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SOLUTION SETS FOR GAMES ON THE SQUARE

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Summary: Some necessary and sufficient conditions that a pair of non-void weak* closed convex sets of strategies form the solution set of a game with continuous payoff on the square are given.

SOLUTION SETS FOR GAMES ON THE SQUARE

I. Glicksberg and O. Gross

Let K denote the set of all optimal strategies for one player, L the corresponding set for his opponent in a game. We shall refer to K x L, the set of all pairs (f,g), $f \varepsilon K$, $g \varepsilon L$, as the solution set of the game. Any non-void weak* closed convex set K is the set of all optimal strategies for one player in some game with continuous payoff, as was shown in [1], but of course not all pairs K,L of such sets will yield solution sets. By means of constructions similar to those used in [1] we shall determine which pairs do occur in terms of the spectra,[1] $\sigma K, \sigma L$ of these sets and the number of independent containing hyperplanes.

1. Preliminaries. As was shown in [1], any non-void weak* (w^*) closed convex set K of strategies is the intersection

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[1] $\sigma K = \bigcup_{f \varepsilon K} \sigma(f)$, which is easily seen to be a closed set.

of a sequence of half spaces, which we may express by

$$(1) \qquad K = \left\{ f \,|\, (\varphi_n, f) = \int \varphi_n(x)\,df(x) \geq 0,\ n = 1, \ldots \right\}$$

where $\{\varphi_n\}$ is a sequence of continuous functions and we may assume, for each n, $(\varphi_n, f) = 0$ for some f in K. Certain of these φ's will yield $(\varphi, f) = 0$ for all f in K, and these we shall denote by p's. Thus we shall write

$$K = S(\varphi_m; p_n)^{2)}$$

to express the fact that $K = \{ f \,|\, (\varphi_m, f) \geq 0 = (p_n, f) \}$ as well as the fact that (φ_m, K) is a non-degenerate interval. The functions p_m thus define hyperplanes containing K while the φ_m do not. If the set K is the intersection of a set of hyperplanes, one may show exactly as in the proof of (1) that it is the intersection of a sequence of these and one may write $K = S(p_m)$.

What we shall be concerned with in large part in the following constructions will be the hyperplanes containing K. It is immediately evident that if we select from the functions $\{p_n\}$ a maximal subsequence $\{p_n'\}$ which is linearly independent on σK then the relations $(p_n, f) = 0$ are consequences of the relations $(p_n', f) = 0$ for f for which

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2) For the opponent we shall write $L = S(\psi_m; q_n)$ where we take $(\psi_m, g) \leq 0$.

$\sigma(f) \subset \sigma K$. Consequently if we set $p^*(x) = \text{dist}(x, \sigma K)$, then $(p^*, f) = 0$, $(p_n', f) = 0$ all $n \Leftrightarrow (p_n, f) = 0$ all n; thus in most of what follows we shall assume the $\{p_n\}$ to be linearly independent,3) and actually orthonormal:

Suppose we define a measure on σK in the following way: select a sequence $\{x_n\}$ dense in σK and place weight 2^{-n} at x_n. Then clearly we may apply the Gram-Schmidt process to the $\{p_n\}$ to obtain an orthonormal sequence $\{p_n'\}$ of the same length (we take $\{x_n\}$ dense to insure that only the function 0 has the integral of its square zero). Just as clear is the fact that $(p_n, f) = 0$ for all n is equivalent to $(p_n', f) = 0$ for all n.

2. Constructions. We shall now construct payoffs which will have three types of solution sets. That these are the only types which occur will be shown later.

Case I: Suppose $\sigma K = [0,1] = \sigma L$ and K and L are the intersections of the same number of independent hyperplanes. The orthonormal sequences $\{p_n\}$ and $\{q_n\}$ defining K and L are thus of the same length, and if we set

$$M(x,y) = \sum a_n p_n(x) q_n(y),$$

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3) We shall say that the hyperplanes H_n defined by $H_n = \{f \mid (p_n, f) = 0\}$ are independent hyperplanes containing K if the p_n are linearly independent on σK, $K \subset H_n$.

where the a_n are chosen to insure uniform convergence of the series, then for f in K and g in L,

$$\int M df = \sum a_n (p_n, f) q_n(y) = 0 = \sum a_n p_n(x)(q_n, g) = \int M dg;$$

on the other hand, if f is optimal

$$\int M df = \sum a_n (p_n, f) q_n(y) = 0,$$

and in view of the orthogonality of the q_n, f is in K. Similarly every optimal g is in L, and K x L is the solution set.

