

# A TRIDENT SCHOLAR PROJECT REPORT

NO. 443

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**Black Hole Entropy Calculated via Wavefunction Approximations on a  
Schwarzschild Spacetime**

by

Midshipman 1/C Eric A. Swanson, USN

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U.S.N.A. --- Trident Scholar project report; no. 443 (2015)

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USNA-1531-2

<b>REPORT DOCUMENTATION PAGE</b>			<i>Form Approved</i> <i>OMB No. 0704-0188</i>		
Public reporting burden for this collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing this collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden to Department of Defense, Washington Headquarters Services, Directorate for Information Operations and Reports (0704-0188), 1215 Jefferson Davis Highway, Suite 1204, Arlington, VA 22202-4302. Respondents should be aware that notwithstanding any other provision of law, no person shall be subject to any penalty for failing to comply with a collection of information if it does not display a currently valid OMB control number. <b>PLEASE DO NOT RETURN YOUR FORM TO THE ABOVE ADDRESS.</b>					
<b>1. REPORT DATE</b> (DD-MM-YYYY) 05-18-2015		<b>2. REPORT TYPE</b>		<b>3. DATES COVERED</b> (From - To)	
<b>4. TITLE AND SUBTITLE</b> Black Hole Entropy Calculated via Wavefunction Approximations on a Schwarzschild Spacetime			<b>5a. CONTRACT NUMBER</b>		
			<b>5b. GRANT NUMBER</b>		
			<b>5c. PROGRAM ELEMENT NUMBER</b>		
<b>6. AUTHOR(S)</b> Swanson, Eric Aanders			<b>5d. PROJECT NUMBER</b>		
			<b>5e. TASK NUMBER</b>		
			<b>5f. WORK UNIT NUMBER</b>		
<b>7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)</b>			<b>8. PERFORMING ORGANIZATION REPORT NUMBER</b>		
<b>9. SPONSORING / MONITORING AGENCY NAME(S) AND ADDRESS(ES)</b> U.S. Naval Academy Annapolis, MD 21402			<b>10. SPONSOR/MONITOR'S ACRONYM(S)</b>		
			<b>11. SPONSOR/MONITOR'S REPORT NUMBER(S)</b> Trident Scholar Report no. 443 (2015)		
<b>12. DISTRIBUTION / AVAILABILITY STATEMENT</b>  This document has been approved for public release; its distribution is UNLIMITED.					
<b>13. SUPPLEMENTARY NOTES</b>					
<b>14. ABSTRACT</b>  The study of thermodynamics of a black hole is at the interface of general relativity (GR) and quantum mechanics (QM). We will calculate the thermodynamic quantities using a first quantized theory which allows for only forward evolution in time but causes formal complications. The first approximation method used is a semiclassical approximation. To gain further understanding, we use perturbation theory in order to incorporate angular degrees of freedom. The calculations will give us insight on the relationship between GR and QM. We will begin by calculating the dynamics of a free particle in Minkowski (flat) spacetime where there is no gravity. We will then calculate the dynamics of a point particle on a Schwarzschild or black hole space-time. Our use of quantum mechanics as opposed to quantum field theory, as has been done in the past, is a unique approach to this problem. This approach can reproduce the Hawking temperature and Hawking-Bekenstein law at the semiclassical level and quantum gravitational corrections are observed using perturbation theory.					
<b>15. SUBJECT TERMS</b> general relativity, special relativity, thermodynamics, quantum mechanics, partition function, black hole					
<b>16. SECURITY CLASSIFICATION OF:</b>			<b>17. LIMITATION OF ABSTRACT</b>	<b>18. NUMBER OF PAGES</b>  31	<b>19a. NAME OF RESPONSIBLE PERSON</b>
<b>a. REPORT</b>	<b>b. ABSTRACT</b>	<b>c. THIS PAGE</b>			<b>19b. TELEPHONE NUMBER</b> (include area code)

### Abstract

The study of thermodynamics of a black hole is at the interface of general relativity (GR) and quantum mechanics (QM). We will calculate the thermodynamic quantities using a first quantized theory which allows for only forward evolution in time but causes formal complications. The first approximation method used is a semiclassical approximation. To gain further understanding, we use perturbation theory in order to incorporate angular degrees of freedom. The calculations will give us insight on the relationship between GR and QM. We will begin by calculating the dynamics of a free particle in Minkowski (flat) spacetime where there is no gravity. We will then calculate the dynamics of a point particle on a Schwarzschild or black hole space-time. Our use of quantum mechanics as opposed to quantum field theory, as has been done in the past, is a unique approach to this problem. This approach can reproduce the Hawking temperature and Hawking-Bekenstein law at the semiclassical level and quantum gravitational corrections are observed using perturbation theory.

**Keywords:** General Relativity, Special Relativity, Thermodynamics, Quantum Mechanics, Partition Function, Black Hole

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# 1 Introduction

At the turn of the 20th century, two powerful ideas revolutionized physics, a discipline that was thought to be outdated because all physical laws were presumed to be understood and there were only a handful of unexplained phenomena. The two ideas were General Relativity (GR) and Quantum Mechanics (QM). General relativity describes physics on large scales with masses the size of stars and distances ranging from the scale of the solar system to the scale of the universe. Albert Einstein completely overturned the common understanding of gravity. His theory describes how gravitational bodies distort the space around them and the force felt by other masses is a manifestation of this curvature. The other revolutionary concept was quantum mechanics which describes the world of the very small. Physics at the scale of atoms is completely different than what is experienced macroscopically. What we think of as particles with definite position and momentum are replaced by wavefunctions that give a probability distribution of where the particle is and where it would be going if these quantities were measured. The reality described by quantum mechanics is a bizarre, indeterministic world and the theory describing it can only be believed because of its amazing accuracy in predicting what has been verified by experiment.

Much of the effort in 20th century physics was spent reconciling what was already known about physics with the postulates of quantum mechanics. One by one, the known forces and fields were “quantized,” a process where the theory of the force or field is interpreted through quantum mechanics. Richard Feynman’s theory of quantum electrodynamics has been the most successful theory in terms of the accuracy of prediction [1]. It has successfully predicted the value of the magnetic moment of an electron to 13 decimal places [2]. All known interactions have been quantized except for gravity. Einstein’s description of gravity in his theory of general relativity has proven immensely difficult to quantize and has remained an outstanding problem for almost a full century [3]. Several attempts have been made to reconcile gravity with the laws of quantum mechanics but at the moment, the nearest we have come is to understand quantum fields on the “background” of curved spacetime. One treats gravity classically while treating nongravitational fields in that spacetime as quantum fields. In this way, the effect that general relativity has on the quantum world can be observed but we have not been able to describe gravity itself as a quantum field without technical and conceptual difficulties that have yet to be overcome.

## 1.1 History of Black Hole Entropy

One way to probe the quantum effects of gravity is to look at the thermodynamics of a relativistic system. Thermal physics is a discipline that was developed empirically. Concepts such as heat capacity, entropy, temperature, and pressure were developed to match observation and then used as analytic tools in the analysis of machine design and performance. One of the major successes of QM in the beginning of the 20th century was its ability to explain thermal physics from first principles. Throughout the development of QM, thermal physics has provided an empirical testing ground for predictions made mathematically. Several seemingly strange concepts in QM like the Pauli Exclusion Principle or the Heisenberg Uncertainty Principle have observable effects when scaled up to systems with thousands of particles in experiment or scaled up by a factor on the order of Avogadro’s number and observed on a macroscopic scale [4]. The relationship is well established between QM and thermal physics, so the application of quantum mechanics to a relativistic system can be extended to calculate thermodynamic properties of that system. Thermodynamic properties have the imprint of quantum mechanics and serve as a tool for relating the weird world of the very small with our everyday experiences.

The effects of general relativity are not experienced directly in our day to day lives so trying to gain insight into quantum gravity by studying heat capacity and entropy in a lab on earth will not demonstrate noticeable effects. In order to gain an understanding of quantum gravity, a more extreme environment will give more insight. A black hole is at the extremity of general relativity. All masses curve spacetime around them but black holes curve space time around them so severely that, classically, nothing can escape, not even light, which gives the name “black” hole [5]. It is near a black hole extreme spacetime curvature that one might expect to sense effects of quantum gravity.

One thermodynamic quantity of interest which will be explored in this paper is entropy. Entropy is a measure of the amount of disorder in a system and is defined by the number of microstates that correspond to an observed macrostate. For example, if the macrostate were 5 differently numbered blocks lined up in order, there would only be one way for the system to be in that macrostate. If the blocks were not numbered and the macrostate was the five blocks lined up, there would be no way to tell which block is which so order would not matter. This means that all ways of ordering the blocks would produce the same macrostate so there would actually be  $5! = 120$  microstates for this macrostate.<sup>1</sup> The second system has many more arrangements of blocks that produce the same macrostate. If the blocks were somehow able to move around (as atoms in a gas are) while maintaining the same macrostate (having the same thermodynamic properties in the gas example), then the second system could be in any number of microstates and is less ordered and more entropic. Knowing the entropy of the system as a function of the other state variables<sup>2</sup> doesn't give us all the information about the system but it does give us insight. Knowing there are 120 microstates for a system of 5 blocks doesn't give all the information but it does tell us something about the system, especially when compared to the case with 1 microstate.

Black hole entropy has been studied for the past 40 years. A thought experiment first considered by Jacob Bekenstein is to consider the entropy of the universe before and after a cup of coffee is thrown into a black hole [6]. The cup of coffee has entropy and if the black hole didn't, we could just get rid of all the entropy in the universe by throwing things into black holes. So either black holes must have entropy or the second law of thermodynamics must be wrong.<sup>3</sup> Stephen Hawking responded to the idea of black hole entropy by noting that for a body to have entropy, it must have a temperature and to have a temperature, it must radiate. Classically, a black hole cannot radiate. The gravitational pull is so great that nothing, not even radiation, can escape in a classical description [5]. This is a paradox that can be answered by turning to quantum physics. The problem of black hole entropy lies at the intersection of classical statistical mechanics, general relativity and quantum mechanics. Stephen Hawking was the first to perform a calculation of black hole entropy using quantum field theory in 1973 [7, 8]. Quantum field theory is a second quantized theory. It is a theory that allows for antiparticles and propagation backwards in time [9]. While it is a successful theory, it does not maintain a probability amplitude that is positive definite at the level of the single particle. Using a quantum mechanics, a first quantized theory, in the investigation of black hole thermodynamics, preserves forward time evolution, and as will be seen, provides an alternate method for calculating entropy.

Current work in the area of the interface of quantum mechanics and general relativity is focused

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<sup>1</sup>If one were to arrange the blocks, there are five choices for which block to put in the first position, four for the second position and so forth so there are  $5 \times 4 \times 3 \times 2 \times 1 = 5!$  ways to arrange the blocks.

<sup>2</sup>In the case of the blocks, the number of blocks would be our state variable and we gain information knowing that the entropy is independent of the number of blocks, as in system 1, or if it changes, and how it changes, with the number of blocks, as in system 2.

<sup>3</sup>The second law of thermodynamics states that the entropy of the universe must always be increasing. If it somehow disappeared inside a black hole then we have violated this law.

on the “information paradox.” In quantum mechanics, the information about a system is contained in the wavefunction and it can be evolved in time but the information is not lost. When incorporating general relativity, it appears as if the black hole can destroy information which violates one of the tenets of quantum mechanics. One attempt to address this issue is to employ something known as the holographic principle, discussed by J. Barbon [10]. Another attempt by H. Nikolic is to treat space and on the same footing in quantum mechanics [11]. The methods in this paper differ from both and could provide a basis for addressing the information paradox by giving a novel description of quantum mechanics near a black hole.

## 1.2 Definition and Conventions

The following conventions will be used throughout this paper.

