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14. ABSTRACT

We describe enhancements under development for multi-scale methods to be applied to the high-fidelity modeling of spacecraft electric propulsion systems and their environment. Multiple challenges in multi-scale integration, statistical accuracy, and physical modeling are addressed through a variety of innovative numerical methods and mathematical approaches. We emphasize the advances made on a specialized multi-scale time-stepping integrator for finite-Larmor radius particle trajectories, accelerated collisional-radiative non-equilibrium ionization kinetics through Boltzmann equilibrated level groups, and a novel approach to dynamic phase-space reconstruction which can be used to resolve the problems of multi-scale statistics in particle-based kinetic simulations.

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Acceleration of PIC and CR Algorithms for High Fidelity In-Space Propulsion Modeling

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PI: Jean-Luc Cambier[2]

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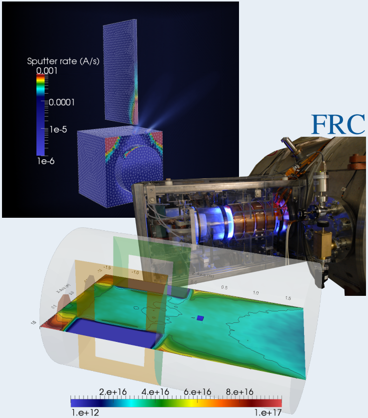
AFOSR Computational Math Annual Review,
Arlington VA,
July 29, 2013

Outline

Spacecraft Propulsion Relevant Plasma:

- From hall thrusters to plumes and fluxes on components
- Complex reaction physics i.e. Discharge and Breakdown in FRC
- Relevant Densities often Span 6+ Orders of Magnitude
- Spatial scales of interest span μm-$100m$ range

Electric Propulsion Plumes



FRC

Chamber Environment

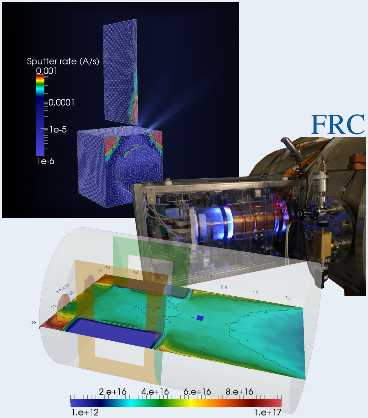
Spacecraft Propulsion Relevant Plasma:

- From hall thrusters to plumes and fluxes on components
- Complex reaction physics i.e. Discharge and Breakdown in FRC
- Relevant Densities often Span 6+ Orders of Magnitude
- Spatial scales of interest span μm-100m range

Solution?

Multi-Scale and Multi-Physics Adaptive Algorithms

Electric Propulsion Plumes



Chamber Environment

Boris Push:

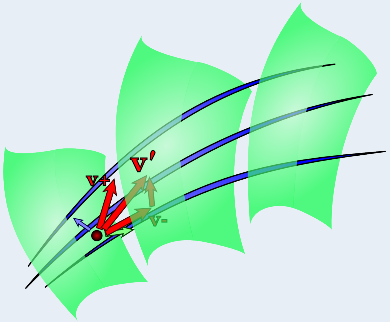
$$v^- = v(t-\Delta t/2) + \frac{q\Delta t}{2m}E(t)$$

$$v' = v^- + \frac{q\Delta t}{2m}v^- \times B(t)$$

$$v^+ = v^- + \frac{\frac{q\Delta t}{2m}}{1+\left(\frac{q\Delta t}{2m}|B|\right)}v' \times B(t)$$

$$v(t+\Delta t/2) = v^+ + \frac{q\Delta t}{2m}E(t)$$

$$r(t+\Delta t) = r(t) + \Delta t v(t+\Delta t/2)$$



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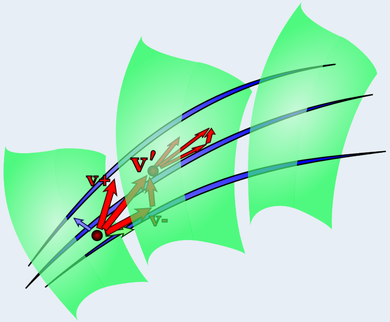
- Leap-Frog in r & v



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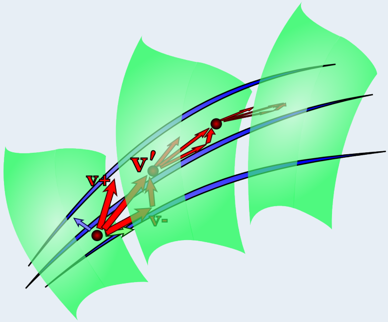
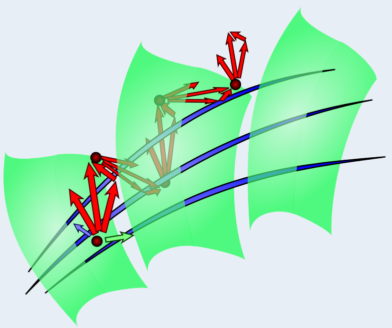
- Leap-Frog in *r* & *v*
- Energy Conservation only via Interpolation in Time



Boris Push:

$$
\begin{aligned}
v^- &= v(t-\Delta t/2) + \frac{q\Delta t}{2m}E(t) \\
v' &= v^- + \frac{q\Delta t}{2m} v^- \times B(t) \\
v^+ &= v^- + \frac{\frac{q\Delta t}{2m}}{1+\left(\frac{q\Delta t}{2m}|B|\right)} v' \times B(t) \\
v(t{+}\Delta t/2) &= v^+ + \frac{q\Delta t}{2m}E(t) \\
r(t{+}\Delta t) &= r(t) + \Delta t v(t+\Delta t/2)
\end{aligned}
$$

