# A Proof of Two Conjectures Related to the Erdös-Debrunner Inequality

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**Abstract**

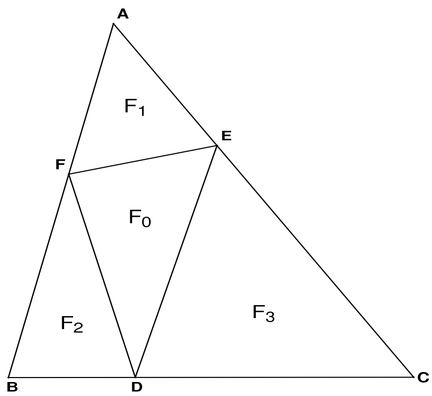
In this paper we prove two conjectures proposed by Janous on an extension to the $p$-th power-mean of the Erdös–Debrunner inequality relating the areas of the four sub-triangles formed by connecting three arbitrary points on the sides of a given triangle.

## 1 Motivation

Given a triangle $ABC$, and three arbitrary points on the sides $AB, AC, BC$, the Erdös-Debrunner inequality [1] states that

$$F_0 \geq \min(F_1, F_2, F_3), \tag{1}$$

where $F_0$ is the area of the middle formed triangle $DEF$ and $F_1, F_2, F_3$ are the areas of the surrounding triangles (see Figure 1).



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14. ABSTRACT
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*Figure 1: Triangle $\triangle ABC$*

The $p$-th power-mean is defined for $p$ on the extended real line by

$$M_p(x_1, x_2, \ldots, x_n) = \begin{cases} \min(x_1, \ldots, x_n), \ if \ p = -\infty, \\ \left(\frac{\sum_{i=1}^n x_i^p}{n}\right)^{1/p}, \ if \ p \neq 0, \\ M_0 = \sqrt[n]{\prod_{i=1}^n x_i}, \ if \ p = 0, \\ \max(x_1, \ldots, x_n), \ if \ p = \infty. \end{cases}$$

It is known (see [2], Chapter 3) that $M_p$ is a non-decreasing function of $p$. Thus, it is natural to ask whether (1) can be improved to:

$$F_0 \geq M_p(F_1, F_2, F_3). \tag{2}$$

The author of [4] investigated the maximum value of $p$, denoted here by $p_{max}$, for which (2) is true, showing that $-1 \leq p_{max} \leq -(\frac{\ln 3}{\ln 2} - 1)$ (and disproving a previously published claim).

Since $p_{max} < 0$, by setting $x = \frac{BD}{AE}\frac{AC}{BC}$, $y = \frac{EC}{FB}\frac{AB}{AC}$, $z = \frac{AF}{DC}\frac{BC}{AB}$, and $q = -p$, it is shown in [4] that (2) is equivalent to

$$f(x, y, z) := g(x, y)^q + g(y, z)^q + g(z, x)^q \geq 3, \tag{3}$$

where $g(x, y) := \frac{1}{x} + y - 1$, $q_{ma}$, the analogue of $p_{max}$, satisfies $\frac{\ln 3}{\ln 2} - 1 \leq q_{max} \leq 1$, and the variables are such that $g(x, y) \geq 0$, $g(y, z) \geq 0$, $g(z, x) \geq 0$ and $x, y, z > 0$.

Let us introduce the natural domain of $f$, say $\mathcal{D}$, to be the set of all triples $(x, y, z) \in \mathbb{R}^3$ with $x, y, z > 0$ and $g(x, y) \geq 0$, $g(y, z) \geq 0$ and $g(z, x) \geq 0$. Since $f(x, y, z) \geq 0$ the function $f$ has an infimum on $\mathcal{D}$. Let us denote this infimum by $m$.

To complete the analysis begun in [4], the author proposed the following two conjectures.

**Conjecture 1.** *For any $q \geq q_0 = \frac{\ln 3}{\ln 2} - 1$, if $f(x, y, z) = m$, then $xyz = 1$.*

**Conjecture 2.** *If $q \geq q_0$, then $m = 3$.*

In this paper we prove (Theorem 3) that for every $q > 0$, the function $f$ has a minimum $m$, and if this infimum is attained for $(x, y, z) \in \mathcal{D}$, then $xyz = 1$. Moreover, we show (Theorem 4) that for every $q > 0$ we have $m = \min\{3, 2^{q+1}\}$. Our results are more general than Conjectures 1 and 2 above, and imply them. After the initial submission of our paper, we learned that the initial conjectures of Janous were also proved by Mascioni [5]. However, our methods are different and our results are slightly stronger.

## 2 Proof of Conjecture 1

We are going to prove the following more general theorem from which Conjecture 1 follows.

