{v}(f_1, f_2) = G(\underline{v}(f_1 + \Gamma_{ij}, f_2 - \Gamma_{ij})).$$

$$\text{(C)} \quad \text{If } \varepsilon \geq 0,$$

$$\underline{v}(f_1 + \varepsilon, f_2 - \varepsilon) \geq \underline{v}(f_1, f_2).$$

In addition

$$\text{(D)} \quad \underline{v}(f_1, f_2) \leq \bar{v}(f_1, f_2).$$

Definition: A strategy for player 2 is called conditionally optimal if the conditional distribution of his strategy at any play of  $\Gamma$ , given the course of the game up to that play, is an optimal strategy for the game  $(\bar{v}(\phi_1 + \Gamma_{ij}, \phi_2 - \Gamma_{ij}))$  where  $(\phi_1, \phi_2)$  is the fortune distribution immediately before the play in question.

From Lemmas  $\bar{1}$  and  $\bar{2}$ , and the analogous Lemmas  $\underline{1}$  and  $\underline{2}$  which we do not write down, we conclude that each player has a conditionally optimal strategy which insures that play ends by the  $n$ -th play with probability

tending to 1 as  $n$  tends to  $\infty$ , uniformly in the opponent's strategy. The first player's strategy insures him  $\bar{v}(f_1, f_2)$  on the average and the second player's strategy insures him  $1 - \underline{v}(f_1, f_2) \geq 1 - \bar{v}(f_1, f_2)$ , on the average. Since together the players can win no more than 1, we get

$$\begin{aligned} 1 &\geq \bar{v}(f_1, f_2) + (1 - \underline{v}(f_1, f_2)) \\ &\geq \bar{v}(f_1, f_2) + (1 - \bar{v}(f_1, f_2)) \\ &= 1. \end{aligned}$$

This means  $\underline{v}(f_1, f_2) = \bar{v}(f_1, f_2) =$  (say)  $v(f_1, f_2)$ , and  $\Omega(\phi_1, \phi_2)$  is inessential with the solution  $\{(v(f_1, f_2), 1 - v(f_1, f_2))\}$ .

$v$  can be characterized as the unique solution of

$$\begin{aligned} 0 \leq v(\phi_1, \phi_2) &= G(v(\phi_1 + \Gamma_{ij}, \phi_2 - \Gamma_{ij})) \leq 1 \\ &\text{if } (f_1, f_2) > (0, 0) \\ &= 0 \text{ if } f_1 \leq 0 \\ &= 1 \text{ if } f_2 \leq 0. \end{aligned}$$

For if  $v^*$  is a solution,

$$\underline{v}^{(0)}(\phi_1, \phi_2) \leq v^*(\phi_1, \phi_2) \leq \bar{v}^{(0)}(\phi_1, \phi_2)$$

by definition, and so by induction, using (A), (B), (A), and (B),

$$\underline{v}^{(n)}(\phi_1, \phi_2) \leq v^*(\phi_1, \phi_2) \leq \bar{v}^{(n)}(\phi_1, \phi_2).$$

Hence

$$v(\phi_1, \phi_2) = \underline{v}(\phi_1, \phi_2) \leq v^*(\phi_1, \phi_2) \leq \bar{v}(\phi_1, \phi_2) = v(\phi_1, \phi_2),$$

giving

$$v(\phi_1, \phi_2) = v^*(\phi_1, \phi_2) ,$$

as was to be proved.

We thus have

Theorem 1:  $\Omega(f_1, f_2)$  is inessential with the solution  $\{(v(f_1, f_2), 1 - v(f_1, f_2))\}$  where  $v$  is the unique solution in  $\{(\phi_1, \phi_2) | \phi_1 > 0 \text{ or } \phi_2 > 0\}$  of

$$0 \leq v(\phi_1, \phi_2) = G(v(\phi_1 + r_{ij}, \phi_2 - r_{ij})) \leq 1$$

if  $(\phi_1, \phi_2) > (0, 0)$

= 0 if  $\phi_1 \leq 0$

= 1 if  $\phi_2 \leq 0$ .

Each player has a conditionally optimal strategy which is optimal and which insures that play ends by the n-th play with probability tending to one uniformly in the opponent's strategies.

