

A Monte Carlo Method for Multi-Objective Correlated Geometric Optimization

by David A. Richie, James A. Ross, Song J. Park, and Dale R. Shires

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14. ABSTRACT Determining the positioning of assets within an unfriendly urban environment subject to complex constraints presents a non-trivial geometric optimization problem. Within realistic scenarios, the risk and success metrics must generally be defined numerically and will lack simple closed-form representations. Moreover, in the case of non-separable objective functions that depend upon correlated positioning of individual assets, the state-space of the system will be of high dimension, requiring computationally intensive algorithms for optimization. This report presents a method developed for solving such systems using a Monte Carlo simulation technique for multi-objective correlated geometric optimization. Once line-of-sight via ray tracing approach is calculated, our algorithm performs a Monte Carlo optimization to provide geospatial intelligence on entity placement using OpenCL framework. The solutions for optimal positioning, calculated through evaluating risk and success objective function with Markov chain Monte Carlo sampling, are presented graphically in this report.					
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1. Introduction

Evaluating the positioning of assets such as Soldiers and equipment within a hostile environment introduces a complex set of constraints defining the expected risk and probability of success in the context of the overall mission objectives. These constraints generally require numerical evaluation and lack the simplicity of closed-form definitions for real-world scenarios. When the objective functions depend on the correlated positions, the result is a complex geometric optimization problem that must be carried out in high-dimensional state space.

As an example, we consider here a prototypical scenario in which hostile “red force” Soldiers are positioned within an urban environment, posing a ballistic (or sniper) threat to the intended positioning of “blue force” Soldiers deployed to accomplish a prescribed mission. The precise threat must be determined using a ray tracing technique to establish the existence and distance of a line-of-sight (LOS) path to a prospective blue force location, with a risk probability defined by a ballistic model as a function of distance (zero when occluded). The mission objective is further defined as a requirement to surveil multiple targets within the urban environment. The quality of the observation from any location in the environment can be defined similarly to the ballistic threat in that it requires a LOS path and depends on distance (ignoring weather effects for now). The optimization of the positioning of the blue force Soldiers must minimize the ballistic threat and maximize the observability of targets for a given scenario. We have developed a method for performing multi-objective correlated geometric optimizations in the context of quasi-static mission planning for tactical scenarios.

This report presents a formulation of a military scenario with red and blue forces to examine a ray tracing technique and Monte Carlo method. The background context and the wording are specific to a military tactical operation for the purposes of connecting this problem to a potential real-world situation. However, this geometric optimization method is not limited only to the presented scenario, but our approach has the possibility to be applicable and expandable to a wider range of GIS intelligence. For instance, the capability of computing timely urban geographical information can be beneficial and effective for first response teams in rescue operations involving exposure to threats and desired targets.

The main contributions of this report are:

- A methodology to perform geospatial analysis of a three-dimensional (3D) urban representation to determine optimal geographical location given threat and target positions, and
- A Monte Carlo method development in the OpenCL programming model for vendor-agnostic architecture support and future processor portability.

The remainder of this report is organized as follows. Section 2 describes background materials involved in the implementation of the geometric optimization. Section 3 lists related work. Following this, the methodology of geometric optimization is discussed in section 4. The output results for the application are revealed in section 5. Finally, section 6 presents conclusions and future work.

2. Background

LOS calculations on a 3D model of an urban environment consisting of terrain and building structures exhibit a characteristic ripe for highly parallel processing. The triangle data representing the geometry of a town was extracted from a COLLABorative Design Activity (COLLADA) XML-based file format, which is used for interactive 3D applications managed by Khronos Group (1). Casting only primary rays, this ray tracing (or casting) variant algorithm provides a field of view from a point selected in the urban area. At its core, it is the problem of determining the first object intersected by a ray as introduced by (4). 3D LOS calculations can further be extended to address ballistic hit probability and visual observation quality for the analysis of threat and exposure. By examining the computational geometry problem involving ray-triangle intersections in a 3D map, situational awareness can be developed to augment the geometric understanding of a region.

The parallel programming framework of OpenCL (2) facilitated the implementation of the geometric optimization problem of determining the placement of blue force Soldiers. The parallelism of processing architectures, which is rampant and unavoidable in modern processors, is embraced from the beginning of the development. As such, LOS calculations and the optimization algorithm are designed for parallel execution efficiency. Details regarding the ray tracing performance, parallel programming, and graphics processor unit acceleration are planned to be presented in a different paper. The focus of this report is to investigate the computation methodology of a Monte Carlo approach in solving a geospatial optimization.