Case II: Suppose $\sigma K = [0,1] \neq \sigma L$, and $K = S(\varphi_m; p_n)$, $L = S(q_n)$ where there are at least as many independent hyperplanes containing K as there are containing L (thus we may assume that a maximal linearly independent set of p_n's is at least as long as the set of q_n's linearly independent on σL). Since σL is not the full unit interval we may select an open interval I which has one end point y_0 in σL. Select a disjoint sequence $\{I_n\}$ of open subintervals of I for which dist $(y_0, I_n) \rightarrow 0$, and an open subinterval I_n^* of each I_n whose closure lies entirely in I_n. Let k_n be a continuous non-negative function which vanishes outside I_n but is non-zero inside I_n, and which assumes the value 1 at a point y_n of I_n^*. Define a continuous function m_n which vanishes at y_n and outside I_n^*, but takes on the values ± 1.

If we then set $q(y) = \text{dist}\,(y, \sigma L \cup \bigcup I_n^*)$, then for every y not in σL one of the non-negative functions q, k_n is non-zero at y.

We now define our payoff as follows: we divide the sequence $\{p_n\}$ into $\{p_n\}$, orthonormal and of the same length as the $\{q_n\}$, and $\{p_n'\}$. If either of the sequences $\{p_n'\}$ or $\{\varphi_n\}$ are finite we use repetitions to form a sequence, and if there are no φ_n's say, we take $\varphi_n = 1$ for all n. We set (for $b_n > 0$, chosen to insure uniform convergence)

$$M(x,y) = \sum a_n p_n(x) q_n(y) + \sum b_n [k_n(y)\varphi_{N_n}(x) + n m_n(y) p'_{N_n}(x)] + q(y)$$

where $\{N_n\}$ is an enumeration of the integers in which each integer occurs infinitely often. For f in K and g in L

$$\int M df = \sum b_n k_n(y)(\varphi_{N_n}, f) + q(y) \geq 0 = \int M dg,$$

so that both are optimal.

Suppose f is optimal; then for y in σL,

$$\sum a_n(p_n, f) q_n(y) = 0$$

whence $(p_n, f) = 0$, and thus

$$0 \leq \sum b_n [k_n(y)(\varphi_{N_n}, f) + n m_n(y)(p_{N_n}, f)] + q(y),$$

and at setting $y = y_n$, $b_n(\varphi_{N_n}, f) \geq 0$, so that $(\varphi_n, f) \geq 0$ for all n. For y in I_n^* we have

$$0 \leq b_n [k_n(y)(\varphi_{N_n}, f) + n m_n(y)(p'_{N_n}, f)]$$

whence $0 \leq (\varphi_{N_n}, f) + nm_n(y)(p'_{N_n}, f)$, and since m_n assumes the values ± 1,

$$(\varphi_{N_n}, f) \geq \pm n(p'_{N_n}, f),$$

hence

$$(\varphi_{N_n}, f) \geq n|(p'_{N_n}, f)|.$$

Since N_n takes on the value n_o infinitely often, $(\varphi_{n_o}, f) \geq n \mid (p'_{n_o}, f)|$ for arbitrarily large n, and $(p'_{n_o}, f) = 0$ for each n_o. Thus f is in K.

If g is optimal, then for for any f in K,

$$Q = \int\int M df dg = \sum b_n(k_n, g)(\varphi_{N_n}, f) + (q, g).$$

But each term of this sum is non-negative ($(k_n, g) \geq 0$ since $k_n \geq 0$) so that surely $(q, g) = 0$. If $(k_n, g) > 0$ for some n then since there is an f in K for which $(\varphi_n, f) > 0$, we would have a contradiction. Thus

$$(q, g) = 0, \ (k_n, g) = 0, \text{ and } (m_n, g) = 0$$

since $(k_n, g) = 0$ implies g places no weight on I_n. Thus $\sigma(g) \subset \sigma L$, and since we now may write

$$0 = \sum a_n p_n(x)(q_n, g), \qquad x \text{ in } \sigma K,$$

and $(q_n, g) = 0$, g is in L.

