U. S. AIR FORCE
PROJECT RAND
RESEARCH MEMORANDUM

A MATHEMATICAL TREATMENT OF LEARNING MODELS

Samuel Karlin

RM-921

2 September 1952

Assigned to ____________________

The RAND Corporation
1700 MAIN ST. • SANTA MONICA • CALIFORNIA

Report Documentation Page

Form Approved
OMB No. 0704-0188

1. REPORT DATE **02 SEP 1952**	2. REPORT TYPE	3. DATES COVERED **00-00-1952 to 00-00-1952**

4. TITLE AND SUBTITLE **A Mathematical Treatment of Learning Models**	5a. CONTRACT NUMBER
	5b. GRANT NUMBER
	5c. PROGRAM ELEMENT NUMBER
6. AUTHOR(S)	5d. PROJECT NUMBER
	5e. TASK NUMBER
	5f. WORK UNIT NUMBER
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) **Rand Corporation,Project Air Force,1776 Main Street, PO Box 2138,Santa Monica,CA,90407-2138**	8. PERFORMING ORGANIZATION REPORT NUMBER
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)	10. SPONSOR/MONITOR'S ACRONYM(S)
	11. SPONSOR/MONITOR'S REPORT NUMBER(S)

12. DISTRIBUTION/AVAILABILITY STATEMENT
Approved for public release; distribution unlimited

13. SUPPLEMENTARY NOTES

14. ABSTRACT

15. SUBJECT TERMS

16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
a. REPORT **unclassified**	b. ABSTRACT **unclassified**	c. THIS PAGE **unclassified**	**Same as Report (SAR)**	**31**	

Standard Form 298 (Rev. 8-98)
Prescribed by ANSI Std Z39-18

SUMMARY

This paper discusses the general linear learning model in one dimension.

A MATHEMATICAL TREATMENT OF LEARNING MODELS. I.

Samuel Karlin

This note presents a complete mathematical analysis of transition operators arising in some learning models introduced by Bush and Mosteller [1], Estes [2] and others. These first two papers shall examine in detail the one dimensional models arising in their theory. They can be described as follows: A particle on the unit interval executes a random walk subject to several impulses. We restrict ourselves in this work to two motions. If it is located at the point $x (0 \leq x \leq 1)$, then with probability $\phi(x)$ the particle moves to the position $(1-\alpha) + \alpha x (0 \leq \alpha \leq 1)$ and with probability $1 - \phi(x)$ the transition takes place from x to $\delta x (0 \leq \delta \leq 1)$. This complete work is devoted to studying all possible models arising from different linear functions taken for $\phi(x)$. In subsequent papers, we shall present the complete n dimensional analogues of these models with also other extensions obtained by removing the linearity restriction imposed on $\phi(x)$.

The method used is to introduce two linear operators, one representing the transition law describing the change of the distribution giving the position of the particle after each experiment and the other operator corresponding to the essential dual of this operator which maps continuous functions into continuous functions. §1 treats the special case where $\phi(x) = x$. This particular model was independently treated by R. Bellman,

T. Harris, and H. N. Shapiro.* This paper includes essentially all their results and presents several new results valid for this model. §2 examines the case where $\phi(x)$ is monotonic increasing with $[\phi(x) - \phi(y)] \leq \mu < 1$. §3 deals with the case where $\phi(x) = 1-x$. This case behaves differently from all the preceding and possesses many interesting new features. §4 considers the case where $\phi(x)$ is linear and monotonic decreasing.

§1. A particle undergoes a random walk on the unit interval subject to the following law: If the particle is at x then $x \to \alpha + (1-\alpha)x$ with probability x and $x \to \sigma x$ with probability $1-x$ where $0 < \alpha, \sigma < 1$. If $F(x)$ represents the cumulative distribution as to the location of x with the understanding that $F(x) \equiv 1$ for $x \geq 1$ and $F(x) = 0$ for $x \leq 0-$, then the new distribution describing the location of the particle is easily seen to be given by

$$(1) \quad G(x) = TF = \int_0^{\frac{x}{\sigma}} (1-t)dF(t) + \int_0^{\frac{x-\alpha}{1-\alpha}} t dF(t) .$$

This represents the transition law for the particular Markoff process on hand.

The transformation T is easily seen to be a continuous bounded mapping of the space of functions of bounded variation on the unit interval into itself. Furthermore, T takes distributions into distributions and is of norm 1. The problem is to study the iterates T^n and to find whatever limiting behavior exists for T^n.

---

* RM-878 "Studies on Functional Equations Occurring in Decision Processes," The RAND Corporation, 1 July 1952.

We consider the mapping U applied to the space of continuous functions defined on the unit interval given by

$$(2) \quad (U\pi)(t) = (1-t)\pi(\sigma t) + t\pi(\alpha+(1-\alpha)t)$$

The operator U has a probabilistic interpretation which we shall speak about later but its relevance to T is given in Theorem 1. The inner product notation $(\pi,F) = \int_0^1 \pi(t)dF(t)$ will b be extensively used.

Theorem 1. The conjugate map U^* to U is T.

Proof: It is necessary to verify that $(U\pi,F) = (\pi,TF)$ for any continuous function $\pi(t)$ and any distribution $F(t)$ with $F(t) \equiv 1$, $t \geq 1$ and $F(t) = 0$ for $t \leq 0-$. Indeed

$$(U\pi,F) = \int (1-t)\pi(\sigma t)dF(t) + \int t\pi(\alpha+(1-\alpha)t)dF(t) .$$

By a change of variable, we get

$$(U\pi,F) = \int(1-\tfrac{t}{\sigma})\pi(t)dF(\tfrac{t}{\sigma}) + \int \pi(t)\frac{t-\alpha}{1-\alpha}dF(\frac{t-\alpha}{1-\alpha})$$

$$= \int \pi(t)dG(t) \quad \text{where} \quad G(t) = TF .$$

The value of Theorem 1 is that by studying the iterates of U^n we deduce results about the conjugate operators T^n. We proceed now to study this operator U. In essence, we should denote the operator by $U_{\sigma\alpha}$ but where no ambiquity arises we shall drop the subscripts. Let W denote the isometry $W\pi(t) = \pi(1-t)$. Clearly $W^{-1} = W$. We now observe the identity

(3) $U_{1-\alpha,\ 1-\sigma} = WU_{\sigma,\alpha}W$.