3. Related Work

Solving for optimal positioning of entities in an urban environment falls under classes of problems found in computational geometry. The LOS importance to the problem allows for some similarity to art gallery or fortress problems. Our problem shares the concern of optimal placement in the facility location problem, but with different optimization criteria. A review of geometric optimization techniques and problems is presented in (3). To simplify the overall objective, we refer to our problem as one of “cooperative guards.”

Observations for running Monte Carlo schemes and Markov chain Monte Carlo algorithms in parallel are provided in (11). Rosenthal argues and advocates that Monte Carlo algorithms are ideal to parallel computing. Lee et al. present a case study of running Monte Carlo methods on graphics cards that indicate potential for faster parallel simulation (7). Our approach, however, is to support a Monte Carlo method for geometric optimization on heterogeneous platforms that harness massively parallel architectures. In addition to mapping for graphics processing units by NVIDIA and AMD, we envision targeting ARM processors, Intel x86 cores, Digital Signal Processor (DSP) architectures, and Field Programmable Gate Arrays (FPGAs).

4. Methodology

To reiterate, the goal of the method is to determine the optimum locations of one or more deployed blue force Soldiers by simultaneously maximizing the mission objective of surveilling one or more targets while minimizing the overall risk posed by one or more enemy red force Soldiers.

Let n be the number of blue Soldiers used to accomplish the overall mission objective. Let the position of each blue Soldier be represented by a continuous position in three-dimensional space, $r_i^{(blue)} \in \underline{R}^{(3)}$.

Let m_{red} be the number of enemy red Soldiers presenting a ballistic threat to the blue Soldiers. Let the position of each red Soldier be represented by a continuous position in three-dimensional space, $r_i^{(red)} \in \underline{R}^{(3)}$.

Let m_{target} be the number of targets that are to be observed as part of the mission objective. Let the position of each target be represented by a continuous position in three-dimensional space, $r_i^{(target)} \in \underline{R}^{(3)}$.

The objective function to be optimized contains two components representing the overall ballistic threat posed by the red Soldiers and the quality of observation over all targets, both of which are functions of the positions of the blue Soldiers. The formulation here will assume that the locations of red Soldiers and targets are fixed during the optimization of the blue Soldier positions (strategic vice tactical).

The overall ballistic threat probability is given by,

$$P(r_0, r_1, \dots, r_{n-1}) = \max_n \left(\max_{m_{red}} p_{ij} \right) \quad (1)$$

where $p_{ij} \in [0, 1]$ is the ballistic threat probability for blue Soldier i from red Soldier j . Here $p = 0$ implies no ballistic threat vulnerability and $p = 1$ implies the highest risk of ballistic threat vulnerability. The rationale for the formulation in equation 1 is that mission risk would be based upon the worst-case threat to any single blue Soldier. This formulation places a premium upon the safety of each individual Soldier that cannot be mitigated by low risk to the group overall. It should be noted that other models could be employed based on different definitions of risk.

The overall quality of observation of all targets is given by,

$$Q(r_0, r_1, \dots, r_{n-1}) = \min_{m_{target}} \left(\max_n q_{ji} \right) \quad (2)$$

where $q_{ji} \in [0, 1]$ is the quality of observation for blue Soldier i observing target j . Here $q = 0$ implies no observability and $q = 1$ implies perfect observability sufficient to perform the necessary observation. The rationale for the formulation of equation 2, which differs in an important but subtle way from that of equation 1, is that the overall quality of observation should be based on the worst case for observing a single target after accounting for the best observation overall from any potential blue Soldier observers. As an example, if a single target is left unobserved, this cannot be mitigated by good observability for the remaining targets. Note that q_{ji} and Q can be interpreted as the probability of making a successful observation.

The objective function that must be minimized to determine an optimal solution is given by,

$$F = w_p P + w_q (1 - Q) \quad (3)$$

where w_p and w_q are weighting coefficients that can be used to set the relative importance of reducing the risk to ballistic threat compared with improving the observation of targets.