Case III: $\sigma K \neq [0,1] \neq \sigma L$. Here we may take any K and L without further restriction, so that $K = S(\varphi_m; p_n)$ and $L = S(\psi_m; q_n)$ $((\psi_m, g) \leq 0$ here, however, in our definitions). We construct functions h_n similar to the k_n of case II, and ℓ_n similar to the m_n, on an interval abutting σK. We set

$$M(x,y) = \sum a_n \left[h_n(x)\psi_{N_n}(y) + n\ell_n(x)q_{N_n}(y) + k_n(y)\varphi_{N_n}(x) + nm_n(y)p_{N_n}(x)\right] .$$

Arguments entirely similar to those used in case II show K x L to be the solution set.

3. Generality. In case I ($\sigma K = [0,1] = \sigma L$) we restricted our attention to the case in which K and L were intersections of the same number of independent hyperplanes. Suppose now that a game with payoff M has as its solution set K x L where σK and σL are the full intervals. K is determined as the set of all f for which

$$\int M(x,y)\,df(x) = 0$$

(for convenience we take the value to be zero), and thus is the intersection of hyperplanes given by the functions $\{M(\cdot,y)\}$, and similarly L is the intersection of the hyperplanes determined by the functions $\{M(x,\cdot)\}$.

If a maximal linearly independent set $\{M(x_i,\cdot)\}$ of the first set, say, is finite, $i = 1,\ldots,n$, then the same is true of the second, indeed there are just as many. For, as is well known, n functions $F_1,\ldots,F_n$ are linearly independent on a set X if and only if there exist $x_1,\ldots,x_n$ in X for which

$$\det (F_i(x_j)) \neq 0;$$

consequently we have $y_1,\ldots,y_n$ for which

$$\det (M(x_i,y_j)) \neq 0, \tag{2}$$

so that the functions $\{M(\cdot,y_j)\}_{j=1,\ldots,n}$ are linearly independent. Of course if $\{M(\cdot,y_j)_{j=1,\ldots,n+1}$ were linearly independent by the same argument we should have an x_{n+1} for which $\{M(x_i,\cdot)\}_{i=1,\ldots,n+1}$ were, which contradicts our assumption, and there are exactly n. Thus the type of solution sets considered in case I are the only type which can occur. (One might note that here finite set of independent containing hyperplanes can only occur in a polynomial-like game, since for every x we have coefficients $a_i(x)$ for which

$$M(x,y) = \sum a_i(x)M(x_i,y),$$

and (2) shows the functions a_i to be continuous.)

In case II, ($\sigma K = [0,1] \neq \sigma L$) we considered only those K and L for which we had as many independent hyperplanes containing K as there are containing L. But if M is the payoff of a game with solution set K x L, $\sigma K = [0,1] \neq \sigma L$, then as before since L is determined by

$$\int M(x,y)dg(y) = 0, \quad \text{all } x,$$

L is just the intersection of hyperplanes. If there are only n independent containing hyperplanes, then, as we shall see in a moment, these must be given by the functions $\{M(x_i,\cdot)\}_{i=1,\dots,n}$ linearly independent on σL, for some set $x_1,\dots,x_n$; consequently there exist $y_1,\dots,y_n$ in σL for which (2) holds, and $\{M(\cdot,y_j)\}_{j=1,\dots,n}$ are linearly independent. Since $\int M(x,y)df(x) = 0$ for y in σL, these functions define n independent hyperplanes containing K.

To see that the n independent hyperplanes containing L arise from functions $M(x_i,\cdot)$ we note that for each x, $M(x,\cdot)$ defines a containing hyperplane since x is in $\sigma K = [0,1]$. Consequently there can be only m points, $m \leq n$, $x_1,\dots,x_m$ for which $\{M(x_i,\cdot)\}$ are linearly independent, so that clearly $L = \{g \mid (M(x_i\cdot),g) = 0,\ i = 1,\dots,m\}$.

If $m < n$, we can find a function q_o for which, denoting $M(x_i,\cdot)$ by q_i, the set $q_o,\dots,q_m$ is linearly independent on σL and $(q_o,g) = 0$ for all g in L. But then the mapping

$$T: g \rightarrow ((q_o,g),\dots,(q_m,g)),$$

of the set S of all strategies into m + 1 space, takes S into a convex subset containing $(0,\dots,0)$ (since L is non-void). But T(S) intersects the line $(t,0,\dots,0)$ in only one point (since $(q_o,g) = 0$ for g in L) — thus $(0,0,\dots,0)$ is a boundary point and we have a supporting hyperplane at this point given by constants (not all zero) $a_o,\dots,a_m$. Thus $\sum_{i=0}^{m} a_i(q_i,g) \geq 0$

for all g in S, hence $\sum a_i q_i(y) \geq 0$ for y in σL. If inequality holds for any y it holds in some neighborhood, and this is, of course, of positive measure with respect to some g in L (from the definition of σL), whence $\sum a_i(q_i, g) > 0$ for some g in L — a contradiction. Thus $\sum a_i q_i = 0$ on σL, which contradicts the linear independence on σL, and we must have m = n.

