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HOW SUSPICION GROWS: EFFECTS OF POPULATION SIZE ON COOPERATION

by

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September 2014

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HOW SUSPICION GROWS: EFFECTS OF POPULATION SIZE ON COOPERATION

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ABSTRACT

We study the interaction between individuals in a population, where each individual encounters one another at random intervals, and in each encounter the two individuals play one round of the game of prisoner's dilemma. By discounting future reward, and allowing for imperfect memory and mobility of the individuals, we study the evolutionary equilibrium strategy to identify situations where cooperation emerges. We find that cooperation among individuals typically emerges when future reward becomes more important, when individuals in the population have better memory, and when the individuals move in and out of the population less frequently. The findings help explain social loafing and free rider commonly seen in towns, corporations, and military units.

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List of Acronyms and Abbreviations

AD always defect
ESS evolutionarily stable strategy
IPD iterated prisoner's dilemma
PD prisoner's dilemma
STFT suspicious tit-for-tat
TFT tit-for-tat
TF2T tit-for-two-tats

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Executive Summary

People in a group typically become less motivated to exert effort for the common good, when the group size grows. This phenomenon is known as social loafing in social psychology, and there has been extensive research on its causes and remedies. The purpose of this thesis is to use a mathematical model to explain social loafing. The mathematical model uses the game of prisoner's dilemma as the building block, and identifies situations when cooperation is likely to emerge. The findings also suggest a few methods that help motivate cooperation.

Specifically, this thesis presents a game-theoretic model for interaction between individuals in a finite population. Each individual encounters one another at random intervals, and in each encounter the two individuals play one round of the game of prisoner's dilemma. The game of prisoner's dilemma has been the canonical example in game theory, where the individually optimal policy (defect) is opposite to the socially optimal policy (cooperate). In other words, a selfish, rational player's optimal action will lead to a less desirable outcome for the entire group. The prisoner's dilemma has seen many applications in arm races, harvest of natural resources, price competition between retail stores, among others. In our context, if the two individuals in the population encounter only once and will not see each other again in the future, then the individually optimal policy is to *defect* (exert minimal effort) rather than to *cooperate* (exert maximal effort).

To study the situations where cooperation is likely to emerge, we introduce three factors into the model. First, we introduce a discount factor so that the utility earned in the future is worth less than the same utility earned today. Second, we introduce the memory factor of an individual, so that each individual will remember past encounters for some random amount of time. If an individual encounters another person repeatedly, then he can use past experience to decide what to do in the future. Third, we allow individuals to move in and out of the population at random intervals.

To make our game-theoretic model mathematically tractable, we restrict each individual to four strategies: tit-for-two-tats (TF2T), tit-for-tat (TFT), suspicious-tit-for-tat (STFT), and always defect (AD). The first three strategies are similar, with the difference being how to play a stranger. When encountering a stranger, TF2T will begin by cooperation twice, TFT by cooperating once, and STFT by defecting once. After the initial moves, in each encounter, the three strategies will mirror what the opponent did in the previous encounter. The fourth strategy considered, AD, simply defects every time. Therefore, TF2T can be viewed as the most cooperative strategy, TFT the second most cooperative, while the AD the least cooperative.

To analyze system equilibrium, we focus on finding the evolutionary equilibrium strategy for a given set of model parameters. A strategy is evolutionary stable—possibly a mixed strategy—if it is optimal for an individual to adopt the same strategy, when the vast majority of the population is playing that strategy. Depending on the model parameters, the evolutionarily stable strategy contains different subsets of the four strategies under consideration. We find that cooperation among individuals typically emerges (TF2T and TFT) when future reward becomes more important, when individuals in the population have better memory, and when the individuals move in and out of the population less frequently. We also find that suspicion (STFT) is a form of behavior that increases the robustness of cooperation under certain situations.

In this thesis, several assumptions may not be entirely realistic. First, each player has to pick a strategy from a set of four options, whereas in real-life people may be more creative in choosing a strategy. Second, we use the geometric distribution to model an individual's memory, and also for the duration of time an individual stays in the population before moving out. Third, we assume that the population is homogeneous; that is, every individual has the same value functions. In real-life, however, the pay-off for cooperation may be higher for some people than for some other people. Fourth, we assume that each individual will encounter every other individual in the population with the same probability, whereas in real-life neighbors may encounter more frequently. Relaxing these assumptions presents challenging future research directions.

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CHAPTER 1:
Introduction

1.1 Background

Social scientists use the term "social loafing" to describe the reduction in cooperation in a group, as the size of the group increases. This phenomenon is surely familiar to anyone who has worked on projects in an academic, commercial or military setting. It is the goal of this study to give a game theoretic explanation to the lack of cooperation in large populations.

The prisoner's dilemma (PD) has been the canonical example in game theory for a game where mutual defection is the only equilibrium, while it is not in the best interest of both players as a group. It was introduced by Merril Flood and Melvin Dresher in 1950, and later revised to the modern form by Albert Tucker (Kuhn, 2009). In this game, two players have been arrested by the police as suspects in a joint crime. Both suspects are under pressure from police investigators to confess, since they cannot produce enough evidence to convict the two of the crime. Should no one confess, they both can expect to receive short prison sentences for some reduced crime. Should one confess, while the other holds out, the one that confessed will go free, while the other bears the maximum prison sentence. If both confess, they are both convicted and spend a significant amount of time in prison, but less than the maximum.

We use standard notation for PD in this thesis. We will call T the pay-off for *Temptation*, or free-riding, R the *Reward* for mutual cooperation, P the *Punishment* for mutual defection and S the *Sucker's* pay-off. These satisfy the inequalities $T > R > P > S$ and $R > \frac{T+S}{2}$. Table 1 presents the game in standard matrix form, where C notates cooperation, which in the case of the prisoners, is to not confess, and D, defection, which for the prisoners is to confess.

As seen in Table 1.1, the game is symmetric between the two players, and D dominates C, since $T > R$ and $P > S$. Hence, the only Nash equilibrium in a single round of PD is for both players to play D, getting pay-off P. However, both players can do better by both playing C to get pay-off R. The players' pursuit of their individual optimum in a single game

	C	D
C	(R,R)	(S,T)
D	(T,S)	(P,P)

Table 1.1: Pay-offs for a two-person prisoner's dilemma.

leads to the Nash equilibrium of both players defecting, which is worse than the social optimum of both players cooperating. This phenomenon has a wide range of applications in human interactions, where pursuit of individual interests leads to sub-optimal results for society. Examples include arms races between countries, harvest of natural resources, price competition between retail stores, among others.

1.2 Literature Review

Iterated prisoner's dilemma (IPD) is a long term game consisting of several rounds, where each round is a PD. The game is a subject of research, as it allows a socially optimal equilibrium to emerge under certain conditions. IPD has been studied extensively theoretically and experimentally. Axelrod (1984) has presented a theoretical framework and a series of experiments conducted in the form of competitions between computer algorithms. He finds that cooperation will be beneficial if "the future" is important enough. Axelrod continues to define and find the value of the "shadow of the future" parameter, also called the discount parameter, for which the strategy of tit-for-tat is collectively stable. Tit-for-tat is a strategy the begins by cooperating, and continues by mirroring the other player's moves, in all subsequent rounds.

$$\alpha_{\text{critical}} = \max\left(\frac{T-R}{T-P}, \frac{T-R}{R-S}\right),$$

where α is the discount parameter.

Cooperation in groups has been researched as an iterated game based on PD. Researchers use the concept of N-person IPD (NIPD). In this game N players are engaged in an IPD. Each round of the game all players play simultaneously. It may be parametrized with b/N as the reward to each player for every player that cooperates and c as the personal cost of cooperation for each player. If all cooperate then each member will gain

$$\frac{bN}{N} - c = b - c$$

It is assumed that $b > c > \frac{b}{N}$, in order to maintain the assumptions of PD (Bendor & Mookherjee, 1987). Cooperation between all is possible under certain conditions using different strategies. Research has been conducted to ascertain limits to this cooperation in groups.