Partial derivatives will be used several times and will be abbreviated by the partial derivative symbol subscripted by the variable with respect to which the derivative is taken:

$$\frac{\partial}{\partial x} = \partial_x. \quad (1)$$

In relativity, time is treated as a coordinate on the same footing as the spatial coordinates but it will often be useful to refer to vectors that include only the three spatial components. A coordinate four-vector is a vector that denotes an event in space time and the temporal component is term labeled by the superscript 0. Greek indices  $\mu, \nu, \dots$  will be used to denote the components of a four vector and they will run from 0 to 3,

$$(x^\mu) = (x^0, x^1, x^2, x^3) = (ct, x^1, x^2, x^3). \quad (2)$$

The factor of  $c$ , the speed of light in a vacuum, is inserted to maintain dimensional consistency. In the case of three-vectors, the vector symbol will be used and the components will be indexed by Roman indices which will run from 1 to 3:

$$\vec{x} = (x^1, x^2, x^3). \quad (3)$$

The Einstein summation convention will be used. This convention does away with the  $\sum$  summation signs when there is no ambiguity about the index. In an expression, if there is an index repeated, raised in one instance and lowered in another, then there is an implied sum over all possible values of the index:

$$x_i x^i \equiv \sum_{i=1}^3 x_i x^i = x_1 x^1 + x_2 x^2 + x_3 x^3. \quad (4)$$

The following definitions will be useful in understanding the processes used in this paper.

Certain functions are defined over phase space. Phase space is the space of all possible coordinates and momenta: each point represents a different state of the system. A function defined on phase space uses coordinates and momenta as the fundamental variables.

We will move from classical physics to quantum physics through a process called quantization. This is a process where all dynamical variables are promoted to operators which act on the states of a system. A general state will be denoted by a “ket”  $|\Psi\rangle$ . The operators of interest will be position  $\hat{r}$ , linear momentum  $\hat{p}$ , and angular momentum  $\hat{L}$ . Other operators can be constructed as functions of these operators. The position operator, when in a coordinate basis, operates by multiplication on the state:  $\hat{r}\Psi(\vec{r}) = \vec{r}\Psi(\vec{r})$ . The linear momentum operator acts by differentiation with respect to

its conjugate generalized coordinate  $\hat{p}_x \Psi(\vec{r}) = \frac{\hbar}{i} \partial_x \Psi(\vec{r})$ . Classically, angular momentum is defined as  $\vec{L} = \vec{r} \times \vec{p}$  [13]. The angular momentum operator is the vector product between the position operator and the linear momentum operator  $\hat{\vec{L}} = \hat{\vec{r}} \times \hat{\vec{p}}$ . These operators can be written in three dimensions in a compact way by using the del operator<sup>4</sup>  $\vec{\nabla} = \partial_x \hat{i} + \partial_y \hat{j} + \partial_z \hat{k}$ ,

$$\hat{\vec{p}} = \frac{\hbar}{i} \vec{\nabla}; \hat{\vec{L}} = \frac{\hbar}{i} \vec{r} \times \vec{\nabla} \quad (5)$$

In a general spacetime, the way distances are measured is encoded in the metric. Certain “flat” coordinates use the Euclidean metric which is what we are familiar with from Euclidean geometry. The distance between two points is defined by  $\Delta r = \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}$ . A curved space time makes this expression more complicated. It is analogous to trying to measure distances between two points on the Earth’s surface using longitude and latitude as coordinates. The measurement of distance incorporates functions of the coordinates themselves. The symbol used to represent the metric is  $g_{\mu\nu}$  [5]. This term tells us the contribution to the length of the displacement from the product of the change in the  $x^\mu$  and the  $x^\nu$  coordinates. The differential line element, the distance between one point and a nearby point whose coordinates differ by  $(dx^\mu)$ , is defined by  $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ . The Lorentz invariant “flat” space is called Minkowski spacetime and the metric is given by

$$g_{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} [5]. \quad (6)$$

It is often written in the more compact form  $[\eta_{\mu\nu}] = \text{diag}(1, -1, -1, -1)$ . The other metric that will be used in this paper is the Schwarzschild metric which is the metric for a spacetime in the presence of a static black hole [5]. The metric is given by

$$g_{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} 1 - \frac{2\mu}{r} & 0 & 0 & 0 \\ 0 & -(1 - \frac{2\mu}{r})^{-1} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2(\theta) \end{pmatrix}. \quad (7)$$

The distortion of spacetime due to the black hole is seen in the first two terms along the diagonal. In this expression for the metric  $\mu$  is defined as  $\frac{GM}{c^2}$  where  $G$  is Newton’s gravitational constant,  $M$  is the mass of the black hole and  $c$  is the speed of light.

With these metrics, it is possible to have length squared values,  $s^2$ , which are not positive. Spacetime intervals are defined as the distance between an initial point  $\vec{x}_1$  at time  $t_1$  and a different point  $\vec{x}_2$  at time  $t_2$  and is measured using the metric.

For this paper, any time a three-vector dot product is used, a Euclidean metric is implied. Explicitly,  $\vec{x} \cdot \vec{x} = x^2 + y^2 + z^2$ .

In Minkowski spacetime, the spacetime intervals,  $s$ , are given by the equation  $s^2 = (ct)^2 - |\vec{x}|^2$ . If  $s^2$  is positive, it is a timelike interval meaning something could travel between the two events without going faster than the speed of light. The case  $s^2 = 0$  is a null interval and corresponds to

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<sup>4</sup>The  $\hat{i}$ ,  $\hat{j}$ , and  $\hat{k}$  vectors are the unit vectors in the direction of the  $x$ ,  $y$ , and  $z$  axes, respectively. Here the “hat” means unit vector, not operator. This is a notational ambiguity but will not be an issue for the remainder of this paper.

something traveling at the speed of light. The case where  $s^2$  is negative is a spacelike interval and classically, nothing can travel from one event to another because it would have to be moving faster than the speed of light. In general, the value of  $s$  can be obtained by integrating the line element along the appropriate parameter and the sign of the value  $s^2$  has the same interpretation with a positive value corresponding to timelike intervals, a negative value corresponding to spacelike intervals and the  $s^2 = 0$  case corresponding to a null interval [5].

### 1.3 Method of Investigation

In this paper, we will be looking for insight on the quantum nature of black holes by investigating the effect a black hole has on a quantum particle. There are proposed theories of quantum gravity yet no full theory has yet to be accepted [12]. While we are not trying to develop a theory of quantum gravity, our method will “probe” the quantum nature of gravity by looking at the effect of a classical black hole on the quantum states of a particle.

We will be using quantum mechanics, the first quantized theory of a point particle. This is a more basic theory than quantum field theory but it contains mathematical complications that are absent in the more advanced theory. As will be seen, there is a square root in the Hamiltonian that complicates the calculation. One way to remove the square root more precisely is to square both sides of the equation but this makes the transition into quantum field theory and one loses the guarantee of forward propagation in time. The approach of this paper is to perform a calculation of the entropy and deal with the mathematical difficulties without squaring the Hamiltonian. This is important because it maintains strict forward propagation in time for the particle used in the calculation of black hole thermodynamics which is unique. It also provides a method for using a more exact approach to the calculation that increases the accuracy of the value for entropy that shows that the quantum particle is affected by quantum gravity, even with gravity is introduced at the classical level.

In quantum mechanics, we are interested in the solving the Schrodinger Equation (SE)

$$i\hbar\partial_t\Psi = \hat{H}\Psi \quad (8)$$

whose solutions are the wavefunction,  $\Psi$ , that describes the system. The left side of the equation includes  $i$ , the imaginary number,  $\hbar$ , the reduced Planck’s constant, and  $\partial_t$ , a partial time derivative. The classical Hamiltonian,  $H$ , depends on the system and is, in general, a function of the momentum  $\vec{p}$  and position  $\vec{x}$  and time  $t$  of a particle which upon quantization is promoted to an operator in the SE. The SE, Eq. (8), is a first order linear differential equation. The wavefunctions,  $\Psi$ , are the solutions to the equation and represent the possible states for a particle which can be expanded in the eigenstates of the Hamiltonian operator. The eigenstates of an operator are the states which return a constant multiplied by the state when operated on by the operator. In the equation  $\hat{A}\Psi = a\Psi$ ,  $\Psi$  is an eigenstate of the operator  $\hat{A}$  with the associated eigenvalue  $a$ . It is a property of linear differential equations that the solutions will form a complete set meaning that a general state can be expressed as a linear combination of the eigenstates.<sup>5</sup>

The general solution to the SE, Eq. (8), for a time-independent Hamiltonian  $H(\vec{x}, \vec{p}, t) = H(\vec{x}, \vec{p})$

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<sup>5</sup>The eigenstates of a system can be thought of like the modes of vibration on a string. The wave on the string can have one, two or any integer number of antinodes but it cannot have one and a half so there are only certain “states” allowed. Also, any configuration that the string could be in while still fixed at both ends can be represented by a sum of the allowed modes. This is analogous to representing any state as a linear combination of eigen or basis states.

is

$$\Psi(\vec{x}, t) = e^{\frac{i\hat{H}}{\hbar}t}\Psi(\vec{x}, 0) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{i\hat{H}t}{\hbar}\right)^n \Psi(\vec{x}, 0) \quad (9)$$

where  $\Psi(\vec{x}, 0)$  is the initial state of the system. The exponential of the operator is defined by its power series. This equation describes how the wavefunction of the system evolves in time and will be used later in calculating the thermodynamic properties of the system.

In the attempt to make quantum mechanics relativistic, a conflict arises. In the SE, Eq. (8), time is given a different role than the spatial coordinates.<sup>6</sup> In a Lorentz invariant theory such as special relativity, space and time appear on the same footing. Time is not absolute and is experienced differently for different observers so there is ambiguity in what “time” to use as the evolution parameter in the SE.<sup>7</sup> Some of the possible choices for parameter<sup>8</sup> are proper time  $\tau$  and coordinate time  $t$ . Proper time  $\tau$  is the time as experienced by the particle while coordinate time  $t$  is the time experience by an observer “at infinity” where the effects of gravity are negligible [5]. Leaving the parameter undefined or using proper time  $\tau$  as the parameter would lead to a vanishing Hamiltonian in going through the procedure used below. A vanishing Hamiltonian has the interpretation that there is no change over time in the system as the Hamiltonian is the generator of time evolution for a system, seen in Eq. (8). In this paper we will choose coordinate time,  $t$ , as our parameter which is the time experienced by a stationary observer “at infinity” or far enough away to not experience the effects of gravity.

When introducing gravity into the problem, we need to use the theory of general relativity. Gravity is not a force like electric attraction but a manifestation of how spacetime bends as a result of the presence of matter. It changes the geometry of the system by curving the spacetime around the mass similar to a bowling ball curving a trampoline when placed in the center. It can be thought of in two dimensions like the deformation of a trampoline around a bowling ball at the center. If one attempts to analyze motion on the trampoline, one has to take into account how distances are stretched by the presence of the bowling ball. The “straight” path of a marble rolling across the stretched trampoline will look as if it curves as if under a force if one looks from above with a 2-dimensional perspective. The effects of gravity will be incorporated in the SE, Eq. (8), by altering the metric used in the Hamiltonian.

## 2 Point Particle in a General Spacetime

We will begin by calculating the Hamiltonian for a particle propagating on a general spacetime to be used in the SE, Eq. (8). This process will first be done in full generality with regard to the metric. This will allow us to first calculate the wavefunction in flat, Minkowski spacetime and then make the calculation in a black hole spacetime by inserting the correct metric terms. We will start with the action for the particle which will define the Lagrangian [14].

The action  $S$  for a free particle in a general metric is given by the length of the worldline multiplied by its mass  $m$  and the speed of light in a vacuum  $c$  to maintain the correct dimensions.

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<sup>6</sup>The left side contains a single derivative in time while the Hamiltonian on the right side depends on position and momentum in the static case. There is a different role given to space and time in QM.

<sup>7</sup>Time is “dilated” for a moving observer so a clock on a spaceship traveling near the speed of light will appear to be running slowly as determined by a stationary observer.

<sup>8</sup>The parameter is what we choose to label or “count” the events on the worldline of the particle. Different parameters are like counting by twos or by threes. I can count to twelve as 6 two’s or as 4 three’s. It looks different but it doesn’t change the value of twelve and it just requires attention to what I am counting by.

