

Complexity Analysis of the Cost-Table Approach to the Design of Multiple-Valued Logic Circuits

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Abstract—We analyze the computational complexity of the cost-table approach to designing multiple-valued logic circuits that is applicable to F²L, CCDs, current-mode CMOS, and RTDs. We show that this approach is NP-complete. An efficient algorithm is shown for finding the exact minimal realization of a given function by a given cost-table.

Index Terms—Computational complexity, cost-table, cost function, logic design, minimization, multiple-valued logic, NP-complete, synthesis.

1 INTRODUCTION

THE first demonstration that a logic synthesis problem is NP-complete occurred as the result of two insights. To find the minimal sum-of-products expression for a logic function, one can produce the set S of all prime implicants and then use a minimal subset of S to cover all minterms of the function. The latter step is a specific case of the set covering problem. Because it is a specific case, it is possible that it is not as complex as the general set covering problem. However, Gimpel [2] showed that this is not true. He showed that any instance of the set covering problem occurs as an instance of the sum-of-products problem. Subsequently, Karp [3] proved that the set covering problem is NP-complete; thus, proving that extracting a minimal sum-of-products expression is NP-complete.¹ While complexity questions have frequently occurred in *multiple-valued* logic (e.g., [1], [7]), there has been no classification of the synthesis of multiple-valued functions complexity classes, e.g., NP-completeness.

The need for design techniques for multiple-valued CCD circuits, [5], inspired interest in the cost-table approach, e.g., [1], [6], [7]. In the cost-table approach, a given function is realized by selecting functions from a table and combining them. Associated with each chosen function is a cost, which can represent chip area, power dissipation, speed, etc. The cost of a realization is the sum of the costs of the component functions plus the cost of combining them. Usually, there is more than one way to realize a given function, and the goal of the design is to find a realization of lowest cost. This is called the *Cost-table Realization* problem. The question posed and answered in this paper is “How does the time to solve the cost-table realization problem depend on the size of the cost-table?” We show that this problem is NP-complete.

1. Keutzer and Richards [4] point out that there has been misunderstanding in certain papers on the complexity of the sum-of-products extraction problem. That is, the problem of finding a sum-of-products expression with no more than some given number of terms is NP-complete if the function is expressed as a truth table, but co-NP hard if the function is expressed as a sum-of-products expression.

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2 BACKGROUND AND NOTATION

A function $f(X)$ is a mapping $f: D^n \rightarrow R$, where $D = \{0, 1, \dots, d-1\}$ and $R = \{0, 1, \dots, r-1\}$.

When $n = 1$, it is convenient to represent $f(X)$ in the form $\langle f(0), f(1), \dots, f(d-1) \rangle$. For example, if $d = r = 4$, then $f(X) = \langle 3, 2, 1, 0 \rangle$ is the four-variable *complement* function. The set of all r -valued functions of n d -valued variables is $U_{d,n}^r$. Let $c(f)$, the *cost function*, be a mapping $c: U_{d,n}^r \rightarrow \mathbf{R}^{0+}$, where $\mathbf{R}^{0+}$ is the set of nonnegative real numbers. For example, the cost function $c(f)$ introduced by Kerkhoff and Robroek [6] for the design of four-valued CCD logic circuits correlates closely with the chip area occupied by the most compact implementation of f .

Given a function $f(X)$ to be realized using a cost-table, we seek a representation of the form $f(X) = f_1(X) + f_2(X) + \dots + f_m(X)$, where $+$ is ordinary addition with logic values viewed as integers and f_i is a cost-table function. For example, if $f_1(X) = \langle 0, 1, 2, 3 \rangle$ and $f_2(X) = \langle 3, 2, 1, 0 \rangle$, then $f_1(X) + f_2(X) = \langle 3, 3, 3, 3 \rangle$. In our analysis, it is convenient to assume that the sum of two logic values does not exceed the highest logic value, $r-1$. Thus, $+$ can be implemented as the sum mod r or as truncated sum, for example. The latter is more common in practice, since it is easily implemented, e.g., in CCD or current-mode logic. The effect of this assumption is not to restrict the operations possible, but the synthesis technique. For example, $f_1 + f_1$ is not a realization of the synthesis technique because two components sum to a value greater than $r-1$. Let σ be the cost of realizing the sum of two functions. The cost of the realization $f = f_1 + f_2 + \dots + f_m$ is

$$c(f_1) + c(f_2) + \dots + c(f_m) + (m-1)\sigma,$$

where σ is the cost of combining two cost-table functions.