Joshi (1987) studies tit-for-tat strategies in NIPD. Several strategies were explored with varying degrees of leniency towards defection. Joshi characterizes the strategies according to the proportion of cooperators in the group needed for a certain player to cooperate. The least lenient of these, the strategy requiring total cooperation by everyone in the group for continued collaboration he dubs "hard" and finds it to be stable against defection. It was found, however, that as the number of players in the game increased, the threshold for the frequency of the "hard TFT" strategy to become dominant rose. This suggests that evolution of cooperation in this scenario is more difficult in larger groups.

Bendor & Mookherjee (1987) find that in NIPD "If relative to group size n the punishment phase is sufficiently long and the members do not discount the future very heavily, there is an equilibrium in which all members cooperate every period. However, given the discount rate α there is an upper limit n^* beyond which cooperative outcome cannot be upheld." They find the value for the critical group size

$$n^* = \frac{\alpha b - b}{\alpha b - c},$$

where b/n is the contribution of cooperation by any player to the pay-off of others and c is the personal cost of cooperating. In a study of common goods problems with various utility functions, Pecorino (1999), finds similar results for cooperation being feasible, yet increasingly hard to achieve as the number of players in a game grows.

A different approach towards studying iterated games in large populations, is to treat a grid of players each interacting only with his neighbours, called spatial IPD. Applying this approach has led Nowak & Jay (1993) to conclude that the dynamics of such games are significantly different than homogeneous IPD and may be chaotic.

“Social loafing” is a term coined by social psychologists as the deterioration of cooperation as group size increases (Karau & Williams, 1993). We have described two game-theoretic approaches, the N-person iterated prisoner’s dilemma and the spatial IPD, that may offer explanations for this type of observational research.

1.3 Contribution

In this thesis a model of iterated games in finite populations is used to analyze dynamics of large populations. The model allows for specific targeting of punishment against defectors and may allow a more accurate prediction of conditions under which cooperation is beneficial. This model may offer an appropriate explanation to situations of cooperation in large groups. Specifically, we make the following contribution to the literature.

1. Develop a game-theoretic model for individuals who meet randomly in a population, where in each encounter the two individuals interact according to PD.
2. For a given set of model parameters, develop a method to mathematically identify the equilibrium states of the game-theoretic model.
3. Explain the influence of several factors on the feasibility of cooperation. These factors include (1) population size, (2) memory and mobility of individuals in the population, and (3) discount factor.
4. Explain the importance of suspicion to the robustness of cooperation.

CHAPTER 2:
The Model

Consider a finite population with N individuals. The game proceeds one round at a time, and goes on indefinitely. In each round, every member of the population is randomly paired with another member for a single round of prisoner's dilemma. In other words, each member in the population faces a steady stream of members who are randomly chosen from a fixed population with replacement. In a small population, a player may expect to see the same players again and again at small intervals, while for larger populations a player may expect to meet players few times in the foreseeable future. For each encounter between two players a standard prisoner's dilemma (PD) is played with the pay-off shown in Table 1.

In this model, the concept of a "foreseeable future" is governed by two factors: a discount factor α and a memory factor β. The discount factor α signifies that a reward 1 received in the next round is only worth α as in the present round. The memory factor β models how well each individual remembers encounters with the other individuals from the past. Specifically, if two players recognize each other at the current round, then they will recognize each other in the next round with probability β, or forget about each other in the next round with probability $1-\beta$. In other words, the number of turns before memory of an encounter between two players is lost is a geometric random variable. The memory factor β can also be interpreted as a mobility factor. In each round, with probability β, an individual in the population may move out and be replaced by another individual moving in.

The model is governed by seven parameters, the four standard parameters of a PD, two parameters for the foreseeable future and a single parameter for the population size. The size of the population will be referred to as N, future discount as α, memory retention as β and the notation for pay-offs in the PD are T, R, P and S.

2.1 Model Strategies

The model presented so far is rather general and there exists a myriad of player strategies. However, for the purpose of studying the emergence and failure of cooperation we choose to introduce four simple strategies. The pool of four strategies will be assumed to be the only options available to the N players in the population. Each player may select a strategy out of the pool and play accordingly for the remainder of the game. The four strategies are tit-for-tat (TFT), tit-for-two-tats (TF2T), suspicious-tit-for-tat (STFT) and always defection (AD). This assumption limits the generality of our results.

2.1.1 Always Defection

A strategy that defects unconditionally in every round of the game, as such it is designated *mean*. This is the dominant strategy for a single round of prisoner's dilemma.

2.1.2 Tit-For-Tat

The TFT strategy, first presented by Anatol Rapaport for PD computer tournaments held by Robert Axelrod (Axelrod, 1984), has been studied extensively and shown to be an effective strategy in many situations. The strategy is extremely simple. If it meets an opponent whom it has never played before (or doesn't remember playing before), it plays cooperate. From that point on it mirrors the opponent's previous move. This strategy is a *nice* strategy since it will never defect first (Axelrod, 1984). If both players use TFT, then cooperation occurs with each player receiving a stream of rewards.

2.1.3 Suspicious-Tit-For-Tat

This strategy is a variation of standard TFT. The difference lies in the strategy's action against an unknown opponent. This strategy will begin an interaction by defecting and will then mirrors the opponent's moves. This strategy may be said to be *mean* as it will never cooperate first. STFT is important to this paper, as we conjecture that use of *suspicion* by a population is a method for increasing the robustness of cooperation.

2.1.4 Tit-For-Two-Tats

This strategy was suggested by Robert Axelrod for the second of his computer tournaments as a solution to the problem of two retaliatory strategies (such as TFT) entering a cycle of alternating cooperation and defection (Axelrod, 1984). This *nice* strategy begins by cooperating twice and then continues to mirror it's opponents moves. In that case, it will begin to defect until the opponent cooperates once.

2.2 The Game in Standard Matrix Form

In order to examine the dynamics among the four strategies in this finite-population model, the first step is to define an objective function for each member in the population. In this thesis, we assume that the objective for each member is to maximize his expected total discounted utility. In other words, if X_t represents the utility received in round t, then a member's objective function is to maximize

$$\sum_{t=0}^{\infty} \alpha^t E[X_t]. \tag{2.1}$$

An alternative approach to compute the objective function in (2.1) is to first compute the expected total discounted utility collected through one *match-up*. A *match-up* is a sequence of encounters between two players until the two players forget about each other. In other words, if a player meets someone whom he does not recognize (an opponent he has never played before, or an opponent he has played before but forgets), then a new match-up begins. If a player meets someone he recognizes from previous encounters, then that round is part of an ongoing match-up.

Let $a_{i,j}$ denote the total expected discounted utility collected through one match-up, if a member plays strategy i against another member who plays strategy j, for $i, j \in \{\text{TF2T}, \text{TFT}, \text{STFT}, \text{AD}\}$. The matrix A is shown explicitly in the following.

$$A = \begin{array}{c} \\ \text{TF2T} \\ \text{TFT} \\ \text{STFT} \\ \text{AD} \end{array} \begin{array}{c} \begin{array}{cccc} \text{TF2T} & \text{TFT} & \text{STFT} & \text{AD} \end{array} \\ \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{pmatrix} \end{array} \tag{2.2}$$

In this matrix, $a_{2,3}$, for example, is the discounted total expected utility a TFT player collects through a match-up, if he meets a stranger who plays STFT.