The negative sign in the definition is a convention.[5]

$$S = -mc \int cd\tau = -mc \int \sqrt{g_{\mu\nu}dx^\mu dx^\nu}. \quad (10)$$

The classical solution to a particle's path through phase space is path which minimizes the action. This is the principle which guides classical mechanics allows to make the transformation between the Lagrangian and the Hamiltonian,

$$S = -(mc) \int \sqrt{g_{\mu\nu}\dot{x}^\mu(t)\dot{x}^\nu(t)}dt = \int Ldt \quad (11)$$

where  $\dot{x}^i = \frac{dx^i}{dt}$ . We have now explicitly shown the parameterization with respect to coordinate time,  $t$ . Next, we perform a 3 + 1 split of the action into temporal and spatial parts.<sup>9</sup> This yields the following expression for the Lagrangian

$$L = -(mc)\sqrt{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} = -(mc)\sqrt{g_{00}c^2 + 2cg_{0i}\dot{x}^i + g_{ij}\dot{x}^i\dot{x}^j}. \quad (12)$$

The expression on the right comes from writing out the 0 index term explicitly but leaving the sum from 1 to 3 written with the summation convention.<sup>10</sup> The Hamiltonian is defined by the Legendre transformation of the Lagrangian [14]:

$$H(x, p, t) = \dot{x}_i \partial_{\dot{x}_i} L - L = p_i \dot{x}^i - L, \quad (13)$$

where  $p_i$  is the momentum canonically conjugate to  $x^i$  defined as

$$p_i = \frac{\partial L}{\partial \dot{x}_i}. \quad (14)$$

To get the Hamiltonian, which is a function defined on phase space we must eliminate all occurrences of the velocity  $\dot{x}^i$  in favor of the momenta  $p_i$  in Eq. (13). For the Lagrangian in Eq. (12), this conjugate momentum from Eq. (14) is

$$p_i = \frac{(mc)^2}{L} (cg_{0i} + g_{ij}\dot{x}^j). \quad (15)$$

Contracting both sides of Eq. (15) with respect to covariant metric<sup>11</sup>  $g^{ij}$ , we can solve for the velocities  $\dot{x}^i$  in terms of the momenta and coordinates,

$$\dot{x}^i = g^{ik} g_{kj} \dot{x}^j = g^{ij} \left( p_i \frac{L}{(mc)^2} - cg_{0i} \right). \quad (16)$$

Inserting Eq. (16) into Eq. (13), we arrive at the Hamiltonian

$$H = \left( g^{ij} \frac{p_i p_j}{(mc)^2} - 1 \right) L - cg^{ij} g_{0j} p_i. \quad (17)$$

<sup>9</sup>This appears ostensibly to be contrary to the idea of spacetime covariance of relativity but it is necessary in order to identify a time to be used for the Hamiltonian that governs time evolution of the quantum mechanical system.

<sup>10</sup>Both radicals contain 16 terms if the double summation was written explicitly.

<sup>11</sup>We are using a property of the metric that  $g^{ik} g_{kj} = \delta_j^i$  which is 1 if  $i = j$  and zero if not. Multiplying by  $\delta_j^i$  has the effect of switching the index from  $i$  to  $j$  [5].

In order to complete the Legendre transformation, the Lagrangian must also be solved for explicitly in terms of momenta and position with no velocities  $\dot{x}^i$  present. By substituting Eq. (16) into Eq. (12), this yields

$$L = m^2 c^3 \frac{\sqrt{g_{00} - g^{ij} g_{0i} g_{0j}}}{\sqrt{(mc)^2 - g^{ij} p_i p_j}} \quad (18)$$

The required form of the Hamiltonian, after inserting Eq. (18) into Eq. (17), for a point particle coupled to a general spacetime metric is

$$H = c \left[ \sqrt{(mc)^2 - g^{ij} p_i p_j} \sqrt{g_{00} - g^{ij} g_{0i} g_{0j}} + g^{ij} g_{0i} p_j \right]. \quad (19)$$

This gives an expression for the relativistically correct Hamiltonian for a free particle propagating on any spacetime metric. We will now examine the thermodynamics of the point particle on specific spacetimes by inserting the correct metric terms.

### 3 Point Particle on Minkowski Spacetime

We will use the expression for the Hamiltonian in Eq. (19) to calculate the partition function for a free particle in Minkowski space time. Using this spacetime that is free from the curvature caused by gravity will provide a first test of our method. It should reproduce the thermodynamics that we experience on earth. We will first examine how the Hamiltonian acts on wavefunctions and then make the appropriate statistical mechanics transformation. This is the simplest Lorentz invariant spacetime geometry given in Eq. (6) as  $g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$ . In this spacetime, the Hamiltonian simplifies

$$H = c \sqrt{(mc)^2 + \vec{p} \cdot \vec{p}} \quad (20)$$

We would like to compute the relativistic particle propagator  $K(x, y; t)$ . The propagator can be used as a transitional step in going from quantum mechanics to statistical mechanics because there is formal similarity between the propagator and the partition function, the function which describes the thermodynamics of a system. The transition to statistical mechanics amounts to doing a variable substitution once the propagator is calculated. The propagator gives the probability amplitude that a particle is localized at position  $x$  at time  $t$  given that it was localized at position  $y$  at time 0<sup>12</sup>

$$K(x, y; t) = \langle x | e^{-(i/\hbar)t\hat{H}} | y \rangle, \quad (21)$$

and with the momenta promoted to operators by  $p_i \rightarrow \hat{p}_i = i\hbar\partial_{x^i}$ , the Hamiltonian operator becomes  $\hat{H} = c\sqrt{-\hbar^2\nabla^2 + m^2c^2}$ . The  $\nabla^2$  is the Laplacian operator defined by the sum of all of the conjugate momenta squared and is the operator equivalent of  $-\frac{|\vec{p}|^2}{\hbar^2}$ . In the case of Cartesian coordinates,  $\nabla^2 = \partial_x^2 + \partial_y^2 + \partial_z^2$ .

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<sup>12</sup>The expression  $e^{-(i/\hbar)t\hat{H}}$  is the time evolution operator as seen in Eq. (9).  $|y\rangle$  is a position basis state. The time evolution operator action on it evolves the state over time  $t$ . When one multiplies by  $\langle x|$ , the value of this inner product is the probability that the evolved state is now in position state  $|x\rangle$ .

### 3.1 The Nonrelativistic Limit

We will first consider the nonrelativistic, or Newtonian, limit where  $mc \gg |\vec{p}|$ . This limit is valid when the particle is traveling at speeds much less than the speed of light so the rest mass energy term is much greater than the momentum term. A binomial expansion of Eq. (20) allows us to make the expression simpler while maintaining a good degree of accuracy when we stay within the nonrelativistic limit.<sup>13</sup> In this limit to second order in momentum, the Hamiltonian is

$$H = mc^2 + \frac{|\vec{p}|^2}{2m}. \quad (22)$$

The error when making this approximation is on the order of  $\frac{(\vec{p} \cdot \vec{p})^2}{m^3 c^2}$  which is very small at low speeds. When quantizing the Hamiltonian, we promote the variables to operators,

$$\hat{H} = mc^2 - \frac{\hbar^2}{2m} \nabla^2. \quad (23)$$

Eq. (23) is the expression for the Hamiltonian of a free particle in nonrelativistic quantum mechanics, found in any introduction to quantum mechanics text, with the addition of the rest mass energy  $mc^2$  [13]. This result of taking the Newtonian limit validates that the expression Eq. (19) is correct in this limit.<sup>14</sup> We will use this Hamiltonian, Eq. (22) to evaluate the propagator in Eq. (21). We express Eq. (21) in a basis which diagonalizes the Hamiltonian operator, namely via momentum space plane waves, without loss of generality, and for simplicity we take  $y = 0$ . To transition into the momentum basis, we insert a complete set of momentum eigenstates which amounts to acting on the state by the identity operator [9]<sup>15</sup>

$$I = \frac{1}{(2\pi)^3} \int d^3\vec{p} |\vec{p}\rangle \langle \vec{p}|. \quad (24)$$

We are doing a Fourier transform into momentum space and an inverse Fourier transform back to position space. The benefit to doing this is that in momentum space, the momentum operator is just  $\vec{p}$  which acts by multiplication. In position space, the momentum operator is a derivative operator. So while we are in momentum space, we act on the state with the Hamiltonian so the terms in the exponent are algebraic, not differential. This is simpler to deal with and in this case allows for an exact solution,

$$K(\vec{x}, 0; t) = \int d^3\vec{p} \langle x | e^{-(\frac{i}{\hbar})t\hat{H}} | \vec{p} \rangle \langle \vec{p} | 0 \rangle = \frac{1}{(2\pi)^3} \int d^3\vec{p} e^{(\frac{i}{\hbar})\vec{p} \cdot \vec{x}} \exp\left[-\frac{it}{\hbar} \left(\frac{\vec{p} \cdot \vec{p}}{2m} + mc^2\right)\right] \quad (25)$$

The term  $e^{(i/\hbar)\vec{p} \cdot \vec{x}}$  is the momentum basis state in the coordinate representation. It is equal to  $\langle \vec{p} | \vec{x} \rangle$  in momentum space. The integral in Eq. (25) can be calculated by separating each component of the momentum. Using Cartesian momentum coordinates, Eq. (25) becomes<sup>16</sup>

$$K(\vec{x}, 0; t) = e^{\frac{-ime^2t}{\hbar}} \left[ \frac{1}{2\pi} \int dp_x \exp\left[\frac{i}{\hbar} p_x x - \frac{i}{\hbar} t \left(\frac{p_x^2}{2m} + mc^2\right)\right] \right]^3 \quad (26)$$

<sup>13</sup>The approximation used is that  $(1+x)^{\frac{1}{2}} \approx 1 + \frac{x}{2}$  for small values of  $x$ .

<sup>14</sup>The rest mass energy will not change the form of the wavefunction, it will only add a constant to the value of the particle energy.

<sup>15</sup>Multiplying a state by the identity operator returns the same state in the same way that multiplying a number by 1 will return the same number.

<sup>16</sup>Eq. (25) can be written as the product of three integrals, each evaluated separately. With the choice of Cartesian coordinates, the three integrals are identical which is why we are calculating the value of one and then taking it to the third power.

which is in the form of a Gaussian integral and can be calculated simply:<sup>17</sup>

$$K(\vec{x}, 0; t) = \left( \frac{m\hbar}{2\pi it} \right)^{3/2} \exp\left[-\frac{i}{\hbar}(mc^2 t - \frac{2m}{t}|\vec{x}|^2)\right]. \quad (27)$$

### 3.2 Partition Function and Thermodynamic Properties

The transition to thermodynamics can be made by calculating the partition function which is formally similar to the propagator. It is calculated by taking the trace<sup>18</sup> of the propagator and relabeling the constants appropriately to transition from the propagator, Eq. (21), and the definition of the partition function [4].<sup>19</sup> Similar to the wavefunction in a quantum mechanical system, the partition function contains all possible information about the thermodynamic properties of a system,

$$Z = \text{tr}K = K(x, x; \beta) = \int d^3x \langle x | e^{-\beta \hat{H}} | x \rangle. \quad (28)$$

This expression defines the propagator for the same starting and ending coordinates after changing the variable from time to temperature. The trace is calculated by integrating over all space. In the expression from Eq. (27), setting the starting and ending coordinates equal, one can read off the partition function for the free particle as

$$Z = \left( \frac{mk_B T}{2\pi} \right)^{3/2} e^{-\frac{mc^2}{k_B T}} \int d^3x = V \left( \frac{mk_B T}{2\pi} \right)^{3/2} e^{-\frac{mc^2}{k_B T}} \quad (29)$$

which is in the same form as calculated in classical statistical mechanics with the identification of  $\int d^3x = V$ . The expression is infinite. Since there were no boundaries placed on this system, the integration must go over all space and thus there is an infinite term. This is rectified by putting the particle in a box of volume  $V$ . This constrains the system to a finite volume. The expression for the partition function, Eq. (29), differs from the form given in [4] by the term  $e^{-\frac{mc^2}{k_B T}}$  which comes from adding the rest mass to each energy term which can be seen from the definition of the partition function

$$Z = \sum_i \exp\left[\frac{-\epsilon_i}{k_B T}\right] = \sum_i \exp\left[\frac{-\epsilon_{\text{nonrel}}^i + mc^2}{k_B T}\right] = e^{-\frac{mc^2}{k_B T}} Z_{\text{nonrel}}. \quad (30)$$

This result validates the method used in the nonrelativistic limit. From this partition function one can recover the ideal gas law, internal energy, and entropy using the relations from statistical mechanics [4]. This is the partition function for a single particle, labeled  $Z_{SP}$ . In the case of  $N$  indistinguishable particles, the partition function for the whole system is given by  $Z_N = \frac{Z_{SP}^N}{N!}$  [4]. From this partition function, we can calculate the internal energy,

$$U = k_B T^2 \frac{\partial \ln Z_N}{\partial T} = N k_B T^2 \partial_T \left( -\frac{mc^2}{k_B T} + \frac{3}{2} \ln T + \ln(\dots) \right) = Mc^2 + \frac{3}{2} N k_B T \quad (31)$$

<sup>17</sup>A Gaussian integral is an integral in the form  $\int dx e^{-ax^2+bx+c}$  which has well known solutions.