A *basis function* f has the property that $f(A)$ is 1 for exactly one assignment A of values to X and is 0 for all other assignments. Let BT be the set of all basis functions plus $\mathbf{0}$, the function that is 0 for all assignments of values to the variables (e.g., $\langle 0, 0, 0, 0 \rangle$). BT is called the *basis cost-table*. F is a *cost-table* if and only if $BT \subseteq F \subseteq U_{d,n}^r$. Note that all functions in BT are needed in F . Indeed, if the function f to be realized has the property $f \in BT$, then f cannot be realized, unless $f \in F$. Of all the ways to realize a given function f using cost-table F , one realization, $f = f_1 + f_2 + \dots + f_m$, where $f_i \in F$, has a cost that is lower than or equal to the cost of all other realizations of f using F . Denote realization $f = f_1 + f_2 + \dots + f_m$ as a *minimal cost realization* of f . Note that there may be more than one such realization. Its cost, $c(f_1) + c(f_2) + \dots + c(f_m) + (m-1)\sigma$, is the *cost of realizing* $f \in U_{d,n}^r$ using cost-table F , and will be denoted as $c_F(f)$. Thus, whenever we seek the cost of realizing a given function f using a given cost-table F , we assume that, of all the ways to realize a function f using cost-table F , we choose the lowest cost realization. Formally,

$$c_F(f) = \min_{\substack{f_1, f_2, \dots, f_m \in F \\ f = f_1 + f_2 + \dots + f_m}} \{c(f_1) + c(f_2) + \dots + c(f_m) + (m-1)\sigma\},$$

The *total cost*, $T(F)$, of cost-table F is

$$T(F) = \sum_{f \in U_{d,n}^r} c_F(f).$$

F is a *minimal* cost-table if $T(F) \leq T(F')$, for all F' , such that $|F| = |F'|$, where $|F|$ is the cardinality of F . The term “minimal” describes the cost over all realizations of a cost-table.

The (Minimal) Cost-table Realization, (MCR) CR, problem is:

Given a (minimal) cost-table F , a function f , and a cost function c , find a minimal cost realization $f = f_1 + f_2 + \dots + f_m$, where $f_i \in F$.

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The (Minimal) Cost-table Decision, (MCD) CD, problem is:

Given a (minimal) cost-table F , a function f , a cost function c , and a target cost P , does there exist a realization $f = f_1 + f_2 + \dots + f_m$, such that $c(f_1 + f_2 + \dots + f_m) \leq P$, where $f_i \in F$?

Let $(MCD(F, f, c, P))$ $CD(F, f, c, P)$ denote an instance of this problem. $(MCD(F, f, c, P))$ $CD(F, f, c, P)$ is said to be satisfied if and only if such a realization exists. The size K of an instance of $(MCD(F, f, c, P))$ $CD(F, f, c, P)$ is $d^n |F|$. K accounts for both the function size, as well as the cost-table size. Since the $MCD(F, f, c, P)$ is a special case of the $CD(F, f, c, P)$, there is the possibility that it is not as complex. We show, however, that this is not the case.

3 COMPLEXITY OF THE COST-TABLE REALIZATION PROBLEM

The main results are presented in two theorems.

THEOREM 1. *The Cost-table Decision problem is NP-complete.*

THEOREM 2. *The Minimal Cost-table Decision problem is NP-complete.*

We proceed by first showing that these two problems are within NP; that is, we show in Lemma 1 that there exists a non-deterministic Turing Machine that calculates each problem in time polynomial in the size of the problem.

Next, in Lemma 2, we show that there is a polynomial time transformation of the Knapsack problem to the (Minimal) Cost-table Decision Problem, where the former is satisfied iff the latter is satisfied. Since the Knapsack problem is known to be NP-complete, this shows that the (Minimal) Cost-table Decision problem is NP-complete.