Suppose the population plays a mixed strategy (q_1, q_2, q_3, q_4). That is, each time a member meets someone he does not recognize (an opponent he has never played before, or an opponent he has played before but forgets), the opponent will play strategy j with probability

q_j, for $j = 1,2,3,4$. By letting $I_t = 1$ if the member does not recognize the opponent in round t (beginning of a new match-up), and $I_t = 0$ if the member recognizes the opponent in round t (part of an ongoing match-up), then an equivalent expression for the objective function in (2.1) is

$$\sum_{t=0}^{\infty}\left(\alpha^t E[I_t]\left(\sum_{i=1}^{4} q_j a_{i,j}\right)\right) = \left(\sum_{i=1}^{4} q_j a_{i,j}\right)\left(\sum_{t=0}^{\infty}\alpha^t E[I_t]\right).$$

Since $(\sum_{t=0}^{\infty}\alpha^t E[I_t])$ is a constant regardless the strategy played by the member, to maximize the preceding it is equivalent to choose i to maximize

$$\sum_{j=1}^{4} q_j a_{i,j}.$$

Consequently, an equivalent objective function for a member is to choose a strategy that maximizes the total discounted expected utility collected in a match-up. We next explain how to compute $a_{i,j}$ in (2.2).

2.2.1 Match-up Between Two Nice Strategies

The match-up between two nice players will be repeated cooperation, as both players will never defect first. At the beginning of a round, denote the total expected discounted utility collected in the remainder of this match-up by $h(R)$, if two players will always play C, so that in each encounter a player receives R. In the present round, the two players will meet with probability $1/N$. If they meet, then each player gets a reward R; otherwise, there is no immediate reward. With probability β, the two players will remember each other in the next round, and will collect $h(R)$ in the next round, which is discounted by α. If they do not meet, with probability $1 - 1/N$, then they will just get $h(R)$ in the next round, if they remember each other in the next round. Therefore, we can set up one equation involving $h(R)$ as follows:

$$h(R) = \frac{1}{N}(R + \alpha\beta h(R)) + \left(1 - \frac{1}{N}\right)\alpha\beta h(R).$$

Solving for $h(R)$ gives

$$h(R) = \frac{R}{N(1-\alpha\beta)}.$$

The function $h(R)$ can be interpreted as the total expected, discounted utility in the remainder of the match-up, if a player receives R from each future encounter. This can then be used in the expected utility of a match-up involving only TF2T and TFT.

$$a_{1,1} = a_{2,2} = a_{2,1} = a_{1,2} = R + \alpha\beta h(R) = R + \alpha\beta \left(\frac{R}{N(1-\alpha\beta)} \right). \tag{2.3}$$

2.2.2 Match-up Between Two Mean Strategies

The match-up between mean strategies will never have cooperation introduced, so the expected utility is derived exactly like the previous section, except for the per round pay-off being P instead of R. Replacing R in equation (2.3) with P gives

$$a_{4,4} = a_{3,3} = a_{3,4} = a_{4,3} = P + \alpha\beta h(P) = P + \alpha\beta \left(\frac{P}{N(1-\alpha\beta)} \right).$$

2.2.3 Match-up Between TFT and AD

The first encounter of TFT and AD, in a match-up, will see TFT getting the sucker's pay-off, while AD gets the temptation pay-off. Every following encounter will be mutual defection, resulting in an expected pay-off of $h(P)$. Similar to equation (2.3), we can conclude that

$$a_{2,4} = S + \alpha\beta \left(\frac{P}{N(1-\alpha\beta)} \right),$$
$$a_{4,2} = T + \alpha\beta \left(\frac{P}{N(1-\alpha\beta)} \right).$$

2.2.4 Match-up Between TF2T and STFT

In the first encounter, TF2T gets S, while STFT gets T. In the second encounter and all following encounters, both strategies will get the steady stream R. Similar to equation (2.3), we can conclude that

$$a_{1,3} = S + \alpha\beta h(R) = S + \alpha\beta\left(\frac{R}{N(1-\alpha\beta)}\right).$$

Similarly, the pay-off for the STFT strategy is

$$a_{3,1} = T + \alpha\beta h(R) = T + \alpha\beta\left(\frac{R}{N(1-\alpha\beta)}\right).$$

2.2.5 Match-up Between TF2T and AD

In the case of TF2T against the AD strategy, the TF2T will allow two defections against it before the interaction collapses into constant defection on both sides. We calculate by conditioning on the number of rounds between the first encounter and second encounter in the match-up.

$$\begin{aligned} a_{1,4} &= S + \sum_{i=1}^{\infty}\left(1-\frac{1}{N}\right)^{i-1}\left(\frac{1}{N}\right)(\alpha\beta)^i a_{2,4} \\ &= S + \frac{1}{N-1}\left(S + \alpha\beta\frac{P}{N(1-\alpha\beta)}\right)\frac{\alpha\beta\left(1-\frac{1}{N}\right)}{1-\alpha\beta\left(1-\frac{1}{N}\right)}. \end{aligned}$$

The pay-off for AD in this match-up will be similar with the exception of T replacing S. That is,

$$a_{4,1} = T + \frac{1}{N-1}\left(T + \alpha\beta\frac{P}{N(1-\alpha\beta)}\right)\frac{\alpha\beta\left(1-\frac{1}{N}\right)}{1-\alpha\beta\left(1-\frac{1}{N}\right)}.$$

2.2.6 Match-up Between TFT and STFT

A match-up between TFT and STFT will always start by a defection on the side of the suspicious strategy and continue with an endless cycle of defection on alternating sides. This means that the expected utilities of the two strategies in this match-up are entwined. We use x to mark the expected value of an alternating T and S match-up starting with the sucker's pay-off S, and y for the same match-up starting with the temptation pay-off T. We condition on the number of rounds between the first encounter and second encounter to set up two equations:

$$y = T + \sum_{i=1}^{\infty} x \frac{1}{N}\left(1 - \frac{1}{N}\right)^{i-1} (\alpha\beta)^i = T + \frac{x}{N}\frac{\alpha\beta}{1 - \alpha\beta\left(1 - \frac{1}{N}\right)},$$

$$x = S + \sum_{i=1}^{\infty} y \frac{1}{N}\left(1 - \frac{1}{N}\right)^{i-1} (\alpha\beta)^i = S + \frac{y}{N}\frac{\alpha\beta}{1 - \alpha\beta\left(1 - \frac{1}{N}\right)}.$$

Solving the two equations for x and y leads to the following

$$a_{2,3} = S + \alpha\beta y = S + \alpha\beta \frac{T\left(1 - \alpha\beta + \frac{\alpha\beta}{N}\right) + S\frac{\alpha\beta}{N}}{N(1 - \alpha\beta)\left(1 - \alpha\beta + 2\frac{\alpha\beta}{N}\right)},$$

$$a_{3,2} = T + \alpha\beta x = T + \alpha\beta \frac{S\left(1 - \alpha\beta + \frac{\alpha\beta}{N}\right) + T\frac{\alpha\beta}{N}}{N(1 - \alpha\beta)\left(1 - \alpha\beta + 2\frac{\alpha\beta}{N}\right)}.$$

2.2.7 Game Matrix in Normal Form

Bringing together all the results of the previous section gives us the following game matrix in normal form.