<sup>18</sup>The trace of a matrix is the sum of the diagonal elements of the matrix. In making the extension to a continuous basis, it is the sum over all spatial coordinates with the same initial and final point.

<sup>19</sup>The propagator and the partition function have a very similar dependence on the Hamiltonian except the prefactor. In the propagator, the prefactor is  $\frac{i\hbar}{h}$  and for the partition function the prefactor is  $\frac{1}{k_B T}$  so we only need to make the identification  $\beta \equiv \frac{i\hbar}{h} = \frac{1}{k_B T}$  where  $T$  is the temperature at thermal equilibrium.

where (...) represents constants that do not depend on  $T$  and so do not contribute to the derivative and  $M = Nm$  is the total mass of the system. This is exactly what would be predicted from adding the rest mass energy to the classical internal energy of an ideal monatomic gas. The pressure is the change in the Helmholtz free energy,  $F = -k_B T \ln Z_N$ , through an isothermal expansion,

$$P \equiv -\left(\frac{\partial F}{\partial V}\right)_T = -k_B T \left(\frac{\partial \ln Z_N}{\partial V}\right)_T = \frac{Nk_B T}{V}. \quad (32)$$

The last equality can be seen from the linear dependence of  $Z_N$  on  $V$ . Both sides of Eq. (32) can be multiplied by  $V$  and the result is the ideal gas law,  $PV = Nk_B T$ .

While this result is not new, it provides verification of our method and provides a framework for addressing the more complicated system.

### 3.3 Relativistic Free Particle Propagator and Partition Function

Using an analogous procedure, we will calculate the partition function without taking the non-relativistic limit in the Hamiltonian. We will be using the exact form of the Hamiltonian that is accurate to all orders in momentum:  $\hat{H} = c\sqrt{-\hbar^2 \nabla^2 + m^2 c^2}$  where we have not made the assumption that momentum is small. Using this expression for the Hamiltonian, the propagator has the form

$$\int d^3 p \langle x | e^{-(i/\hbar)t\hat{H}} | p \rangle \langle p | y \rangle = \frac{1}{(2\pi)^3} \int d^3 p e^{(i/\hbar)\vec{p}\cdot\vec{x}} e^{-(i/\hbar)ct\sqrt{|\vec{p}|^2 + m^2 c^2}} e^{-(i/\hbar)\vec{p}\cdot\vec{y}}. \quad (33)$$

We have again used plane waves as a complete set of states that diagonalizes the Hamiltonian. In the momentum basis, the Laplacian operator,  $\nabla^2$ , is replaced with  $-\vec{p} \cdot \vec{p}$  which operates by multiplication<sup>20</sup>. We will again take  $y = 0$  so the integral reduces to

$$\frac{1}{(2\pi)^3} \int d^3 p \exp\left[\frac{i}{\hbar}(\vec{p} \cdot \vec{x} - ct\sqrt{|\vec{p}|^2 + m^2 c^2})\right] \quad (34)$$

This is not in the form of the Gaussian as for the previous case so we will evaluate the integral using the method of steepest descents [15].<sup>21</sup>

The general expression for the Taylor expansion with three variables to second order, after making the variable substitution  $\vec{p}' = \vec{p} - \vec{p}_0$ ,<sup>22</sup> is

$$S(\vec{p}) \simeq S(\vec{p}_0) + \sum_i (p'_i) \left. \frac{\partial S}{\partial p_i} \right|_{\vec{p}_0} + \frac{1}{2} \sum_{i,j} (p'_i)(p'_j) \left. \frac{\partial^2 S}{\partial p_i \partial p_j} \right|_{\vec{p}_0}. \quad (35)$$

A function can be Taylor expanded around any point where it is continuous. One possible choice for the point of expansion is  $\vec{p} = 0$  but for our purposes, we would rather expand about the critical point (where the first derivative is zero). We will see that this is a physically motivated choice in

<sup>20</sup>The method of going into the momentum basis simplifies the calculation because there is no explicit position dependence in the Hamiltonian. It will not be the case later with a more complicated metric.

<sup>21</sup>In the method of steepest descents, the argument of the exponent is expanded in a Taylor series around its critical point to second order. This expansion will produce an approximation of the argument that will express the integral as a Gaussian integral. A Taylor expansion is a way of representing a continuous function as a sum of polynomials. One starts with the function evaluated at a point and then calculates the derivatives of the function at that point. Successive terms in the series are of higher degree. We will pick the point where the first derivative is zero and calculate it to second order.

<sup>22</sup>The value  $\vec{p}_0$  is the point of expansion.

interpreting the result. Define  $S$  to be the argument of the exponent in Eq. (34). The critical point is found by setting the first derivative of  $S$  to zero,

$$\left(\frac{\partial S}{\partial p_i}\right) = x_i - \frac{p_i(ct)}{\sqrt{|\vec{p}|^2 + m^2c^2}} = 0 \quad (36)$$

This can be solved for the momentum which defines the critical point which we will label as  $\vec{p}_0$ .

$$p_i = \frac{x_i}{(ct)} \sqrt{|\vec{p}|^2 + m^2c^2} \equiv p_{0i} \quad (37)$$

This expression needs to be solved so that the momentum only appears on one side of the equation. The algebraic steps are omitted but the result, with  $\vec{p}_0$  denoting the critical point, is

$$p_{0i} = \frac{x_i mc}{\sqrt{(ct)^2 - |\vec{x}|^2}}. \quad (38)$$

The argument evaluated at the critical point, which will be used in the Taylor expansion, is

$$S(\vec{p}_0) = -mc\sqrt{(ct)^2 - |\vec{x}|^2}. \quad (39)$$

We need to calculate the second derivative of the argument,  $S$ ,

$$\frac{\partial^2 S}{\partial p_i \partial p_j} = \frac{ct}{\sqrt{(mc)^2 + |\vec{p}|^2}} \left( \frac{p_i p_j}{(mc)^2 + |\vec{p}|^2} - \delta_{ij} \right) \quad (40)$$

The second derivative needs to be evaluated at the critical point so the expression for  $\vec{p}_0$  from Eq. (38) is inserted into the second derivative,

$$\left( \frac{\partial^2 S}{\partial p_i \partial p_j} \right) \Big|_{\vec{p}_0} = - \left( \frac{\sqrt{(ct)^2 - |\vec{x}|^2}}{(mc)} \right) \left( \delta_{ij} - \frac{x_i x_j}{(ct)^2} \right) \quad (41)$$

The value of  $\vec{p}_0$  was chosen so that the first derivative is zero at that point so only the first and third term in Eq. (35) remain. To quadratic order in momentum, the action is approximated as

$$S(\vec{p}) \approx -mc\sqrt{(ct)^2 - |\vec{x}|^2} - \frac{1}{2} \left( \frac{\sqrt{(ct)^2 - |\vec{x}|^2}}{(mc)} \right) \left( \delta_{ij} - \frac{x_i x_j}{(ct)^2} \right) p'_i p'_j \quad (42)$$

When doing the integration in Eq. (34), the variable substitution to  $\vec{p}'$  does not cause an issue because it is integrated over all values of momentum.<sup>23</sup> The integral in Eq. (34) now becomes

$$\frac{1}{(2\pi)^3} e^{-\frac{imc}{\hbar} \sqrt{(ct)^2 - |\vec{x}|^2}} \int d^3 p e^{-\frac{i}{\hbar} M_{ij} p'_i p'_j} \quad (43)$$

where the factor in front of the integral comes from the  $S(\vec{p}_0)$  term in the expansion and the factor  $M_{ij}$  is given by the matrix [5]

$$[M_{ij}] = \frac{1}{2} \left( \frac{\sqrt{(ct)^2 - |\vec{x}|^2}}{(mc^3 t^2)} \right) \begin{pmatrix} c^2 t^2 - x_1^2 & -x_1 x_2 & -x_1 x_3 \\ -x_1 x_2 & c^2 t^2 - x_2^2 & -x_2 x_3 \\ -x_1 x_3 & -x_2 x_3 & c^2 t^2 - x_3^2 \end{pmatrix}. \quad (44)$$

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<sup>23</sup>The measure of integration is invariant under translation and the expression is integrated over all values so this variable substitution does not change the form of the integral.

One of the eigenvectors of the matrix is  $\vec{x} = x\hat{i} + y\hat{j} + z\hat{k} = \vec{r}$  with the eigenvalue  $c^2t^2 - |x|^2$ . The other eigenvalue,  $c^2t^2$ , is doubly degenerate and corresponds to the plane perpendicular to the position vector so the corresponding eigenvectors can be chosen to be  $\hat{\phi}$  and  $\hat{\theta}$  where  $\phi$  and  $\theta$  are the azimuthal and polar angles in spherical coordinates, respectively. Eq. (43) can be evaluated simply as it is in the general form of a Gaussian integral. The propagator is approximately

$$K(\vec{x}, 0; t) \approx \frac{1}{(2\pi)^3} \left( \frac{8\pi^3 \hbar^3 m^3 c^5 t^2}{i^3 ((c^2 t^2 - |x|^2)^{5/2})} \right)^{1/2} e^{-\frac{imc}{\hbar} \sqrt{(ct)^2 - |\vec{x}|^2}} \quad (45)$$

We want to see how the propagator behaves in the three cases where the spacetime intervals are spacelike, null, and timelike.

The propagator, Eq. (45), exhibits oscillatory behavior in the case of a timelike interval. The probability of finding the particle at the initial point decreases over time as the particle ‘‘dissipates’’ over time. In the case of a null interval, the propagator is undefined and can only be finite if the particle is massless. This has the interpretation that only massless particles, like photons, can travel at the speed of light. In the case of a spacelike interval, the propagator is still defined but has exponentially decaying behavior. This is classically not allowed because it has the interpretation that there is a non-zero probability of finding the particle outside the lightcone.<sup>24</sup> The probability falls off quickly outside the lightcone but it means that in quantum mechanics, a particle has a probability of being somewhere that light could not travel.

We need to calculate the trace of the propagator in order to calculate the partition function,

$$\text{tr}K = \int d^3\vec{x} K(\vec{x}, \vec{x}, t) = V \left( \frac{\hbar m}{2\pi i t} \right)^{3/2} e^{\frac{imc^2 t}{\hbar}}. \quad (46)$$

We will perform the variable transformation between time and temperature to arrive at the single particle partition function.

$$Z = V \left( \frac{mk_B T}{2\pi} \right)^{3/2} e^{\frac{-mc^2}{k_B T}} \quad (47)$$

This result is the same as the nonrelativistic solution which is consistent with what is expected since the initial nonrelativistic calculation was done to second order in momentum. The reproduction of the same expression provides verification for the method and the benefit of this method is that the extension to higher orders of momentum can be calculated by continuing the Taylor expansion in Eq. (35). Using more terms in the expansion will become more complicated but it is straightforward and is an area for future work on the topic. It is also in a similar form to the integral representation of the Bessel functions so calculating an exact solution may be possible. Again, this is an area for future work.