Consider the solution of $(MCD(F, f, c, P))$ $CD(F, f, c, P)$ by a nondeterministic algorithm that scans F , choosing as many as $r - 1$ copies of each function for each of the d^n possible assignments of values to the variables. This can be done in no more than $O((r - 1)d^n |F|)$ time. This algorithm can check whether the chosen function is a realization of f in $O(d^n)$ time. Also, it can check whether the cost is less than or equal to P in $O((r - 1)|F|)$ time. Since the size of an instance of this problem is $K = d^n |F|$, this proves the following:

LEMMA 1. *There exists a nondeterministic algorithm that solves $(MCD(F, f, c, P))$ $CD(F, f, c, P)$ in time that is polynomial in its size.*

The Knapsack Decision problem can be stated as follows:

Given a set Q of objects, a size function $s : Q \rightarrow \mathbf{Z}^+$, a value function $v : Q \rightarrow \mathbf{Z}^+$, a size S , and a value V , is there a subset $Q' \subseteq Q$ such that $\sum_{u \in Q'} v(u) \geq V$ and $\sum_{u \in Q'} s(u) \leq S$, where $\mathbf{Z}^+$ is the set of positive integers?

Let $KD(Q, s, v, S, V)$ be an instance of the Knapsack Decision problem. $KD(Q, s, v, S, V)$ is said to be satisfied if and only if such a subset Q' exists. The size of an instance of this problem is $|Q|$.

DEFINITION. Let Φ be a transformation from any instance of the Knapsack Decision problem to an instance of the (Minimal) Cost-table Decision problem

$$\Phi(KD(Q, s, v, S, V)) = (MCD(F, f, c, P)) CD(F, f, c, P),$$

with F , f , c , and P defined as follows:

- 1) The cost-table F consists of r -valued functions on one d -valued variable, where $r = S + 1$ and $d = |Q| + 1$. Besides the $d + 1$ functions in BT , there are $d - 1$ nonbasis functions $f_1, f_2, \dots, f_{d-1}$, where f_i corresponds to u_i , the i th element in Q . Specifically, $f_i(0) = s(u_i)$, $f_i(i) = 1$, and $f_i(j) = 0$, for $1 \leq j \leq d - 1, j \neq i$. We have

$$f_1 = \langle s(u_1), 1, 0, 0, \dots, 0 \rangle$$

$$f_2 = \langle s(u_2), 0, 1, 0, \dots, 0 \rangle$$

$$\vdots$$

$$f_{d-1} = \langle s(u_{d-1}), 0, 0, 0, \dots, 1 \rangle.$$

- 2) Function f has the form

$$f = \langle S, 1, 1, 1, \dots, 1 \rangle.$$

Since $f(i) = 1$ for $1 \leq i \leq d - 1$, each f_i can be used at most once in the realization of f . This corresponds to the restriction that each element $u_i \in S$ is used at most once in the Knapsack Decision problem. Also, since $f(0) = S$, the sum $\sum f_i(0)$ over the f s used in a realization of f (i.e., $s(u_i)$) must be less than or equal to S .

- 3) Let $c(f_i) = s(u_i)$, for $1 \leq i \leq d - 1$. Let the cost of functions in BT be defined as follows:

$$c(b_j) = \begin{cases} 0 & \text{if } j = 0 \\ v(u_j) & \text{otherwise,} \end{cases}$$

where $b_j(j) = 1$ and $b_j(i) = 0$ for $i \neq j$. That is, the cost of $\langle 1, 0, \dots, 0 \rangle$ is 0, while the cost of all other basis functions is the value of some object in Q . The cost of the constant function $\langle 0, 0, \dots, 0 \rangle$ is 0. Let the cost, σ , of combining two functions be 1.

If Φ is a transformation to $CD(F, f, c, P)$, we allow any specification of the cost of a function g , such that $g \notin F$. If Φ is a transformation to $MCD(F, f, c, P)$, we make the additional specification that, for $g \notin F$, $c(g) = \infty$. In this way, F is a minimal cost-table; i.e., no interchange of functions outside F with functions inside F that preserves the size of the cost-table yields a total cost lower than $T(F)$.

- 4) P is defined by

$$P = \sum_{u_i \in Q} v(u_i) - V + (S + d - 2) \quad (1)$$

EXAMPLE. Consider a knapsack defined as follows: Let $Q = \{u_1, u_2, u_3\}$, and let $s(u_i)$ and $v(u_i)$ be specified as in Table 1. Let $S = 5$ and $V = 6$.