$$A = \begin{array}{c} \\ \end{array}\begin{pmatrix} \text{TF2T} & \text{TFT} & \text{STFT} & \text{AD} \\ R + \alpha\beta\left(\frac{R}{N(1-\alpha\beta)}\right) & R + \alpha\beta\left(\frac{R}{N(1-\alpha\beta)}\right) & S + \alpha\beta\left(\frac{R}{N(1-\alpha\beta)}\right) & S + \frac{\alpha\beta\left(S + \alpha\beta\frac{P}{N(1-\alpha\beta)}\right)}{N\left(1-\alpha\beta\left(1-\frac{1}{N}\right)\right)} \\ R + \alpha\beta\left(\frac{R}{N(1-\alpha\beta)}\right) & R + \alpha\beta\left(\frac{R}{N(1-\alpha\beta)}\right) & S + \alpha\beta\frac{T\left(1-\alpha\beta+\frac{\alpha\beta}{N}\right)+S\frac{\alpha\beta}{N}}{N(1-\alpha\beta)\left(1-\alpha\beta+2\frac{\alpha\beta}{N}\right)} & S + \alpha\beta\frac{P}{N(1-\alpha\beta)} \\ T + \alpha\beta\left(\frac{R}{N(1-\alpha\beta)}\right) & T + \alpha\beta\frac{S\left(1-\alpha\beta+\frac{\alpha\beta}{N}\right)+T\frac{\alpha\beta}{N}}{N(1-\alpha\beta)\left(1-\alpha\beta+2\frac{\alpha\beta}{N}\right)} & P + \alpha\beta\left(\frac{P}{N(1-\alpha\beta)}\right) & P + \alpha\beta\left(\frac{P}{N(1-\alpha\beta)}\right) \\ T + \frac{\alpha\beta\left(T + \alpha\beta\frac{P}{N(1-\alpha\beta)}\right)}{N\left(1-\alpha\beta\left(1-\frac{1}{N}\right)\right)} & T + \alpha\beta\frac{P}{N(1-\alpha\beta)} & P + \alpha\beta\left(\frac{P}{N(1-\alpha\beta)}\right) & P + \alpha\beta\left(\frac{P}{N(1-\alpha\beta)}\right) \end{pmatrix} \quad (2.4)$$

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CHAPTER 3:
Analysis

In this thesis, we wish to find the connection between population size and other parameters of a game and the possibility of cooperation in a stable steady state. To achieve this goal, we first define a stable steady state. The common definition for such a state is the Nash equilibrium. A Nash equilibrium of a game is a profile of actions (strategy), such that each player's action is optimal against the actions of the other players. In other words, no player can increase his expected utility by switching to a different strategy, given that all the other players play their Nash equilibrium strategy (Osborne & Rubinstein, 1994).

Whereas Nash equilibrium specifies a stable state involving several players, an evolutionarily stable strategy (ESS) is a refinement of Nash equilibrium that applies to a population. Specifically, an ESS is a (possibly) mixed strategy that, if adopted by a population, cannot be invaded by other strategies (Smith, 1974). Write $E[U,V]$ for the expected utility of the strategy U in a match-up against the strategy V. We say S is an ESS if either of the following two conditions hold:

1. $E[S,S] > E[T,S]$, for all $T \neq S$; or
2. $E[S,S] = E[T,S]$ and $E[S,T] > E[T,T]$, for all $T \neq S$.

In this thesis, we seek to determine the ESS for our game-theoretic model introduced in Chapter 2, as we attempt to ascertain the conditions required for a long term cooperative equilibrium in a dynamic population. In this thesis, we focus on finding sets of game parameters that allow a cooperative evolutionary stable (mixed) strategy to emerge.

3.1 Determine Evolutionary Stable Strategy

In this section, we analyze the two-person non-zero-sum game with pay-offs given in (2.4), to determine the ESS for given parameters. To begin, first we recognize the following string of inequalities, which will be helpful in the following analysis.

$$a_{31} > a_{11} = a_{12} = a_{22} = a_{21} > a_{33} = a_{34} = a_{44} = a_{43} > a_{24} > a_{14} \tag{3.1}$$

We seek to find the ESS. In general, the population may be playing all four strategies with proportions expressed in the tuple x. Let $x = (x_{\text{TF2T}}, x_{\text{TFT}}, x_{\text{STFT}}, x_{\text{AD}})$, where $\sum_{i \in I} x_i = 1$ and $I = \{\text{TF2T}, \text{TFT}, \text{STFT}, \text{AD}\}$. This problem may have at most 15 evolutionary stable strategies, including four involving one pure strategy, six involving two pure strategies, four involving three pure strategies, and one involving all four pure strategies.

For each subset $\tilde{I} \subseteq I$, we want to determine whether there exists an ESS that involves all strategies in $\tilde{I}$ and none in $I \setminus \tilde{I}$. Write $\tilde{A}$ for the sub matrix of A by removing rows and columns not in $\tilde{I}$. If there exists an ESS that involves all strategies in $\tilde{I}$, then the population distribution among the strategies, denoted by $\tilde{x}$, must be chosen so that for each strategy in $\tilde{I}$, the expected utility against the population distribution $\tilde{x}$ must be the same. Hence, a necessary condition for $\tilde{x}$ to constitute an ESS is for $\tilde{x}$ to satisfy the following equation, where v is a constant.

$$\tilde{A}\tilde{x} = \tilde{A} \begin{pmatrix} \tilde{x}_1 \\ \vdots \\ \tilde{x}_m \end{pmatrix} = \begin{pmatrix} v \\ \vdots \\ v \end{pmatrix} \tag{3.2}$$

The solution $\tilde{x}$ to the preceding equation is only a necessary condition for $\tilde{x}$ to be an ESS, but not a sufficient condition. For $\tilde{x}$ to be an ESS, we need to verify that it will not be invaded by the strategies in any strategy $i \notin \tilde{I}$. In other words, we need to verify that the expected utility for any strategy $i \notin \tilde{I}$ against $\tilde{x}$ is strictly less than v. Mathematically, we need to verify that

$$\sum_{j \in \tilde{I}} a_{ij} \tilde{x}_j < v \tag{3.3}$$

for $i \neq \tilde{I}$.

3.2 ESS Involving One Pure Strategy

A pure strategy constitutes an ESS, if it is best to play that pure strategy when everyone else in the population plays that same pure strategy. In other words, a pure strategy is an ESS, if and only if a diagonal entry in the matrix A is the largest in its column.

3.2.1 AD Pure ESS

The AD constitutes an ESS solution for this game, since against a population of players always defecting, a single player can do no better than to defect along with everyone else.

3.2.2 TFT Pure ESS

For TFT to be a stable pure strategy, the pay-off to a player playing TFT needs to be the greatest among four strategies, given that everyone else is playing TFT. In other words, we require

$$a_{2,2} > a_{4,2} \qquad \text{and} \qquad a_{2,2} > a_{3,2}.$$

Solving for N gives two conditions for the limiting population, for which cooperation can be maintained, as long as either condition is met.

$$P+R-S-T > 0 \qquad \text{and} \qquad N < \frac{\alpha\beta}{1-\alpha\beta}\frac{R-P}{T-R},$$
$$P+R-S-T < 0 \qquad \text{and} \qquad N < \frac{\alpha\beta}{1-\alpha\beta}\frac{2R-S-T}{T-R}.$$

This means that depending on the sign of $P+R-S-T$, if the population is less than the appropriate threshold, everyone plays TFT. However, if the population should grow past the threshold then all players would immediately revert to AD.

3.2.3 STFT Pure ESS

For STFT to be a stable pure strategy, the pay-off to a player playing STFT needs to be the greatest among four strategies, given that everyone else is playing STFT. In other words, we require

$$a_{1,3} < a_{3,3} \qquad \text{and} \qquad a_{2,3} < a_{3,3}.$$

Solving for N gives two conditions for the limiting population, for which cooperation can be maintained, as long as either condition is met.

$$S+T-R-P>0 \qquad \text{and} \qquad N>\frac{\alpha\beta}{1-\alpha\beta}\frac{T+S-2P}{P-S},$$
$$S+T-R-P<0\leq\frac{S+T}{2} \qquad \text{and} \qquad N>\frac{\alpha\beta}{1-\alpha\beta}\frac{R-P}{P-S}.$$

A large population may be all playing STFT in a suspicious ESS. However, if the population decreases beyond a certain threshold, the population may be invaded by TF2T players, who will begin cooperation.

