

ON THE FIRST DIGITS OF THE FIBONACCI NUMBERS AND THEIR EULER FUNCTION

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ABSTRACT. Here, we show that given any two finite strings of base b digits, say s_1 and s_2 , there are infinitely many Fibonacci numbers F_n such that the base b representation of F_n starts with s_1 and the base b representation of $\phi(F_n)$ starts with s_2 .

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1. Introduction

Let $b \geq 2$ be and integer. Let $s_1 = \overline{c_1 \cdots c_{k(b)}}$ be a positive integer s_1 written in base b . Washington [4] proved that there exist infinitely many Fibonacci numbers F_n whose base b representation starts with s_1 . In fact, the first digits of the Fibonacci sequence obey Benford's law in that the proportion of the positive integers n such that F_n starts with s_1 is precisely $\log((s_1 + 1)/s_1)/\log b$. Here, we take this one step further. Let $\phi(m)$ be the Euler function of the positive integer m . We put $s_2 = \overline{d_1 \cdots d_{\ell(b)}}$ for some other positive integer written in base b and prove the following theorem.

Theorem. *Given positive integers $s_1 = \overline{c_1 \cdots c_{k(b)}}$ and $s_2 = \overline{d_1 \cdots d_{\ell(b)}}$ written in base b , there exist infinitely many positive integers n such that the base b representation of F_n starts with the digits of s_1 and the base b representation of $\phi(F_n)$ starts with the digits of s_2 .*

We use the fact that with $(\alpha, \beta) = ((1 + \sqrt{5})/2, (1 - \sqrt{5})/2)$ the Binet formula

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{holds for all} \quad n \geq 0.$$

For a positive real number x we write $\log x$ for the natural logarithm of x , and $\lfloor x \rfloor$, respectively, $\{x\}$, for the integer part, respectively, fractional part of x .

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2. The proof

By replacing s_1 with $s_1 b^m$ for some positive integer m , if needed, whose effect is adding m zeros at the end of the base b representation of s_1 , we may assume that $s_1 > s_2$. By replacing s_1, s_2 by $s_1 b^m, s_2 b^m$ for an arbitrary positive integer m , we may assume that the length of the base b representation of s_1 , that is k , is as large as we wish. In Section 4 of [2], it is shown that $\phi(F_n)/F_n$ is dense in $[0, 1]$. So, we take $\varepsilon \in (0, 1/(15b^{2k}))$ and choose a positive integer a such that

$$\frac{\phi(F_a)}{F_a} \in \left(\frac{s_2}{s_1} + \varepsilon, \frac{s_2}{s_1} + 2\varepsilon \right). \quad (1)$$

Now we take any prime $p > F_a$ and look at F_{ap} . Since $p > F_a$, it follows that

$$F_{ap} = F_a \left(\frac{F_{ap}}{F_a} \right),$$

and the two factors F_a and F_{ap}/F_a on the right above are coprime (indeed, the only common prime factor of these two numbers could be p , which is not the case since $p > F_a$). Any prime factor q of F_{ap}/F_a is a *primitive prime factor* of F_{dp} for some divisor d of a . Recall that a prime number q is said to be a primitive prime factor of F_n if q divides F_n , but does not divide any F_m for $1 \leq m < n$. One of the properties of primitive prime factors q of F_n when $n > 5$ is that $q \equiv \pm 1 \pmod{n}$. In particular, every prime factor q of F_{ap}/F_a is congruent to $\pm 1 \pmod{p}$.

Let $q_1, \dots, q_t$ be all the prime factors of F_{ap}/F_a . Then

$$(2p-1)^t \leq q_1 \cdots q_t \leq \frac{F_{ap}}{F_a} \leq F_{ap} \leq \alpha^{ap}.$$

Thus, $t = O(p/\log p)$. Then

$$\begin{aligned} \frac{\phi(F_{ap})}{F_{ap}} &= \left(\frac{\phi(F_a)}{F_a} \right) \prod_{i=1}^t \left(1 - \frac{1}{q_i} \right) \\ &= \frac{\phi(F_a)}{F_a} \exp \left(- \sum_{i=1}^t \frac{1}{q_i} + O \left(\sum_{q \geq q_1} \frac{1}{q^2} \right) \right) \\ &= \frac{\phi(F_a)}{F_a} \exp \left(O \left(\frac{t}{q_1} \right) \right) \\ &= \frac{\phi(F_a)}{F_a} \exp \left(O \left(\frac{1}{\log p} \right) \right) \end{aligned}$$

$$= \frac{\phi(F_a)}{F_a} \left(1 + O\left(\frac{1}{\log p}\right) \right).$$

It implies that, if $p > \exp(\kappa\varepsilon^{-1})$, where $\kappa > 0$ is some absolute constant, then

$$\frac{\phi(F_{ap})}{F_{ap}} \in \left(\frac{s_2}{s_1} + 0.5\varepsilon, \frac{s_2}{s_1} + 1.5\varepsilon \right). \quad (2)$$