TABLE 1
SIZES AND VALUES OF ELEMENTS OF THE KNAPSACK

	$s(u_i)$	$v(u_i)$
u_1	3	4
u_2	2	3
u_3	2	2

Of the eight ways to choose subsets of Q , there are two that satisfy $KD(Q, s, v, S, V)$, as shown in Table 2

TABLE 2
THE TWO SOLUTIONS TO THE KNAPSACK DECISION PROBLEM

$Q_1 = \{u_1, u_2\}$	$\sum_{u \in Q_1} v(u) = 7 \geq V = 6$ $\sum_{u \in Q_1} s(u) = 5 \leq S = 5$
$Q_2 = \{u_1, u_3\}$	$\sum_{u \in Q_2} v(u) = 6 \geq V = 6$ $\sum_{u \in Q_2} s(u) = 5 \leq S = 5$

Applying the transformation yields a cost-table where $r = 6$ and $d = 4$ with functions as shown in Table 3. The function to be synthesized is $f = \langle 5, 1, 1, 1 \rangle$, and $P = 10$. The instance of the cost-table decision problem, $CD(F, f, c, P)$ so formed, is satisfied by exactly two realizations of f , as shown in Table 4. These two realizations match left to right with $\{u_1, u_2\}$ and $\{u_1, u_3\}$, the subsets satisfying $KD(Q, s, v, S, V)$. Note that of the two realizations of $\langle 5, 1, 1, 1 \rangle$, one is uniquely minimal, that given in the left hand column of Table 4.

TABLE 3
COST-TABLE AS TRANSFORMED
FROM THE KNAPSACK DECISION PROBLEM

Function	Cost	
$\langle 0, 0, 0, 0 \rangle$	0	0
$\langle 1, 0, 0, 0 \rangle$	0	0
$\langle 0, 1, 0, 0 \rangle$	4	$v(u_1)$
$\langle 0, 0, 1, 0 \rangle$	3	$v(u_2)$
$\langle 0, 0, 0, 1 \rangle$	2	$v(u_3)$
$\langle 3, 1, 0, 0 \rangle$	3	$s(u_1)$
$\langle 3, 0, 1, 0 \rangle$	2	$s(u_2)$
$\langle 2, 0, 0, 1 \rangle$	2	$s(u_3)$

TABLE 4
TWO SOLUTIONS TO THE COST-TABLE DECISION PROBLEM

Function	Cost	Function	Cost
$\langle 3, 1, 0, 0 \rangle$	3	$\langle 3, 1, 0, 0 \rangle$	3
$\langle 2, 0, 1, 0 \rangle$	2	$\langle 2, 0, 0, 1 \rangle$	2
$\langle 0, 0, 0, 1 \rangle$	2	$\langle 0, 0, 1, 0 \rangle$	3
Additions	2	Additions	2
Total	9	Total	10

We can make the following general statement.

LEMMA 2. Φ is a polynomial time transformation of the Knapsack Decision problem to the (Minimal) Cost-table Decision problem, such that $KD(Q, s, v, S, V)$ is satisfied if and only if $(MCD(F, f, c, P))$ $CD(F, f, c, P) = \Phi(KD(Q, s, v, S, V))$ is satisfied.

PROOF. The proof is divided into three parts. First, it is shown that Φ takes polynomial time. Next, it is shown that if $KD(Q, s, v, S, V)$ is satisfied, then $\Phi(KD(Q, s, v, S, V))$ is satisfied (only if). Finally, it is shown that, if $\Phi(KD(Q, s, v, S, V))$ is satisfied, then $KD(Q, s, v, S, V)$ is satisfied (if).

To form the cost-table $F \subseteq U_{d,1}^r$, Φ generates $d - 1 = |Q|$ nonbasis functions, d basis functions, and the constant function $\langle 0, 0, \dots, 0 \rangle$. Each function can be described by a truth table with $d = |Q| + 1$ entries. An entry in the truth table can be made in constant time. Thus, the total time needed to generate F is $O(|Q|^2)$. A cost is then assigned to each function requiring constant time per function. Since $s(u_i)$ can be computed in constant time, the target function f can be formed in $O(|Q|)$ time. Finally, P requires the summation of all $v(u_i)$, which also takes $O(|Q|)$ time. Since each step takes at most polynomial time, the entire transformation takes polynomial time.