The mapping $\sigma,\alpha \rightarrow (1-\alpha, 1-\alpha)$ has the effect of mapping the portion of the unit square bounded above by $1-\alpha-\sigma = 0$ into the other triangle located in the unit square. Consequently, in the future we restrict ourselves to the case where $1-\alpha-\sigma \geq 0$. Corresponding results for the circumstance where $1-\alpha-\sigma < 0$ are deduced easily by virtue of the identity (3).

The next two theorems which we state for completeness, are immediate from (2).

Theorem 2. The operator U preserves the values at 0 and 1.

Theorem 3. The operator U is positive i.e., it transforms positive continuous functions into positive continuous functions.

In particular, if $\pi_1 \geq \pi_2$ i.e., for all t, then $U\pi_1 \geq U\pi_2$.

Theorem 4. If $\pi, \pi', \ldots \pi^{(n)} \geq 0$, then $U\pi, (U\pi)' \ldots (U\pi)^{(n)} \geq 0$.

Proof: A simple calculation yields

$$(U\pi)^{(n)} = (1-t)\sigma^n \pi^{(n)}(\sigma t) + t(1-\alpha)^n \pi^{(n)}(\alpha+(1-\alpha)t)$$

$$+ n(1-\alpha)^{n-1} \quad \pi^{(n-1)}(\alpha+(1-\alpha)t)$$

$$- n\sigma^{(n-1)} \qquad \pi^{(n-1)}(\sigma t)$$

Since $6t < t < \alpha + (1-\alpha)t$, we conclude as $\pi^{(n-1)}(t)$ is monotonic increasing that $\pi^{(n-1)}(\alpha+(1-\alpha)t) \geq \pi^{(n-1)}(6t) \geq 0$. The assumption $1-\alpha \geq 6$ implies that $(1-\alpha)^{n-1} \geq 6^{n-1}$. As $\pi^{(n)}(t) \geq 0$ it follows that $(U\pi)^{(n)} \geq 0$. The same conclusion and argument apply to $(U\pi)^{(i)}$ for $0 \leq i \leq n-1$.

In particular, U transforms positive monotonic convex functions into functions of the same kind. Although in the proof of Theorem 4 we assumed the existence of derivatives, the argument can be carried through routinely at the expense of elegance using the general definitions of convexity and monotonicity.

Theorem 5. If $c \geq \pi^{(i)}(t) \geq 0$ for $0 \leq i \leq n$, then $(U^r\pi)^{(i)}(1) \leq K_i$ for $0 \leq i \leq n$, and hence $(U^r\pi)^{(i)}(t) \leq K_i$.

Proof: The proof is by induction. Suppose we have established the result for the i^{th} derivative with $0 \leq i \leq n-1$. Equation (4) yeilds

$$(5) \quad (U\pi)^{(n)}(1) - \pi^{(n)}(1) = c_1(\alpha)\pi^{(n-1)}(1) - c_2(6)\pi^{(n-1)}(6)$$

$$+ \left[(1-\alpha)^n - 1\right]\pi^{(n)}(1)$$

where $c_1(\alpha)$ and $c_2(6)$ are constants depending only on α and σ respectively and on n. If $\pi^{(n)}(1) > M(\alpha, \sigma, c)$ where M is a constant sufficiently large, then (5) yeilds $(U\pi)^{(n)}(1) < \pi^{(n)}(1)$. Since $c_1(\alpha)$ and $c_2(6)$ do not depend on k and by the induction hypotheses $|(U^k\pi)^{n-1}(x)| \leq M$ uniformly in k and x, we find in general that when $(U^k\pi)^{(n)}(1)$ becomes larger than $M(\alpha, \sigma, c)$, then $(U^{k+1}\pi)^{(n)}(1) < (U^k\pi)^{(n)}(1)$. Consequently, the iterates

$(U^k\pi)^{(n)}(1)$ for $k \geq k_0$ is bounded by $M(\alpha,\sigma,c) + c_1(\alpha)M + c_2(\sigma)M$. This trivially implies the conclusion of Theorem 5.

Theorem 6 (Bellman) There exists at most one continuous solution $U\pi = \pi$ for which $\pi(0) = 0$ and $\pi(1) = 1$. We present the proof for completeness.

Proof: (By contradiction) Let π_1 and π_2 denote two solutions with the prescribed boundary conditions. Put $\pi_0 = \pi_1 - \pi_2$, then $\pi_0(0) = \pi_0(1) = 0$. Let t_0 be a point where π_0 achieves its maximum. Since

$$\pi(t_0) = (1-t_0)\pi(\sigma t_0) + t_0\,\pi(\alpha+(1-\alpha)t_0)$$

we deduce that σt_0 is also a maximum point. Iterating, we find by continuity that $\pi(0) \equiv 0$ is the maximum value of $\pi(t)$. A similar argument shows that $0 = \min \pi(t)$ which implies that $\pi_1 = \pi_2$.

Theorem 7. For any function $\pi(t) = t^r$ with $\infty > r \geq 1$, $U^n(t^r)$ converges uniformly as $n \to \infty$.

Proof: Clearly $t \geq t^r > p(t)$ where $p(t) = \begin{cases} 0 & \text{for } 0 \leq t \leq t_0 \\ \frac{t-t_0}{1-t_0} & \text{for } t_0 \leq t \leq 1 \end{cases}$

where t_0 is close to 1 with r fixed. Since $U(t)$ is convex and the values at 0 and 1 are fixed, we find that $t \geq Ut$. Hence, $U^n t \geq U^{n+1} t \geq 0$ and $\lim U^n t = \theta(t)$ for every t. Since $\theta(t)$ is convex and by Theorem 5 the derivatives of $U^n t$ at 1 are uniformly bounded, we conclude that $\theta(t)$ is continuous. By Dini's Theorem the convergence of $U^n t$ to $\theta(t)$ is uniform. Trivially $U\theta = \theta$. On the other hand, if t_0 is close to 1, then $(Up)'(1) < p'(1)$ (see the proof of Theorem 5). Since Theorem 4 guarantees the convexity of Up and the slope at 0 is 0, it follows that $U^n p \leq U^{n+1} p$ and hence $\lim U^n p = \phi(t)$. Again $\phi(t)$ is a continuous fixed point and therefore by Theorem 6 we infer that $\phi(t) = \theta(t)$. On account of $U^n t \geq U^n t^r \geq U^n p$, we deduce that $\lim U^n t^r = \phi(t)$ with the convergence being uniform.