3.2.4 TF2T Pure ESS

TF2T by itself can never be a ESS since the best one player can do against TF2T is STFT, as seen by $a_{3,1} > a_{2,1}$ in (3.1).

3.3 ESS Involving Two Pure Strategies

In this section, we use equation (3.2) to determine a necessary condition on the population size N for ESSs that involves two pure strategies. There are $\binom{4}{2} = 6$ cases. Recall that we need to use equation (3.3) to determine whether an ESS does exist, which will be done in Chapter 4.

3.3.1 TF2T and TFT

For this strategy pair the game matrix becomes degenerate with all four elements of $\tilde{A}$ being equal. In this case, it is clear that any distribution of the population between the two strategies is a candidate for ESS. For each population distribution, one needs to use equation (3.3) to verify whether it is indeed an ESS.

3.3.2 STFT and AD

For this strategy pair the game matrix also becomes degenerate with all four elements of $\tilde{A}$ being equal. Thus any distribution of the population between the two strategies is a candidate for ESS. For each population distribution, one needs to use equation (3.3) to verify whether it is indeed an ESS.

3.3.3 TF2T and STFT

The game matrix for this strategy pair is

$$\tilde{A} = \begin{array}{c} \\ \text{TF2T} \\ \text{STFT} \end{array} \begin{array}{c} \begin{array}{cc} \quad\quad\text{TF2T} & \quad\quad\quad\quad\text{STFT} \end{array} \\ \begin{pmatrix} R+\alpha\beta\left(\frac{R}{N(1-\alpha\beta)}\right) & S+\alpha\beta\left(\frac{R}{N(1-\alpha\beta)}\right) \\ T+\alpha\beta\left(\frac{R}{N(1-\alpha\beta)}\right) & P+\alpha\beta\left(\frac{P}{N(1-\alpha\beta)}\right) \end{pmatrix} \end{array}$$

Since $\tilde{a}_{11} < \tilde{a}_{21}$, if $\tilde{a}_{12} \leq \tilde{a}_{22}$, then STFT dominates TF2T. In that case an ESS does not exist with only these two strategies active. As a necessary condition for an ESS for the population with TF2T and STFT strategies active, we solve $\tilde{a}_{12} > \tilde{a}_{22}$, or equivalently,

$$S+\alpha\beta\frac{R}{N(1-\alpha\beta)} > P+\alpha\beta\frac{P}{N(1-\alpha\beta)}.$$

Solving for a threshold population value gives a necessary condition, for which an ESS *may* exist as the following

$$N < \frac{\alpha\beta}{1-\alpha\beta}\frac{R-P}{P-S}. \tag{3.4}$$

Below this threshold there may be an ESS with some in the population playing TF2T and the rest playing STFT. Above the threshold the entire population would revert to playing STFT, as suspicion will be their best option. We emphasize that equation (3.4) is only a necessary condition for an ESS, and one needs to use equation (3.3) to verify whether it is indeed an ESS.

3.3.4 TFT and STFT

The game matrix for this strategy pair is

$$\tilde{A} = \begin{array}{c} \\ \text{TFT} \\ \text{STFT} \end{array} \begin{array}{c} \begin{array}{cc} \quad\quad\text{TFT} & \quad\quad\quad\quad\quad\quad\text{STFT} \end{array} \\ \begin{pmatrix} R+\alpha\beta\left(\frac{R}{N(1-\alpha\beta)}\right) & S+\alpha\beta\frac{T\left(1-\alpha\beta+\frac{\alpha\beta}{N}\right)+S\frac{\alpha\beta}{N}}{N(1-\alpha\beta)\left(1-\alpha\beta+2\frac{\alpha\beta}{N}\right)} \\ T+\alpha\beta\frac{S\left(1-\alpha\beta+\frac{\alpha\beta}{N}\right)+T\frac{\alpha\beta}{N}}{N(1-\alpha\beta)\left(1-\alpha\beta+2\frac{\alpha\beta}{N}\right)} & P+\alpha\beta\left(\frac{P}{N(1-\alpha\beta)}\right) \end{pmatrix} \end{array}$$

For an ESS with both TFT and STFT to be active at the same time there can be two cases:

1. $\tilde{a}_{1,1} > \tilde{a}_{2,1}$ and $\tilde{a}_{1,2} < \tilde{a}_{2,2}$. This can be written in the following two inequalities.

$$R + \alpha\beta\left(\frac{R}{N(1-\alpha\beta)}\right) > T + \alpha\beta\frac{S\left(1-\alpha\beta+\frac{\alpha\beta}{N}\right)+T\frac{\alpha\beta}{N}}{N(1-\alpha\beta)\left(1-\alpha\beta+2\frac{\alpha\beta}{N}\right)}$$

and

$$P + \alpha\beta\left(\frac{P}{N(1-\alpha\beta)}\right) > S + \alpha\beta\frac{T\left(1-\alpha\beta+\frac{\alpha\beta}{N}\right)+S\frac{\alpha\beta}{N}}{N(1-\alpha\beta)\left(1-\alpha\beta+2\frac{\alpha\beta}{N}\right)}.$$

These can be simplified to give the following necessary condition for ESS.

$$T+S-R < P < \frac{S+T}{2} \quad \text{and} \quad \frac{\alpha\beta}{1-\alpha\beta}\frac{T+S-2P}{P-S} < N < \frac{\alpha\beta}{1-\alpha\beta}\frac{2R-T-S}{T-R},$$

$$P > \frac{S+T}{2} \quad \text{and} \quad N < \frac{\alpha\beta}{1-\alpha\beta}\frac{2R-T-S}{T-R}.$$

2. $\tilde{a}_{1,1} < \tilde{a}_{2,1}$ and $\tilde{a}_{1,2} > \tilde{a}_{2,2}$. These are the same inequalities, with the inequality signs reversed. This is simplified to find the following necessary condition on an ESS.

$$P < T+S-R \quad \text{and} \quad \frac{\alpha\beta}{1-\alpha\beta}\frac{2R-S-T}{T-R} < N < \frac{\alpha\beta}{1-\alpha\beta}\frac{T+S-2P}{P-S}.$$

3.3.5 TF2T and AD

The game matrix for this strategy pair is

$$\tilde{A} = \begin{array}{c} \\ \text{TF2T} \\ \text{AD} \end{array} \begin{array}{c} \begin{array}{cc} \text{TF2T} & \text{AD} \end{array} \\ \begin{pmatrix} R+\alpha\beta\left(\frac{R}{N(1-\alpha\beta)}\right) & S+\frac{\alpha\beta\left(S+\alpha\beta\frac{P}{N(1-\alpha\beta)}\right)}{N\left(1-\alpha\beta\left(1-\frac{1}{N}\right)\right)} \\ T+\frac{\alpha\beta\left(T+\alpha\beta\frac{P}{N(1-\alpha\beta)}\right)}{N\left(1-\alpha\beta\left(1-\frac{1}{N}\right)\right)} & P+\alpha\beta\left(\frac{P}{N(1-\alpha\beta)}\right) \end{pmatrix} \end{array}$$

Since $\tilde{a}_{12} < \tilde{a}_{22}$, if $\tilde{a}_{11} \leq \tilde{a}_{21}$ also holds, then AD dominates TF2T. Therefore, a mixed strategy ESS is possible only if $\tilde{a}_{11} > \tilde{a_{21}}$, or equivalently,

$$R+\alpha\beta\left(\frac{R}{N(1-\alpha\beta)}\right) > T + \frac{\alpha\beta\left(T+\alpha\beta\frac{P}{N(1-\alpha\beta)}\right)}{N\left(1-\alpha\beta\left(1-\frac{1}{N}\right)\right)}.$$

Solving for a threshold population value gives a necessary condition that an ESS may exist.

$$N < \frac{\alpha\beta}{1-\alpha\beta}\left(\sqrt{\frac{T-P}{T-R}}-1\right).$$

In this case an ESS *may* exist with some playing AD, while the rest play TF2T. Above the threshold, the entire population reverts to playing AD.