As preparation for the next two parts, consider

$$g_i = \begin{cases} f_i & \text{if } u_i \in Q' \text{ and } 1 \leq i \leq d-1 \\ b_i & \text{if } u_i \notin Q' \text{ and } 1 \leq i \leq d-1 \\ b_0 & \text{if } d \leq i \leq m \end{cases}$$

where Q' is the subset of Q that satisfies the Knapsack Decision problem and $m = S - S' + |Q|$, for $S' = \sum_{u_i \in Q'} s(u_i)$. We

now show that $g_1 + g_2 + \dots + g_m = f$. Consider $g_1 + g_2 + \dots + g_m$, when the variable value is 0.

$$\begin{aligned} \sum_{i=1}^m g_i(0) &= \sum_{u_i \in Q'} f_i(0) + \sum_{u_i \notin Q'} b_i(0) + \sum_{i=d}^m b_0(0) \\ &= \sum_{u_i \in Q'} s(u_i) + 0 + (m - d + 1) = S' + 0 + (S - S') = f(0). \end{aligned}$$

When the variable value is not 0, $g_1 + g_2 + \dots + g_m$ is evaluated as follows: By the definition of f_i and b_i , $g_i(j) = 0$ if $i \neq j$ and $1 \leq j$. Therefore, $\sum_{i=1}^m g_i(j) = 1 = f(j)$, for $1 \leq j \leq d-1$.

This proves that $g_1 + g_2 + \dots + g_m = f$.

The cost of realization $f = g_1 + g_2 + \dots + g_m$ is

$$\sum_{u_i \in Q'} s(u_i) + \sum_{u_i \notin Q'} v(u_i) + 0 + (m - 1)$$

or

$$S' + \sum_{u_i \in Q} v(u_i) - V' + (S - S' + |Q| - 1),$$

where $V' = \sum_{u_i \in Q'} v(u_i)$. From (1), the cost of this realization is $P - V' + V$.

(only if) Assume $KD(Q, s, v, S, V)$ is satisfied by Q' . The size of this collection is $S' = \sum_{u_i \in Q'} s(u_i)$, and the value is V' .

Since Q' satisfies $KD(Q, s, v, S, V)$, $S' \leq S$ and $V' \geq V$. Now consider $c_F(f)$, the minimal cost realization of f in cost-table F . Because the cost of the realization $g_1 + g_2 + \dots + g_m$ is an upper bound on the minimal cost realization, $c_F(f) \leq P - V' + V$. Since $V' \geq V$, $c_F(f) \leq P$. If F is a minimal cost-table, then $MCD(F, f, c, P)$ is satisfied. Else, $CD(F, f, c, P)$ is satisfied.

(if) Assume $\Phi(KD(Q, s, v, S, V)) = (MCD(F, f, c, P))$ $CD(F, f, c, P)$ is satisfied by the realization $f = h_1 + h_2 + \dots + h_l$, where $h_i \in F$. Then, $\sum_{i=1}^l c(h_i) + (l - 1) \leq P$. We show that the Knapsack Decision problem is satisfied for

$$Q' = \{u_i \mid h_i \notin BT\}.$$

To calculate the "size" of the solution, consider the function evaluated at 0; that is, $\sum_{i=1}^l h_i(0) = f(0)$. We can write

$$\sum_{u_i \in Q'} h_i(0) + \sum_{u_i \notin Q'} h_i(0) = S,$$

where the functions in the right sum are in BT , while those in the left sum are not. Since $h_i(0) \geq 0$, the right sum in the above equation is nonnegative. Therefore, $\sum_{u_i \in Q'} h_i(0) \leq S$ and thus,

$$\sum_{u_i \in Q'} s(u_i) \leq S.$$

To calculate the "value" of the solution, consider the cost of the realization $f = h_1 + h_2 + \dots + h_l$. Because this is a solution to $(MCD(F, f, c, P))$ $CD(F, f, c, P)$,

$$\sum_{i=1}^l c(h_i) + (l-1) \leq P.$$

Inserting the definitions of P and $c(h_i)$ into this equation yields

$$\sum_{u_i \in Q'} s(u_i) + \sum_{u_i \notin Q'} v(u_i) + l - 1 \leq \sum_{u_i \in Q} v(u_i) - V + (S + d - 2).$$

Rearranging yields

$$V + \left[l - \left[(d-1) + S - \sum_{u_i \in Q'} s(u_i) \right] \right] \leq \sum_{u_i \in Q'} v(u_i).$$

We show that the term in large brackets is 0. Thus, $V \leq \sum_{u_i \in Q} v(u_i)$, and so the Knapsack Decision problem has a solution. Each of the 1 terms in $f = \langle S, 1, 1, \dots, 1 \rangle$ is realized by either a b_i or an f_i , for $1 \leq i \leq d-1$. The f_i terms contribute $\sum_{u_i \in Q'} s(u_i)$ to $f(0)$. Thus, $S - \sum_{u_i \in Q'} s(u_i)$ copies of b_0 are needed. It follows that $l = (d-1) + S - \sum_{u_i \in Q'} s(u_i)$.