We denote the unique fixed point of U by $\phi_{\sigma,\alpha}(t)$ or $\phi(t)$ whenever no ambiguity arises.

Theorem 8. The iterates U^n converge strongly.

Proof: The constant function are fixed points of U^n. Consequently by Theorem 7, $U^n q$ converges uniformly for any function $q(t)$ in the linear space L spanned by $(1,t^r)$. The set L is dense in the space of continuous functions. Moreover, as $\|U^n\| = 1$, by a well-known theorem of Banach U^n converges strongly when applied to any continuous function $q(t)$.

The acutal limit is easily seen to be given by

$$(6) \quad \lim_{n\to\infty} U^n q(t) = q(1)\, \phi_{\sigma,\alpha}(t) + q(0)\left[1-\phi_{\sigma,\alpha}(t)\right]$$

This is an immediate consequence of the fact that the fixed points of U consist of the two dimensional space spanned by the constants and $\phi_{\sigma,\alpha}$. Equation (6) shows that two functions q_1 and q_2

which agree at 0 and 1 have the same limit. This enables us to show

Theorem 9. If $q(t)$ is any bounded function continuous at 0 and 1, then $U^n q$ converges strongly.

Proof: Let $q(t)$ in addition to being continuous at 0 and 1 possess finite derivatives at 0 and 1. Then clearly there exists two continuous functions $h_1(t)$ and $h_2(t)$ with $h_1(t) \geq q(t) \geq h_2(t)$ where $h_1(0) = h_2(0)$ and $h_1(1) = h_2(1)$. We condlude the result from this using the argument of Theorem 7 and equation (6). If now $q(t)$ is only continuous at 0 and 1 then we can find for any ϵ a $q_\epsilon(t)$ satisfying the properties assumed about $q(t)$ in the first part of the proof with $|q(t) - q_\epsilon(t)| \leq \epsilon$. As $\|U^n\| = 1$, the conclusion of the theorem now follows by a standard argument.

Theorem 10. If $q^{(i)}(t) \geq 0$ for $0 \leq i \leq n$ and $q^{(i)}(1) \leq c_i$, then $\lim_{m\to\infty} [U^m q(t)]^{(r)}$ converges uniformly for $0 \leq r \leq n-2$.

Proof: We prove the result only for $(U^m q)^{(1)}$ since the proof for general (r) is similar. Theorem 4 shows that the functions $(U^m q)^{(1)}$ are all convex monotonic and positive. By Helly's selection theorem we select a sequence $(U^{n_i} q)^{(1)} \to m(t)$. Again, $m(t)$ is convex monotonic and positive and by virtue of Theorem 5 is continuous for all $0 \leq t \leq 1$. This implies easily that $(U^{n_i} q)^{(1)}$ converges uniformly to $m(t)$. As $U^{n_i} q$ converges uniformly to a fixed point ψ, we find that $m(t) = \psi'(t)$. Since the limit is the same for any subsequence the conclusion follows.

Theorem 11. If $q(t)$ belongs to C^n (n continuous derivative), then $\lim_{m\to\infty} [U^m q(t)]^{(r)}$ converges uniformly for $0 \leq r \leq n-2$.

Proof: This follows directly from Theorem 10 since we can add to $q(t)$ a polynomial of n^{th} degree with large positive coefficients so that the hypothesis of Theorem 10 are satisfied.

Theorem 12. The fixed point $\phi_{\sigma,\alpha}$ is analytic for $0 \leq t < 1$ with $\phi^{(r)}_{\sigma,\alpha} \geq 0$.

Proof: Let $p(t)$ denote a function infinitely differentiable with $p^{(r)}(t) \geq 0$ and $p(0) = 0$ $p(1) = 1$. By virtue of Theorem 11 and Theorem 4 we deduce that $\lim_{n\to\infty} (U^n p)^{(r)} = \phi^{(r)}_{\sigma,\alpha} \geq 0$. Therefore $\phi^{(r)}$ is absolutely monotonic and hence by a well-known theorem is analytic.

Theorem 13. The functions $\phi_m(t) = \sum_{n=m}^{\infty} U^n(t(1-t))$ converge geometrically to 0.

Proof: It is immediate from (6) that $U^n(t(1-t)) = \psi_n(t)$ tends uniformly to zero. Since the derivative at 0 and 1 of $t(1-t)$ is $\frac{1}{2}$ and $-\frac{1}{2}$, we conclude by Th. 11 that for n sufficiently large there exists an $n_0(\lambda)$ with $U^{n_0}(t(1-t)) \leq \lambda t(1-t)$ with $\lambda < 1$. Let kn_0 denote the last interger k for which $kn_0 \leq m$. We obtain,

$$0 \leq \phi_m(t) \leq \phi_{kn_0}(t) \leq \frac{\lambda^k}{1-\lambda} \sum_{i=0}^{n_0-1} U^i(t(1-t)) \leq C\lambda^k$$

$$\leq C\rho^{(n_0+1)k} < C\rho^m \quad \text{where} \quad \rho = \lambda^{\frac{1}{n_0+1}} < 1.$$

Theorem 14. If $q(t)$ is continuous, $|q'(1)| < \infty$ and $|q'(0)| < \infty$, then $\lim U^n[q(t)]$ converges geometrically.

Proof: We first establish the result for the special functions t^r with $1 \leq r < \infty$. A simple calculation shows that

$$-Ct(1-t) \leq U(t^r) - t^r \leq Ct(1-t)$$

For $n < m$, we obtain upon continued application of U and summation that

$$-C \sum_{i=m}^{n} U^i(t(1-t)) \leq U^n(t^r) - U^m(t^r) \leq C \sum_{i=m}^{n} U^i(t(1-t)) \; .$$

The conclusion now follows from Theorem 13. The general function $q(t)$ satisfying the hypothesis of Theorem 14 can be bounded from above and below by two polynomials $P_1(t)$ and $P_2(t)$ which agree at 0 and 1. The result now follows directly from this fact and the first part of this proof.

We observe easily the identity $Ut - t = (\alpha+\sigma-1)t(1-t)$. Applying successively U and adding, we obtain

$$(7) \quad \phi_{\sigma,\alpha} = \lim_{n\to\infty} U^m t = t + (\alpha+\sigma-1) \sum_{n=1}^{\infty} U^n_{\sigma,\alpha} t(1-t)$$

This is useful for purposes of calculation.