3.3.6 TFT and AD

The game matrix for this strategy pair is

$$\tilde{A} = \begin{array}{c} \\ \text{TFT} \\ \text{AD} \end{array} \begin{array}{c} \begin{array}{cc} \text{TFT} & \text{AD} \end{array} \\ \begin{pmatrix} R+\alpha\beta\frac{R}{N(1-\alpha\beta)} & S+\alpha\beta\frac{P}{N(1-\alpha\beta)} \\ T+\alpha\beta\frac{P}{N(1-\alpha\beta)} & P+\alpha\beta\left(\frac{P}{N(1-\alpha\beta)}\right) \end{pmatrix} \end{array}$$

Since $\tilde{a}_{12} < \tilde{a}_{22}$, an ESS with both strategies may exist only when $\tilde{a}_{11} > \tilde{a}_{21}$, which results in the following condition for the population size.

$$N < \frac{\alpha\beta}{1-\alpha\beta}\frac{R-P}{T-R}.$$

Similar to the previous sections, this equation gives a threshold below which cooperation is preferred to defection. We may expect the population to have an ESS in which at least a certain proportion are cooperating by using TFT. Above the threshold, all will revert to playing AD.

3.4 ESS Involving Three Pure Strategies

The full game matrix A in (2.4) contains two 2x2 sub-matrices along the main diagonal whose elements are all equal. Removing any strategy from the game will leave a 3x3 matrix, which we call $\tilde{A}$. The sub-matrix $\tilde{A}$ contains at least one of the two degenerate sub-matrices. A three-strategy ESS can be represented by probability vector $\tilde{x} \in \mathbb{R}^3$ whose sum is 1. According to equation (3.2), the probabilities must satisfy the condition that

$$\tilde{A}\tilde{x} = \begin{pmatrix} v \\ v \\ v \end{pmatrix}, \tag{3.5}$$

for some $v \in \mathbb{R}$. In particular, the two rows that contain the degenerate 2x2 sub-matrix must be equal to each other when weighted by $\tilde{x}$. Consider an example by removing the AD strategy.

$$\tilde{A} = \begin{array}{c} \\ \\ \end{array} \overset{\quad\text{TF2T}\qquad\qquad\qquad\text{TFT}\qquad\qquad\qquad\text{STFT}}{\begin{pmatrix} R+\alpha\beta\left(\frac{R}{N(1-\alpha\beta)}\right) & R+\alpha\beta\left(\frac{R}{N(1-\alpha\beta)}\right) & S+\alpha\beta\left(\frac{R}{N(1-\alpha\beta)}\right) \\ R+\alpha\beta\left(\frac{R}{N(1-\alpha\beta)}\right) & R+\alpha\beta\left(\frac{R}{N(1-\alpha\beta)}\right) & S+\alpha\beta\frac{T\left(1-\alpha\beta+\frac{\alpha\beta}{N}\right)+S\frac{\alpha\beta}{N}}{N(1-\alpha\beta)\left(1-\alpha\beta+2\frac{\alpha\beta}{N}\right)} \\ T+\alpha\beta\left(\frac{R}{N(1-\alpha\beta)}\right) & T+\alpha\beta\frac{S\left(1-\alpha\beta+\frac{\alpha\beta}{N}\right)+T\frac{\alpha\beta}{N}}{N(1-\alpha\beta)\left(1-\alpha\beta+2\frac{\alpha\beta}{N}\right)} & P+\alpha\beta\left(\frac{P}{N(1-\alpha\beta)}\right) \end{pmatrix}}$$

The equality between the first two rows of the condition expressed in (3.5) is then

$$\tilde{x}_1\tilde{a}_{1,1} + \tilde{x}_2\tilde{a}_{1,2} + \tilde{x}_3\tilde{a}_{1,3} = \tilde{x}_1\tilde{a}_{2,1} + \tilde{x}_2\tilde{a}_{2,2} + \tilde{x}_3\tilde{a}_{2,3}.$$

But since $\tilde{a}_{1,1} = \tilde{a}_{1,2} = \tilde{a}_{2,1} = \tilde{a}_{2,2}$, the preceding simplifies to

$$\tilde{a}_{1,3} = \tilde{a}_{2,3}.$$

The result is simply an *equality* that can be solved for N as a function of $\alpha, \beta, T, R, P, S$. An ESS with any three of the strategies, at best, may only exist for a precise population size, making it extremely unstable with relation to population changes.

3.5 ESS Involving Four Pure Strategies

An ESS with all four strategies active at the same time may exist only if there is no dominance between the rows in matrix A. Due to the degeneracy in the two 2x2 sub-matrices and the fact that $a_{2,4} > a_{1,4}$, one of the following must hold for there to be no dominance between rows.

1. $a_{1,3} > a_{2,3}$ and $a_{3,2} > a_{4,2}$ and $a_{4,1} > a_{3,1}$. These inequalities do not result in any feasible solutions for N.
2. $a_{1,3} > a_{2,3}$ and $a_{3,2} < a_{4,2}$ and $a_{4,1} < a_{3,1}$. With some algebra, we can show that an ESS involving all four pure strategies may exist if either of the following two equations is true.

$$T+S-R < P < \frac{S+T}{2} \qquad \text{and} \qquad \frac{\alpha\beta}{1-\alpha\beta}\frac{T+S-2P}{P-S} < N < \frac{\alpha\beta}{1-\alpha\beta}\frac{R-P}{T-R},$$

$$P > \frac{S+T}{2} \qquad \text{and} \qquad N < \frac{\alpha\beta}{1-\alpha\beta}\frac{R-P}{T-R}.$$

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CHAPTER 4:
Case Studies

In this section, we present a numerical analysis of the ESS for several case studies. For each case study, we choose the four pay-off values T, R, P, S, as defined in Table 1.1. Since we obtain an equivalent game when shifting these pay-off values by the same amount, or scaling these pay-off values by the same positive constant, for any set of T, R, P, S we can always normalize these values so that $R = 1$ and $P = 0$. The temptation pay-off $T > R = 1$ specifies how much it pays to *cheat*, and the pay-off $S < P = 0$ specifies how much it hurts to get *cheated*.

4.1 Case Study with Large Temptation and Large Sucker's Payoff

In this example, we use the following parameters for the pay-off of the basic PD game:

$$T = 1.14,$$
$$R = 1,$$
$$P = 0,$$
$$S = -0.29.$$

The temptation pay-off T is substantially higher than R, and the sucker pay-off S is substantially lower than P.

For these parameters, we use equations (3.2) and (3.3) to determine the ESS, based on the population size N and the compound discount factor $\alpha\beta$. The result is shown in Figure 4.1. It may be noted, in this case, that the population in which some type of collaborative ESS exists tends to be small.

For example, consider a standard army platoon of $N = 42$ men with a compound discount factor of $\alpha\beta = .95$. The platoon will be in a stable ESS of cooperation based on all the men playing TFT, in the bottom-right yellow region of 4.1. However, should the replacement rate grow, perhaps due to a low retention rate, then β decreases. The value of the future will then be discounted more heavily. As $\alpha\beta < .934$, a phase shift will occur, which will introduce suspicion and defection into the platoon.

At this point, we may expect that some men will request a transfer out of the platoon, as the ESS is rather poor to live with. As a result, the future may be further discounted. The increase in β, in turn, may bring an increase in the proportion of men playing AD until an ESS of constant defection is reached. This case study shows the importance of a low mobility rate in small, cooperative groups.