**Theorem 3.** *For every $q > 0$, the function $f$ defined by (3) has a minimum $m$ and if $f(x, y, z) = m$ for some $(x, y, z) \in \mathcal{D}$ then $xyz = 1$.*

*Proof.* Since $f(1, 1, 1) = 3$ and $f(2, 1/2, 1) = 2^{q+1}$ we see that

$$0 \leq m \leq \min\{3, 2^{q+1}\}.$$

Since $g(x, y) > y - 1$ we see that if $y > 1 + 3^{\frac{1}{q}} =: a$ then $f(x, y, z) > 3$. Similarly, $f(x, y, z) > 3$ if $x$ or $z$ is greater than $a$. On the other hand, if $x < \frac{1}{a}$ then $g(x, y) > 1/x - 1 > a - 1 = 3^{1/q}$ which implies $f(x, y, z) > 3$ again. Clearly, if $y$ or $z$ are less than $1/a$ we also have $f(x, y, z) > 3$. Hence, we can introduce the compact domain

$$\mathcal{C} := \left\{ (x, y, z) \Big| \frac{1}{a} \leq x, y, z \leq a,\ g(x, y) \geq 0,\ g(y, z) \geq 0 \text{ and } g(z, x) \geq 0 \right\},$$

which has the property that

$$m = \inf\{f(x, y, z) | (x, y, z) \in \mathcal{C}\}. \tag{4}$$

Because any continuous function defined on a compact set attains its infimum, we infer that $m$ is a minimum for $f$. Moreover every point at which $f$ takes the value $m$ must be in $\mathcal{C}$.

Let us assume now that we have such point $(x, y, z)$ as in the statement of Theorem 3: $f(x, y, z) = m$. We will consider first the case in which $(x, y, z)$ is in the interior of $\mathcal{C}$.

By the first derivative test (sometimes called Fermat's principle) for local extrema, this point must be a critical point. So, $\frac{\partial f(x,y,z)}{\partial x} = 0$, which is equivalent to

$$x^2 = \frac{g(x, y)^{q-1}}{g(z, x)^{q-1}}.$$

Hence the system

$$\nabla f(x, y, z) = (0, 0, 0) \tag{5}$$

is equivalent to

$$x^2 = \frac{g(x,y)^{q-1}}{g(z,x)^{q-1}};\ y^2 = \frac{g(y,z)^{q-1}}{g(x,y)^{q-1}};\ z^2 = \frac{g(z,x)^{q-1}}{g(y,z)^{q-1}}. \tag{6}$$

Multiplying the equalities in (6) gives $xyz = 1$, and this proves the theorem when the infimum occurs at an interior point of $\mathcal{C}$.

Now let us assume that the minimum of $f$ is attained at a point $(x, y, z)$ on the boundary of $\mathcal{C}$. Clearly the boundary of $\mathcal{C}$ is

$$\{(x, y, z) \in \mathcal{C} | \{x, y, z\} \cap \{a, 1/a\} \neq \emptyset \textit{ or } g(x, y)g(y, z)g(z, x) = 0\}.$$

We distinguish several cases.

*Case* 1: First, if $x = a$, since $1/z > 0$, we have

$$f(x, y, z) \geq (1/z + x - 1)^q > (a - 1)^q = 3 \geq m.$$

Thus, we cannot have $f(x, y, z) = m$ in this situation. Similarly, we exclude the possibility that $y$ or $z$ is equal to $a$.

*Case* 2: If $x = 1/a$, because $y > 0$, it follows that

$$f(x, y, z) \geq (1/x + y - 1)^q > (a - 1)^q = 3 \geq m.$$

Again this implies that $f(x, y, z) = m$ is not possible. Likewise, we can exclude the cases in which $y$, or $z$ is $1/a$.

*Case* 3: Let us consider now the case in which $g(x, y) = 0$, that is $y = \frac{x-1}{x}$ (observe that we need $x > 1$). Therefore, $f(x, y, z) = f(x, \frac{x-1}{x}, z)$ becomes the following function of two variables

$$\begin{aligned} k(x, z) &= \left(\frac{x}{x-1} + z - 1\right)^q + \left(\frac{1}{z} + x - 1\right)^q \\ &= \left(z + \frac{1}{x-1}\right)^q + \left(\frac{1}{z} + x - 1\right)^q. \end{aligned}$$

Hence, using the arithmetic-geometric inequality, we obtain

$$\begin{aligned} \left(z + \frac{1}{x-1}\right)^q + \left(\frac{1}{z} + x - 1\right)^q &\geq 2\sqrt{\left(\frac{1}{x-1} + z\right)^q \left(\frac{1}{z} + x - 1\right)^q} \\ &= 2\sqrt{\left[2 + z(x-1) + \frac{1}{z(x-1)}\right]^q} \geq 2^{q+1}, \end{aligned} \tag{7}$$

where we have used $X + 1/X \geq 2$ (for $X > 0$). We observe that if $m = 2^{q+1}$ (this is equivalent to $q \leq q_0$), since $f(x, y, z) = m$, we must have equality in (7), which, in particular, implies $z = \frac{1}{x-1}$, that is, $xyz = 1$. If $m < 2^{q+1}$ then (7) shows that we cannot have $f(x, y, z) = m$. Either way, the conjecture is also true in this situation. The other cases are treated in a similar way. □