Thus, a solution to $KD(Q, s, v, S, V)$ exists, such that $\sum_{u_i \in Q'} s(u_i) \leq S$ and $\sum_{u_i \in Q'} v(u_i) \geq V$. $\square$

Since the Knapsack Decision problem is NP-complete, Lemmas 1 and 2 prove the main result.

4 AN ALGORITHM FOR FINDING MINIMAL COST

In this section, we present an algorithm, MIN_COST, for solving the cost-table problem. Next, we analyze the time complexity of MIN_COST, showing how the number of steps depends on K , the size of the problem. We show that for smaller cost-tables, the complexity is exponential, while for larger cost-tables, the complexity is polynomial in the size of the problem.

4.1 MIN_COST

We present an algorithm, MIN_COST, to find the minimal cost realization of a function f using the cost-table technique. Specifically, MIN_COST (F, f) finds a realization of f with minimum cost, $c_F(f)$, given any cost-table $F \subseteq U_{d,n}^r$ and any function $f \in U_{d,n}^r$. No other published algorithm is known. It is superior to the exhaustive search algorithm used in [7]. The algorithm for solving CD given in Section 3 is the nondeterministic version of a deterministic algorithm that searches exhaustively over all combinations of cost-table functions for a realization with a cost less than a given threshold. Searching for the *least cost* realization yields behavior that is identical to MIN_COST.

However, it is not necessary to search over all cost-table functions. Given two functions, f and e , let $e \leq f$ mean that for every assignment A of values to the variables, $e(A) \leq f(A)$. It follows that, unless $e \leq f$, e will never be used in a realization of f . Let $E = \{e \mid e \leq f\}$. $(E, \leq)$ be a partially ordered set i , and the elements in E can be indexed such that, for all $e_j, e_k \in E$, if $e_j \leq e_k$, then $j \leq k$.

Then, $e_0 = \mathbf{0}$ (the constant 0 function) and $e_{|E|-1} = f$. Let $I = (F \cap E) - BT$. I consists of all functions in cost-table F that are potentially in the minimal realization of f , excluding functions in BT . MIN_COST forms a sequence of cost-tables $BT = F_0 \subset F_1 \subset \dots \subset F_{|I|}$, such that $F_i - F_{i-1} = \{f_i\}$, where $f_i \in I$. MIN_COST begins by initializing $c_{F_0}(e_j)$ to $c_{BT}(e_j)$, for $0 \leq j < |E|$. Then, for each cost-table F_i , where $1 \leq i \leq |I|$, $c_{F_i}(e_j)$ is computed for each $e_j \in E$. When MIN_COST

reaches $F_{|I|}$, it has found a minimal cost realization of the given function f in cost-table F .

MIN_COST only checks for one use of f_i in the realization of any e_j . A complication arises if f_i is required more than once in the minimal realization of some function e_j . Consider the case where $e_k = f_i + f_i + e_r$, and $e_s = f_i + e_r$. Since $e_r \leq e_s \leq e_k$, the ordering over E requires that $r \leq s \leq k$. So $c_{F_k}(e_k)$ will be calculated using $c_{F_i}(f_i)$ and $c_{F_i}(e_s)$, but the cost of e_s will have already been updated using the functions f_i and e_r . Therefore, algorithm MIN_COST correctly computes the cost of functions which use multiple copies of cost-table functions.

4.2 The Time Complexity of MIN_COST

4.2.1 The Time Complexity for a Single Function

MIN_COST consists of two steps. First, the cost of each $e_j \in E$ using the basis cost-table is computed by summing over all functions in BT , requiring d^n operations or $\mathbf{O}(d^n |E|)$ operations for all e_j . Second, for each cost-table F_i , the new cost of each e_j is computed, requiring at most $\mathbf{O}(d^n |E|)$ operations per cost-table. Since there are $|I|$ cost-tables, the entire algorithm has time complexity $\mathbf{O}(d^n |I| |E|)$. (The algorithm is shown in Table 5.)

In [7], cost-tables for one-variable four-valued functions were analyzed in order to study heuristics for finding minimal cost-tables. We can conclude that MIN_COST works well for cost-tables for such functions with sizes as small as five and as large as 256.