- Leap-Frog in r & v
- Energy Conservation only via Interpolation in Time
- Numerically Stable but Drifts and Fails when $\Delta t > \omega_c/2$



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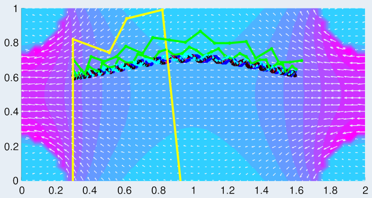
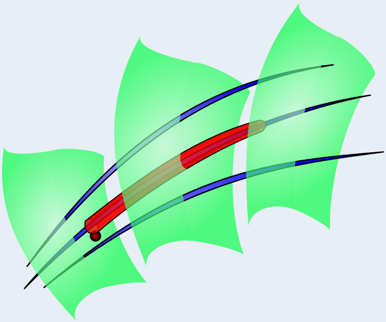


Magnetic Mirror Test

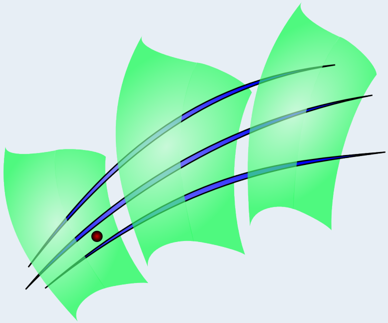
Gyrokinetic Push:

- Effective for High-B Plasma (i.e. Magnetic-Fusion)
- Assumes $\lambda_c \ll dx$
- Looses Phase Information
- Assumes Phase-Scatter Diffusion Negligible



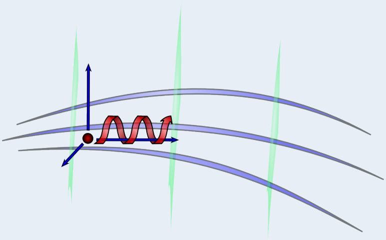
B-Frame Push:

- Coordinates Rotated and Aligned to *B*-Field



B-Frame Push:

- Coordinates Rotated and Aligned to *B*-Field
- Motion Decomposed into Rotation and Drift
- Exact Solution in Constant Field



B-Frame Push:

- Coordinates Rotated and Aligned to *B*-Field
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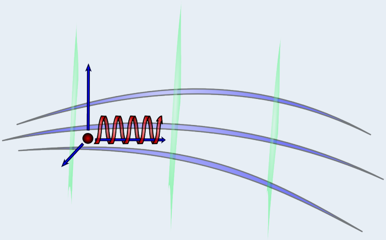
- Phase Preserved Across Many-ω_c



B-Frame Push:

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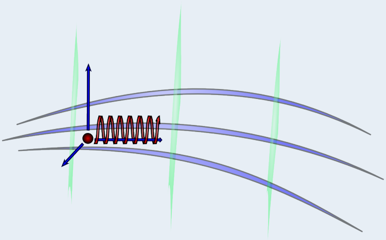
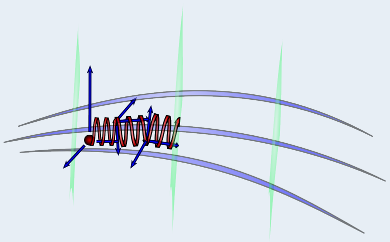
What about Variable Fields?



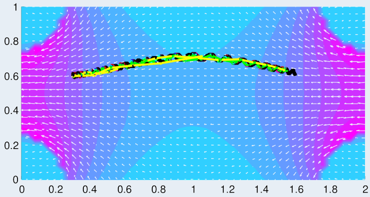
B-Frame Push in Variable Fields:

- Consistent Solution Recovered with Multiple Steps



B-Frame Push in Variable Fields:

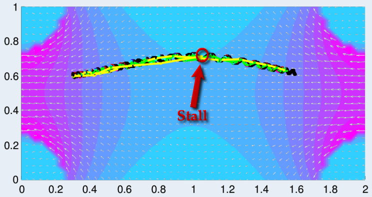
- Consistent Solution Recovered with Multiple Steps
- Convergence Order is Low



Magnetic Mirror Test

B-Frame Push in Variable Fields:

- Consistent Solution Recovered with Multiple Steps
- Convergence Order is Low
- Solution Stalls when $dt > \omega_c$



Magnetic Mirror Test

MULTI-SCALE PARTICLE ADVANCEMENT

B-Frame Push in Variable Fields:

- Consistent Solution Recovered with Multiple Steps
- Convergence Order is Low
- Solution Stalls when $dt > \omega_c$
- Consider the Push in $\hat{B}$-Aligned Coordinates...
- No Mechanism for $v_\perp \to v_\parallel$ in Step?
  (No Bounce in Magnetic Mirror?)
- Variable $\vec{B}$ and $\vec{E}$ via Taylor Series?

$$\Delta\vec{v} = \mathbf{D}_0 \cdot \vec{v} + \frac{q\Delta t}{m}\mathbf{D}_1 \cdot \vec{E}$$

$$\Delta\vec{x} = \mathbf{D}_1 \cdot \vec{v}\Delta t + \frac{q\Delta t^2}{m}\mathbf{D}_2 \cdot \vec{E}$$

$$\mathbf{D}_k = \hat{\mathbf{R}}^{-1} \cdot \Delta_k \cdot \hat{\mathbf{R}}$$

$$\Delta\mathbf{v}^R = \underbrace{\begin{pmatrix} -C_0 & S_0 & 0 \\ -S_0 & C_0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{\Delta_0} \cdot \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} + \underbrace{\begin{pmatrix} S_1 & C_1 & 0 \\ -C_1 & S_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\Delta_1} \cdot \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}$$

$$\Delta\mathbf{x}^R = \underbrace{\begin{pmatrix} S_1 & C_1 & 0 \\ -C_1 & S_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\Delta_1} \cdot \begin{bmatrix} v_x\Delta t \\ v_y\Delta t \\ v_z\Delta t \end{bmatrix} + \underbrace{\begin{pmatrix} C_2 & -S_2 & 0 \\ S_2 & C_2 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}}_{\Delta_2} \cdot \begin{bmatrix} a_x\Delta t^2 \\ a_y\Delta t^2 \\ a_z\Delta t^2 \end{bmatrix}$$