Another benefit of this method is that it gives a justification of the Hamilton-Jacobi approximation that will be used to incorporate the black hole metric into the partition function calculation.

### 3.4 Hamilton-Jacobi Approximation

The argument of the exponent in Eq. (34),  $\vec{p}\cdot\vec{x} - Ht$  is in fact the phase-space action of the system describing a free particle moving from some initial point to a final point. This identification is why

<sup>24</sup>The lightcone is a term used to describe all the spacetime points that are within a timelike interval from an initial spacetime point, or event. Anything outside the lightcone classically cannot be influenced by the initial event as this would imply something is traveling faster than light between the two events.

the variable  $S$  was chosen to represent the argument of the exponent.<sup>25</sup> Since the Hamiltonian is a constant of the motion (has no time dependence), we can rewrite the argument of the exponent in integral form by first taking the derivative with respect to time and then integrating with respect to time [16]

$$\vec{p} \cdot \vec{x} - Ht = \int (\vec{p} \cdot \dot{\vec{x}} - H) dt = \int L dt = S. \quad (48)$$

The second equality comes from the definition of the inverse Legendre transform, which can be found by solving Eq. (13) for  $L$ . In order to get the propagator into the form of a Gaussian integral, we expanded it about the critical point. The critical point is where the action is stationary and is the classical solution according to Hamilton's Principle[14] so  $S(\vec{p}_0) \equiv S_0$  in Eq. (38) is in fact the classical action of a particle moving from point a to point b. So to the semiclassical order, the wavefunction can be approximated as

$$\Psi \approx e^{\frac{i}{\hbar} S_0} \quad (49)$$

where  $S_0$  is the action evaluated by classical methods.

If the propagator were to be evaluated only to semiclassical order, then we would calculate the internal energy to be just the rest-mass energy as the only terms left in Eq. (47) would be the exponential term with no prefactors which would come from approximating the energy of the system as just the rest mass energy. In the non-relativistic limit, the internal energy is dominated by the rest mass energy term so Eq. (31) is a good approximation. The classical statistical mechanics solution comes from the second order approximation of the propagator.

## 4 Point Particle Propagating on a Schwarzschild Black Hole Metric

We will now treat the case of the particle in a black hole spacetime which will give permit us to calculate the temperature and entropy of the black hole, the goal of this investigation. The method we will employ has been validated in its application to the flat space case and we will extend the method into the more complicated system that includes the classical black hole. We will change the background spacetime and the test particle will now be propagating on a black hole space time that is static and spherically symmetric with mass  $M$  where the metric is given by the Schwarzschild metric defined in the introduction. The line element is given by [5]

$$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 - \frac{dr^2}{1 - \frac{2GM}{c^2 r}} - r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (50)$$

We are now using spherical coordinates to describe position. The radial coordinate  $r$  is the distance from origin in flat spacetime, the angular coordinates  $\theta$  and  $\phi$  are the polar and azimuthal coordinates respectively. They are analogous to the longitude and latitude when describing position on the earth's surface. This defines the metric tensor  $g_{\mu\nu}$ , with  $g_{0i} = 0$ , and thus the Hamiltonian from Eq. (19) becomes

$$H = c\sqrt{g_{00}(m^2 c^2 - g^{ij} p_i p_j)}$$

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<sup>25</sup> $S$  is the conventional variable used to denote the action.

$$= c\sqrt{\left(1 - \frac{2GM}{c^2 r}\right)\left((mc)^2 + \left(1 - \frac{2GM}{c^2 r}\right)p_r^2 + \frac{1}{r^2}p_\theta^2 + \frac{1}{r^2 \sin^2\theta}p_\phi^2\right)}. \quad (51)$$

The solution to the wavefunction to semiclassical order was shown above and is given by

$$\Psi = e^{\frac{i}{\hbar}S_0} \quad (52)$$

The extra terms in the Hamiltonian make the calculating much more difficult so we will restrict ourselves to the semiclassical order in the case of the black hole spacetime. In order to use the approximation in Eq. (52), it remains to calculate  $S_0$ , the classical action. The classical action is given in Eq. (11). When using the Schwarzschild metric, this becomes very complicated as there would be differentials of all coordinates under the same radical. Any attempt at a Taylor expansion would lead to mixed differentials. The way to address this issue is by restricting the motion to a single degree of freedom. The two cases we will treat is a static particle that is evolving in time. The second case will be a particle that is constrained to move radially.

#### 4.1 Wavefunction for a Particle at Rest

We will treat the case of a particle at rest and look at its evolution in time. In this case  $\dot{x}^i = 0$  and the Lagrangian from Eq. (12) reduces to

$$L = -mc^2\sqrt{1 - \frac{2GM}{c^2 r}} \quad (53)$$

which contains no time dependence. So integration of the Lagrangian with respect to time yields the action given by Eq. (48),

$$S = -mc^2 t \sqrt{1 - \frac{2GM}{c^2 r}} \rightarrow \Psi \approx e^{-\frac{imc^2 t}{\hbar} \sqrt{1 - \frac{2GM}{c^2 r}}}. \quad (54)$$

The energy of the particle is given by the equation  $i\hbar\partial_t\Psi = E\Psi$ . The energy is then read from Eq. (54),  $E = mc^2\sqrt{1 - \frac{2GM}{c^2 r}}$ . In the limit of large  $r$  (far outside the event horizon),  $E \rightarrow mc^2 - \frac{GMm}{r}$  which is the Newtonian limit with the addition of rest mass energy. The effects of the black hole can also be seen as a redshift. The factor multiplying  $(it)$  in Eq. (54) is the frequency  $\omega$  of the de Broglie matter wave. In comparison to the flat-space solution, the particle is redshifted by the factor  $\sqrt{g_{00}} = \sqrt{1 - \frac{2GM}{c^2 r}}$  which is the same redshift that occurs for a photon in a Schwarzschild spacetime [5].<sup>26</sup> Within the event horizon,  $r < 2GM/c^2$ , the argument of the exponent in Eq. (54) becomes imaginary. An imaginary action corresponds to a classically not allowed region. The interpretation is that the particle is tunneling inside the black hole.<sup>27</sup>

$$\Psi \approx e^{-\frac{mc^2 t}{\hbar} \sqrt{\frac{2GM}{c^2 r} - 1}} \quad (55)$$

The tunneling probability given by  $|\Psi|^2$  decreases exponentially with time. This wavefunction describes a particle decaying and that decay is more rapid as  $r$  approaches 0.

<sup>26</sup>A redshift is the term for the Doppler Shift with electromagnetic radiation. The term comes from the fact that a source moving away from the observer will shift towards the red part of the spectrum if it starts in the visible spectrum. The curvature of spacetime also causes a gravitational redshift. The redshift in this case is a redshift of the de Broglie matter wave, not of electromagnetic radiation.

<sup>27</sup>In QM, a particle can tunnel through a barrier that it classically would not be allowed to go through. Imagine a ball in a bucket having a small probability of being found out of the bucket when observed. This phenomena explains how nuclei can spontaneously decay as nucleons tunnel outside of the nucleus.

## 4.2 Partition Function and Entropy of a Particle Constrained to Radial Motion

We will next look at the simple case of purely radial motion. In this case the action can be calculated from the original form of the Lagrangian given in Eq. (12). The quantity under the square root reduces to a single term and the action becomes

$$\text{Action} = -mc \int \sqrt{\frac{1}{1 - \frac{2GM}{c^2 r}}} dr \quad (56)$$

which is formally similar to the brachistichrone problem [14]. Making the substitution that  $\frac{c^2 r}{2GM} = \sin^2 u$  we arrive at the solution, inside the event horizon, that

$$S = imc \frac{2GM}{c^2} \left( \sin^{-1} \sqrt{\frac{c^2 r}{2GM}} - \sqrt{\frac{c^2 r}{2GM}} \sqrt{1 - \frac{c^2 r}{2GM}} \right) \Big|_{r_i}^{r_f}. \quad (57)$$

First, we note that the action is imaginary which has the interpretation that the action is classically not allowed. It is similar to the case of spacelike distances in Minkowski space time which is also classically disallowed as this would result in a particle leaving the light cone and there is thus a violation of causality. In quantum mechanics, classically not allowed paths become allowed as a particle can tunnel through a potential barrier [13]. In calculating the partition function, we set  $r_f = r_i$ . One solution is that  $S = 0$  but that gives us no information but the inverse sine is a multivalued function with a period of  $2\pi$  so the action becomes

$$S = imc^2 \frac{2GM}{c^3} 2\pi n \quad (58)$$

for any integer  $n$ . We will consider the  $n = 1$  case, as it is the dominant term in the expression. When inserting this action into the Hamilton-Jacobi expression, we find that the transition amplitude for a particle making one full trip from a starting point to the edge of the black hole and back to the original starting point is given by

$$\Psi = e^{-mc^2 \left( \frac{2GM 2\pi}{c^3 \hbar} \right)} \quad (59)$$

The tunneling probability is given by the square of Eq. (59), the transition amplitude. There is no dependence on time in this expression so the transition between time and temperature is done without changing the argument of the exponent. We have the partition function for a single particle upon integration over all space,

$$\int d^3 \vec{r} |\Psi|^2 = V e^{-mc^2 \left( \frac{8GM 2\pi}{c^3 \hbar} \right)} = Z = V e^{\frac{-E}{k_B T}} \quad (60)$$

The energy,  $E$ , is the rest energy  $mc^2$  or integer multiples of the rest mass energy for higher modes. The temperature can be read off directly as

$$T = \frac{c^3 \hbar}{8\pi GM k_B} = T_H \quad (61)$$

This temperature is the Hawking temperature of a black hole that was calculated in 1973 using a completely different approach [7]. This is another validation of this method and demonstrates that a direct quantum mechanical approach can be used for this problem and one does not need to use quantum field theory.

We are now interested in calculating the entropy of the black hole. One definition of entropy is  $\Delta S = \int \frac{dQ}{T}$  [4]. We will build up the entropy of the black hole from 0 when the black hole has zero mass and the  $dQ$  that is added to the system will be the addition of rest mass energy  $dMc^2$ .<sup>28</sup> The entropy,  $S$ , of the black hole is then

$$S = \int_0^M \frac{dM'c^2}{T} = \int_0^M \frac{c^2 8\pi GM'k_B}{c^3\hbar} dM' = \frac{4\pi GM^2 k_B}{c\hbar}. \quad (62)$$

The radius of the black hole is the Schwarzschild radius  $r = \frac{2GM}{c^2}$  so the surface area of the black hole is  $A = 4\pi r^2 = 4\pi \frac{4G^2 M^2}{c^4}$ . Rewriting Eq. (62) in terms of the area, we find that the entropy of the black hole is

$$S = \frac{Ak_B c^3}{4G\hbar}. \quad (63)$$

This is the Bekenstein-Hawking Law [7] which shows that the entropy of a black hole is proportional to the area of the event horizon. This is an important consequence of the geometry of the black hole spacetime and is one of the laws of black hole thermodynamics [17].

It is significant that, through a vastly different approach, we were able to reproduce the results of Stephen Hawking and Jacob Bekenstein for the temperature and entropy of the black hole. This gives an alternate justification for both of these values. In analyzing the approach, we can see two areas where we simplified the calculation. The first is using a semiclassical approximation for the wavefunction, the second being that we ignored any angular motion, the particle was not allowed to orbit or have any angular momentum. In order to stay fully quantum mechanical (not using a semiclassical approximation) and to incorporate the angular degrees of freedom, we will use a different approach known as perturbation theory.

## 5 Perturbative Approach to Calculating the Partition Function on a Schwarzschild Spacetime

We will next try to gain further insight into the system by employing perturbation theory. The requirement to use this approach is to begin with a system that has known, exact solutions and that the corrections to the system can be identified. The Hamiltonian can be written as a combination of an unperturbed Hamiltonian which has a known eigenbasis and a perturbation potential. In this case, the potential with known solutions will be one with a  $\frac{1}{r}$  potential. Using this potential in the expression of the Hamiltonian gives us an expression in the same form of the nonrelativistic hydrogen atom whose wavefunctions are known exactly. It is also formally the same potential as described by Newtonian gravity. By appropriately renaming the constants, one can import the known solutions from the nonrelativistic hydrogen atom problem. With the introduction of general relativity, it will be seen that the Hamiltonian can be written as the Hamiltonian in the Newtonian limit with some correction terms which will be identified as the perturbation potential.