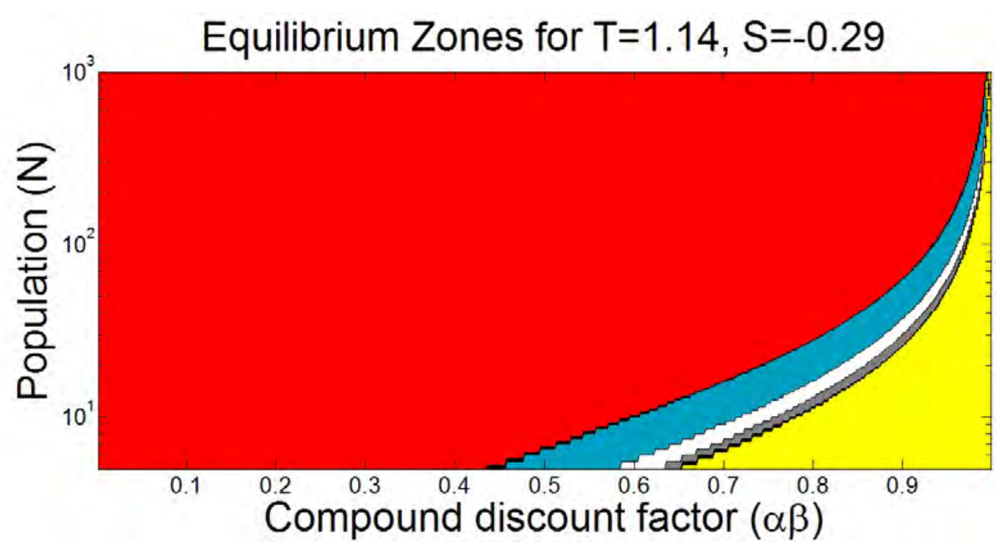

Figure 4.1: The potential ESS zones corresponding to the case of T=1.14, R=1, P=0 and S=-0.29. The red area represents combinations of N and $\alpha\beta$, for which the only stable ESS is constant defection by everyone in the population. The blue, next on the way towards the bottom right corner of the chart, represents an area in which TFT and AD are in a mixed ESS or TFT exists as a pure strategy ESS. In addition to those two states, the white area is a region in which all 4 strategies are an ESS. In the grey, a STFT and TF2T mixed equilibrium exists, as well. In the bottom-right yellow region only a pure TFT or a STFT-TF2T mix occurs.

4.2 Case Study with Small Temptation and Large Sucker's Payoff

In this example, we use the following parameters for the pay-off of the basic PD game.

$$T = 1.02,$$
$$R = 1,$$
$$P = 0,$$
$$S = -0.29.$$

The temptation pay-off T is slightly higher than R, and the sucker pay-off S is substantially lower than P.

For these parameters, we calculate what type of ESS are possible depending on the population size and on the compound discount factor $\alpha\beta$. The result is shown in Figure 4.2. It may be noted, in this case, that the population in which some type of collaborative ESS exists tends to be large.

For example, we consider a town with a population of $N = 500$ and a discount factor of $\alpha\beta = .996$ existing in an ESS of everyone playing TFT. As long as the town does not grow and the population remains stationary, there is no reason for anyone in the town to behave in any way other than cooperative.

Should the population of the town ever increase past $N = 640$ the system may enter an ESS with all four strategies present. While some people will continue to play TFT, some people may see the opportunity to play STFT or AD to obtain the temptation pay-off T. If a substantial fraction of people play STFT, then some people may become motivated to play TF2T in order to reach cooperation. The presence of TF2T in the population will also make AD more attractive, for it will produce two temptation pay-offs at the beginning against a TF2T player. If the population continues to increase, the benefit of playing STFT and AD will also increase. In the end, everyone will play either STFT or AD and no cooperation is possible.

An increase in population mobility, even without changing population size, will reduce the value of the future as it is modeled by a reduced memory retention, causing a decrease in the value of the future and thus a change in ESS. If the value of the discount factor is reduced below .996 the only ESS position for the population will be AD by everyone. If that happens we may expect the town to be abandoned by many citizens, until at some point the population will be reduced enough, to cause another phase change back into the fold of cooperation.

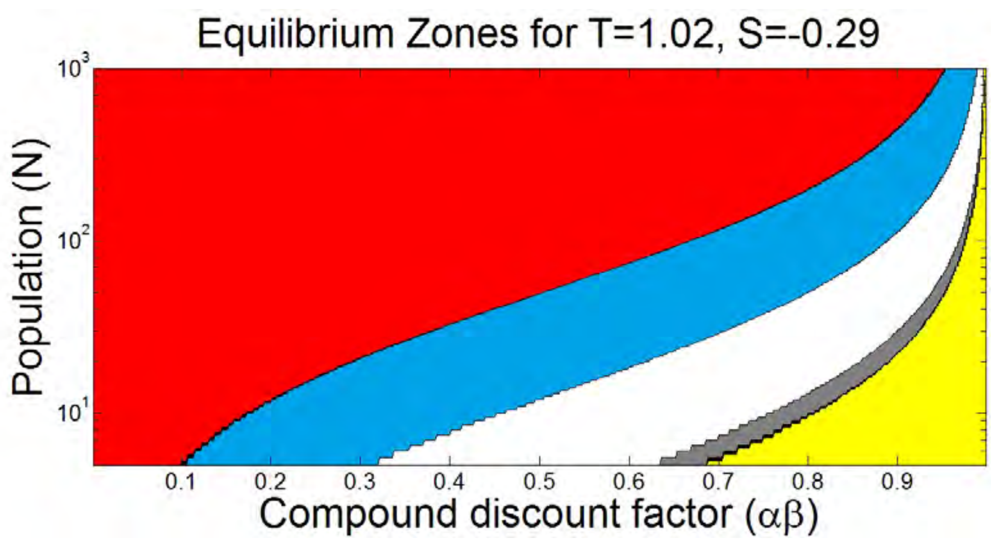

Figure 4.2: The potential ESS zones corresponding to the case of T=1.02, R=1, P=0 and S=-0.29. The red area represents combinations of N and $\alpha\beta$, for which the only stable ESS is constant defection by everyone in the population. The blue, next on the way towards the bottom right corner of the chart, represents an area in which TFT and AD are in a mixed ESS or TFT exists as a pure strategy ESS. In addition to those two states, the white area is a region in which all 4 strategies are an ESS. In the grey, a STFT and TF2T mixed equilibrium exists, as well. In the bottom-right yellow region only a pure TFT or a STFT-TF2T mix occurs.

4.3 Case Study with Small Temptation and Small Sucker's Payoff

In this example, we use the following parameters for the pay-off of the basic PD game.

$$T = 1.02,$$
$$R = 1,$$
$$P = 0,$$
$$S = -0.01.$$

The temptation pay-off T is slightly higher than R, and the sucker pay-off S is slightly lower than P.

For these parameters, we calculate what type of ESS are possible depending on the population size and on the compound discount factor $\alpha\beta$. The result is shown in Figure 4.3. It may be noted, in this case, that the most robust ESS involves the suspicious strategy, STFT.

We consider a start-up company of $N = 50$ employees. Since the future of the enterprise is uncertain we use $\alpha\beta = 0.7$. The managers have managed to raise the required funds to significantly expand their business. They may hire 100 additional employees over the course of a short period of time. As a result, the population of the company would be $N = 150$, and the unanimous cooperation that gave the group their edge, will become an ESS of the suspicious strategy with TFT. If the team manages to pull through the expansion phase, their future may become more certain, bringing the value of the future parameters up. If $\alpha\beta > 0.76$ then cooperation based on pure TFT would once more become the norm.

