ON NUMBERS OF THE FORM $p + 2^n - n$[*]

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Abstract

Here, we show that the set of positive integers of the form $p + 2^n - n$ where p is prime
has a positive lower asymptotic density, thus answering a question of Z.-W. Sun.

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1. Introduction

In 1934, Romanoff [3] showed that the set $\mathcal{R}$ of integers of the form $p + 2^n$ where p is
a prime has positive lower asymptotic density. That is, if we put $\mathcal{R}(x) = \{p + 2^n \leq x\}$, then

$$\liminf_{x \to \infty} \frac{\#\mathcal{R}(x)}{x} > 0.$$

In 2010, Lee [2] studied a variant of Romanoff's problem with the powers of 2 replaced by
the Fibonacci numbers. With $\{F_n\}_{n \geq 1}$ being the Fibonacci sequence given by $F_1 = 1$, $F_2 = 1$
and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 1$, Lee proved that the set of positive integers of the form
$p + F_n$ has a positive lower density. A far reaching generalization of the above two results
was given by Ballot and Luca in [1], where it is shown that if $\{u_n\}_{n \geq 1}$ is a non-degenerate
integral linear recurrent sequence whose characteristic polynomial has simple roots, then
the set of positive integers of the form $p + u_n$ for some prime p and integer $n \geq 1$ has a
positive lower asymptotic density. In the above statement, it is further assumed that if the
recurrence is of order 1 (that is, a geometrical progression) then its only root is of absolute
value larger than 1, just to avoid the case of a bounded sequence $\{u_n\}_{n \geq 1}$ for which the
conclusion would not hold.

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Here, we show that the set of positive integers of the form p+2n????n where p is prime has a positive lower asymptotic density, thus answering a question of Z.-W. Sun.

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The method of attack in the above three results is the same. Namely, for a sequence $\{w_n\}_{n\geq 1}$ of positive integers and a positive integer m put

$$r_w(m) = \#\{(p,n) : m = p + w_n\}.$$

Clearly, m is of the form $p + w_n$ for some prime p and positive integer n if and only if $r(m) \neq 0$ (for simplicity, we write $r(m)$ instead of $r_w(m)$). So, we need to show that we have $\#\{m \leq x : r(m) \neq 0\} \gg x$ for sufficiently large x. Via Cauchy-Schwartz inequality, we have

$$\left(\sum_{m\leq x} r(m)\right)^2 \leq \left(\sum_{\substack{m\leq x \\ r(m)\neq 0}} 1\right)\left(\sum_{m\leq x} r(m)^2\right), \tag{1.1}$$

and we need a lower bound on the first factor in the right–hand side. This we obtain from (1.1) provided that we have a lower bound on the left–hand side and an upper bound on the second factor on the right–hand side.

For the left–hand side, we assume that $\{w_n\}_{n\geq 1}$ has exponential growth; that is, that $\rho_1^n < w_n < \rho_2^n$ holds for all $n > n_0$ with some $1 < \rho_1 < \rho_2$. Then

$$\begin{aligned}
\sum_{m\leq x} r(m) &\geq& \#(\{p : p \leq x/2\} \times \{m : w_m \leq x/2\}) \\
&\geq& \#(\{p : p \leq x/2\} \times \{m > n_0 : \rho_2^m \leq x/2\}) \\
&\geq& \pi(x/2)\left(\lfloor \log(x/2)/\log\rho_2\rfloor - n_0\right) \gg x.
\end{aligned}$$

For the right–hand side, assume further that $\{w_n\}_{n\geq 1}$ is linearly recurrent. It is then shown in all the above works that

$$\sum_{m\leq x} r(m)^2 \ll x \sum_{1\leq n\leq x} \frac{\mu(n)^2}{nz_w(n)}, \tag{1.2}$$

where $z_w(n)$ is the following function defined on the set of squarefree numbers:

(i) In case $w_n = 2^n$, then $z_w(n) = \ell_2(n/\gcd(2,n))$ where $\ell_2(m)$ is the order of 2 modulo m. In particular, if n is squarefree, then $z(n) = \ell_2(n)$ if n is odd and $z(n) = \ell_2(n/2)$ if n is even.

(ii) In case $w_n = F_n$, then $z_w(n) = \max\{e_p(F) : p \mid n\}$, where for a prime p, $e_p(F)$ denotes the length of the period of the Fibonacci sequence modulo p.

(iii) In case $\{w_n\}_{n\geq 1}$ is non-degenerate linearly recurrent and its characteristic polynomial has only simple roots, let $k(p)$ denote the period of $\{w_n\}_{n\geq 1}$ modulo p for a prime p and for any integer y let us put $\nu(p,y) = \#\{1 \leq m \leq k(p) : w_n \equiv y \pmod{p}\}$ for the frequency with which w_n represents y modulo p as n runs through an entire period modulo p. Put $\nu_p = \max\{\nu(p,y) : y = 1,2,\ldots,p\}$ and put $z(p) = k(p)/\nu_p$. Finally, for a squarefree n, let $z(n) = \max\{z(p) : p \mid n\}$.