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<sup>28</sup> $dQ$  is a differential amount of heat. We are using differential rest-mass energy,  $dMc^2$  in its place as we assume that all energy is converted into heat when the particle passes the event horizon.

## 5.1 Eigenstates of the Unperturbed Hamiltonian

The solutions to the unperturbed Hamiltonian are similar to the solutions for the nonrelativistic hydrogen atom with different constants.<sup>29</sup> The hydrogen atom solutions can be expressed in the basis of the principal quantum number,  $n$ , the angular momentum quantum number,  $l$ , and the magnetic quantum number,  $m$ ,

$$|n l m\rangle. \quad (64)$$

The states can be expressed in the coordinate representation by projecting the  $n, l, m$  basis states onto the coordinate basis states,

$$\psi_{nlm}(\vec{r}) = \langle n l m | r \theta \phi \rangle = Y_l^m(\theta, \phi) R_{nl}(r), \quad (65)$$

where  $Y_l^m(\theta, \phi)$  are the spherical harmonics and  $R_{nl}(r)$  is the radial part of the wavefunction, related to the associated Laguerre polynomials by

$$R_{nl} = N_{nl} \left( \frac{2r}{na_0} \right)^l e^{-r/na_0} L_{n-l-1}^{2l+1} \left( \frac{2r}{na_0} \right). \quad (66)$$

$N_{nl}$  is a normalization constant and  $a_0$  is the Bohr radius in the case of the hydrogen atom. For the case of two interacting gravitational bodies, the Bohr radius is defined as<sup>30</sup>

$$a_0 = \frac{\hbar^2}{GMm^2}. \quad (67)$$

The action of the unperturbed Hamiltonian on these states, Eq. (64), is

$$\widehat{H}_0 |n l m\rangle = \left( -\frac{\hbar^2}{2ma_0^2} \frac{1}{n^2} \right) |n l m\rangle = \left( -\frac{G^2 m^3 M^2}{2\hbar^2} \frac{1}{n^2} \right) |n l m\rangle \quad (68)$$

The  $|n l m\rangle$  basis is also an eigenbasis of the angular momentum operator squared,  $\widehat{\vec{L}} \cdot \widehat{\vec{L}}$ , with action

$$\widehat{\vec{L}} \cdot \widehat{\vec{L}} |n l m\rangle = \hbar^2 l(l+1) |n l m\rangle. \quad (69)$$

## 5.2 Perturbation Hamiltonian

We start with the relativistically correct Hamiltonian from equation Eq. (19),

$$H = c\sqrt{g_{00}} \sqrt{m^2 c^2 + g^{ij} p_i p_j} = c\sqrt{1 - \frac{2\mu}{r}} \sqrt{m^2 c^2 + \vec{p} \cdot \vec{p} - \frac{2\mu}{r} p_r p_r}. \quad (70)$$

The second equality comes from inserting the Schwarzschild metric for a black hole. After combining the two radicals, we can identify the terms that appear in the Hamiltonian for a non-relativistic system with a  $\frac{1}{r}$  potential, a rest energy term, and a perturbation potential

<sup>29</sup>The Hamiltonian for the non-relativistic hydrogen atom is  $H = \frac{\vec{p} \cdot \vec{p}}{2m} - \frac{ke^2}{r}$  with  $k$  being Coulomb's constant and  $e$  the fundamental charge, which matches the form of what has been defined as  $H_0$ .

<sup>30</sup>The Bohr radius is the expectation value of the radial coordinate of the electron in the Hydrogen atom (where it is most likely to be found) and is equal to  $a_0 = \frac{\hbar^2}{kme^2}$ . The potential for a Hydrogen atom is  $\frac{ke^2}{r}$  and  $\frac{GMm}{r}$  which gives the relationship between the constants for the wavefunctions.

$$\begin{aligned}
H &= c\sqrt{m^2c^2 + 2m\left(\frac{\vec{p}\cdot\vec{p}}{2m} - \frac{GMm}{r}\right) - \frac{2\mu}{r}(p_r^2 + \vec{p}\cdot\vec{p} - \frac{2\mu}{r}p_r^2)} \\
&= c\sqrt{m^2c^2 + 2m\left(\frac{\vec{p}\cdot\vec{p}}{2m} - \frac{GMm}{r}\right) + \frac{2\mu}{r}\left(\frac{2\mu}{r} - 2\right)\vec{p}\cdot\vec{p} - \frac{2\mu}{r}\left(\frac{2\mu}{r} - 1\right)\frac{\vec{L}\cdot\vec{L}}{r^2}} \\
&= c\sqrt{m^2c^2 + 2mH_0 + \mu A + \mu^2 B}, \tag{71}
\end{aligned}$$

where  $H_0 = \frac{\vec{p}\cdot\vec{p}}{2m} - \frac{GMm}{r}$  is the unperturbed Hamiltonian for the nonrelativistic hydrogen atom and  $A$  and  $B$  from the perturbation are defined as

$$A = \frac{-8mH_0}{r} - \frac{8GMm^2}{r^2} + \frac{2\vec{L}\cdot\vec{L}}{r^3} \text{ and } B = \frac{8mH_0}{r^2} + \frac{8GMm^2}{r^3} - \frac{4\vec{L}\cdot\vec{L}}{r^4}. \tag{72}$$

The first equality in Eq.(71) uses the fact that  $\vec{p}\cdot\vec{p} = p_r^2 + \frac{\vec{L}\cdot\vec{L}}{r^2}$  and the terms  $A$  and  $B$  have been rewritten using the relationship  $\vec{p}\cdot\vec{p} = 2m(H_0 + \frac{GMm}{r})$ . The states that will be used as the basis for the perturbation are eigenstates of the unperturbed Hamiltonian and the square of the angular momentum operator but not of the linear momentum operator.<sup>31</sup> The unperturbed Hamiltonian,  $H_0$ , when promoted to a quantum operator, has the same form as the Hamiltonian for the hydrogen atom whose complete set of eigenstates are known. These eigenstates will be used as the basis for the expansion of the partition function.

In order to facilitate the calculation, we would like to separate the part of the variables  $A$  and  $B$  which are dimensional from the part which are dimensionless and are functions of  $n$  and  $l$ . We will make a change of variable  $\rho = r/a_0$  so the radius is now in terms of the Bohr radius with the appropriate factors,

$$\begin{aligned}
A &= -\frac{\hbar^2}{a_0^3}\left(\frac{4}{n^2}\rho^{-1} + 8\rho^{-2} - 2l(l+1)\rho^{-3}\right) \equiv \frac{\hbar^2}{a_0^3}a; \\
B &= \frac{\hbar^2}{a_0^4}\left(\frac{4}{n^2}\rho^{-2} + 8\rho^{-3} - 2l(l+1)\rho^{-4}\right) \equiv \frac{\hbar^2}{a_0^4}b, \tag{73}
\end{aligned}$$

where  $a$  and  $b$  are the dimensionless versions of the quantities  $A$  and  $B$ . The Hamiltonian must have the dimensions of energy.<sup>32</sup> With the factor of  $c$  outside of the radical in the expression for the Hamiltonian in Eq. (71) makes the unit of the Hamiltonian J.

To gain an understanding of the dimensions of this system, we will rewrite the expression for the Hamiltonian in a more suggestive way,

$$\begin{aligned}
\widehat{H} &= c\sqrt{m^2c^2 + 2m\widehat{H}_0 + \frac{\mu\hbar^2}{a_0^3}\widehat{a} + \frac{\mu^2\hbar^2}{a_0^4}\widehat{b}} \\
&= \frac{\hbar c}{a_0}\sqrt{\frac{a_0^2m^2c^2}{\hbar^2} + \frac{2ma_0^2\widehat{H}_0}{\hbar^2} + \frac{\mu}{a_0}\widehat{a} + \frac{\mu^2}{a_0^2}\widehat{b}}. \tag{74}
\end{aligned}$$

<sup>31</sup>Operators that act on their eigenstates are much easier to work with. We incur a factor of  $\frac{1}{r}$  in one of the terms by making this substitution. The basis that will be used is not an eigenbasis of the inverse position operator but the inverse position operator acts by multiplication not differentiation which simplifies the calculation.

<sup>32</sup>From Eq. (71), it is seen that  $\mu A$  and  $\mu^2 B$  must have the same dimensions as  $m^2c^2$ . The expansion parameter,  $\mu = \frac{GM}{c^2}$ , is half the Schwarzschild radius,  $R_s$ , and has units of length as does the Bohr radius,  $a_0$ , and  $\hbar$  has units of angular momentum or  $\text{kgm}^2\text{s}^{-1}$ . So the overall dimension of  $\mu A$  is  $\text{kg}^2\text{m}^2\text{s}^{-2}$  which is the expected dimension. The  $\mu^2 B$  has an extra unit of length in the numerator but is also divided by another factor of length with the increased power of the Bohr radius in the denominator so its dimension is also consistent.

Everything under the radical on the right hand side of Eq. (74) is dimensionless. Using the definition of the Bohr radius,  $a_0 = \frac{\hbar^2}{GMm^2}$ , rewrite the first term under the radical,

$$\begin{aligned} H &= \frac{\hbar c}{a_0} \sqrt{\left(\frac{\hbar c}{GMm}\right)^2 - \frac{2ma_0^2 H_0}{\hbar^2} + \frac{\mu}{a_0} a + \frac{\mu^2}{a_0^2} b} \\ &= \frac{\hbar c}{a_0} \sqrt{\frac{a_0}{\mu} - \frac{2ma_0^2 H_0}{\hbar^2} + \frac{\mu}{a_0} a + \frac{\mu^2}{a_0^2} b}. \end{aligned} \quad (75)$$

The second term in the radical is indeed dimensionless and if acting in isolation on the basis states it will return  $\frac{2ma_0^2 E_n}{\hbar^2} = -\frac{1}{n^2}$ . This equality cannot be used before the operator acts on one of its eigenstates and the Hamiltonian needs to be written without the square root in order to use this property and so it will just be noted as a suggestive relation for now. Then the operators  $\hat{a}$  and  $\hat{b}$  act on the basis of states, they return the expectation values of the inverse powers of  $\rho$  present in their definition in Eq. (73). The expression for these expectation values are given in Appendix B. Writing the Hamiltonian gives two ratios of significance. The first is  $\frac{\hbar c}{a_0}$  which we will refer to as the Bohr energy  $E_{Bohr}$ . It is the energy of a particle which has a de Broglie wavelength equal to the Bohr radius of the gravitational system [13]. The Bohr energy sets the relevant energy scale of this system, similar to how the rest mass energy was the energy scale for the particle in Minkowski spacetime. The other important ratio is the dimensionless  $\frac{\mu}{a_0}$  which is the ratio of the Schwarzschild radius to the Bohr radius. The Schwarzschild radius is the length scale of general relativity and the Bohr radius is the length scale of quantum mechanics. The ratio gives the relative magnitude of the effect of the two theories.

### 5.3 Partition Function Calculation

We will now calculate the partition function

$$Z = \text{tr } e^{-\beta H} \quad (76)$$

In order to calculate the trace, we will expand the Eq. (76) in a Taylor series using  $\mu$  as the expansion parameter. We will return to the expression for the Hamiltonian given by the last line in Eq. (71). This will provide a cleaner form for the expansion and the dimensions can be treated later. Because of the form of the Hamiltonian, the calculation of the action of the Hamiltonian, once promoted to an operator, will become complicated as operator ordering needs to be preserved when dealing with operators that do not commute and both the radical and the exponent will need to be expressed as infinite series that account for the ordering. A more straightforward way to proceed is to expand the expression for  $Z$  first at the classical level in powers of  $\mu$  and then promote the variables to operators. Proceeding in this manner will avoid any issues with operator ordering. As will be seen below, the only term that is under a radical or in an exponent will be the zeroth order term. The zeroth order term contains the unperturbed Hamiltonian plus a constant so the action of this term, even when promoted to an operator, is to return the eigenvalue and it is a property of operators that in a basis of eigenstates, all continuous functions of operators are defined.