## 3 Results Implying Conjecture 2

We are going to prove a result slightly more general than Conjecture 2:

**Theorem 4.** *Assume the notations of Section 2. Then, for every $q > 0$ we have $m = \min\{3, 2^{q+1}\}$.*

In [4] the truth of Theorem 4 was shown to be true for $\frac{\ln 3}{\ln 2} - 1 \leq q \leq 1$. So we are going to assume without loss of generality that $q < 1$ throughout. Based on what we have shown in Section 2, we can let $z = \frac{1}{xy}$ and study the minimum of the function $h(x, y) = f(x, y, \frac{1}{xy})$ on the trace of the domain $\mathcal{C}$ in the space of the first two variables:

$$\mathcal{H} = \left\{ (x, y) |\ x, y \in [1/a, a] \ \ \textit{and} \ \ \frac{x+1}{x} \geq y \geq \frac{|x-1|}{x} \right\}.$$

Before we continue with the analysis of the critical points inside the domain $\mathcal{H}$ we want to expedite the boundary analysis. We define $A := 1/x + y - 1$, $B := 1/y + 1/(xy) - 1$ and $C := xy + x - 1$. It is a simple matter to show

$$ABC + AB + AC + BC = 4. \tag{8}$$

If $(x, y)$ is on the boundary of $\mathcal{H}$, then either $y = \frac{x+1}{x}$, or $y = \frac{|x-1|}{x}$. The first possibility is equivalent to $B = 0$, and the second is equivalent to $A = 0$ (if $x > 1$), or $C = 0$ (if $x < 1$). Now, if $C = 0$ then $AB = 4$. Hence

$$f(x, y, z) \geq A^q + B^q + C^q = A^q + B^q \geq 2\sqrt{(AB)^q} = 2^{1+q}.$$

Similar arguments can be used for the cases $A = 0$ or $B = 0$. Hence, since $h(1, 2) = 2^{q+1}$ we obtain the following result.

**Lemma 5.** *The minimum of $h$ on the boundary of $\mathcal{H}$, say $\partial\mathcal{H}$, is*

$$\min\{h(x, y) | (x, y) \in \partial\,\mathcal{H}\} = 2^{q+1}. \tag{9}$$

Next, we analyze critical points inside $\mathcal{H}$. By Fermat's principle, these critical points will satisfy $\dfrac{\partial h}{\partial x} = 0, \dfrac{\partial h}{\partial y} = 0$, that is, $-\frac{1}{x^2}qA^{q-1} - \frac{1}{x^2 y}qB^{q-1} + (y+1)qC^{q-1} = 0$, and $qA^{q-1} - \frac{x+1}{xy^2}qB^{q-1} + xqC^{q-1} = 0$. Remove the common factor $q$ in both of these equations to obtain

$$- \quad \frac{1}{x^2}A^{q-1} - \frac{1}{x^2 y}B^{q-1} + (y+1)C^{q-1} = 0 \tag{10}$$

$$A^{q-1} - \frac{x+1}{xy^2}B^{q-1} + xC^{q-1} = 0. \tag{11}$$

Solving for $A^{q-1}$ in (11) and substituting in (10) we get

$$-\frac{x+1}{x^3y^2}B^{q-1} + \frac{1}{x}C^{q-1} - \frac{1}{x^2y}B^{q-1} + (y+1)C^{q-1} = 0$$

or

$$\frac{xy+x+1}{x}C^{q-1} = \frac{x+1+xy}{x^3y^2}B^{q-1}.$$

Since $xy + x + 1 > 0$, $x > 0$, by simplifying the previous equation we obtain

$$C^{q-1} = \frac{B^{q-1}}{x^2y^2}. \tag{12}$$

Moreover, replacing (12) in (11), say, we get $A^{q-1} - \frac{x+1}{xy^2}B^{q-1} + x\frac{B^{q-1}}{x^2y^2} = 0$, which implies

$$\frac{A^{q-1}}{x^2} = \frac{B^{q-1}}{x^2y^2}. \tag{13}$$

Therefore, if we put (12) and (13) together, we obtain

$$\frac{A^{q-1}}{x^2} = \frac{B^{q-1}}{x^2y^2} = C^{q-1}. \tag{14}$$

The equality $\frac{A^{q-1}}{x^2} = C^{q-1}$ is equivalent to $x^{\frac{2}{1-q}}(1/x+y-1) = xy+x-1$. If we introduce the new variable $s = \frac{1+q}{1-q} > 1$ the last equality can be written as $yx(1-x^s) = (x^s+1)(1-x)$.