Let us turn now to the problem of effectively computing an  $\epsilon$ -optimal strategy for  $\Omega(f_1, f_2)$ . This is easy if we are not interested in efficiency. Namely, we need only find an  $n$  such that  $\bar{v}^{(n)}(f_1, f_2) - \underline{v}^{(n)}(f_1, f_2) \leq \epsilon - \delta$  where  $\delta > 0$ . Then a  $\delta$ -optimal strategy for the first player for

$\underline{\Omega}^{(n)}(f_1, f_2)$  provides an  $\epsilon$ -optimal strategy for him for  $\Omega(f_1, f_2)$ . Namely, he can use the strategy on the first  $n$  moves of  $\Omega(f_1, f_2)$  and act arbitrarily thereafter. Similarly, a  $\delta$ -optimal strategy for the second player for  $\bar{\Omega}^{(n)}(f_1, f_2)$  provides an  $\epsilon$ -optimal strategy for him for  $\Omega(f_1, f_2)$ .

If  $\max_{i,j} |\Gamma_{ij}|$  is small enough compared to  $f_1$  and  $f_2$ , there is another class of interesting  $\epsilon$ -optimal strategies. Repeatedly playing an optimal strategy for  $\Gamma$  is an  $\epsilon$ -optimal strategy for  $\Omega$ . More precisely, let us remove the restriction that each  $\Gamma_{ij}$  be a non-zero integer. Let us require instead, say, that  $G(\Gamma) \geq 0$  and that for some optimal strategy  $I$ ,  $\Pr \{ \Gamma_{Ij} > 0 \} > 0$  for all  $j$ . If  $G(\Gamma) = 0$ , we require in addition that for some optimal  $J$ ,  $\Pr \{ \Gamma_{iJ} < 0 \} > 0$  for all  $i$ . Define  $\alpha = G(\Gamma)$ ,  $\beta = \min_j \Pr \{ \Gamma_{Ij} > 0 \}$ ,  $\gamma = \max_{i,j} |\Gamma_{ij}|$ .

We assume that both  $f_1$  and  $f_2$  are positive and define  $f = f_1 + f_2$ . Define for  $\alpha = 0$

$$\begin{aligned}
 p_0(\phi_1) &= \frac{1}{f + \gamma} \phi_1 & \text{if } 0 < \phi_1 < f \\
 &= 0 & \text{if } \phi_1 \leq 0 \\
 &= 1 & \text{if } \phi_1 \geq f
 \end{aligned}$$

and for  $\alpha > 0$

$$\begin{aligned}
 p_{\alpha}(\phi_1) &= \frac{1 - \exp\left\{-\frac{\alpha}{\gamma^2} \phi_1\right\}}{1 - \exp\left\{-\frac{\alpha}{\gamma^2} (f + \gamma)\right\}} && \text{if } 0 < \phi_1 < f \\
 &= 0 && \text{if } \phi_1 \leq 0 \\
 &= 1 && \text{if } \phi_1 \geq f .
 \end{aligned}$$

Lemma  $\bar{3}$ : If player 1 plays I repeatedly, then he can expect at least  $p_{\alpha}(f_1)$  in payoff. (I is any optimal strategy for  $\Gamma$  satisfying  $\Pr(\Gamma_{ij} > 0) > 0$ .)

Proof: Since  $\beta > 0$ , by the method of proof of Lemma  $\bar{2}$ , it follows that if player 1 plays I repeatedly, the probability that the game ends by the  $n$ -th play tends to one as  $n$  tends to  $\infty$ . Hence, in order to prove Lemma  $\bar{3}$ , it is sufficient to show that for all  $N$ ,

$$E p_{\alpha}(F_1^N) \geq p_{\alpha}(f_1) .$$

By induction, this would follow from

$$E \left\{ p_{\alpha}(F_1^{N+1}) | F_1^N \right\} \geq p_{\alpha}(F_1^N) .$$

We prove the latter.