B-Frame Push in Variable Fields:

- Consistent Solution Recovered with Multiple Steps
- Convergence Order is Low
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- No Mechanism for $v_\perp \rightarrow v_\parallel$ in Step?
  (No Bounce in Magnetic Mirror?)
- Variable $\vec{B}$ and $\vec{E}$ via Taylor Series?

But even $\hat{R}$ depends on $\vec{B}$
(And is Partially Arbitrary!)

$$\Delta\vec{v} = \mathbf{D}_0 \cdot \vec{v} + \frac{q\Delta t}{m}\mathbf{D}_1 \cdot \vec{E}$$

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B-Frame Push in Variable Fields (cont.):

- Push as Explicit Matrix Operator:
  (For now Ignoring $\vec{E}$)

$$\mathbb{X}_+ = \mathbb{M} \cdot \mathbb{X}_0$$

$$\mathbb{X}_+ = \begin{bmatrix} \vec{x}_+ \\ \vec{v}_+ \end{bmatrix} = \begin{bmatrix} \vec{x_0} + \Delta\vec{x} \\ \vec{v_0} + \Delta\vec{v} \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbb{I} & \mathbf{D}_1 \\ 0 & \mathbb{I} + \Delta t \mathbf{D}_0 \end{bmatrix}}_{\mathbb{M}} \cdot \begin{bmatrix} \vec{x}_0 \\ \vec{v}_0 \end{bmatrix}$$

B-Frame Push in Variable Fields (cont.):

- Push as Explicit Matrix Operator:
  (For now Ignoring $\vec{E}$)

- What's Known about $\mathbb{M}$:
  - Reversible through Time Inversion...
  - $\mathbb{M}^{-1}$ Must Exist
  - $\mathbb{M}$ is a 1:1 Map of $\mathbb{X}_0 \rightarrow \mathbb{X}_+$
  - $\mathbb{M}$ is Unique (Even if $\hat{R}$ was Not...)
  - If $\mathbb{M}$ is Unique, Elements are Unique
  - $\mathbb{M}$ is Constant if $\vec{B}$ is Constant

$$\mathbb{X}_+ = \mathbb{M} \cdot \mathbb{X}_0$$

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Hypothesis:

Elements of $\mathbb{M}$ Vary Smoothly, $\vec{x} \underset{\text{Ngbh}}{\in} \vec{x}_0$

(Assuming $\vec{B}$ Varies Smoothly with $\vec{x}$)

$$\mathbb{X}_+ = \mathbb{M} \cdot \mathbb{X}_0$$

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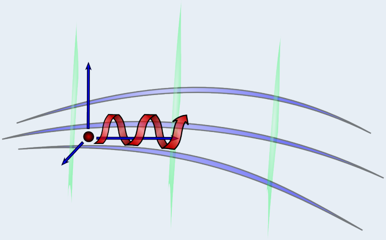
B-Frame Push in Variable Fields (cont.):

- Assuming $\vec{B}$ varies Periodically in ω_c and Linearly in multiples $N \times \omega_c$...?



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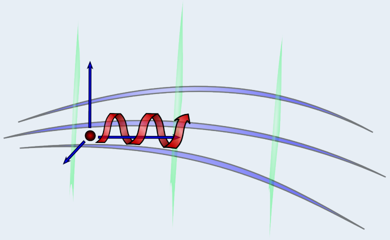
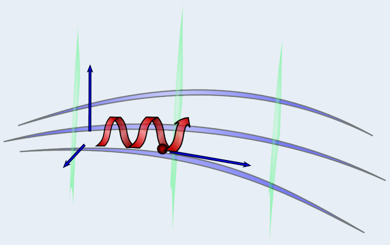


B-Frame Push in Variable Fields (cont.):

- Assuming $\vec{B}$ varies Periodically in ω_c and Linearly in multiples $N \times \omega_c$...?
- Elements of $\mathbb{M}$ take Form:

$$m_{ij} = \overline{m}_{ij} + \frac{n\delta t}{\Delta t}\Delta m_{ij}^{[slow]} + cos(\omega n\delta t)\Delta m_{ij}^{[c]} + sin(\omega n\delta t)\Delta m_{ij}^{[s]}$$



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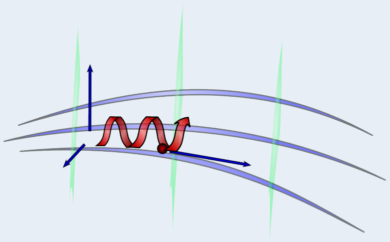
- Coefficients, Δm_{ij}, can be Sampled using Original Push with $\Delta t = [0, \pi/4, 3\pi/4, 2\pi n]\omega_c$



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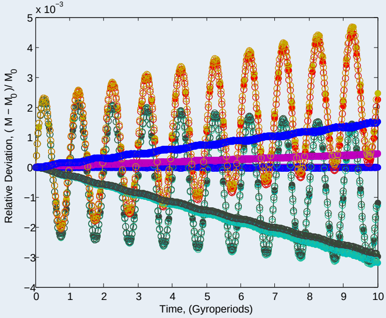
- Coefficients, Δm_{ij}, can be Sampled using Original Push with $\Delta t = [0, \pi/4, 3\pi/4, 2\pi n]\omega_c$
- Comparison of Fit and Sampled $\mathbb{M}$ from Magnetic Mirror