Similarly, the equality $\frac{A^{q-1}}{x^2} = \frac{B^{q-1}}{x^2y^2}$ can be manipulated in the same way to obtain $1/x + y - 1 = y^{\frac{2}{1-q}}(1/y + 1/xy - 1)$, or $(1/x)(1 - y^s) = (1 - y)(1 + y^s)$.

So, the two equations in (14) give the critical points (inside the domain $\mathcal{H}$), which can be classified in the the following way:

- $(C_1)$: (1,1)
- $(C_2)$: $\{(x,1) : x \neq 1 \; satisfies \; x(1-x^s) = (x^s+1)(1-x)\}$
- $(C_3)$: $\{(1,y) : y \neq 1 \; satisfies \; (1-y^s) = (1-y)(1+y^s)\}$
- $(C_4)$: $\left\{(x,y) : y = \dfrac{(x^s+1)(x-1)}{x(x^s-1)} \; and \; x = \dfrac{y^s-1}{(y-1)(y^s+1)}, \, x \neq 1, y \neq 1\right\}$

Let $\phi(t) = \begin{cases} \frac{(t^s+1)(t-1)}{t(t^s-1)} & if \; 1 \neq t > 0, \\ \frac{2}{s} & if \; t = 1, \end{cases}$ which is continuous for all $t > 0$. Since it is going to be useful later, we note that $\phi$ satisfies

$$\phi\left(\frac{1}{t}\right) = t\phi(t), \quad for \; all \; t > 0. \tag{15}$$

Thus $(C_2)$ is the set of all $(x,1)$ $(x \neq 1)$ with $\phi(x) = 1$; $(C_3)$ is the set of all $(1,y)$ $(y \neq 1)$ with $\phi(1/y) = 1$; and $(C_4)$ is the set of all $(x,y)$ $(x \neq 1, y \neq 1)$ with

$$\begin{cases} y = \phi(x) \\ x = \dfrac{1}{\phi(1/y)}. \end{cases} \tag{16}$$

**Remark 6.** *Because of* (15)*, the class* $(C_3)$ *is in fact the set of all points* $(1,y)$*, where* $y = 1/x$ *and* $(x,1)$ *is in* $(C_2)$*.*

To determine the nature of the critical points we compute the second partial derivatives, and analyze the Hessian of $h$ at these critical points. Using relations (14) we obtain:

$$\begin{aligned} \frac{\partial^2 h}{\partial x^2} &= q(q-1)\left[\frac{1}{x^4}A^{q-2} + \frac{1}{x^4y^2}B^{q-2} + (y+1)^2C^{q-2}\right] + q\left[\frac{2}{x^3}A^{q-1} + \frac{2}{x^3y}B^{q-1}\right] \\ &= \frac{2q(1+y)C^{q-1}}{x} - q(1-q)C^{q-1}\left[\frac{1}{x^2A} + \frac{1}{x^2B} + \frac{(y+1)^2}{C}\right] \\ &= \frac{qC^{q-1}}{x}\left[2(1+y) - (1-q)\frac{(A+B)C + x^2(y+1)^2AB}{xABC}\right] \\ &= \frac{q(q+1)}{x^2ABC^{2-q}}\left(ABC(C+1) - \frac{4}{s}\right), \end{aligned} \tag{17}$$

using the fact that $x(y+1) = C+1$.

Similarly we get

$$\begin{aligned}
\frac{\partial^2 h}{\partial y^2} &= q(q-1)\left[A^{q-2} + \frac{(x+1)^2}{x^2y^4}B^{q-2} + x^2C^{q-2}\right] + q\frac{2(x+1)}{xy^3}B^{q-1} \\
&= \frac{2qx(1+x)C^{q-1}}{y} - q(1-q)C^{q-1}\left[\frac{x^2}{A} + \frac{(x+1)^2}{y^2B} + \frac{x^2}{C}\right] \\
&= \frac{qC^{q-1}}{y}\left[2x(1+x) - (1-q)\frac{x^2y^2(A+C)B + (x+1)^2AC}{yABC}\right] \\
&= \frac{q(q+1)x^2}{ABC^{2-q}}\left(ABC(B+1) - \frac{4}{s}\right),
\end{aligned} \tag{18}$$

using $xy(B+1) = x+1$.

Further, the mixed second derivative is

$$\begin{aligned}
\frac{\partial^2 h}{\partial x \partial y} &= q(q-1)\left[-\frac{1}{x^2}A^{q-2} + \frac{x+1}{x^3y^3}B^{q-2} + x(y+1)C^{q-2}\right] + q(\frac{1}{x^2y^2}B^{q-1} + C^{q-1}) \\
&= 2qC^{q-1} - q(1-q)C^{q-1}\left[-\frac{1}{A} + \frac{x+1}{xyB} + \frac{x(y+1)}{C}\right] \\
&= vC^{q-1}(1+q-(1-q)\left[\frac{AC(B+1)+AB-BC}{ABC}\right] \\
&= \frac{q(q+1)}{ABC^{2-q}}\left(ABC - \frac{2}{s}(2-BC),\right)
\end{aligned} \tag{19}$$

using the identities $xy(B+1) = x+1$, and $x(y+1) = C+1$.