Some remarks describing the dependence of $\phi_{\sigma,\alpha}$ on σ and α are in order. We consider the following identity:

$$(8)\quad U^n_{\sigma,\alpha} - U^n_{\sigma'\alpha'} = \sum_{i=0}^{n-1} U^i_{\sigma,\alpha}(U_{\sigma,\alpha} - U_{\sigma',\alpha'})\, U^{n-i-1}_{\sigma',\alpha'}$$

If $f(t)$ is any function with bounded derivatives, then we obtain by the mean value theorem that

$$|(U_{\sigma,\alpha}-U_{\sigma',\alpha'})f| \leq |(1-t)[f(\sigma t)-f(\sigma' t)] + t[f(\alpha+(1-\alpha)t)$$

$$- f(\alpha'+(1-\alpha')t)]| \leq C(|\sigma-\sigma'| + |\alpha-\alpha'|)t(1-t) \ .$$

Applying equation 8 to $f(t) = \phi_{\sigma'\alpha'}$, and remembering that inequalities are preserved

$$|U^n_{\sigma,\alpha}\phi_{\sigma'\alpha'} - \phi_{\sigma'\alpha'}| \leq C(|\sigma-\sigma'| + |\alpha-\alpha'|) \sum_{i=0}^{n-1} U^i(t(1-t))$$

Allowing n to go to ∞, we have easily that

$$|\phi_{\sigma,\alpha}-\phi_{\sigma'\alpha'}| \leq K(|\sigma-\sigma'| + |\alpha-\alpha'|)$$

where $K(\delta)$ is finite provided that $0<\delta<\alpha,\alpha'\sigma,\sigma'<1-\delta<1$. It is worthwhile to discuss the nature of $\phi_{\sigma,\alpha}$ for (σ,α) lying on the boundary of the unit square. First, we observe by direct verification that when $\alpha+\sigma = 1$ that $\phi_{\sigma,\alpha}(x) = x$. Next let $\alpha = 0$ and $\sigma < 1$, then

$$U\phi = (1-x)\ \phi(\sigma x) + x\phi(x)$$

Therefore, if ϕ is a fixed point with $\phi(0) = 0$ and $\phi(1) = 1$ then for $x \neq 1$ we have that $\phi(x) = \phi(\sigma x)$ and hence $\phi(x) \equiv \phi(0)=0$ $(0 \leq x<1)$ provided that ϕ is continuous at 0. Similarly, when $\sigma = 1$ and $\alpha < 1$ then the only fixed point ϕ continuous at 1 and $\phi(0) = 0$, $\phi(1) = 1$ is $\phi(x) \equiv 1$ for $0 < x \leq 1$. On the other two boundaries of the unit square the solutions are easily calculated and turn out as follows: If $0 < \sigma < 1$ is arbitrary and $\alpha = 1$, then $\phi_{\sigma,1} = 1 - \prod_{r=0}^{\infty} (1-\sigma^r x)$ while when $\sigma = 0$, $0 < \alpha < 1$, then

$$\phi_{0,\alpha} = \prod_{r=0}^{\infty} L^r x$$

where $L^0 = I$ and the operation L applied to x gives $\alpha + (1-\alpha)x$. Finally for $\alpha = 0$, $\sigma = 1$ the operator U reduces to the identity mapping. We now investigate the dependence of $\phi_{\sigma,\alpha}$ on σ and α as we allow σ and α to tend to the boundary. We limit ourselves for definiteness to studying the case where $(\sigma,\alpha) \rightarrow (\sigma_0,0)$ with $\sigma_0 < 1$ and we show that $\phi_{\sigma,\alpha}$ converges pointwise to 0 for $0 \leq x < 1$ and $\phi_{\sigma,\alpha}(1) \equiv 1$ otherwise. Moreover, the convergence is uniform in any interval $0 \leq x \leq 1-\delta < 1$. Let $(\sigma_n,\alpha_n) \rightarrow (\sigma_0,0)$, then without loss of generality we may assume that $1-\sigma_n - \alpha_n > 0$. Therefore, ϕ_{σ_n,α_n} are convex, monotonic increasing and positive with $\phi_{\sigma_n\alpha_n}(0) = 0$. Also, for any interior interval $0 \leq x \leq 1 - \delta < 1$, the first derivatives are $\phi'_{\sigma_n, \alpha_n}$ uniformly bounded. Since this implies $\phi_{\sigma_n, \alpha_n}$ are equi-continuous over the sub interval and as $0 \leq \phi_{\sigma_m, \alpha_n} \leq 1$, we can select a subsequence which may be denoted as $\phi_{\sigma_r \alpha_r}$ converging to $\psi(t)$ uniformly for any interval of the form $0 \leq x \leq 1 - \delta < 1$. As $\phi_{\sigma_r, \alpha_r}(1) = 1$, we get $\psi(1) = 1$ and similarly $\psi(0) = 0$. The uniform convergence of $\phi_{\sigma_r, \alpha_r}$ guarantees the continuity of ψ at zero. Put $U_r = U_{\sigma_r, \alpha_r}$, $U_0 = U_{\sigma_0, 0}$ and $\phi_r = \phi_{\sigma_r, \alpha_r}$. We consider the following identity:

$$\psi - U_0\psi = (\psi - \phi_r) + (\phi_r - U_r\psi) + (U_r\psi - U_0\psi) = I_1 + I_2 + I_3$$

We consider a fixed $x < 1$, then trivially $|I_1| + |\psi - \phi_r| \leq \epsilon$ when r is sufficiently large. Also $|I_2| = |\phi_r - U_r\psi| = |U_r\phi_r - U_r\psi|$ $= |(1-x)\,[\phi_r(\sigma_r x) - \psi(\sigma_r x)] + x[\phi_r(\alpha_r + (1-\alpha_r)x) - \psi(\alpha_r + (1-\alpha_r)x]|$

But for $x = x_0 < 1$ fixed, we observe that $\alpha_r + (1-\alpha_r)x_0$ varies in an interval $\leq 1 - \delta$ as $\alpha_r \to 0$ and the same applies to $\sigma_r x$. The uniform convergence of $\phi_r \to \psi$ inside $0 \leq x \leq 1-\delta$ yields that $|I_2| \leq \epsilon$. By construction, $|I_3| \leq \epsilon$ for r large. Thus we infer the equality $\psi = U_0\psi$ for $0 \leq x < 1$ and by direct verification for 1. However, the fixed point to the equation $U_0\psi = \psi$ with $\psi(0) = 0$, $\psi(1) = 1$ and ψ continuous at 0 is $\psi(x) = 1$ for $0 \leq x < 1$ and $\psi(1) = 1$. Thus the limit function ψ is the same for every subsequence of $\phi_{\sigma_n, \alpha_n}$ and hence we deduce that

$\phi_{\sigma_n, \alpha_n} \to 0$ pointwise. We furthermore note that ψ is independent of $\sigma_0 < 1$. A similar analysis applies to the case where $(\sigma,\alpha) \to (1,\alpha)$ $(\alpha > 0)$. The other two boundaries yeild to simpler analysis. Summarizing, we have established the following theorem.