We now follow Washington's argument [4] to prove that there exist infinitely many primes p such that the base b representation of F_{ap} starts with s_1 . For this, it is enough to show that

$$F_{ap} = s_1 b^N + \zeta_{ap} \quad \text{for some integer} \quad 0 \leq \zeta_{ap} \leq b^N - 1. \quad (3)$$

Note that since $q_1 \geq 2p - 1$, it follows that if p is sufficiently large (say, $p > b^k$), then F_{ap} cannot equal $s_1 b^N$, and in particular, if in the above formula (3) we have $\zeta_{ap} \geq 0$, then in fact $\zeta_{ap} \geq 1$. The above formula (3) yields

$$\alpha^{ap} = \sqrt{5} s_1 b^N + \sqrt{5} \zeta_{ap} + \beta^{ap} = \sqrt{5} s_1 b^N (1 + x_{ap}).$$

Since $\zeta_{ap} \geq 1$, it follows that $\sqrt{5} \zeta_{ap} + \beta^{ap} > \sqrt{5} - 1 > 1$, and

$$0 < x_{ap} = \frac{\sqrt{5} \zeta_{ap} + \beta^{ap}}{\sqrt{5} s_1 b^N}.$$

So, if $x_{ap} \in (0, 1/b^k)$, and $p > b^k$ is sufficiently large, it then follows that $\zeta_{ap} < b^N$, which is what we want. Thus,

$$ap \log \alpha = \log(\sqrt{5} s_1) + N \log b + \log(1 + x_{ap}),$$

or

$$ap \frac{\log \alpha}{\log b} - N - \left\lfloor \frac{\log(\sqrt{5} s_1)}{\log b} \right\rfloor = \left\{ \frac{\log(\sqrt{5} s_1)}{\log b} \right\} + \frac{\log(1 + x_{ap})}{\log b}. \quad (4)$$

Observe that $\log(\sqrt{5} s_1)/\log b$ is never an integer. Assume that k is sufficiently large such that

$$\frac{1}{b^k \log b} < 1 - \left\{ \frac{\log(\sqrt{5} s_1)}{\log b} \right\}.$$

Then putting

$$\delta = \frac{\log(1 + 1/b^k)}{\log b},$$

we see that a relation like (4) with $x_{ap} \in (0, 1/b^k)$ holds provided that

$$\left\{ p \left(\frac{a \log \alpha}{\log b} \right) \right\} \in \left(\left\{ \frac{\log(\sqrt{5} s_1)}{\log b} \right\}, \left\{ \frac{\log(\sqrt{5} s_1)}{\log b} \right\} + \delta \right). \quad (5)$$

The number $\gamma = a \log \alpha / \log b$ is irrational. By a result of Vinogradov [3], the sequence of fractional parts $\{p\gamma\}_{p \text{ prime}}$ is uniformly distributed. In particular, containment (5) holds for a positive proportion of primes p , and therefore certainly for infinitely many of them. So, indeed relation (3) holds. Relation (2) now shows that

$$\phi(F_{ap}) = s_2 b^N + \theta,$$

where

$$\theta \in \left(\zeta_{ap} \left(\frac{s_2}{s_1} + 0.5\varepsilon \right) + 0.5\varepsilon b^N, \zeta_{ap} \left(\frac{s_2}{s_1} + 1.5\varepsilon \right) + 1.5\varepsilon s_1 b^N \right).$$

Since $\varepsilon < 1/(15b^{2k})$, the above upper bound is

$$\begin{aligned} \zeta_{ap} \left(\frac{s_2}{s_1} + 1.5\varepsilon \right) + 1.5\varepsilon s_1 b^N &< (b^N - 1) \left(\frac{b^k - 1}{b^k} + \frac{0.1}{b^k} \right) + \frac{0.1(b^k - 1)}{b^{2k}} b^N \\ &< b^N - 1, \end{aligned}$$

where the last inequality above is implied by

$$\frac{1}{9} < \frac{b^N - 1}{b^N},$$

which holds true for all $b \geq 2$ and $N \geq 1$. This completes the proof of the theorem.

3. Comments

It was shown in [1] that with $\sigma(m)$ being the sum of divisors of the positive integer m , the ratio $\sigma(F_n)/F_n$ is dense in $[1, \infty)$. The present method now shows that there are infinitely many positive integers n such that the base b representation of F_n starts with the digits of s_1 and the base b representation of $\sigma(F_n)$ starts with the digits of s_2 . Also, one may replace the Fibonacci sequence F_n in the above statements with some other sequence u_n for which it has been proved that $\phi(u_n)/u_n$ and $u_n/\sigma(u_n)$, respectively, are dense in $[0, 1]$. For example, one can take $u_n = 2^n - 1$ (see [1]) and the main result of this paper still holds provided that b is not a power of 2. We give no further details.

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