The discriminant (determinant of the Hessian) $D := \frac{\partial^2 h}{\partial x^2} \cdot \frac{\partial^2 h}{\partial y^2} - (\frac{\partial^2 h}{\partial x \partial y})^2$ can be calculated using (17), (18) and (19) to obtain

$$
\begin{aligned}
D &= \frac{q^2(q+1)^2}{A^2B^2C^{4-2q}}\left[A^2B^2C^2((B+1)(C+1)-1)\right. \\
&\quad \left. -\frac{4}{s}ABC(B+C+2-(2-BC))+\frac{4}{s^2}(4-(2-BC)^2)\right] \\
&= \frac{q^2(q+1)^2}{A^2B^2C^{4-2q}}\left(A^2B^2C^2((BC+B+C)\right. \\
&\quad \left. -\frac{4}{s}ABC(BC+B+C)+\frac{4}{s^2}(4BC-B^2C^2))\right).
\end{aligned}
$$

Now, by (8) we have $4BC-B^2C^2 = BC(4-BC) = BC(ABC+AB+AC) = ABC(BC+B+C)$ and so we have the factor $ABC(BC+B+C)$ in all the terms above. This implies that the discriminant of $h$ (at the critical points, that is, assuming relations (14)) can be simplified to

$$
D = \frac{q^2(q+1)^2}{ABC^{3-2q}}(BC+B+C)\left[ABC+\frac{4}{s^2}-\frac{4}{s}\right]. \tag{20}
$$

Our next lemma classifies the critical point $(1,1)$.

**Lemma 7.** *For $q \geq 1/3$, the point $(1,1)$ is a local minimum. For $q < 1/3$ the critical point $(1,1)$ is not a point of local minimum.*

*Proof.* If $q = 1/3$, $h(1,1) = 3$, so, since $h(x,y) = f(x,y,\frac{1}{xy}) \geq 3$ by equation (3), we establish that $(1,1)$ is a local minimum point of $h$. Assume $q \neq 1/3$. For $x = 1$ and $y = 1$ the formulae established above become $\frac{\partial^2 h}{\partial x^2}(1,1) = \frac{\partial^2 h}{\partial y^2}(1,1) = 2q(3q-1) > 0$, $\frac{\partial^2 h}{\partial x \partial y}(1,1) = q(3q-1) > 0$ and $D = 3q^2(3q-1)^2$. So, the Hessian is positive definite and so we have a local minimum at this point (cf. [3, Theorem 2.9.7, p. 74]). For the second part observe that $D(1,1) > 0$, but $\frac{\partial^2 h}{\partial x^2}(1,1) < 0$ if $q < 1/3$, and so $(1,1)$ is not a local minimum if $q < 1/3$. □

**Theorem 8.** *If $q \neq 1/3$, there exists only one solution $x_0$ of $\phi(x) = 1$, $0 < x \neq 1$, such that*

*(a)* $x_0 \in \left(\frac{1}{2}, \frac{s}{2(s-1)}\right)$ *if $q > 1/3$ $(s > 2)$;*

*(b)* $x_0 \in \left[2^{\frac{1}{s-1}} - \frac{s^{\frac{s}{s-1}} - s}{2(s-1)}, 2^{\frac{1}{s-1}} - \frac{1}{2(s-1)}\right)$ *if $q < 1/3$ $(1 < s < 2)$.*

*Furthermore, there is only one solution* $y_0 = 1/x_0$ *to* $\phi(1/y) = 1$, $0 < y \neq 1$. *If* $q = 1/3$ $(s = 2)$, *there are no positive solutions for* $\phi(x) = 1$, $0 < x \neq 1$, *or* $\phi(1/y) = 1$, $0 < y \neq 1$.

*Proof.* First, assume $q = 1/3$. Then $s = 2$. It is straightforward to show that $(x, 1)$ is in $(C_2)$ implies $x = 1$. However, $x = 1$ is not allowed. Similarly, $(1, y)$ is in $(C_3)$ implies $y = 0$, or 1, which are not allowed. Thus, if $q = 1/3$, there are no positive solutions for $\phi(x) = 1$, $0 < x \neq 1$, or $\phi(1/y) = 1$, $0 < y \neq 1$.