The above formulation puts no spatial constraints upon the locations of entities in three-dimensional space. In practice, realistic scenarios will constrain the motion of the blue Soldiers to a quasi-two-dimensional surface Ω that can be traversed by each Soldier. This surface can be mapped to a two-dimensional plane $\underline{R}^{(2)}$, with an elevation associated with each point of the plane. In order to make the problem numerically tractable, $\underline{R}^{(2)}$ can be discretized to form a uniform two-dimensional lattice $\underline{L}^{(2)}$. Thus, the location of each blue Soldier can be represented as a point $x_i \in \underline{L}^{(2)}$ since there is a one-to-one mapping from $\underline{L}^{(2)} \rightarrow \underline{R}^{(3)}$ for a given surface Ω .

In practice, there will be constraints placed on the accessibility of locations within the domain. Allowable movement over the surface Ω can be described by links between adjacent points in the lattice $\underline{L}^{(2)}$. Starting with a uniform and fully linked lattice, specific links are removed using a simple criterion for acceptable movement of a dismounted Soldier. If the change in elevation between adjacent points is greater than some limit, the link between those points is removed. By constraining the allowable motion over this lattice, each point will retain one to four links to adjacent points. Based on an initial location representing an accessible location for a Soldier, only the subset of points in the lattice that can be connected to the initial location are retained as the domain of the problem. Let this domain be given by $\underline{D}^{(2)} \subset \underline{L}^{(2)}$.

As an example, figure 1 shows the elevation map for an urban environment and figure 2 shows the resulting domain $\underline{D}^{(2)}$, where the yellow region represents a constrained two-dimensional linked lattice. The domain is constructed by starting with a three-dimensional representation of the urban environment and choosing an initial location at ground level. The connected lattice corresponds to all accessible locations from this initial point. Locations specifically excluded are rooftops and other locations that would be inaccessible to a dismounted Soldier without extreme effort.

The optimization problem reduces to determining the set of positions $x_i \in \underline{D}^{(2)}$ that minimize the objective function F defined by equation 3. Based on the definitions in equation 1 and equation 2, this objective function is not separable over independent calculations for each blue Soldier, but is instead directly dependent upon the combined state $X = \{x_0, x_1, \dots, x_{n-1}\} \in \{\underline{D}^{(2)}\}^n$. The state space that X belongs to is a $2n$ -dimensional space formed by the n outer-products of $\underline{D}^{(2)}$.

In order to determine an optimal value for X in a high dimensional space, a Markov Chain Monte Carlo sampling technique is employed. Using N “walkers” over the sample space, with initial positions sampled randomly from $\{\underline{D}^{(2)}\}^n$, a Markov Chain is constructed for each walker using

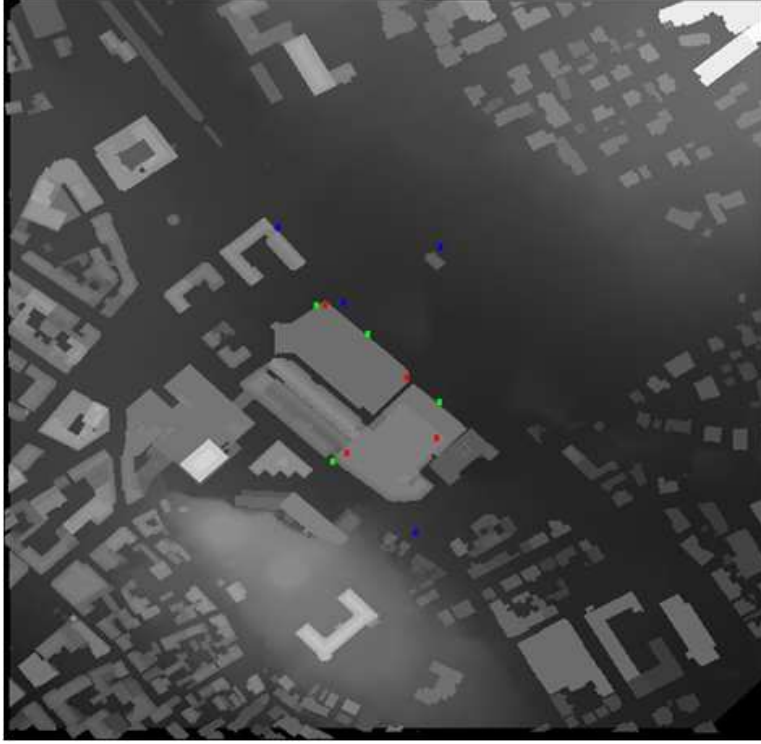


Figure 1. Elevation map showing the input locations of four enemy shooters located on rooftops (red) and four targets located at entryways to buildings (green).