For simplicity in the calculation, it will be useful to define the zeroth order term as

$$O = m^2 c^2 + 2mH_0. \quad (77)$$

So the Hamiltonian can be written as

$$H = c\sqrt{O + \mu A + \mu^2 B}. \quad (78)$$

We are now looking for the Taylor series of

$$F = F(\mu, r) = e^{-c\sqrt{O+\mu A+\mu^2 B}}. \quad (79)$$

We will regard this as a function of  $r$  parametrized by the parameter  $\mu$ . Then the desired expansion is given by

$$F(\mu, r) = \sum_{N=0}^{\infty} \frac{\mu^N}{N!} \left( \frac{\partial^N F(q, r)}{\partial q^N} \right)_{q=0}. \quad (80)$$

It suffices to evaluate the derivatives to the desired order of the expansion. This expansion can be done before taking the trace and before promoting the variables to operators. Doing the expansion before the quantization allows us to treat the Hamiltonian as a smooth function. The zeroth order term is

$$F(0, r) = e^{-\beta c\sqrt{O}}. \quad (81)$$

To calculate the partition function in the form given by Eq. (28), we need to make the transition to quantum mechanics with the dynamical variables being promoted to operators. We will also insert the identity operator in order to project the eigenstates of the unperturbed Hamiltonian onto the initial state. Inserting this basis of states allows us to use the properties of operators acting on their eigenstates,<sup>33</sup>

$$\int d^3\vec{r}' \langle \vec{r}' | e^{-\beta \hat{H}} | \vec{r} \rangle = \int d^3\vec{r}' \langle \vec{r}' | e^{-\beta \hat{H}} \sum_{l,n,m} |n l m\rangle \langle n l m | \vec{r}' \rangle. \quad (82)$$

So to zeroth order (setting  $\mu$  equal to zero), the partition function becomes

$$Z_0 = \sum_{n,l,m} \int d^3\vec{r}' \langle \vec{r}' | e^{-\beta c\sqrt{\hat{O}}} |n l m\rangle \langle n l m | \vec{r}' \rangle. \quad (83)$$

In this expression, the zeroth order part of the Hamiltonian,  $O$ , has been promoted to the operator;

$$\hat{O} = m^2 c^2 + 2m\hat{H}_0. \quad (84)$$

When acting on the  $|n l m\rangle$  basis states, it returns the eigenvalue of the unperturbed Hamiltonian plus the term that can be associated with the rest mass energy,

$$\hat{O}|n l m\rangle = E_n |n l m\rangle \equiv \left( m^2 c^2 - \frac{G^2 M^2 m^4}{\hbar^2} \frac{1}{n^2} \right) |n, l, m\rangle \quad (85)$$

Thus the expression for the partition function to zeroth order in  $\mu$  is given by

$$Z^0 = \sum_{m,l,n} e^{-\beta c\sqrt{E_n}} \int d^3\vec{r}' \psi_{nlm}^*(\vec{r}') \psi_{nlm}(\vec{r}') = \sum_n e^{-\beta c\sqrt{E_n}} \quad (86)$$

This is the partition function of the non-relativistic system of a point mass interacting with a gravitational body but it neglects the effects of general relativity. The effect of the black hole on the partition function, and thus the thermodynamics of the system, are not represented in this

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<sup>33</sup>The result of projecting the  $|n l m\rangle$  state onto the position basis is given by  $\langle \vec{r}' | n l m\rangle = \psi_{nlm}^*(\vec{r}')$  and  $\langle n l m | \vec{r}' \rangle = \psi_{nlm}(\vec{r}')$

zeroth order term. The series does not converge because in the limit as  $n$  goes to infinity, the terms themselves not go to zero so the sequence of partial sums diverges. For term that is first order in  $\mu$ , we need the first derivative

$$\frac{\partial F}{\partial \mu} = \frac{\partial}{\partial \mu} e^{-\beta c \sqrt{O + \mu A + \mu^2 B}} = \frac{1}{2} (O + \mu A + \mu^2 B)^{-1/2} (A + 2\mu B) e^{-\beta c \sqrt{O + \mu A + \mu^2 B}}. \quad (87)$$

Evaluated at  $\mu = 0$ , this is

$$\left( \frac{\partial F}{\partial \mu} \right)_{\mu=0} = -\frac{\beta c A}{2O^{1/2}} e^{-\beta c \sqrt{O}}. \quad (88)$$

Note that the zeroth order term is simply the exponential part of (88). Moving on to the second derivative for order  $\mu^2$  we must differentiate (87). So we have

$$\begin{aligned} \frac{\partial^2 F}{\partial \mu^2} &= \frac{\partial}{\partial \mu} \left[ -\frac{1}{2} \beta c (O + \mu A + \mu^2 B)^{-1/2} (A + 2\mu B) e^{-\beta c \sqrt{O + \mu A + \mu^2 B}} \right] \\ &= \left[ \frac{\beta c}{4} (O + \mu A + \mu^2 B)^{-3/2} (A + 2\mu B)^2 - \frac{\beta c}{2} (O + \mu A + \mu^2 B)^{-1/2} (2B) \right. \\ &\quad \left. + \frac{\beta^2 c^2}{4} (O + \mu A + \mu^2 B)^{-1} (A + 2\mu B)^2 \right] e^{-\beta c \sqrt{O + \mu A + \mu^2 B}}. \end{aligned} \quad (89)$$

Evaluated at  $\mu = 0$ , this is

$$\left( \frac{\partial^2 F}{\partial \mu^2} \right)_{\mu=0} = \left[ \beta c \left( \frac{A^2}{4O^{3/2}} - \frac{B}{O^{1/2}} \right) + (\beta c)^2 \frac{A^2}{4O} \right] e^{-\beta c \sqrt{O}}. \quad (90)$$

Again, note how the zeroth order term appears as the exponent in (90), a feature that will be common to all terms, which facilitates the interpretation of the expansion about the  $\mu = 0$  partition function. The general term will be of the form

$$\left( \frac{\partial^N F}{\partial \mu^N} \right)_{\mu=0} = \left[ \sum_{k=0}^N (\beta c)^k f_{N,k}(A, B, O) \right] e^{-\beta c \sqrt{O}}, \quad (91)$$

So to second order,

$$e^{-\beta H} \simeq (1 - \mu \left[ \frac{(\beta c) A}{2O^{1/2}} \right] + \frac{\mu^2}{2!} \left[ \beta c \left( \frac{A^2}{4O^{3/2}} - \frac{B}{O^{1/2}} \right) + (\beta c)^2 \frac{A^2}{4O} \right]) e^{-\beta c \sqrt{O}}. \quad (92)$$

It is at this point that we should now carry out a quantization of Eq. (92) by promoting the variables to operators. Operator ordering is no longer an issue as the operators and factors of  $r$  are no longer in an exponent or a radical. For each term we can bring out any  $O$  dependence, replacing it by its eigenvalue  $E_n$  and likewise for the angular momentum squared terms. Then we are left with expectation values of the form  $\langle \frac{1}{\rho^N} \rangle$ , which is well-defined (see Appendix B). The partition function is given by

$$Z = \sum \frac{\mu^N}{N!} Z_N = Z_0 + \mu Z_1 + \frac{\mu^2}{2!} Z_2 + \dots, \quad (93)$$

where the first few terms are given by

$$\begin{aligned} Z_0 &= \sum_n e^{-\beta c \sqrt{E_n}}; \\ Z_1 &= \frac{(\beta c)}{2} \sum_{n,l} \frac{\langle A \rangle_{n,l}}{\sqrt{E_n}} e^{-\beta c \sqrt{E_n}}; \text{ and} \\ Z_2 &= \frac{(\beta c)}{4} \sum_{n,l} \left( \frac{\langle A^2 \rangle_{n,l}}{E_n^{3/2}} - \frac{\langle B \rangle_{n,l}}{E_n^{1/2}} \right) e^{-\beta c \sqrt{E_n}} + \frac{(\beta c)^2}{4} \sum_{n,l} \frac{\langle A^2 \rangle_{n,l}}{E_n} e^{-\beta c \sqrt{E_n}}. \end{aligned} \quad (94)$$

While it is not obvious, each term in the series in Eq. (93), aside from the first, is finite.<sup>34</sup> The values of  $l$  in the sum are bounded by  $n$  so each sum over  $l$  has a finite number of terms and as  $n$  goes to infinity, the  $E_n$  term goes to a constant, namely  $m^2 c^2$ , and so it is a question of whether the expectation value of  $\langle A \rangle$  and  $\langle B \rangle$  converge. Each term in the expectation value has constants multiplied by a factor of  $n^2$  or higher in the denominator and so the series converge.

## 5.4 Calculation of the Entropy

From the partition function, we can calculate entropy.<sup>35</sup> The entropy is defined as<sup>36</sup>

$$S = -k_B \sum_i P_i \ln P_i = -k_B \left[ \beta \frac{\partial}{\partial \beta} + 1 \right] \ln Z \quad (95)$$

where  $P_i$  is the probability that the system is in the  $i^{\text{th}}$  microstate and is defined as  $P_i = \frac{e^{-\beta E_i}}{Z}$ . The second term in the calculation of the entropy,  $-k_B \ln Z$ , can be calculated from the definition of  $Z$  above. The value of the entropy will still be infinite yet the contribution to the entropy from the black hole is finite.

The derivative in the first term in the expression for entropy,  $-k_B \beta \frac{\partial}{\partial \beta} \ln Z = \frac{-k_B \beta}{Z} \frac{\partial Z}{\partial \beta}$ , can be calculated at the level of the definition of the partition function, Eq. (76), as

$$\frac{\partial Z}{\partial \beta} = \text{tr}(-H e^{-\beta H}). \quad (96)$$

In the same process as above, we will expand this expression in a Taylor expansion around  $\mu = 0$ ,<sup>37</sup>

$$\begin{aligned} H e^{-\beta H} &= c\sqrt{O} e^{-\beta c\sqrt{O}} + \frac{\mu}{2} \left[ \frac{cA}{\sqrt{O}} - \beta c^2 A \right] e^{-\beta c\sqrt{O}} \\ &+ \frac{\mu^2}{2!} \left[ \frac{cB}{\sqrt{O}} - \frac{cA^2}{4O^{3/2}} - \beta c^2 B - \frac{-\beta c^2 A^2}{O} - \frac{\beta^2 c^3 A^2}{\sqrt{O}} \right] e^{-\beta c\sqrt{O}} + \dots \end{aligned} \quad (97)$$

The trace of this expression is taken in the same way as above where we integrate the propagator from position  $\vec{r}$  to position  $\vec{r}$  over all space. The calculation is done by inserting the complete set of basis states of the unperturbed Hamiltonian and acting on the wavefunction with the dynamical variables promoted to operators. In order to stay organized, we will identify the variable  $K$  with  $H e^{-\beta H}$  and  $K_0, K_1, K_2$  etc. will be the coefficients in the Taylor expansion

$$K = \sum \frac{\mu^N}{N!} K_N = K_0 + \mu K_1 + \frac{\mu^2}{2!} K_2 + \dots \quad (98)$$

<sup>34</sup>These expectation values are taken with respect to the state  $|n l m\rangle$ , and in general carry the labels  $n$  and  $l$ .

<sup>35</sup>An alternative definition of the partition function is  $Z = \sum_i e^{-\beta E_i}$  where  $\beta = \frac{1}{k_B T}$  [4].

<sup>36</sup> $P_i$  is the probability of finding a particle in macrostate  $i$  which is defined by  $\frac{e^{-\beta E_i}}{Z}$  [4].

<sup>37</sup>This done with the same process as calculating the partition function to second order. The steps involved with the derivative were omitted for brevity as it is straight forward calculus. The expression is more complicated and only the result will be needed for our purposes.