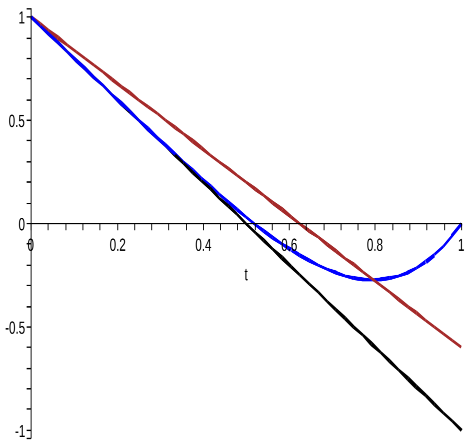
Now we shall assume throughout that $q \neq 1/3$. Let us observe that the equation $\phi(x) = 1$ can be written equivalently as $\psi(x) = 0$ $(x \neq 1)$ where

$$\psi(t) := t^s - 2t + 1, \quad t \geq 0.$$

We first assume that $q > 1/3$, which is equivalent to $s > 2$. The derivative of $\psi$ is $\psi'(t) = st^{s-1} - 2$ which has only one critical point $t_0 = (2/s)^{\frac{1}{s-1}}$. Since $s > 2$ we obtain that $t_0 < 1$. We have $\psi(0) = 1$, $\psi(1) = 0$ and then automatically

$$\psi(t_0) = (2/s)^{\frac{s}{s-1}} - 2(2/s)^{\frac{s}{s-1}} + 1 = 1 - (s-1)(2/s)^{\frac{s}{s-1}} < 0.$$

The second derivative of $\psi$ is: $\psi''(t) = s(s-1)t^{s-2}$. This shows that $\psi$ is a convex function and so its graph lies above any of its tangent lines and below any secant line passing through its graph, as in the figure below.




We conclude that $x_0$ is between the intersection of the tangent line at $(0, 1)$ with the $x$-axis and the intersection between the secant line connecting $(0, 1)$ and $(t_0, \psi(t_0))$ with the $x$-axis.

Since $\psi'(0) = -2$, the equation of the tangent line is $y - 1 = -2x$ and so its intersection with the $x$-axis is $(1/2, 0)$. The equation of the secant line through $(0, 1)$ and $(t_0, \psi(t_0))$ is $y - 1 = \frac{1-\psi(t_0)}{-t_0}x$, or $y = 1 - \frac{(s-1)2}{s}x$. This gives the intersection with the $x$-axis: $(\frac{s}{2(s-1)}, 0)$. Therefore the first part of our theorem is proved. The last claim is shown similarly. □

**Remark 9.** *As $q$ approaches 1 from below, $s$ becomes large and the interval around $x_0$ (part (a) in Theorem 8) is very small.*

**Theorem 10.** *The critical points in $(C_2)$ and $(C_3)$ are not points of local minimum for $h$.*

*Proof.* We show that the Hessian of $h$ is not positive semi-definite by showing that the discriminant $D$ is less than zero.

We will treat only the critical points of type $(C_2)$, since the case $(C_3)$ is similar. We get $A = A(x_0, 1) = 1/x_0$, $B = B(x_0, 1) = 1/x_0$ and $C = C(x_0, 1) = 2x_0 - 1$.

The condition $D < 0$ is the same as

$$\frac{2x_0 - 1}{x_0^2} + \frac{4}{s^2} - \frac{4}{s} < 0,$$

which is equivalent to

$$s^2(2x_0 - 1) - 4x_0^2(s - 1) = (s - 2x_0)(2(s - 1)x_0 - s) < 0$$

or

$$\begin{aligned} &x_0 \in (-\infty, \tfrac{s}{2}) \cup (\tfrac{s}{2(s-1)}, \infty),\ \mathit{if}\ q \le 1/3\ (1 < s \le 2)\ \ \mathit{and} \\ &x_0 \in (-\infty, \tfrac{s}{2(s-1)}) \cup (\tfrac{s}{2}, \infty)\ \mathit{if}\ q > 1/3\ (s > 2) \end{aligned} \tag{21}$$

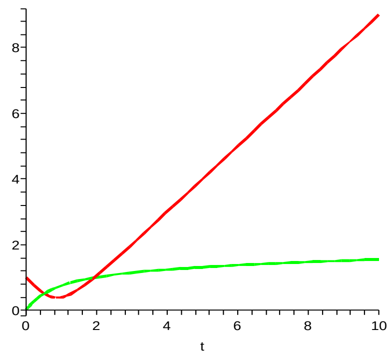
By Theorem 8 parts (a) and (b), and the inequality $2^{\frac{1}{s-1}} - \frac{s^{\frac{s}{s-1}} - s}{2(s-1)} > \frac{s}{2(s-1)}$ that can be easily checked, we see that $D < 0$ which ends this proof. □

Next, we define the two functions

$$\gamma_1(t) := \frac{(t-1)(1+t^s)}{t^s - 1};\ \gamma_2(t) := \left(\frac{t^{s+1} - t^s}{t^s - t}\right)^{1/s}, t > 0, t \neq 1, \tag{22}$$

extended by continuity at $t = 0$ and $t = 1$.