a standard Metropolis algorithm (9, 8, 5, 6). A trial state $X' = \{x'_0, x'_1, \dots, x'_{n-1}\}$ is constructed from X by drawing n random numbers σ_i and updating each of the state components individually using the following rule,

$$x'_i = \begin{cases} x_i \rightarrow north & 0 \leq \sigma_i < 0.25 \\ x_i \rightarrow south & 0.25 \leq \sigma_i < 0.5 \\ x_i \rightarrow east & 0.50 \leq \sigma_i < 0.75 \\ x_i \rightarrow west & 0.75 \leq \sigma_i < 1 \end{cases} \quad (4)$$

where the directions refer to the links to adjacent points. In the event that a given link no longer exists within the constrained lattice, the action will return the original point.

The objective function F' is then evaluated for this trial state, and the trial state is accepted based on the standard Metropolis criterion. A random number σ is drawn from a distribution, and if $\sigma < F/F'$, then the trial state is accepted and $X = X'$. This has the effect of accepting any trial state with a lower objective function, and with some probability accepts trial functions that

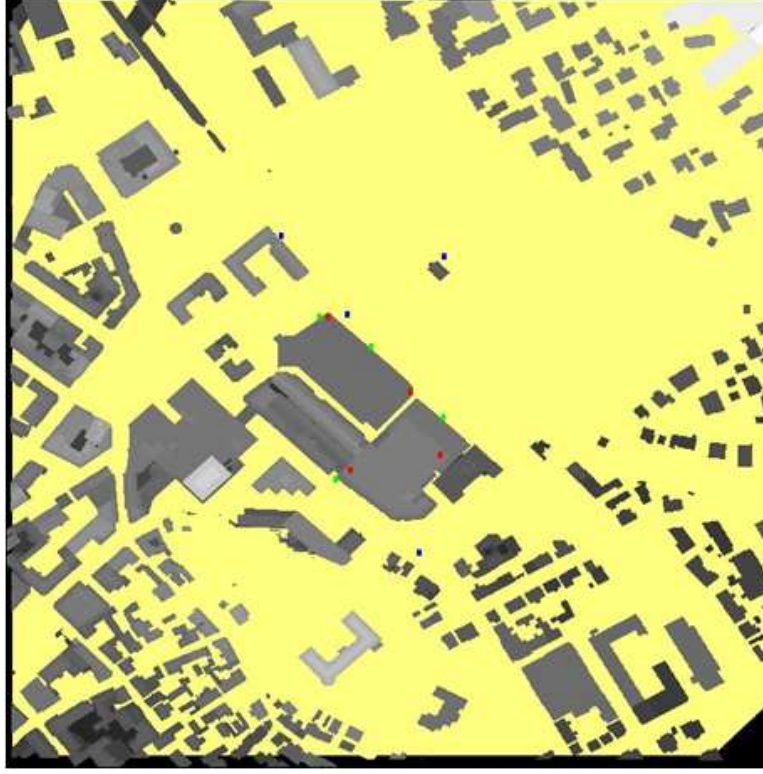


Figure 2. Shown in yellow is the ground terrain that represents all accessible points within the scenario accessible by movement on the ground. This region was discretized and used as a linked lattice for Monte Carlo minimization.

actually increase the objective function to avoid local minima. The optimum state is taken to be that which minimizes F over all states in the Markov Chain of the N walkers.

In order to evaluate the objective function for fixed red Soldiers and targets, the probability of ballistic threat and the quality of observation is pre-calculated for each point within the constrained linked lattice $\underline{D}^{(2)}$. This proves to be efficient since it is presumed that each point within this domain will be sampled at least once and each calculation is costly.

The ballistic threat posed by each of the red Soldiers is calculated as a probability field projected upon $\underline{D}^{(2)}$. There will be m_{red} such fields used in the calculation of P . Equation 1 can be re-expressed as,

$$P(X) = \max_n \left(\max_{m_{red}} p_j(x_i) \right) \quad (5)$$

where $p_j(x_i)$ is the ballistic threat probability from red Soldier j at lattice point $x_i \in \underline{D}^{(2)}$. The probability fields $p_j(x_i)$ are pre-calculated using a three-dimensional ray-traced LOS method

(10), with a polynomial representation for hit probability as a function of LOS distance. Examples of individual ballistic threat probability fields are shown in figure 3.