<u>Theorem</u> 15. The fixed points $\phi_{\sigma,\alpha}$ satisfy the following continuity properties: For $0 < \delta \leq \alpha, \alpha' \leq 1$ and $0 \leq \sigma, \sigma' \leq 1 - \delta$, then $|\phi_{\sigma,\alpha} - \phi_{\sigma',\alpha'}| \leq K(\delta)[|\sigma-\sigma'| + |\alpha - \alpha'|]$. If $(\sigma,\alpha) \to (\sigma_0, 0)$ with $\sigma_0 < 1$, then $\phi_{0,\alpha}(x) \to 0$ pointwise for $0 \leq x < 1$ and $\phi_{\sigma,\alpha}(1) \equiv 1$. If $(\sigma,\alpha) \to (1,\alpha_0)$ with $\alpha_0 > 0$, then $\phi_{\sigma,\alpha}(x) \to 1$ pointwise for $0 \leq x \leq 1$.

Finally a word concerning convergence of $U^n\pi$ for π continuous when the parameter values lie on the boundary. When $\alpha = 0$ $\sigma < 1$, then $U^n\pi$ converges pointwise. The same comclusion holds when $\alpha > 0$ and $\sigma = 1$. On the other two boundaries the convergence is uniform for $U^n\pi$. We omit the proofs.

We now return to the study of the operator T.

<u>Theorem</u> 16. For any distribution the iterates T^nF converge in the sense of distributions to the distribution

$$G(x) = I_1(x) \int \phi_{\sigma,\alpha} dF + I_0(x) \int (1-\phi_{\sigma,\alpha}) dF$$

where $I_0(x)$ and $I_1(x)$ are the distributions concentrating fully at 0 and 1 respectively.

<u>Proof</u>: From the convergence of $U^n\pi$ for any continuous function π and Theorem 1 follows the weak* convergence of T^nF. This is equivalent to the convergence of T^nF in the sense of distributions.

The actual form of $\lim_{u\to\infty} T^nF = G$ as given in the theorem follows directly from (6).

§2. In this second model the random walk is described as follows: If the particle is at x, then $x \to \alpha + (1-\alpha)x$ with probability $\phi(x)$ and $x \to \sigma x$ with probability $1 - \phi(x)$ where $|\phi(x)-\phi(y)| \leq \mu < 1$. The analogous transition operator to (1) becomes

$$(9)\quad G(x) = TF = \int_0^{\frac{x}{\sigma}} (1-\phi(t))dF(t) + \int_0^{\frac{x-\alpha}{1-\alpha}} \phi(t)dF(t)$$

with the same understandings concerning F applying as before. Let

$$(10)\quad U\pi = [1-\phi(t)]\pi(\sigma t)+\phi(t)\pi(\alpha+(1-\alpha)t)$$

In this section, we take $0<\alpha, \sigma < 1$ although the boundary values for α and σ are easy to handle but not of great interest. The spaces on which they operate are the same as in section 1. Again, in a similar manner to Theorem 1 we obtain

Theorem 17. The operator T is conjugate to the operator U.

We now further assume that $\phi(t)$ is monotonic increasing.

Theorem 18. The operator U preserves positivity and positive monotonic increasing functions.

Proof: Direct verification.

Since the hypothesis on $\phi(t)$ implies either $\phi(1) < 1$ or $\phi(0) > 0$. We analyze the case where $\phi(1) < 1$. The other circumstance can be treated in an analogous manner. Furthermore, we now assume that if $\phi(0) = 0$, then $\phi'(0)$ exists and is finite.

Theorem 19. If $\pi(t)$ is monotonic increasing bounded and positive, then $U^n\pi$ converges uniformly to a constant.

Proof: We observe first that

$$(11)\quad \begin{aligned} U\pi(1) - \pi(1) &= [1-\phi(1)][\pi(\sigma) - \pi(1)] \leq 0 , \\ U\pi(0) - \pi(0) &= \phi(0)[\pi(\alpha) - \pi(0)] \geq 0 . \end{aligned}$$

On account of Theorem 18, we conclude that $0 \leq U^n\pi(1) \leq U^{n-1}\pi(1)$ and $C \geq U^n\pi(0) \geq U^{n-1}\pi(0)$ and hence $\lim_{n\to\infty} U^n\pi(1)$ and $\lim_{n\to\infty} U^n\pi(0)$ exist. Since $1-\phi(1) > 0$. (11) shows that $\lim_{n\to\infty} U^n\pi(\sigma) = \lim_{n\to\infty} U^n\pi(1)$. As $U^n\pi$ are monotonic increasing and bounded, by virtue of Helly's Theorem we get a subsequence $U^{n_i}\pi(t)$ convergent to a limit $g(t)$ which is monotonic increasing. The above argument shows that $g(\sigma) \equiv g(1)$ and hence $g(t) \equiv g(1)$ for $\sigma \leq t \leq 1$. By (10), it follows that $\lim_{i\to\infty} U^{n_i}\pi(\sigma^2) = g(1)$ and hence $g(t) \equiv g(1)$ for $\sigma^2 \leq t \leq 1$. Continuing in this way, we get that $g(t) \equiv g(1)$ for $0 < t \leq 1$. If $\phi(0) > 0$, then by (11) it follows that

$\lim_{n\to\infty} U^n\pi(1) = \lim_{n\to\infty} U^n\pi(\alpha)$. On the other hand when $\phi(0) = 0$ as $0 \leq \phi'(0) < \infty$, we find that

$$(U\pi)'(0) = \pi'(0) = (\sigma-1)\,\pi'(0) + \phi'(0)\,[\pi(\alpha) - \pi(0)]\ .$$

Consequently, if $\pi'(0) > \frac{\phi'(0)}{1-\sigma} C$, then $U\pi'(0) < \pi'(0)$ and therefore we infer easily that $(U^n\pi)'(0)$ are uniformly bounded. From this we conclude easily that $\lim_{n\to\infty} U^n\pi(0) = g(1)$. The limit function $g(t) \equiv g(1)$ is thus uniquely determined by $\lim_{n\to\infty} U^n\pi(1)$ and is therefore independent of the subsequence chosen. Since $g(t)$ is continuous we obtain as $U^n\pi$ are monotonic that $U^n\pi(t)$ converges uniformly to the constant $g(1)$.