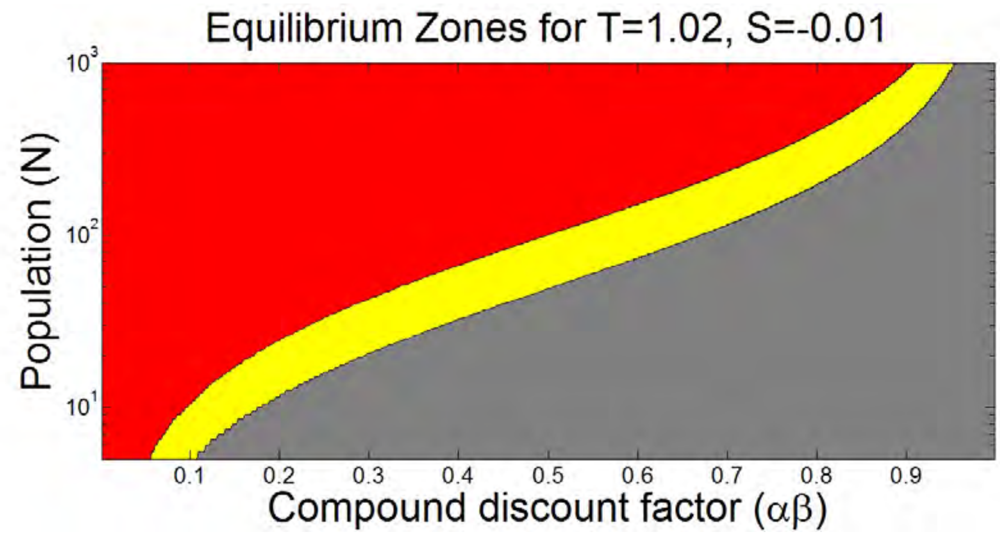

Figure 4.3: The potential ESS zones corresponding to the case of T=1.02, R=1, P=0 and S=-0.01. The red area represents combinations of N and $\alpha\beta$, for which the only stable ESS is constant defection by everyone in the population. The yellow, next on the way towards the bottom right corner of the chart, represents an area in which TFT and STFT are in a mixed ESS. In the grey region, a STFT and TF2T mixed equilibrium exists, as well as a pure strategy ESS of TFT.

4.4 Case Study with Large Temptation and Small Sucker's Payoff

In this example, we use the following parameters for the pay-off of the basic PD game.

$$T = 1.14,$$
$$R = 1,$$
$$P = 0,$$
$$S = -0.01.$$

The temptation pay-off T is substantially higher than R, and the sucker pay-off S is slightly lower than P.

For these parameters, we calculate what type of ESS are possible depending on the population size and on the compound discount factor $\alpha\beta$. With these parameters the region of suspicious ESS is clearly much larger than in the previous section, as seen by a comparison of Figure 4.4 and Figure 4.3.

We return to consider the start-up company from the previous section. A group operating under these pay-offs would expect to become suspicious at a much earlier phase of recruitment. For a value of compound discount factor of $\alpha\beta = 0.7$ even a population of $N > 14$ would begin to exhibit suspicion. However, this set of parameters is more robust with respect to complete loss of cooperation. It can be seen that this situation may support a suspicious collaboration even for a group as large as $N = 250$. This extension of cooperation by use of suspicion may be of advantage in certain situations.

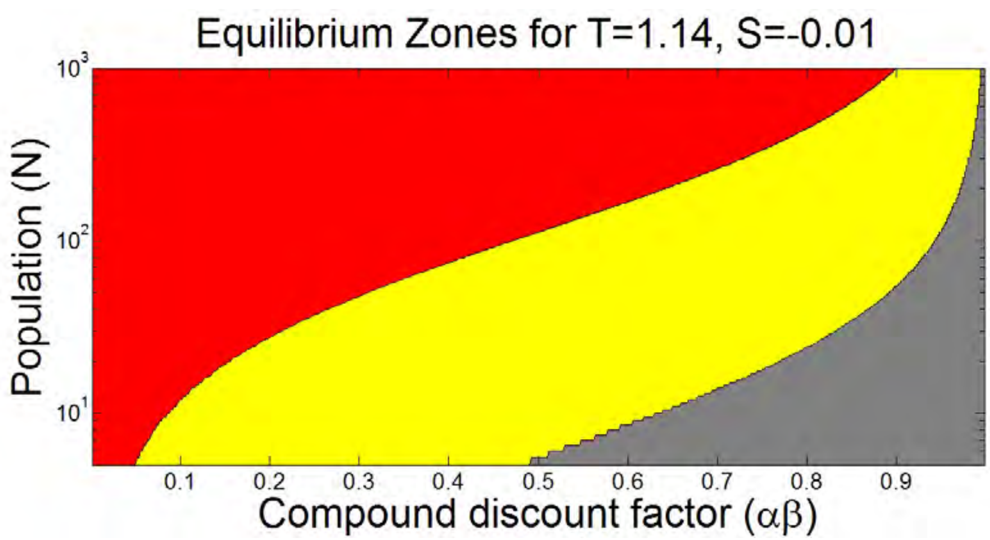

Figure 4.4: The potential ESS zones corresponding to the case of T=1.14, R=1, P=0 and S=-0.01. The red area represents combinations of N and $\alpha\beta$, for which the only stable ESS is constant defection by everyone in the population. The yellow, next on the way towards the bottom right corner of the chart, represents an area in which TFT and STFT are in a mixed ESS. In the grey region, a STFT and TF2T mixed equilibrium exists, as well as a pure strategy ESS of TFT.

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CHAPTER 5:
Conclusion

Research has shown that cooperation is constrained with relation to group size. In this thesis, we present a model for interaction in a finite population of size N, engaged in iterative rounds of prisoner's dilemma. The model allows us to explore conditions for the viability of cooperative equilibrium. We are also able to explain the importance of suspicion, as a form of behavior that inhibits uncooperative behavior, under certain conditions. Among the factors used to explain conditions for cooperation, a model for memory in the population is presented, as a geometric random variable, which can also be used to model mobility of individuals in the population.

Several strategies are used by the population in the model. Tit-for-two-tats, tit-for-tat, suspicious-tit-for-tat and constant defection allow a study of some social phenomenon. There are 15 different possible combinations of strategies, each involving either one, two, three or four pure strategies, that may be prevalent in the population. We find the different necessary conditions for combinations of strategies to become equilibrium states, using for this purpose the concept of an evolutionarily stable strategy (ESS).

For any parameters of the game an upper limit on population size was found, above which, the only ESS is constant defection by everyone in the population. All the different conditions found for a population size, that allow a collaborative ESS, include a term that depends on the compound value of the future, $\alpha\beta$:

$$N \propto \frac{\alpha\beta}{1-\alpha\beta}.$$

This shows that increased population mobility results in significant lowering of viability of cooperation. The different conditions on population, also depend on a ratio of differences, of the pay-offs, of the single round prisoner's dilemma. For some sets of parameter, the collaborative ESS, that remains viable in the largest population, employs the suspicious STFT strategy. This justifies the existence of suspicion in observed behavior, as a means to ensure cooperation remains viable, despite temptation to defect.

In this thesis, several assumptions may not be entirely realistic. First, each player has to pick a strategy, which he uses for the whole game, from a set of four options, whereas in real-life people may be more creative in choosing a strategy. Second, we use the geometric distribution to model an individual's memory, and also for the duration of time an individual stays in the population before moving out. Third, we assume that the population is homogeneous; that is, every individual has the same value functions. In real-life, however, the pay-off for cooperation may be higher for some people than for some other people. Fourth, we assume that each individual will encounter every other individual in the population with the same probability, whereas in real-life neighbors may encounter more frequently. Relaxing these assumptions presents challenging future research directions.

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