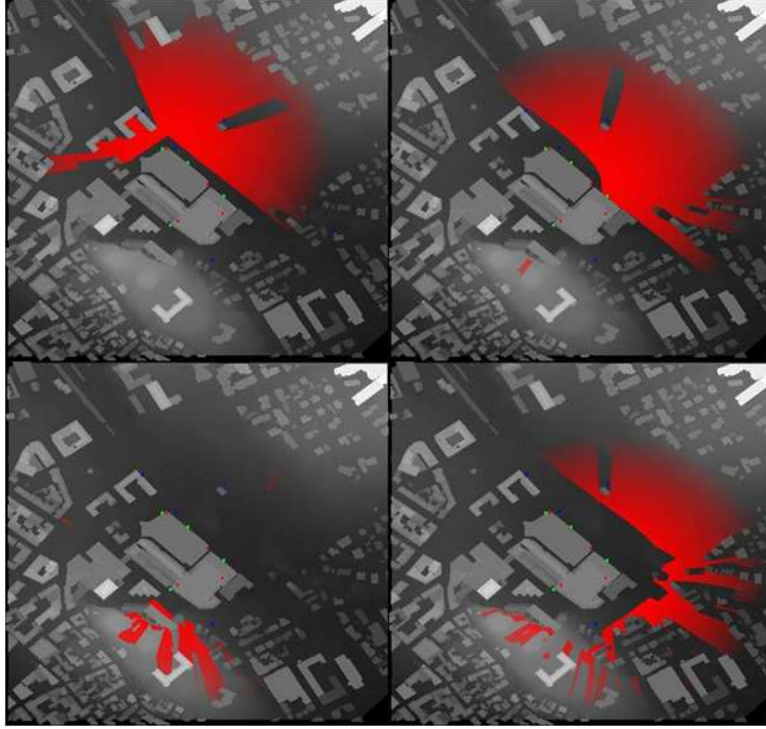


Figure 3. Ballistic threat from each of the four shooters as determined by a full three-dimensional line-of-sight ray-tracing calculation using a mathematical model for the ballistic hit probability.

Likewise, the quality of visual observation for each target is calculated as a probability field projected upon $\underline{D}^{(2)}$. There will be m_{target} such fields used in the calculation of Q . Equation 2 can be re-expressed as,

$$Q(X) = \min_{m_{target}} \left(\max_n q_i(x_j) \right) \quad (6)$$

each of the observation probability fields $q_i(x_j)$ is pre-calculated using a three-dimensional ray-traced LOS, method where the quality of visual observation is represented as a piecewise linear function of LOS distance with both a minimum and maximum cut-off. The minimum cut-off is introduced to prevent placing blue Soldiers too close to the targets since this would not make sense from an operational perspective. Examples of individual observability fields are shown in figure 4.

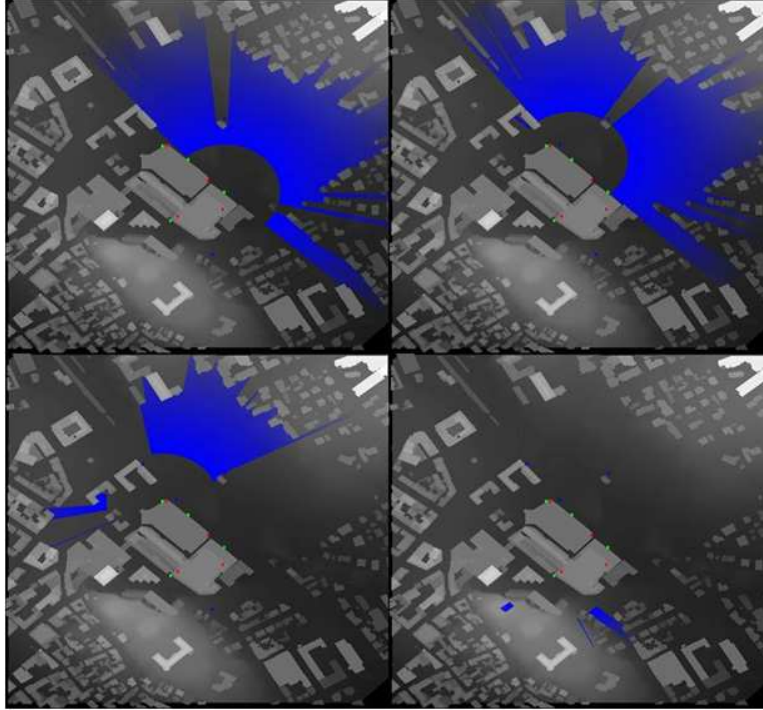


Figure 4. Quality of visual observation for each of the targets based on a full three-dimensional ray-tracing calculation using a model for the quality of sight as a function of distance.