Thus the theme of things is as follows: The necessary and sufficient condition that K x L be the solution set for a game with continuous payoff on the square (where K and L are non-void ω^* closed convex sets of strategies) is that one of the following hold:

(a) $\sigma K = [0,1] = \sigma L$ and K and L are the intersection of the same number (finite if and only if the game is polynomial-like) of independent containing hyperplanes

(b) $\sigma K = [0,1] \neq \sigma L$, L is the intersection of hyperplanes and K has as many independent containing hyperplanes as L

(c) $\sigma K \neq [0,1] \neq \sigma L$.

The constructions we have used can be duplicated in in the case of a game with continuous payoff played on a pair of infinite compact metric spaces; the character of solution sets, however, involves slightly different conditions:

$\sigma K = [0,1] = \sigma L$ must be replaced by σK, σL open,
$\sigma K = [0,1] \neq \sigma L$ by σK open, σL not open,
$\sigma K \neq [0,1] \neq \sigma L$ by σK and σL not open. In the case of a unique optimal strategy forming K and another forming L we are thus guaranteed a game having K x L as the solution set, which generalizes the result of [2].

As a final remark, we note that solution sets for symmetric games on the square (where $M(x,y) = -M(y,x)$) can be easily described. For such games the value is always zero and any optimal strategy for one player is optimal for his opponent, so that a solution set is of the form K x K. The necessary and sufficient condition that K x K be the solution set of a symmetric game is that either

(a) $\sigma K = [0,1]$ and K is the intersection of an <u>even</u> (we take ∞ as even) number of independent hyperplanes, or

(b) $\sigma K \neq [0,1]$.

For if $\sigma K = [0,1]$ and K is the intersection of an even number of independent hyperplanes given by functions $\{p_n\}$ (which we may take orthonormal), then, dividing these into two sets $\{p_n\}$, $\{p'_n\}$ of equal cardinality, we may set

$$M(x,y) = \sum a_n [p_n(x) p'_n(y) - p'_n(x)\, p_n(y)],$$

which is easily seen to have K x K as its solution, and is symmetric. On the other hand, if K x K is the solution set of a game with payoff M and $\sigma K = [0,1]$, then K is, of course, the intersection of a set of hyperplanes. If only a finite number of these are independent, then, as before, M is polynomial-like, that is,

$$M(x,y) = \sum_{n=1}^{k} \varphi_n(x)\ \psi_n(y),$$

where $\{\varphi_n\}$ and $\{\psi_n\}$ are linearly independent sets of functions. Since M is symmetric

$$M(x,y) = -M(y,x) = -\sum_{n=1}^{k} \varphi_n(y)\,\psi_n(x),$$

so

$$M(x,y) = \frac{1}{2}\sum_{n=1}^{k} [\varphi_n(x)\ \psi_n(y) - \varphi_n(y)\,\psi_n(x)].$$

If the functions $\{\varphi_n, \psi_n\}$ are not a linearly independent set, replacement of a dependent φ or ψ again yields a sum of the same type, and we finally obtain a similar expression for M in which the set $\{\varphi_n, \psi_n\}$ is linearly independent; however, there are an even number of terms in the resulting sums, and thus there must be an even number of independent hyperplanes determining K.

In case (b), $K = S(\varphi_m;p_n)$, and we may set

$$M(x,y) = \sum a_n [k_n(y)\varphi_{N_n}(x) + nm_n(y)p_{N_n}(x) - k_n(x)\varphi_{N_n}(y) - nm_n(x)p_{N_n}(y)]$$

to obtain a symmetric game in which K x K is the solution set.

REFERENCES

1. I. Glicksberg and O. Gross, Optimal Sets for Games over the Square, RM-889.
2. I. Glicksberg and O. Gross, Continuous Games with Given Unique Solutions, RM-620.