The hypothesis on $\phi(t)$ easily yields the fact that the only continuous fixed points of $U\pi = \pi$ are constant functions. This **fact directly connects with the result of Theorem 21 below. First,** we complete the proof of the convergence of $U^n\pi$ for any continuous function $\pi(t)$.

Theorem 20. The operators $U^n\pi$ converge uniformly for any continuous function.

Proof: Since $\| U^n \| = 1$ and the space of all monotonic positive continuous functions span a dense subset of the set of all continuous functions the theorem follows a wlll-known theorem of Banach.

Theorem 21. For any distribution F the distributions T^nF converge as distributions to a unique distribution G for which $TG = G$ which is independent of F.

Proof: The weak* convergence of T^nF follows directly from Theorem 20 and Theorem 16. To complete the proof we must establish that if $\lim T^n F = G$ and $\lim T^n H = K$, then $G = K$. Indeed, let ψ denote any continuous function. We have that

$$(\psi, G-K) = \lim_{n\to\infty} (\psi, T^n(F-H))$$

$$= \lim_{n\to\infty} (U^n\psi, F-H) = a \left(\int dF - \int dH\right) \equiv 0$$

as F and H are distributions. Hence, $\int \psi(t)dG(t) = \int \psi(t)dK(t)$ for any continuous function ψ and therefore $G = K$.

It seems extremely difficult to determine the nature of this unique fixed distribution G. We denote it through $F_{\sigma,\alpha}$

Theorem 22. The distribution $F_{\sigma,\alpha}$ is a continuous function of σ,α i.e. if $(\sigma_n,\alpha_n) \to (\sigma,\alpha)$ with $0 < \sigma,\alpha < 1$, then $F_{\sigma_n,\alpha_n} \to F_{\sigma,\alpha}$ at every point of continuity of $F_{\sigma,\alpha}$.

Proof: Let $(\sigma_n,\alpha_n) \to (\sigma,\alpha)$ and by Helly's theorem we choose a subsequence $F_r = F_{\sigma_{n_r}\alpha_{n_r}}$ converging to the distribution F at every continuity point. Denote T_r for $T_{\sigma_{n_r},\alpha_{n_r}}$ and T for $T_{\sigma,\alpha}$. Let $\pi(t)$ denote any fixed continuous function. We consider the quantity

$$(\pi,F-TF) = (\pi,F-F_r) + (\pi,F_r) - (\pi,TF_r) + (\pi,TF_r-TF)$$

Since $F_r \to F$ as distributions, we find for r sufficiently large that $|(\pi,F-F_r)| < \epsilon$. Now, we note that

$$|(\pi,F_r)-(\pi,TF_r)| = |(\pi,T_rF_r) - (\pi,TF_r)| = |(U_r\pi-U\pi,F_r)| .$$

Since $U_r = U_{\sigma_{n_r},\alpha_{n_r}}$ converges strongly to $U = U_{\sigma,\alpha}$, it follows that $U_r\pi$ converges uniformly to $U\pi$. Whence, as F_r are distributions, we infer that $|(U_r\pi-U\pi,F_r)| \leq \max_t |U_r\pi-U\pi| \leq \epsilon$ when r is chosen large enough. Evidently, with r large we get as before that

$$|(\pi,T(F_r-F))| = |(U\pi, F_r-F)| \leq \epsilon .$$

Therefore, we obtain for r large that $|(\pi,F-TF)| \leq 3\epsilon$ and hence $(\pi,F) = (\pi,TF)$. Since π is any continuous function, we infer $F = TF$ and therefore $F = F_{\sigma,\alpha}$ by Theorem 21. Consequently, as any limit distribution of F_{σ_n,α_n} must be $F_{\sigma,\alpha}$ the conclusion of Theorem 22 is now immediate.

§3 The model considered in this section is as before with $\phi(x) = 1-x$. The operator U becomes

$$(12) \quad U\pi(t) = t\pi(\sigma t) + (1-t)\,\pi(1-\alpha+\alpha t)$$

Note that we have replaced α by $1-\alpha$. This is only for convenience in Theorem 28. In this model the closer the particle moves to the ends 0 and 1 the greater probability there is of moving back into the interior. Again, it is easy to show that the only continuous fixed points $U\pi = \pi$ are the constant function. Therefore, we shall find as in section 2 that the distributions describing the position of the particle converge to a limit distribution independent of the initial distribution. We first proceed to analyze convergence properties of U^n. In this case it is no longer true that U preserves the class of positive monotonic functions. Only positivity is conserved by the mapping U. However, a new quality as described in Theorem 23 serves here well.

Throughout this section in order to avoid changes of proof and different results at times, we suppose that $0 < \alpha, \sigma < 1$.

Theorem 23. If $\pi(t)$ has a continuous derivative, then $\underset{t}{\mathrm{Max}} |(U\pi)'(t)| \leq \underset{t}{\mathrm{Max}} |\pi'(t)|$ with equality holding if and only if $\pi(t)$ is linear.