5. Application and Results

The method has been implemented using C++ and OpenCL in a code suitable for execution on modern multi-core processors and many-core general-purpose Graphics Processing Units (GPUs). Multiple GPUs can be used to evaluate solutions at interactive speeds, thus enabling more tactical response times to dynamic situations. The details of the implementation will be presented elsewhere. As a functional demonstration of the method, the scenario shown in figure 1 was constructed to include four enemy red force Soldiers placed on rooftops and four targets (green) representing entrance points to a set of buildings. For each red force Soldier and target, the ballistic threat and observability fields, respectively, were calculated using a ray-traced LOS algorithm, with the full three-dimensional city map describing the urban environment. These fields are shown in figures 3 and 4, respectively. The code was then used to perform a static optimization to place four blue Soldiers in a position to observe the targets while minimizing the ballistic threat posed by enemy red Soldiers.

A set of 4,096 walkers in 8-dimensional space (2D coordinates for four blue Soldiers) were used to sample the ground plane linked-lattice, conveyed in figure 2, in order to determine optimal positioning. A total of 409,600,000 samples were generated using the conventional Metropolis algorithm. Figure 5 illustrates an early state of walkers in the Monte Carlo sampling, from which shows sampling distribution coverage of the ground plane and beginning indications of higher density locations. The final optimum positioning solution, taken as the state configuration optimizing the objective function, is depicted in figure 6 as the collection of four blue Soldier locations.

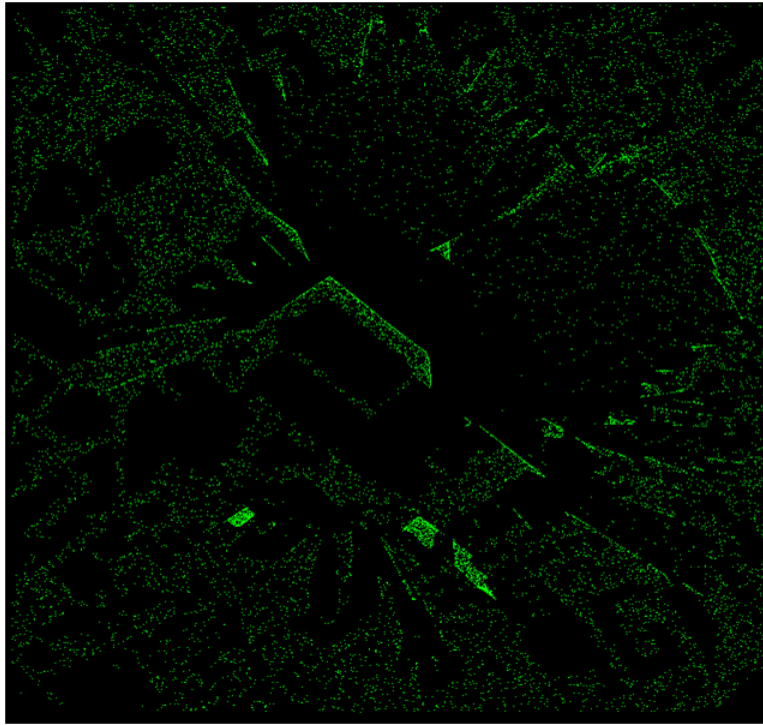


Figure 5. Early snapshot from a Markov Chain Monte Carlo optimization taken after 1,000 steps using 4096 walkers. The 8-dimensional state coordinate for the 4 blue Soldiers is projected onto the 2-dimensional lattice. At this early stage of the optimization, samples can already be seen to collect near the locations that will eventually be identified as optimal locations for the four blue Soldiers.