Suppose first that  $\alpha = 0$ . If  $0 < F_1^N < f$ , then for all  $(i, j)$ , since  $\Gamma_{ij} \leq \gamma$ ,

$$p_0(F_1^N + \Gamma_{ij}) \geq \frac{1}{f + \gamma} (F_1^N + \Gamma_{ij}) .$$

Hence, if  $0 < F_1^N < f$ ,

$$\begin{aligned} E\left\{p_0(F_1^{N+1}) | F_1^N\right\} &\geq \min_j E p_0(F_1^N + r_{1j}) \\ &\geq \frac{1}{f + \gamma} \min_j E(F_1^N + r_{1j}) \\ &\geq \frac{1}{f + \gamma} F_1^N \\ &= p_0(F_1^N) . \end{aligned}$$

Since if  $F_1^N \leq 0$  or  $F_1^N \geq f$  our proposition is trivial, we have disposed of the case  $\alpha = 0$ .

Suppose now that  $\alpha > 0$ . Again we need only consider  $0 < F_1^N < f$ . Then

$$p_{\alpha}(F_1^N + r_{1j}) \geq \frac{1 - \exp\left\{-\frac{\alpha}{\gamma^2} (F_1^N + r_{1j})\right\}}{1 - \exp\left\{-\frac{\alpha}{\gamma^2} (f + \gamma)\right\}} .$$

Hence,

$$\begin{aligned} E\left\{p_{\alpha}(F_1^{N+1}) | F_1^N\right\} &\geq \min_j E p_{\alpha}(F_1^N + r_{1j}) \\ &\geq \min_j \frac{1 - E \exp\left\{-\frac{\alpha}{\gamma^2} (F_1^N + r_{1j})\right\}}{1 - \exp\left\{-\frac{\alpha}{\gamma^2} (f + \gamma)\right\}} \\ &\geq \frac{1 - M \exp\left\{-\frac{\alpha}{\gamma^2} F_1^N\right\}}{1 - \exp\left\{-\frac{\alpha}{\gamma^2} (f + \gamma)\right\}} \end{aligned}$$

where

$$\begin{aligned}
 M &= \max_j E \exp \left\{ -\frac{\alpha}{\gamma^2} \Gamma_{1j} \right\} \\
 &\leq \max_j \left\{ 1 - \frac{\alpha}{\gamma^2} E \Gamma_{1j} + (e-2) \left( \frac{\alpha}{\gamma} \right)^2 \right\} \\
 &\leq \left\{ 1 - \frac{\alpha^2}{\gamma^2} + (e-2) \frac{\alpha^2}{\gamma^2} \right\} \\
 &< 1 .
 \end{aligned}$$

Hence

$$\begin{aligned}
 E \left\{ p_{\alpha}(F_1^{N+1}) | F_1^N \right\} &\geq \frac{1 - \exp \left\{ -\frac{\alpha}{\gamma^2} F_1^N \right\}}{1 - \exp \left\{ -\frac{\alpha}{\gamma^2} (f + \gamma) \right\}} \\
 &= p_{\alpha}(F_1^N) ,
 \end{aligned}$$

as was to be proved.

By symmetry, if  $\alpha = 0$  we conclude

Lemma 2. If  $\alpha = 0$ , and if player 2 plays J repeatedly, then he can expect at least  $p_0(f_2)$  in payoff. ( $J$  is any optimal strategy for  $\Gamma$  satisfying  $\Pr(\Gamma_{1j} < 0) > 0$ .)

If  $\alpha = 0$ , Lemmas 1 and 2 give us, whenever  $\Omega(f_1, f_2)$  is inessential with solution  $\{(v(f_1, f_2), 1 - v(f_1, f_2))\}$ ,

$$p_0(f_1) \leq v(f_1, f_2) \leq 1 - p_0(f_2) = p_0(f_1) + \frac{\gamma}{f + \gamma} .$$

Thus repeating I is  $(\frac{\gamma}{f + \gamma})$ -optimal for player 1, and repeating J is  $(\frac{\gamma}{f + \gamma})$ -optimal for player 2. If  $\alpha > 0$ , Lemma 3 gives us whenever  $\Omega(f_1, f_2)$  is inessential with solution  $\{(v(f_1, f_2), 1 - v(f_1, f_2))\}$ ,

$$1 - \exp\left\{-\frac{\alpha}{\gamma^2} f_1\right\} \leq p_\alpha(f_1) \leq v(f_1, f_2) \leq 1.$$

Thus repeating I is  $\exp\left\{-\frac{\alpha}{\gamma^2} f_1\right\}$ -optimal for player 1 and any strategy is  $\exp\left\{-\frac{\alpha}{\gamma^2} f_1\right\}$ -optimal for player 2.