The first few terms are

$$\begin{aligned}
K_0 &= \sum_n c\sqrt{E_n}e^{-\beta c\sqrt{E_n}}; \\
K_1 &= \sum_{n,l} \frac{1}{2} \left[ \frac{c\langle A \rangle}{\sqrt{E_n}} - \beta c^2 \langle A \rangle \right] e^{-\beta c\sqrt{E_n}}; \text{ and} \\
K_2 &= \sum_{n,l} \left[ \frac{c\langle B \rangle}{\sqrt{E_n}} - \frac{c\langle A^2 \rangle}{4E_n^{3/2}} - \beta c^2 \langle B \rangle - \frac{\beta c^2 \langle A^2 \rangle}{E_n} - \frac{\beta^2 c^3 \langle A^2 \rangle}{\sqrt{E_n}} \right] e^{-\beta c\sqrt{E_n}}
\end{aligned} \tag{99}$$

We will now examine the dimensionality each of the terms. It will be convenient to rewrite the expectation values in terms of a constant that contains the dimension times the part that is dimensionless using the relationship from Eq. (73),

$$E_n = \left(\frac{\hbar^2}{a_0^2}\right)\epsilon_n; \quad \langle A \rangle_{n,l} = \frac{\hbar^2}{a_0^3} \langle a \rangle_{n,l}; \quad \text{and} \quad \langle B \rangle_{n,l} = \frac{\hbar^2}{a_0^4} \langle b \rangle_{n,l}. \tag{100}$$

The expectation values of  $A$  and  $B$  depend on the values of  $n$  and  $l$  of the state and constitute the terms of the series. Making the substitutions defined above, we can rewrite the first few orders of  $K$ ,

$$\begin{aligned}
K_0 &= \sum_n c\left(\frac{\hbar}{a_0}\right)\sqrt{\epsilon_n}e^{-\beta c\left(\frac{\hbar}{a_0}\right)\sqrt{\epsilon_n}}; \\
K_1 &= \sum_{n,l} \frac{1}{2} \left[ \frac{\hbar c \langle a \rangle}{a_0^2 \sqrt{\epsilon_n}} - \beta c^2 \frac{\hbar^2}{a_0^3} \langle a \rangle \right] e^{-\beta c\left(\frac{\hbar}{a_0}\right)\sqrt{\epsilon_n}}; \text{ and} \\
K_2 &= \sum_{n,l} \left[ \frac{\hbar c \langle b \rangle}{a_0^3 \sqrt{\epsilon_n}} - \frac{\hbar c \langle a^2 \rangle}{4a_0^3 \epsilon_n^{3/2}} - \beta c^2 \frac{\hbar^2}{a_0^4} \langle b \rangle - \frac{\beta c^2 \hbar^2 \langle a^2 \rangle}{a_0^3 \epsilon_n} - \frac{\beta^2 c^3 \hbar^3 \langle a^2 \rangle}{a_0^5 \sqrt{\epsilon_n}} \right] e^{-\beta c\left(\frac{\hbar}{a_0}\right)\sqrt{\epsilon_n}}
\end{aligned} \tag{101}$$

When bringing in the factors of  $\mu$  that multiply the coefficients  $K_N$ , we find that the dimension of each of the terms in the series is of the form

$$\left(\frac{\mu}{a_0}\right)^n \left(\frac{\hbar c}{a_0}\right)^\alpha \beta^{\alpha-1} = E_{Bohr} \left(\frac{\mu}{a_0}\right)^n \left(\frac{E_{Bohr}}{k_B T}\right)^{\alpha-1} \tag{102}$$

which has units of energy. When inserted into the definition of entropy in Eq. (95), it gains a factor of  $k_B \beta$  so the resulting dimension is energy per temperature, the correct units for entropy.

Similar to the partition function, this expression for entropy diverges but the infinite part comes from the zeroth order term which comes from the unperturbed Hamiltonian. The contribution to the entropy from the black hole is finite. While it is not obvious, for the same reasons that the higher order terms of the partition function converge, the higher order terms of the entropy converge.

## 5.5 Interpretation of the Partition Function and Entropy

The expressions for both the partition function and the entropy can be expressed as functions of the two ratios discussed earlier when defining the perturbed Hamiltonian,

$$Z = Z\left(\frac{\mu}{a_0}, \frac{\hbar c}{a_0}\right) \quad \text{and} \quad S = S\left(\frac{\mu}{a_0}, \frac{\hbar c}{a_0}\right) \tag{103}$$

The arguments of these functions have an interesting interpretation. As discussed in Appendix A, the length and energy scales where the effects of quantum gravity would be dominant are so extreme

they would exist only at the Big Bang but the imprints of quantum gravity are already seen on the thermodynamics of a point particle interacting with a black hole. While the gravitational field itself was not quantized, the quantum particle that exists in the classically treated black hole responds to quantum gravity. The ratio  $\frac{\mu}{a_0}$  has a significant meaning when evaluating the constants,

$$\frac{\mu}{a_0} = \left( \frac{GMm}{\hbar c} \right)^2 = \left( \frac{Mm}{M_{Pl}^2} \right)^2 = \frac{E_{Bohr}}{k_B T_H}. \quad (104)$$

The third term in the equation is the ratio of the product of the mass of the particle and the black hole to the square of the Planck mass. The ratio gives a measure of how close we are to the Planck scale. The last term is the ratio of the Bohr energy to the thermal energy of the black hole where the thermal energy of the black hole is given by Boltzmann's constant times the Hawking temperature of the black hole.

Another interesting consequence of the calculation is that there are stable states within the event horizon of the black hole. These stable states lead to an area of future study on this topic. The expansion was done assuming  $\mu$  is small and thus  $\frac{\mu}{a_0} > 1$  which is only the case when the particle is outside of the black hole. Within the black hole, the dominant terms in the Hamiltonian are different than when outside of the black hole. The value of the ratio  $\frac{\mu}{a_0}$  is now greater than 1. It can be seen from Eq. (75) that the terms that were considered small become large in comparison to the rest mass energy and nonrelativistic energy terms,

$$H = \frac{\hbar c}{a_0} \sqrt{\frac{a_0}{\mu} - \frac{2ma_0^2 H_0}{\hbar^2} + \frac{\mu}{a_0} a + \frac{\mu^2}{a_0^2} b}. \quad (105)$$

In this case, the Taylor expansion truncated at second order around  $\mu = 0$  is less accurate. While it still gives insight, a different expansion parameter should be used to give a more accurate result.

## 6 Conclusion

This method of investigation has yielded an important result; the thermodynamics of a black hole *can* be studied from the perspective of quantum mechanics. However, one needs to deal with the complicated expression for the Hamiltonian in an appropriate way. The square root in the definition of an operator presents nontrivial mathematical challenges.

The first result, the partition function of a nonrelativistic point particle, matched the empirically developed partition function for monatomic gases. The approach in this paper was nonempirical and yet, starting from quantum mechanics, we were able to reproduce the standard result from statistical mechanics. Repeating this procedure using the full Hamiltonian in the expression for the propagator led to the same result when taken to second order in the Taylor expansion of the momentum. A new result was not produced but having reproduced a known result gives merit to this method of calculating the entropy. Moreover, the process gave insight into the origin of the different terms in the result for the partition function. It also indicates a direct path forward for calculating higher order correction terms. The method was truncated at second order but the mathematical structure enables one to calculate the partition function any order desired.

When making the same calculations to find the thermodynamic properties in the Schwarzschild metric, the method of going into momentum space in order to simplify the Hamiltonian operator does not work with the more complicated metric. We used the intermediate result from the previous method that the wavefunction can be approximated using classical action. In the calculation of the propagator, the leading order term in the Taylor expansion was the phase space action evaluated

on the classical solution. In looking at the expression for the internal energy of the free particle, the rest mass energy term came from the classical action. This gives an idea of the validity of this approximation. The rest mass energy is orders of magnitude larger than the thermal energy (which was found by taking the expansion to second order). While there would be important insight that comes from evaluating the higher order terms, the zeroth order gives the dominant effects. When using this semiclassical treatment of the wavefunction for a particle trapped within a black hole, we found tunneling solutions to the wave equation which indicate that the particle has a nonzero probability of being within the black hole. It is these particle states that allowed us to calculate the Hawking temperature through a completely different method. The entropy calculated with this wavefunction is again the entropy that has been calculated before by different methods. There were higher order terms that were omitted in this method so the value we have for temperature and entropy are good approximations to semiclassical order.

In order to gain further insight into quantum mechanics, we turned to perturbation theory. In doing this calculation, we were able to get expressions for the particle states and the partition functions that had corrections from general relativity which have imprints of quantum gravity. It can be seen from the dimensional analysis of the correction to the wavefunctions that the Bohr energy,  $\frac{hc}{a_0}$ , is the energy scale of the corrections that general relativity makes to the thermodynamic properties of the system, similar to how the rest mass energy is the energy for the free particle case, and the expansion parameter relates the masses of the system to the Planck mass, the mass scale of quantum gravity. This implies that the correction terms are proportional to how close the system is to a fully quantum gravitational system.

A full theory of quantum gravity requires that the gravitational field itself be quantized which is beyond the scope of this paper. This is a century old problem in physics. This paper investigated quantized gravity indirectly by looking at the influence of general relativity on quantum and statistical mechanics. What was found was that even at this level, the corrections to the nonrelativistic case contain imprints of quantum gravity. It is interesting that this quantum particle extracts information about quantum gravity even when the gravity itself was treated classically.

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## 7 Appendix A: Planck Length and Planck Mass

The effects of quantum mechanics dominate at certain scales while they are nearly absent on large length scales. For example, the world of atoms is dominated by quantum mechanics and what we understand as classical physics has almost no meaning. In contrast, at the scale we experience day to day, we do not experience quantum effects. A general guideline is that at distances on the order of the Compton wave length of the object quantum effects become important [13]. The Compton wave length is defined by  $\lambda_C = \frac{2\pi\hbar}{mc}$ . So the length scale where we would see a black hole exhibiting quantum effects is when the Compton wavelength is the same as the Schwarzschild radius,  $R_S = \frac{2GM}{c^2}$ . Setting these two values equal to each other,

$$M_{Pl} \equiv \sqrt{\frac{\hbar c}{G}} \quad (106)$$

There is a factor of  $\sqrt{\pi}$  that is not included in the definition of the Planck mass. We are only looking for an order of magnitude so this is not an issue. To calculate the Planck length, we insert the Planck mass into either the Compton wave length or the Schwarzschild radius.

$$\ell_{Pl} \equiv \frac{2G\sqrt{\frac{\hbar c}{G}}}{c^2} = \sqrt{\frac{\hbar G}{c^3}} \quad (107)$$

The numerical value of the Planck length is  $\ell_{Pl} = 1.6 \times 10^{-35}$  cm and the value of the Planck mass is  $M_{Pl} = 1.2 \times 10^{19}$  GeV/ $c^2 = 2.2 \times 10^{-8}$  kg. While this value seems very small, the density of such an object would be way beyond the limits of anything that would occur outside of the conditions present at the Big Bang. With scales of this magnitude, quantum gravity is beyond the reach of any experiment and is most likely beyond the reach of any of the conditions present in the universe.

## 8 Appendix B: Tabulation of $\langle \frac{1}{\rho^N} \rangle$ Values

The expectation values of the inverse powers of  $\rho$  are used in the definition of the expectation value the operators  $\hat{A}$  and  $\hat{B}$  in Eq. (73). The first two inverse powers are given below [13];

$$\langle \rho^{-1} \rangle = \frac{1}{n^2};$$

$$\langle \rho^{-2} \rangle = \frac{2}{n^3(2l+1)} \quad (108)$$

One can use Kramers' recursion relation to calculate the value of the expectation value for any power of  $\rho$  [13],

$$\frac{k+1}{n^2} \langle \rho^k \rangle - (2k+1)a_0 \langle \rho^{k-1} \rangle + \frac{ka_0^2}{4} \left[ (2l+1)^2 - k^2 \right] \langle \rho^{k-2} \rangle = 0. \quad (109)$$