Closed Circle - Original Push Stepping m_{ij}
Open Circle - Estimated m_{ij} from Sampling

Compounded Push Operator:

- Uniform Field Push can be Subdivided

Compounded Push Operator:

- Uniform Field Push can be Subdivided

$$\mathbb{X}(t+\Delta t) = \mathbb{M}(\Delta t)\cdot\mathbb{X}_0$$
$$= \mathbb{M}^{(p-1)}(\Delta t/p)\cdot\mathbb{X}_0$$

Compounded Push Operator:

- Uniform Field Push can be Subdivided
- Can be Written as Product of δt Substeps

$$
\begin{aligned}
\mathbb{X}(t+\Delta t) &= \mathbb{M}(\Delta t)\cdot \mathbb{X}_0 \\
&= \mathbb{M}^{(p-1)}(\Delta t/p)\cdot \mathbb{X}_0 \\
&= \prod_{k=0}^{p-1} \mathbb{M}^{[k]}(\delta t)\cdot \mathbb{X}_0
\end{aligned}
$$

Compounded Push Operator:

- Uniform Field Push can be Subdivided
- Can be Written as Product of δt Substeps
- Also True if $\mathbb{M}^{[k]}$ varies with t – Equivalent to Explicit Stepping with Smaller Δt

$$
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\end{aligned}
$$

Compounded Push Operator:

- Uniform Field Push can be Subdivided
- Can be Written as Product of δt Substeps
- Also True if $\mathbb{M}^{[k]}$ varies with t – Equivalent to Explicit Stepping with Smaller Δt
- Estimator for $\mathbb{M}^{[k]}$ Reduces Field Evaluations, but not Push Multiplier Operations...

$$\begin{aligned}
\mathbb{X}(t+\Delta t) &= \mathbb{M}(\Delta t)\cdot\mathbb{X}_0 \\
&= \mathbb{M}^{(p-1)}(\Delta t/p)\cdot\mathbb{X}_0 \\
&= \prod_{k=0}^{p-1}\mathbb{M}^{[k]}(\delta t)\cdot\mathbb{X}_0
\end{aligned}$$

Operation Reduction:

- Given the Gyro-Estimated Compounded Operator $\widetilde{\mathbb{M}}$...

$$\widetilde{\mathbb{M}} = \prod_{k=0}^{p-1} \mathbb{M}^{[k]}$$

Operation Reduction:

- Given the Gyro-Estimated Compounded Operator $\widetilde{\mathbb{M}}$...
- Can be Split into Mean and Deviation

$$\widetilde{\mathbb{M}} = \prod_{k=0}^{p-1} \mathbb{M}^{[k]}$$
$$= \prod_{k=0}^{p-1} \left(\overline{\mathbb{M}} + \Delta\mathbb{M}^{[k]}\right)$$

Operation Reduction:

- Given the Gyro-Estimated Compounded Operator $\widetilde{\mathbb{M}}$...
- Can be Split into Mean and Deviation
- The Operator is Linearized around $\overline{\mathbb{M}}$

$$\widetilde{\mathbb{M}} = \prod_{k=0}^{p-1} \mathbb{M}^{[k]}$$
$$= \prod_{k=0}^{p-1} \left(\overline{\mathbb{M}} + \Delta\mathbb{M}^{[k]}\right)$$
$$= \prod_{k=0}^{p-1} \left(\mathbb{L}\Lambda\mathbb{R} + \Delta\mathbb{M}^{[k]}\right)$$

Operation Reduction:

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- Can be Split into Mean and Deviation
- The Operator is Linearized around $\overline{\mathbb{M}}$

$$\begin{aligned}\widetilde{\mathbb{M}} &= \prod_{k=0}^{p-1} \mathbb{M}^{[k]} \\ &= \prod_{k=0}^{p-1} \left(\overline{\mathbb{M}} + \Delta\mathbb{M}^{[k]}\right) \\ &= \mathbb{L}\left[\prod_{k=0}^{p-1}\left(\Lambda + \mathbb{R}\Delta\mathbb{M}^{[k]}\mathbb{L}\right)\right]\mathbb{R}\end{aligned}$$

Operation Reduction:

- Given the Gyro-Estimated Compounded Operator $\widetilde{\mathbb{M}}$...
- Can be Split into Mean and Deviation
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- And Assuming $||\Lambda|| \gg ||\mathbb{R}\Delta\mathbb{M}^{[k]}\mathbb{L}||$

$$
\begin{aligned}
\widetilde{\mathbb{M}} &= \prod_{k=0}^{p-1} \mathbb{M}^{[k]} \\
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&= \mathbb{L}\left[\prod_{k=0}^{p-1}\left(\Lambda + \mathbb{R}\Delta\mathbb{M}^{[k]}\mathbb{L}\right)\right]\mathbb{R} \\
&= \mathbb{L}\left[\Lambda^{(p-1)} + \sum_{k=0}^{p-2}\left(\Lambda^{(p-2)-k}\mathbb{R}\Delta\mathbb{M}^{[k]}\mathbb{L}\Lambda^{k}\right) + O(\Delta\mathbb{M}^2)\right]\mathbb{R}
\end{aligned}
$$

Operation Reduction:

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- Summations to Series Solution for Linear and Oscillatory Parts of $\Delta M \rightarrow$ Direct Eval of m_{ij}

$$\widetilde{\mathbb{M}} = \prod_{k=0}^{p-1} \mathbb{M}^{[k]}$$

$$= \prod_{k=0}^{p-1} \left(\overline{\mathbb{M}} + \Delta\mathbb{M}^{[k]}\right)$$

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$$m_{ij} \approx \mathbb{L} \left[\lambda_i^{(p-1)} + \lambda_i^{(p-2)} \mathbb{Q}_{ij} \sum_{k=0}^{p-2} \left[f_Q(k) \lambda^k_{j\setminus i} \right] \right] \mathbb{R}$$

$$\approx \mathbb{L} \left[\lambda_i^{(p-1)} + \lambda_i^{(p-2)} \mathbb{Q}_{ij} \begin{cases} \sum k\lambda^k, \\ \sum cos(\alpha k)\lambda^k, \\ \sum sin(\alpha k)\lambda^k \end{cases} \right] \mathbb{R}$$

Operation Reduction:

- Given the Gyro-Estimated Compounded Operator $\widetilde{\mathbb{M}}$...
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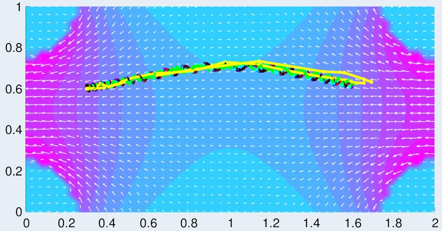


Magnetic Mirror Test

Elementary processes (electron-impact only)

- Collisional excitation/deexcitation

$$H(i) + e^- \leftrightarrow H(j) + e^-$$

- Collisional ionization/recombination

$$H(i) + e^- \leftrightarrow H^+ + 2e^-$$

- Bound-bound radiative transition

$$H(i) + h\nu \leftarrow H(j)$$

Rate Equation

$$K_{ji}^{rad.} = A_{ji}, \quad K_{ji}^{coll.} = N_e \int_{E_{ij}}^{\infty} \sigma_{ij}(\varepsilon) v f(\varepsilon) d\varepsilon$$

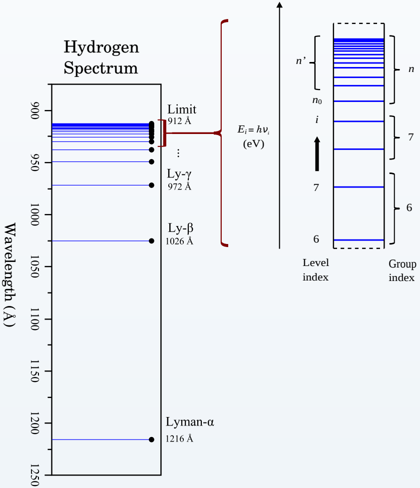
$$\frac{dN_i}{dt} = \sum_{R \in \mathrm{Rxn}} \sum_{j \neq i} (K_{ij}^R N_j - K_{ji}^R N_i)$$

$$dN_i = \sum_{j \neq i} \tilde{\mathbf{K}}_{ij} N_j - \tilde{\mathbf{J}}_i \circ N_i$$

Implicit formulation

$$dN_j = \mathbb{A}_{ij}^{-1} \left[\sum_{j \neq i} \tilde{\mathbf{K}}_{ij} N_j - \tilde{\mathbf{J}}_i \circ N_i \right]$$

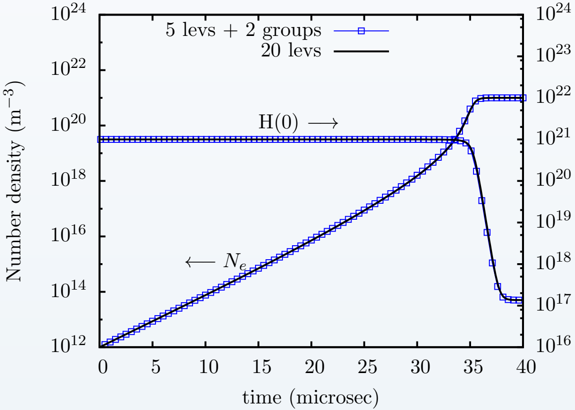
Level Grouping



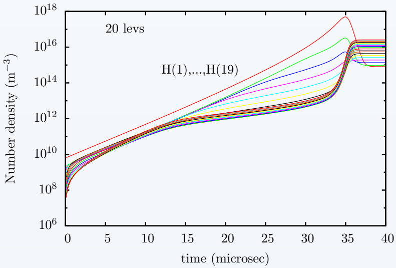
n,m: bin indices; i,j: level indices

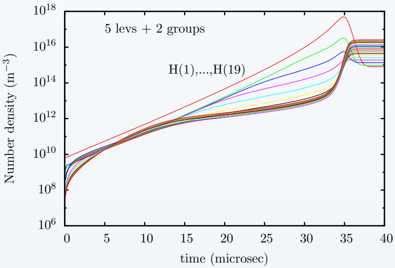
Model	Cons. vars.	Size	Lev. Pop.
Full CR	N_i	N_{level}	
Reduced CR (Uniform)	$\Sigma_n = \sum_{i\in n} N_i$	N_{group}	$N_{i\in n} = \frac{g_i \Sigma_n}{\sum_{j\in n} g_j}$
Reduced CR (Boltzmann)	N_{n_0} $\Sigma'_n = \sum_{i\in n'} N_i$	$2 \times N_{\text{group}}$	$N_{i\in n} = \frac{g_i N_{n_0} e^{-\Delta E_i/kT_n}}{g_{n_0}}$ or $N_{i\in n} = \frac{g_i \Sigma'_n e^{-\Delta E_i/kT_n}}{Q'_n}$