Proof: By direct computation, we obtain

$$U\pi'(t) = t\sigma\pi'(\sigma t) + (1-t)\alpha\pi'(1-\alpha+\alpha t) + \pi(\sigma t) - \pi(1-\alpha+\alpha t) .$$

Hence, with the aid of the mean value Theorem we get

$$(13)\ \underset{t}{\mathrm{Max}} |U\pi'(t)| = \max_t \left|t\sigma\pi'(\sigma t) + (1-t)\alpha\pi'(1-\alpha+\alpha t) + \left(\frac{\pi(\sigma t)-\pi(1-\alpha+\alpha t)}{\sigma t - (1-\alpha)-\alpha t}\right)\right.$$

$$\leq [t\sigma + (1-t)\alpha + 1-\alpha-(\sigma-\alpha)t] \max_t |\pi'(t)| = \max_t |\pi'(t)| .$$

If equality holds, then let t_0 denote a point where

$$\max_t |\pi'(t)| = |\pi'(t_0)|$$

It follows easily from (13) that

$$(14)\ \mathrm{Max} |\pi'(t)| = |\pi'(\sigma t_0)| = |\pi'(1-\alpha+\alpha t_0)| = \left|\frac{\pi(\sigma t_0) - \pi(1-\alpha+\alpha t_0)}{\sigma t_0 - (1-\alpha)-\alpha t_0}\right| .$$

This yields that $\pi(t)$ is linear for $\sigma t_0 \leq t \leq 1-\alpha+\alpha t_0$ or otherwise somewhere between σt_0 and $1-\alpha+\alpha t_0$ the slope has greater magnitude than the slope of the chord subtended by $\pi(t)$ at these points. Equation (14) also yields that σt_0 and $(1-\alpha+\alpha t_0)$ are maximum points of $\pi'(t)$. Thus repeating this argument then shows that equality in (13) requires $\pi(t)$ to be linear.

Theorem 24. If $\pi(t)$ belongs to C^m ($\pi(t)$ possesses m continuous derivatives), then $\max_t |(U^n\pi)^{(r)}(t)|$ are uniformly bounded in n for each r $(0 \leq r \leq m)$.

Proof: The proof is by induction. For $r = 1$ the theorem is an easy consequence of Theorem 23. For $r = 0$, the result is trivial since U preserves positivity and the constant functions are fixed points of U. Suppose we have established the result for $r = m-1$. We note that $(U\pi)^{(m)}(t) = t\sigma^m\pi^{(m)}(\sigma t) + (1-t)\alpha^m\pi^{(m)}(1-\alpha+\alpha t)$

$$+ m\sigma^{m-1}\pi^{(m-1)}(\sigma t) - m\alpha^{m-1}\pi^{(m-1)}(1-\alpha+\alpha t) .$$

This easily yields with the mean value theorem that

$$(15) \quad \max_t |U\pi^{(m)}(t)| \leq [t\sigma^m + (1-t)\alpha^m + m\alpha^{m-1}(1-\alpha+(\sigma-\alpha)t] \max |\pi^{(m)}(t)|$$

$$+ |m\sigma^{m-1} - m\alpha^{m-1}| \max_t |\pi^{m-1}(t)| .$$

The coefficient $L(t)$ of $\operatorname{Max} |\pi^{(m)}(t)|$ in (15) being linear in t achieves its maximum at 0 or 1. We get

$$L(0) = \alpha^m + m\alpha^{m-1}(1-\alpha) = m\alpha^{m-1} - (m-1)\alpha^m < 1$$

for any fixed $\alpha < 1$. If $\sigma > \alpha$, then

$$L(1) = \sigma^m + m\alpha^{m-1}(1-\sigma) \leq m\sigma^{m-1} - (m-1)\sigma^m < 1$$

In the case where $\alpha > \sigma$ the inequality (15) remains valid with the coefficient of $\max |\pi^{(m)}(t)|$ equal to $t\sigma^m + (1-t)\alpha^m + m\sigma^{m-1}[1-\alpha+(\sigma-\alpha)t]$ and the estimates go through as before. Thus

$$\max_t |U\pi^{(m)}(t)| \leq \lambda \max |\pi^{(m)}(t)| + C \max_t |\pi^{m-1}(t)$$

where C denotes the constant $|m\sigma^{m-1} - m\alpha^{m-1}|$ and $\lambda < 1$. Therefore,

$$\max_t |(U^k\pi)^{(m)}(t)| \leq \lambda \max |(U^{k-1}\pi)^{(m)}(t)| + C \max_t |(U^{k-1}\pi)^{m-1}(t)|$$

$$\leq \lambda \max |(U^{(k-1)}\pi)^{(m)}(t)| + K$$

using our induction hypothesis. Iterating this last inequality gives that

$$\max_t |(U^k\pi)^m(t)| \leq \sum_{i=0}^{k-1} \lambda^i K + \lambda^k \max_t |\pi^{(m)}(t)| \leq M.$$

This establishes the theorem.

Theorem 25. If $\pi(t)$ possesses two continuous derivatives and $\sigma \neq \alpha$, then $U^n\pi$ converges uniformly to a constant.

Remark: The reason why the two cases $\sigma = \alpha$ and $\sigma \neq \alpha$ are distinguished and necessarily so will be explained later.

Proof: In view of Theorem 23 and Theorem 24 the first and second derivatives of $U^n\pi$ are uniformly bounded. Thus $U^n\pi$ and $(U^n\pi)'$ constitute equi-continuous families of functions. We can thus select a subsequence n_i such that $U^{n_i}\pi$ converges uniformly to $\phi(t)$

and $(U^{n_i}\pi)'$ converges uniformly to $\phi(t)$. It follows trivially that $U^{n_i+1}\pi$ tends uniformly to $U\phi$ and $U^{n_i+2}\pi \Rightarrow U^2\phi$. Moreover, by virtue of Theorem 23,

$$\max_t |(U^{n_i}\pi)'| \geq \max_t |(U^{n_i+1}\pi)'| \geq \max_t |(U^{n_{i+1}}\pi)'|$$

Hence,

$$\lim_{i\to\infty} \max_t |(U^{n_i}\pi)'| = \lim_{i\to\infty} \max_t |(U^{n_i+1}\pi)'| = \lim_{i\to\infty} \max_t |(U^{n_i+2}\pi)'| .$$

Therefore, by the uniform convergence of the derivatives we secure

$$\max_t |\phi'(t)| = \max_t |(U\phi)'(t)| = \max_t |(U^2\phi)'(t)|$$

Invoking theorem 23 yields that $\phi(t)$ and $U\phi(t)$ are linear. However, if $\alpha \neq \sigma$ and $\phi(t)$ contains a term with t, then $U\phi$ is quadradic. This impossibility forces $\phi(t)$ to be identically a constant . Let i be chosen sufficiently large so that $|U^{n_i}\phi - c| \leq \epsilon$. Then,

$$|U^{n_i+1}\phi - c| \leq t\,|U^{n_i}\phi(\sigma t) - c| + (1-t)|U^{n_i}(1-\alpha+\alpha t) - c| < \epsilon .$$

Repeating this argument shows that $|U^{n_i+p}-c| \leq \epsilon$ for any p. Hence, this establishes that $U^n\pi$ converges uniformly to c.