What if, instead of repeating I, player 1 repeated a  $\delta$ -optimal I $\delta$ , where  $\delta$  is the smallest number for which I $\delta$  is  $\delta$ -optimal? If  $\alpha > \delta$  no great harm is done, since it can be verified by precisely the proof given above that this is an  $\exp\left\{-\frac{\alpha - \delta}{\gamma^2} f_1\right\}$ -optimal strategy for player 1. If, however,  $\alpha < \delta$ , player 2 could expect at least  $1 - \exp\left\{-\frac{\delta - \alpha}{\gamma^2} f_2\right\}$  in payoff. When  $\frac{\delta - \alpha}{\gamma^2} f_2$  is large, this payoff is close to one, so I $\delta$  is not a good strategy. Thus if  $\alpha = 0$ , no matter how small  $\gamma$  is, it is not enough to repeat a  $\delta$ -optimal strategy for sufficiently small  $\delta$ . On the other hand, suppose that  $(I_n)$  is a sequence of strategies for player 1 whose  $n$ -th member is  $\delta^n$ -optimal for  $\Gamma$  and satisfies

$$\min_j \Pr\{\Gamma_{I_n j} > 0\} \geq \beta' > 0,$$

where  $\beta'$  does not depend on  $n$ . Then

Lemma 4: If  $\alpha = 0$  and player 1 plays  $I_n$  at the  $n$ -th stage, then he can expect at least  $p_\alpha(f_1) - \frac{\delta}{(1-\delta)(f+\gamma)}$  in payoff.

Proof: The proof is almost identical with that of Lemma 3 where instead of proving

$$E p_\alpha(F_1^N) \geq p_\alpha(f_1)$$

one proves

$$E p_\alpha(F_1^N) \geq p_\alpha(f_1) - \frac{\delta + \dots + \delta^{N-1}}{f + \gamma} .$$

It is an easy step (left to the reader) now to

Theorem 2. If  $G(\Gamma) = \alpha > \delta$  and  $\Omega(f_1, f_2)$  is inessential, repeating a strategy which is  $\delta$ -optimal for  $\Gamma$  is  $\exp\left\{-\frac{\alpha-\delta}{\gamma^2} f_2\right\}$ -optimal for  $\Omega(f_1, f_2)$ . Let  $G(\Gamma) = 0$ , and let  $(I_n)$  be a sequence of strategies for player 1 whose  $n$ -th member is  $\delta^n$ -optimal for  $\Gamma$  and satisfies

$$\min_j \Pr \{ \Gamma_{I_n j} > 0 \} \geq \beta' > 0 ,$$

where  $\beta'$  does not depend on  $n$ . Then playing  $I_n$  at the  $n$ -th stage is a  $\left(\frac{\gamma}{f+\gamma} + \frac{2\delta}{(1-\delta)(f+\gamma)}\right)$ -optimal strategy for player 2.

The reader will observe that when each  $\Gamma_{ij} \neq 0$ , say  $|\Gamma_{ij}| \geq C$ , we automatically have for a  $\delta^n$ -optimal  $I_n$ , when  $\delta$  is sufficiently small,

$$\min_j \Pr \{ \Gamma_{I_n j} > 0 \} \geq \frac{C - \delta^n}{C + \gamma} \geq \frac{C - \delta}{C + \gamma} > 0 .$$

In closing, we wish to point out that the method of proof leading to Theorem 1 is trivially sufficient to handle the following generalized game of survival in which the result of a play is a random state instead of a definite number. However, the method is apparently insufficient to handle more than a finite number of possible states, or the possibility of "zeros." A finite set  $\Sigma$  with two distinguished points  $\sigma_1$  and  $\sigma_2$  is given.  $\Sigma$  is partially ordered by  $<$ , which satisfies for some fixed  $n$  and all  $\{x_i | 1 \leq i \leq n\}$ ,

$$x_1 < x_2 < \dots < x_{n-1} < x_n \longrightarrow x_1 = \sigma_2, x_n = \sigma_1 .$$