Let us take a look at the graphs of these two functions for $s = 6$:




The following two lemmas will be crucial for our final argument.

**Lemma 11.** *For every $s > 1$, the function $\gamma_1$ is convex and the function $\gamma_2$ is concave.*

*Proof.* For $\gamma_1$, one can readily check that

$$\gamma_1''(t) = \frac{2st^{s-2}}{(t^s - 1)^3}\beta_1(t)$$

where $\beta_1(t) = (s-1)(t^{s+1}-1)-(s+1)(t^s-t)$. Next we observe that $\beta_1'(t) = (s+1)\beta_2(t)$ where $\beta_2(t) = (s-1)t^s - st^{s-1}+1$ and observe that $\beta_2'(t) = s(s-1)[t^{s-1}-t^{s-2}] = s(s-1)t^{s-2}(t-1)$. The sign of $\beta_2'$ is then easily determined showing that $\beta_2$ has a point of global minimum at $t = 1$. Hence $\beta_2(t) \geq \beta_2(1) = 0$. This implies that $\beta_1$ is strictly increasing. Since $\beta_1(1) = 0$ we see that sign of $\beta_1$ is exactly as the the sign of $(t^s - 1)^3$. This means that $\gamma_1''(t) > 0$ for all $t > 0$. At $t = 1$ the limit is $\frac{s^2-1}{3s} > 0$ also).

In order to deal with $\gamma_2$ we rewrite it as $\gamma_2(t) = \left(\frac{t^r(t-1)}{t^r - 1}\right)^{\frac{1}{r+1}} = \theta(t)^{\frac{1}{r+1}}$ where $r := s - 1 > 0$. Because

$$\gamma_2''(t) = \frac{1}{(r+1)\theta(t)^{\frac{2r+1}{r+1}}}\left(\theta(t)\theta''(t) - \frac{r}{r+1}\theta'(t)^2\right),$$

we have to show that $\delta(t) := \theta(t)\theta''(t) - \frac{r}{r+1}\theta'(t)^2 < 0$ for all $t > 0$.

The first and second derivatives of $\theta$ are given by

$$\theta'(t) = \frac{t^{2r} - (r+1)t^r + rt^{r-1}}{(t^r - 1)^2},$$

and

$$\theta''(t) = \frac{r[(r-1)t^{2r-1} - (r+1)t^{2r-2} + (r+1)t^{r-1} - (r-1)t^{r-2}]}{(t^r-1)^3}$$

These two expressions substituted into $\delta(t)$ yield

$$\delta(t) = -\frac{rt^{2r-2}}{(r+1)}\delta_1(t)$$

where the sign of $\delta$ is determined by $\delta_1(t) = t^{2r+2} - (t^{r+2}+t^r)(r+1)^2 + t^{r+1}(2r^2+4r)+1$. But $\delta_1(1) = 0$ and $\delta_1'(t) = (r+1)t^{r-1}\delta_2(t)$ where

$$\delta_2(t) = 2t^{r+2} - ((r+2)t^2+r)(r+1) + (2r^2+4r)t.$$

Again, observe that $\delta_2(1) = 0$ and $\delta_2'(t) = 2(r+2)\delta_3(t)$ where $\delta_3(t) = t^{r+1} - (r+1)t + r$. Finally, $\delta_3(1) = 0$ and $\delta_3'(t) = (r+1)(t^r-1)$. Now $\delta_3$ has only a single critical point at $t = 1$ which is a global minimum. Thus $\delta_3(t) \geq \delta_3(1) = 0$. This shows $\delta_2$ is strictly increasing on $(0, \infty)$ and is zero at $t = 1$. Therefore, $\delta_1(t)$ has a minimum at $t = 1$ implying $\delta_1(t) \geq 0$ with its only zero at $t = 1$. Hence $\delta(t) < 0$ for all $t \neq 1$. This, and $\lim_{t\to 1}\delta(t) = -\frac{(r+1)(r+2)}{12r^2}$ show that $\gamma_2$ is a concave function and complete the proof. □

We shall need the following well-known result which may be formulated with weaker hypotheses. For convenience, we include it here.