Comparison of ground state and electron number densities

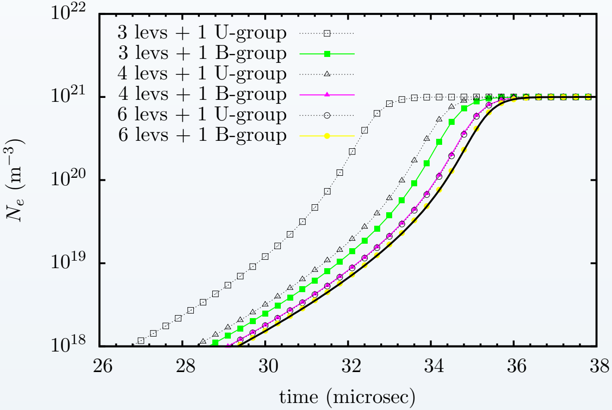


Comparison of excited states number densities





Uniform vs. Boltzmann



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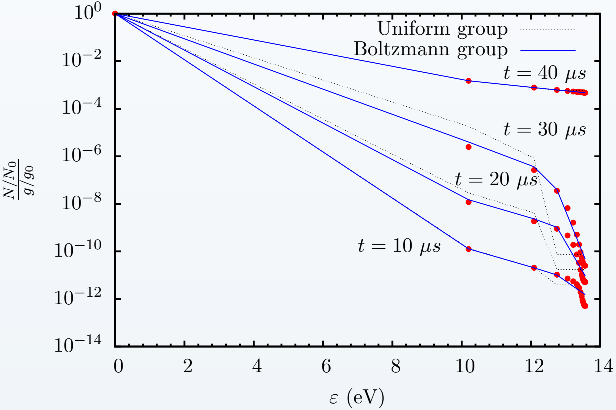

Numerous Previous Merge Methods:

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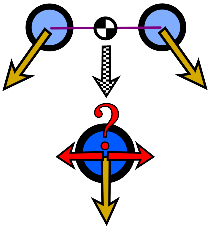
- 2:1 - Cannot Conserve Energy
  (Lapenta & Brackbill, JCP 1994)



$$w_m = \sum w_i$$
$$\vec{v}_m = \sum w_i \vec{v}_i / w_m$$

$$w_m v_m^2 < \sum w_i v_i^2 \;!!!$$

Numerous Previous Merge Methods:

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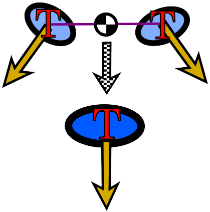
  (Lapenta & Brackbill, JCP 1994)
- Complex Macro-particles with Internal Energy
  (Hewett, JCP 2003)



$$w_m = \sum w_i, \quad \vec{v}_m = \sum w_i \vec{v}_i / w_m$$

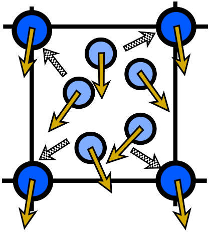
$$T_m^{(int)} = \left(\sum w_i T_i^{(int)} + \sum w_i v_i^2 - w_m v_m^2 \right) / w_m$$

$$w_m \left(v_m^2 + T_m^{(int)} \right) = \sum \left(w_i v_i^2 + w_i T_i^{(int)} \right) \checkmark$$

Particle Push with $T^{(int)}$?...
Split if $(T^{(int)})^{1/2} \gg v_m$?...
In What Coord-System?...

Numerous Previous Merge Methods:

- 2:1 - Cannot Conserve Energy
  (Lapenta & Brackbill, JCP 1994)
- Complex Macro-particles with Internal Energy
  (Hewett, JCP 2003)
- Merge to Grid
  (Assous et al., JCP 2003, Welch et al., JCP 2007)



Conserve Arbitrary Moments to Grid

Not Explicitly Conserved Lost?
Entropy Generation?
Shape Functions?
Grid Dependence?

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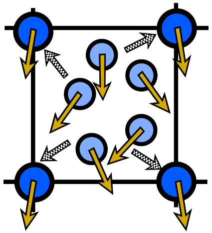
All Introduce Significant Error and/or Complexity



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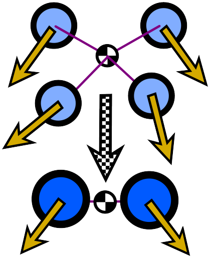
Grid Dependence?

Merge to Pair → DOF for Conservation:

- (n+2):2 yields Exact Mass, Momentum, and Kinetic Energy Conservation
- Applied Spatially also Shown to Conserve Electrostatic Energy
- Though Energy Conserving, Still Thermalizes VDF

(AFOSR Review 2006)



$$w_m = \sum_i^{(n+2)} w_i$$
$$\bar{\vec{v}} = \left(\sum_i^{(n+2)} w_i \vec{v}_i\right) / w_m$$
$$\overline{V^2} = \left(\sum_i^{(n+2)} w_i \left(\vec{v}_i - \bar{\vec{v}}\right)^2\right) / w_m$$

$$w_{(a/b)} = w_m / 2$$
$$\vec{v}_{(a/b)} = \bar{\vec{v}} \pm \hat{\mathcal{R}} \sqrt{\overline{V^2}}$$
$$\left(\text{Similarly: } \vec{x}_{(a/b)} = \bar{\vec{x}} \pm \hat{\mathcal{R}} \sqrt{\overline{X^2}}\right)$$

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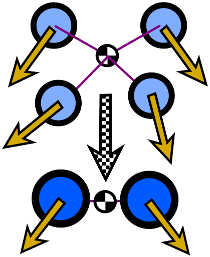
Selection of Near Neighbors in VDF Limits Thermalization

(≈ Near Neighbor Pairs in 2:1 Merges that Limit Numerical Cooling)



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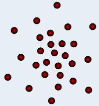
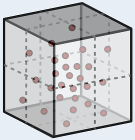
Phase-Space Decomposition

- Given a Set of Particles...