<u>Theorem</u> 26. If $\pi(t)$ is continuous and $\sigma \neq \alpha$, then $U^n\pi$ converges uniformly.

<u>Proof</u>: The space of all functions with two continuous derivatives span linearly a dense subset of the space of all continuous functions.

Since $\|U^n\| = 1$, we obtain the result using Theorem 25 and a well known theorem of Banach.

In the next two theorems we establish the uniform convergence of $U^n\pi$ for the case where $1 > \sigma = \alpha > 0$. We note in this case the interesting fact that U applied to a polynomial does not increase its degree. Particularly, $Ux^n = [\alpha^n - n\alpha^{n-1}(1-\alpha)]\ x^n + P_{n-1}(x)$

where $P_{n-1}(x)$ denotes a polynomial of degree $n - 1$.

<u>Theorem</u> 27. If $P(t)$ is any polynomial, then $U^k P$ converges uniformly to a constant.

<u>Proof</u>: The proof is by induction on the degree of the polynomial. Clearly if P is a constant = c then $U^k P \equiv c$. Suppose we have shown for any polynomial P_{n-1} of degree $\leq n-1$ that the iterates $U^k P_{n-1}$ converge uniformly. To complete the proof it is enough to verify that $U^k x^n$ converges uniformly. Let $\lambda = \alpha^n - n\alpha^{n-1}(1-\alpha)$, then $|\lambda| < 1$ since $1 > \alpha > 0$. We obtain

$Ux^n = \lambda x^n + P_{n-1}(x)$. Repeating, we get for $k \geq 1$

$U^k x^n = \lambda^k x^n + \sum_{r=0}^{k-1} \lambda^r U^{k-r-1} P_{n-1}$. This last sum is of the form

$c_k = \sum_{r=0}^{k} a_r b_{k-r}$ with $\sum |a_r| < \infty$ and $\lim_{k=\infty} b_k(x)$ exists.

It is a well known theorem that $\lim c_k(x)$ exists uniformly whenever $b_k(x) = U^{k-1} P_{n-1}$ converges uniformly. Thus, $U^k x^n$ converges uniformly to a fixed point which must be a constant function.

Theorem 28. If $\pi(t)$ is continuous and $\sigma = \alpha > 0$, then $U^n\pi$ converges uniformly.

Proof: Similar to Theorem 26 since the set of all polynomials are dense.

We now note the important example that when $\alpha = \sigma = 0$ it is no longer true that $U^n\pi$ converges. It is easily verified that in this case $U^{2n}\pi$ and $U^{2n+1}\pi$ converge separately but that a periodic phenomenum occurs otherwise. The argument of Theorem 27 breaks down in this case as the quantity λ is -1. We only mention that other difficult convergence behavior occurs when α, σ traverse the boundary of the unit square for this model.

We return now to the hypothesis $0 < \alpha, \sigma < 1$.

Theorem 29. If $\pi(t)$ belongs to C^m, then $(U^k\pi)^{(r)}(t)$ for $0 \leq r \leq m$ converges uniformly.

Proof: This follows easily from Theorems 24, 26 and 28.

Let $TF = \int_0^{\frac{x}{\sigma}} t\,dF(t) + \int_0^{\frac{x+\alpha-1}{\alpha}} (1-t)\,dF(t)$. This represents the the transition law for the distribution describing the position of the particle. By using arguments analagous to the preceeding sections we can establish the following theorems using the conjugate relationship between T and U.

Theorem 30. For any distribution F the distributions T^nF converge as distributions to a unique distribution $F_{\sigma,\alpha}$ for which $TF_{\sigma,\alpha} = F_{\sigma,\alpha}$ which is independent of F.

<u>Theorem</u> 31. The distributions $F_{\sigma,\alpha}$ constitute a continuous family of distributions in the sense of Theorem 22.

Again it seems very difficult to determine any more explicit information about $F_{\sigma,\alpha}$.

§4. The model examined here is of the form where $1-\phi(x) = \lambda x + \mu$ with $\lambda + \mu \leq 1$ and at least $1 > \lambda$ or $0 < \mu$. The operator U has the form

(16) $U\pi = (\lambda x+\mu)\pi(\sigma x) + (1-\lambda x-\mu)\pi(1-\alpha+\alpha x)$

Of course, as before $0 < \alpha, \sigma < 1$. Convergence questions for $U^n\pi$ turn out to be very elementary in this case in view of the following theorem:

<u>Theorem</u> 32. If $\pi(x)$ has a bounded derivative, then

$$\max_x |(U\pi)'(x)| \leq a \max_x |\pi'(x)|$$

with $a < 1$.

<u>Proof</u>: By direct computation, we get

$$(U\pi)' = (\lambda x+\mu)\sigma\pi'(\sigma x) + \alpha(1-\lambda x-\mu)\pi'(1-\alpha+\alpha x) + \lambda[\pi(\sigma x) - \pi(1-\alpha+\alpha x)]$$

Using the mean value Theorem, we secure

$$\max_x |(U\pi)'(x)| \leq [(\lambda x+\mu)\sigma + \alpha(1-\lambda x-\mu) + \lambda(1-\alpha-\sigma x+\alpha x)] \max |\pi'(x)|$$

$$= a \max_x |\pi'(x)|$$

where $a = \mu\sigma + \alpha(1-\mu) + (1-\alpha)\lambda$. The assumptions on μ and λ yeild that $0 < a < 1$. Indeed, the max of a as a function of α is achieved when α is 0 or 1. But, $a(0) = \mu\sigma + \lambda < 1$ as $\mu + \lambda \leq 1$, $\mu > 0$ and $0 < \sigma < 1$ while $a(1) = \mu\sigma + (1-\mu) < 1$ since $0 < \sigma < 1$ and $\mu > 0$.

An immediate consequence of Theorem 32 is that $(U^k\pi)'$ converges geometrically to 0. This easily yields that $U^k\pi$ converges geometrically to a constant. Let T denote the transition operator for this model. In the standard way, we obtain

Theorem 33. For any distribution the distributions T^nF converge to the distribution $F_{\sigma,\alpha}$ which is a continuous function of (σ,α) and $TF_{\sigma,\alpha} = F_{\sigma,\alpha}$. Moreover, $F_{\sigma,\alpha}$ is independent of F.

mhb