**Lemma 12.** *The graphs of two functions $f$ and $g$ twice differentiable on $[a,b]$, $f$ convex ($f'' > 0$) and $g$ concave ($g'' < 0$) cannot have more than two points of intersection.*

*Proof.* Suppose by way of contradiction that they have at least three points of intersection. We thus assume $(x_1, f(x_1)) = (x_1, g(x_1))$, $(x_2, f(x_2)) = (x_2, g(x_2))$, $(x_3, f(x_3) = (x_3, g(x_3))$, with $a \leq x_1 < x_2 < x_3 \leq b$ are such points. Next, we look at the expression

$$E = \frac{f(x_2)-f(x_1)}{x_2-x_1} - \frac{f(x_3)-f(x_2)}{x_3-x_2} = \frac{g(x_2)-g(x_1)}{x_2-x_1} - \frac{g(x_3)-g(x_2)}{x_3-x_2}.$$

By the Mean Value Theorem applied twice to $f$ and $f'$ the expression $E$ is equal to

$$E = f'(c_1) - f'(c_2) = f''(c)(c_1-c_2) < 0, c_1 \in (x_1,x_2), c_2 \in (x_2,x_3), c \in (c_1,c_2)$$

and applied to $g$ and $g'$ gives

$$E = g'(\xi_1) - g'(\xi_2) = g''(\xi)(\xi_1-\xi_2) > 0, \xi_1 \in (x_1,x_2), \xi_2 \in (x_2,x_3), \xi \in (\xi_1,\xi_2)$$

which is a contradiction. □

Let us observe that if $x_0$ is a solution of the equation $\phi(x_0) = 1$ then $(1/x_0, x_0)$ is a solution of the system (16).

**Theorem 13.** *If $q \neq 1/3$, then the only critical points of $h$ are $(1,1)$, $(x_0, 1)$, $(1, \frac{1}{x_0})$, $(\frac{1}{x_0}, x_0)$, where $x_0$ is as in Theorem 8. If $q = 1/3$, $(1,1)$ is the only critical point.*

*Proof.* Start with $q = 1/3$. Then Lemma 7 and Theorem 8 imply the claim that $(1,1)$ is the only critical point.

Next, for $q \neq 1/3$, we consider the following system in the variables $x$ and $y$:

$$\begin{cases} \dfrac{1}{x} = \dfrac{(y-1)(1+y^s)}{y^s - 1} \\ \dfrac{1}{x} = \left(\dfrac{y^{s+1} - y^s}{y^s - y}\right)^{1/s} \end{cases}. \tag{23}$$

In what follows next we show that every solution of $(C_4)$ is a solution of (23). Indeed, if $(x,y)$ is in $(C_4)$ then it satisfies

$$x = \frac{(x^s+1)(x-1)}{y(x^s-1)}, \ x = \frac{y^s - 1}{(y-1)(1+y^s)}.$$

This implies that

$$\frac{(x^s+1)(x-1)}{y(x^s-1)} = \frac{y^s-1}{(y-1)(1+y^s)},$$

or

$$(x^s+1)x(y-1)(1+y^s) - (x^s+1)(y-1)(1+y^s) = y(x^s-1)(y^s-1).$$

Now, use $x(y-1)(y^s+1) = y^s - 1$ to simplify the first term of the previous equality and derive

$$(x^s+1)(y^s-1) - (x^s+1)(y-1)(1+y^s) - y(x^s-1)(y^s-1) = 0.$$

Finally, we solve for $x^s$ to obtain $x^s(y^s - 1 - y^{s+1} - y + 1 + y^s - y^{s+1} + y) = y + y^{s+1} - 1 - y^s + y - y^{s+1} - y^s + 1$, which is equivalent to $x^s(2y^s - 2y^{s+1}) = 2y - 2y^s$. So, if $y \neq 1$ this implies $x^s = \dfrac{y^s - y}{y^{s+1} - y^s}$ which implies that $\dfrac{1}{x} = \left(\dfrac{y^{s+1} - y^s)}{y^s - y}\right)^{1/s}$.

We observe that $(1, 1/x_0)$, $(1/x_0, x_0)$ are solutions of (23). By Lemmas 11 and 12 these two points are the only solutions of this system which proves our theorem. □

Using Lemma 11 and Theorem 13 we infer the next result.

**Theorem 14.** *The point in* $(1/x_0, x_0)$ *in* $(C_4)$ *is not a point of minimum.*

*Proof.* Since at this point, $A = 2x_0 - 1$, $B = 1/x_0$, $C = 1/x_0$ we see that $ABC = \frac{2x_0-1}{x_0^2}$ and the discriminant $D$ takes the same form as in Theorem 10. Hence the proof here follows exactly in the same way as in Theorem 10. $\square$

Putting together Lemmas 5, 7, and Theorems 10, 13, and 14, we infer the truth of Theorem 4.

In terms of our original problem we have obtained the following theorem.

**Theorem 15.** *Given the points* $D, E, F$ *on the sides of a triangle* $ABC$*, and* $F_0, F_1, F_2, F_3$ *the areas as in Figure* 1*, then*

$$F_0 \geq C_p M_p(F_1, F_2, F_3),$$

*where* $C_p = \min\left(1, \left(\frac{2^{1-p}}{3}\right)^{1/p}\right)$*, for all* $p < 0$*.*

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