Performance of the Markov chain Monte Carlo optimization OpenCL kernel was collected for the Intel Xeon X5675 and AMD Radeon HD 6970 devices. For the reported CPU performance, the execution time reflects a measurement for employing two physical Xeon CPU chips equipped in the test workstation. As for the GPU results, execution times are provided for one, two, and four



Figure 6. The solution of positions for four reconnaissance Soldiers to observe the targets (blue). The locations of the reconnaissance Soldiers were determined using the optimization based on minimizing ballistic threat while maximizing line-of-sight observation.

graphics cards inside the testbed workstation. Figure 7 presents our performance findings for the geometric optimization algorithm. The CPUs with 12 cores and 24 threads at 3.07 GHz exhibit higher performance when compared to a single Radeon card. Depending on an algorithm's specific characteristics, optimal architecture can vary with an elusive answer. However, maximum performance will depend on effectively exploiting the hardware parallelism of processors. The scaling issue that is observed with GPU cards can be attributed to the lack of parallelism for our test case of 4,096 walkers. Not enough computational work is applied to fully engage the massively parallel hardware design.

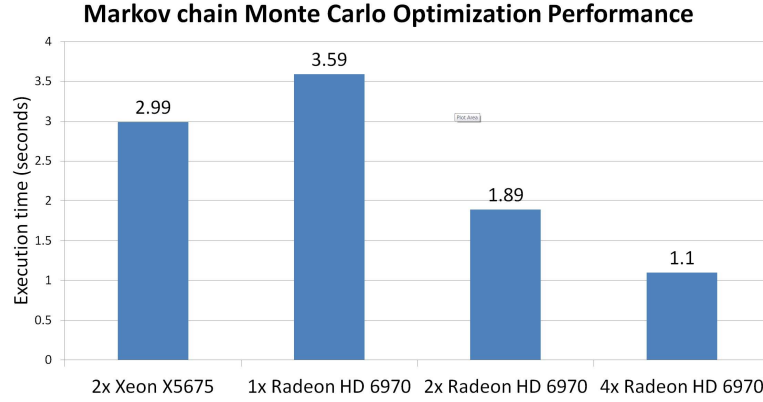


Figure 7. Execution time measured for two Intel processors (Xeon X5675 six-core) and one, two, and four AMD graphics cards (Radeon HD 6970).

6. Conclusions and Future Work

This report describes the theory and demonstrates the proof of concept for calculating the geospatial optimization of objective-constrained placement via Monte Carlo method to enhance tactical intelligence in an urban setting. We have presented a novel method coupling an advanced, portable, heterogeneous parallel computing framework and a highly parallel ray tracing algorithm, all under the auspices of an optimization methodology, itself, facilitated by massive parallelism. The end product is an approach allowing for greater situational awareness through advanced geographical and geospatial processing. Threat and visibility analysis in a 3D urban environment was performed using a first-hit ray tracing algorithm. Once fields-of-view are generated for the input entities, the numerical optimization phase calculates the best coordinates in the map that minimize threat while maximizing visibility.

A variant of a ray tracing algorithm occupies the core of the algorithm. Here, the line-plane intersection computation was extended to calculate ballistic threat and quality of observation, but this ray tracing approach can be applicable to analyzing radiowave propagation in urban operations. Although a specific scenario was presented in this study, the core algorithms can serve as a foundation to compute and extract myriad of time-critical intelligence in a 3D environment.

To allow these approaches to be used as more than just a before-action planning tool and work more in a real-time operational environment, future work also entails performing analysis on

execution speed and examining differing processor architectures to achieve near real-time operation. With this in mind, the source code implementation was developed in OpenCL to conduct performance research on diverse parallel processors and accelerators. Given the unclear roadmap for what might become a dominant processing architecture, indicated by the rise of ARM systems and accelerators in HPC centers, our development path decided to embrace parallel computing, vendor agnostic portability, and heterogeneous platforms.

Additional code execution speed improvements will pave the way for a more robust application. Planning of extensions to dynamic scenario optimization is underway where scenario assets are not completely stationary within the map. Optimization will then consider determining routes through the map that will satisfy specific mission objectives, thus increasing realism of the problem. Determining the minimum cardinality of the set of cooperative guards is also planned for future exploration. This will be invaluable in determining not only placement strategies, but also in defining the minimum number of guards or Soldiers required based on mission parameters. Here we will attempt to map more completely the cooperative guard problem to the fortress problem, and develop solutions with response times integrated with dynamic mission unfolding and streaming situational awareness.

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