For each  $x \in \Sigma$ , there is a set of random variables on  $\Sigma$ ,  $\{Y_{ij}(x) | 1 \leq i \leq i_0, 1 \leq j \leq j_0\}$  such that for all  $i$  and  $j$   $Y_{ij}(\sigma_1) = \sigma_1$ ,  $Y_{ij}(\sigma_2) = \sigma_2$ , and for  $x \neq \sigma_1, \sigma_2$

$$\Pr \{ Y_{ij}(x) < x \} = 0 \longrightarrow \Pr \{ x < Y_{ij}(x) \} = 1$$

$$\Pr \{ x < Y_{ij}(x) \} = 0 \longrightarrow \Pr \{ Y_{ij}(x) < x \} = 1 .$$

In addition, for  $x \neq \sigma_1, \sigma_2$ , for each  $i$ , there is a  $j$  such that

$$\Pr \{Y_{ij}(x) < x\} > 0 ,$$

and for each  $j$ , there is an  $i$  such that

$$\Pr \{x < Y_{ij}(x)\} > 0 .$$

Define  $\prod_{n=1}^N Y_{i_n j_n}^{(n)}(x)$  by induction by

$$\prod_{n=1}^{M+1} Y_{i_n j_n}^{(n)}(x) = Y_{i_{M+1} j_{M+1}}^{M+1} \left( \prod_{n=1}^M Y_{i_n j_n}^{(n)}(x) \right) ,$$

where  $\left\{ \left( Y_{ij}^{(n)}(x) \mid 1 \leq i \leq i_0, 1 \leq j \leq j_0, x \in \Sigma \right) \right\}$  is a set of

independent random variables, each distributed like  $(Y_{ij}(x))$ .

Then we finally require that  $x < x'$  implies that for all  $N$

$$\Pr \left\{ \prod_{n=1}^N Y_{i_n j_n}^{(n)}(x') = \sigma_1 \right\} \geq \Pr \left\{ \prod_{n=1}^N Y_{i_n j_n}^{(n)}(x) = \sigma_1 \right\} ,$$

$$\Pr \left\{ \prod_{n=1}^N Y_{i_n j_n}^{(n)}(x') = \sigma_2 \right\} \leq \Pr \left\{ \prod_{n=1}^N Y_{i_n j_n}^{(n)}(x) = \sigma_2 \right\} .$$

All that we have said about  $\{\Omega(f_1, f_2)\}$  up to Theorem 1, trivially modified, applies to the games  $\{\Omega(x)\}$  in which two players repeatedly and simultaneously choose integers  $i_n$  and  $j_n$

at each time  $n$ , until  $\prod_{n=1}^N Y_{i_n j_n}^{(n)}(x) = \sigma_1$  or  $\sigma_2$ , or ad infinitum

if this never occurs. The payoff is  $(1, 0)$  if the game ends in the state  $\sigma_1$ , and  $(0, 1)$  if the game ends in the state  $\sigma_2$ .

If the game goes on indefinitely, then the payoff is  $(\alpha(C), \beta(C))$  where  $(\alpha(C), \beta(C)) \leq (1, 1)$  and  $\alpha(C) + \beta(C) \leq 1$ , where  $(\alpha(C), \beta(C))$  can depend on the course of the game,  $C$ .

Similarly, Theorem 2 can be generalized by the use of expected values to the situation where  $\Sigma$  is a set of reals and for  $\sigma_1 > x > \sigma_2$

$$\begin{aligned}
 Y_{ij}(x) &= x + a_{ij} & \text{if } \sigma_2 < x + a_{ij} < \sigma_1 \\
 &= \sigma_1 & \text{if } \sigma_1 \leq x + a_{ij} \\
 &= \sigma_2 & \text{if } \sigma_2 \geq x + a_{ij} ,
 \end{aligned}$$

where  $a_{ij}$  is a real-valued random variable whose distribution depends on  $(i, j)$ .