Phase-Space Decomposition

- Given a Set of Particles...
- Particles Binned in Octants



Phase-Space Decomposition

- Given a Set of Particles...
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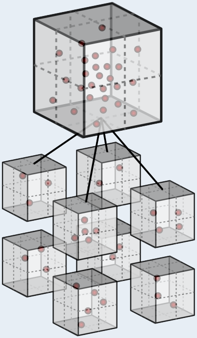
- Octants Recursively Sub-Divided



Phase-Space Decomposition

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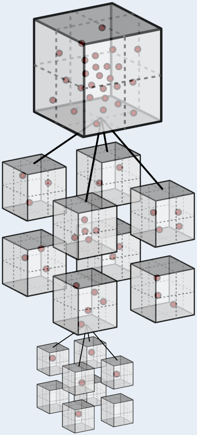

Octree Binning

Phase-Space Decomposition

- Given a Set of Particles...
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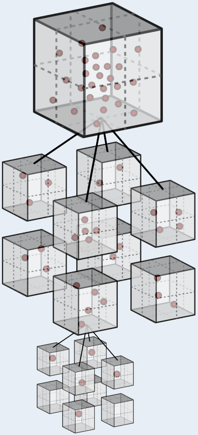
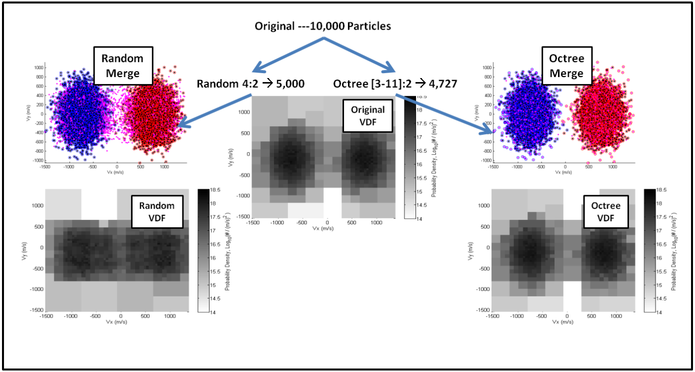
Restricts Phase-Space Diffusion to Within Local Bins



Comparison of Random vs. Octree Merge Partner Selection

(Note: Mass, Momentum, and Kinetic Energy Both Exactly Conserved)



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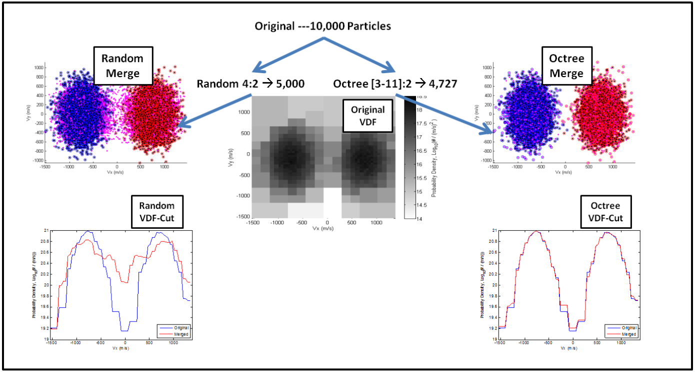
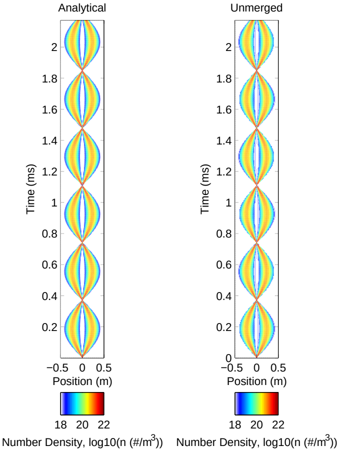


Collisionless Particles in Well

- 6000 Unmerged Particles
- Reproduces 3-4 Orders of Magnitude



Collisionless Particles in Well

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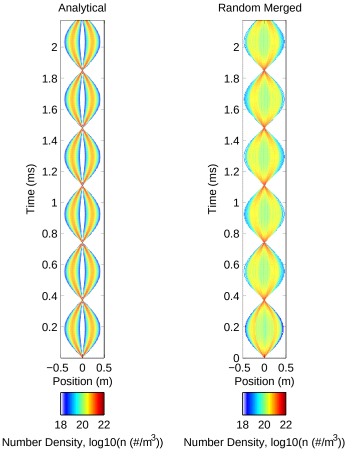
- Random Merge -> Thermalization
- 3000 First Point, 1500 First Cross
- Bi-Maxwellian Specifically Difficult



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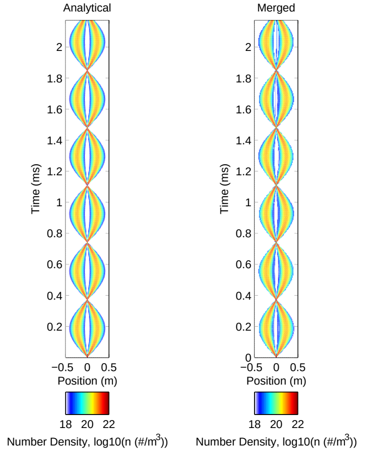

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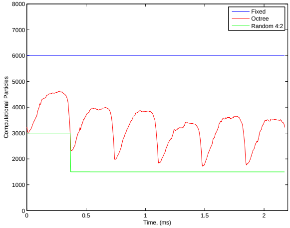