4.2.2 The Time Complexity as a Function of Input Size

From the previous analysis, the time complexity of MIN_COST is polynomial in $|E|$. We now consider the relationship between $|E|$ and the size of the Cost-table Decision problem $K = d^n |F|$. Let F be a cost-table of size one larger than the basis cost-table; therefore $|F| = d^n + 2$. Let f , the function whose cost we wish to minimize, be the constant $r-1$ function, so $E = U_{d,n}^r$, and $|I| = 1$. In this case, the time complexity of MIN_COST is $\mathbf{O}(d^n, r^{d^n})$, while the size of the problem is $K = d^n(d^n + 2)$. Thus, MIN_COST's time complexity is $\mathbf{O}(\sqrt{K} r^{\sqrt{K}})$.

As the size of the cost-table $|F|$ increases, the time complexity of MIN_COST becomes polynomial in $|F|$. In the limit, $F = U_{d,n}^r$, and the time complexity of MIN_COST becomes $\mathbf{O}(d^n r^{d^n} r^{d^n})$, while the size of the problem is $K = d^n r^{d^n}$. Thus, MIN_COST's time complexity, $\mathbf{O}(K^2/d^n)$, is polynomial in the size of the problem when the cost-table is sufficiently large (approaching $U_{d,n}^r$).

4.2.3 The Time Complexity for All Functions

In the process of finding a minimal cost of function f , MIN_COST finds a minimal cost realization for all functions $e_j \in E$. If f is chosen to be the constant $r-1$ function, then $e \leq f$ for all functions $e \in U_{d,n}^r$, so $E = U_{d,n}^r$. Using the previous analysis, a minimal cost realization of all functions can be found in $\mathbf{O}(d^n |F - BT| r^{d^n})$ time by MIN_COST. Thus, MIN_COST provides a more efficient alternative to exhaustive search algorithms, as demonstrated in analyzing various cost-tables [7].

TABLE 5
FORMAL DESCRIPTION OF MIN_COST, AN ALGORITHM FOR FINDING THE MINIMAL COST REALIZATION
OF A GIVEN FUNCTION FROM A GIVEN COST-TABLE

Algorithm MIN_COST
<pre> { Compute costs of $e_i \in E$ (and thus f) using the basis cost-table } $c_{F_0}(0) := c(0)$ for $j := 1$ to $E - 1$ do $c_{F_0}(e_j) := \sum_{\substack{b \in BT \\ b(A)=1}} e_j(A) c(b) + \sum_{\substack{b \in BT \\ b(A)=1}} e_j(A) \sigma - \sigma$ {where σ is the cost of adding two functions and $e_j(A)$ is the value (viewed as an integer) of e_j for the assignment of values A such that $b(A) = 1$. The left sum represents the costs of basis functions, while the right sum less σ represents the costs of adders.} { Compute costs of $e_i \in E$ (and thus f), using F_i, the next cost-table in the sequence } for $i := 1$ to I do begin { for f_i in I, where $\{f_i\} = F_i - F_{i-1}$. } for $j := 0$ to $E - 1$ do {set the cost of a function e_j using F_i to the cost of e_j using F_{i-1} } $c_{F_i}(e_j) := c_{F_{i-1}}(e_j)$ if $c(f_i) < c_{F_{i-1}}(f_i)$ then begin { update the cost of e_j using F_i if it is less than the cost of e_j in F_{i-1} } $c_{F_i}(f_i) := c(f_i)$ for $j := 0$ to $E - 1$ do if $f_i = e_j$ then $c_{F_i}(e_j) = \min\{c_{F_{i-1}}(e_j), c(f_i)\}$ else if $f_i \leq e_j$ then begin find h such that $h + f_i = e_j$ $NEW_COST := c_{F_i}(h) + c(f_i) + \sigma$ $c_{F_i}(e_j) := \min\{c_{F_{i-1}}(e_j), NEW_COST\}$ end end { update the cost of e_j in F_i if it is less than the cost of e_j in F_{i-1} } end { for f_i in I, where $\{f_i\} = F_i - F_{i-1}$. } </pre>

5 CONCLUDING REMARKS

During the past 15 years of research on cost-tables, there has been no computationally tractable algorithm for finding minimal cost realizations of given functions. We show that this problem is NP-complete. We also show that restricting the cost-tables to be minimal (the total cost of realizations by such cost-tables is minimal) produces no relief; the problem is still NP-complete. This result represents compelling evidence for the value of heuristic methods for cost-tables.

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