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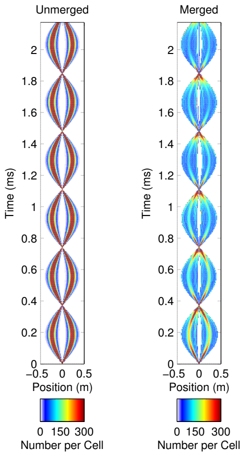
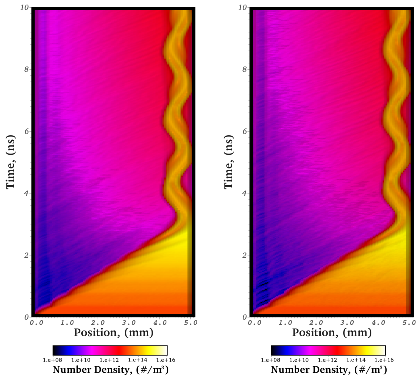
- Computational Particles per Cell Vastly Different



DC-Diode Test Case:

- Full 3D Electrostatic-PIC
- Averaged to 1D XT-Plot

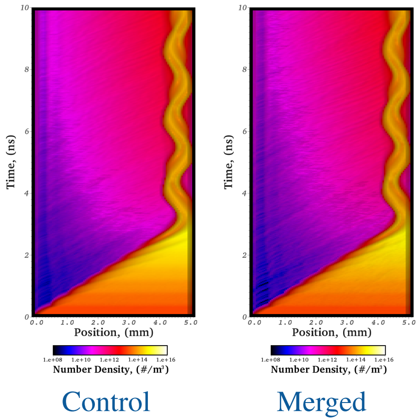


Control

Merged

DC-Diode Test Case:

- Full 3D Electrostatic-PIC
- Averaged to 1D XT-Plot
- 250V Cathode → Anode
- MCC-Ionization Collisions
- Secondary Emission at Cathode



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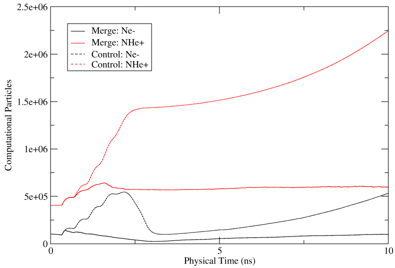

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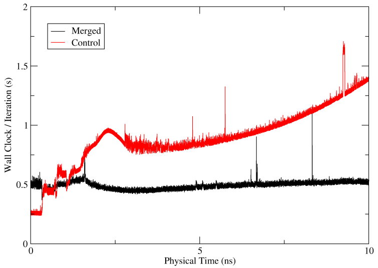
- Merge Overhead Rapidly Negligible



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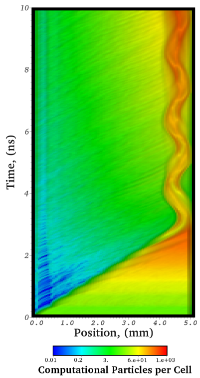
- Merge Overhead Rapidly Negligible
- Control: Parts/Cell $\propto$ Density



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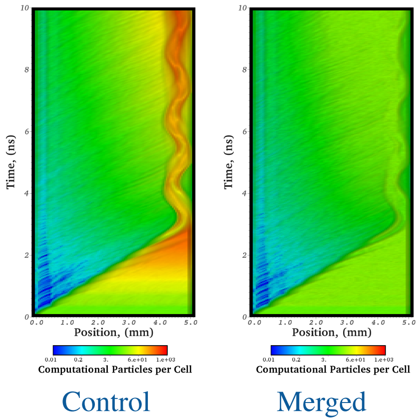
- Merge Overhead Rapidly Negligible
- Control: Parts/Cell $\propto$ Density
- Merge: Parts/Cell Much Reduced



DC-Breakdown

DC-Diode Test Case:

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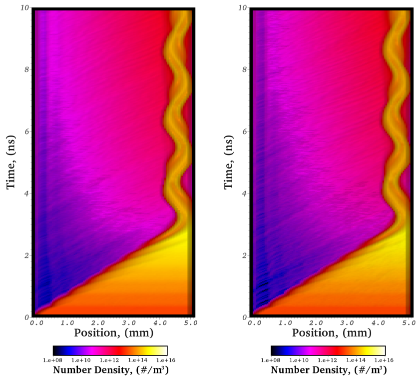
- Chain-Branching Needs Merge
- Merge Overhead Rapidly Negligible
- Control: Parts/Cell $\propto$ Density
- Merge: Parts/Cell Much Reduced
- Despite Identical Densities



Control Merged

Integrate R&D w/ Production TODO:

- High-Order Fluid/MHD GPU Models

  (Le/Cole*/Bilyeu PhD Research)

- GPU Accelerated Chemical Kinetics / CR Ar-Ne-Xe-Molecular Models

  (Le/Cole*/Kapper* PhD Research)

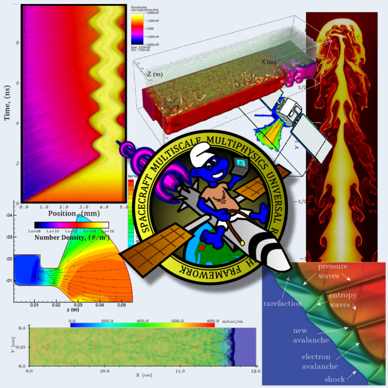
- Phase-Space Reconstruction/Vlasov

  (Martin/Bilyeu/TBD)

- Implicit / Multiscale GPU-Accelerated PIC

  (Lederman/Gimelsheins/Martin/TBD)

*Note: Former Co-op Student Work to be Integrated into Framework



Thank You

Questions?