So, in all three cases, the conclusion was reached by proving that

$$\sum_{n\geq 1} \frac{\mu(n)^2}{nz_w(n)} = O(1). \tag{1.3}$$

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In the general approach of Ballot and Luca, the function $z_w(n)$ is defined in the way explained at *(iii)* above for all linearly recurrent sequences $\{w_n\}_{n\geq 1}$, and then estimate (1.2) holds. Ballot and Luca also proved that estimate (1.3) holds for all non degenerate linearly recurrent sequences w_n whose characteristic polynomial has only simple roots. So, even if $\{w_n\}_{n\geq 1}$ does not have only simple roots, one can still obtain the same conclusion as in the theorems of Romanoff, Lee, as well as, Ballot and Luca concerning the set of numbers of the form $p + w_n$ provided that one can prove the convergence of series (1.3) for that particular sequence $\{w_n\}_{n\geq 1}$.

This is what we do in the remaining of this note for the sequence of general term $w_n = 2^n - n$ and thus obtain the following result.

Theorem 1.1. *The set of integers of the form $\{p + 2^n - n\}$ (where p is a prime number) is of positive lower density.*

2. Proof of Theorem 1.1

2.1. Preliminaries

Here, we make explicit some of the parameters $k(p)$ and $z_w(p)$ for the sequence of general term $w_n = 2^n - n$.

Lemma 2.1. *We have $k(2) = 2$ and $k(p) = \ell_2(p)p$ for all odd primes p.*

Proof. The fact that $k(2) = 2$ it is clear since $2^n - n \equiv n \pmod 2$ for all $n \geq 1$. As for $k(p)$ for an odd p, let k be such that $w_{n+k} \equiv w_n \pmod p$ for all $n \geq 1$. Making $n = 1, 2$ above we get $2^{k+1} - (k+1) \equiv 1 \pmod p$ and $2^{k+2} - (k+2) \equiv 2 \pmod p$. Writing then $a := 2^{k+1}$, $b := k+1$, we get $a - b \equiv 1 \pmod p$ and $2a - b - 1 \equiv 2 \pmod p$. Subtracting the above relations we get $a \equiv 2 \pmod p$, so $2^{k+1} \equiv 2 \pmod p$, therefore $2^k \equiv 1 \pmod p$. This shows that $\ell_2(p) \mid k$. Since now we know that $a \equiv 2 \pmod p$, we get $b \equiv 1 \pmod p$, so $k+1 \equiv 1 \pmod p$, therefore $p \mid k$. Since $\ell_2(p) \mid p - 1$, we have that $\ell_2(p)$ and p are coprime, so $\ell_2(p)p \mid k$. Conversely, it is easy to see that $w_{n+\ell_2(p)p} \equiv w_n \pmod p$ for all $n \geq 1$, so indeed $k(p) = \ell_2(p)p$. $\square$

The following is the analogue of Lemma 10 in [1].

Lemma 2.2. *The diophantine equation*

$$\begin{vmatrix} 1 & 0 & 1 \\ 2^{x_1} & x_1 & 1 \\ 2^{x_2} & x_2 & 1 \end{vmatrix} = 0,$$

has no integer solutions $x_2 > x_1 \geq 1$.

Proof. Expanding the determinant, we get the equation $2^{x_1}(2^{x_2-x_1}x_1 - x_2) = x_1 - x_2$. Thus, $2^{x_1} < x_2$. Therefore

$$2^{x_2-x_1}x_1 - x_2 \geq \frac{2^{x_2}}{2^{x_1}} - x_2 > \frac{2^{x_2}}{x_2} - x_2 \geq 0 \qquad (x_2 \geq 4),$$

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where the last inequality follows because $2^{x_2} \geq x_2^2$ holds for $x_2 \geq 4$, a fact that can be proved by induction. Thus, $2^{x_1}(2^{x_2-x_1}x_1 - x_2) \geq 0$ cannot equal $x_1 - x_2$ which is negative for $x_2 \geq 4$. It follows that $1 \leq x_1 < x_2 \leq 3$, and these cases can be checked by a computation. $\qquad\square$

Let us now define a prime p to be *bad* if

$$\nu_p > 2\lfloor(\ell_2(p)p)^{6/7}\rfloor + 2.$$

Put $\mathcal{P} = \{p : p \text{ bad}\}$ and for a positive real number x let $\mathcal{P}(x) = \mathcal{P} \cap [1,x]$.

Lemma 2.3. *We have*

$$\#\mathcal{P}(x) \ll x^{6/7}. \tag{2.4}$$

In particular,

$$\sum_{p \in \mathcal{P}} \frac{\log p}{p} \tag{2.5}$$

is finite.

Proof. Let $p \in \mathcal{P}(x)$ and y be such that $\nu_p = \nu(p,y)$. Embed the interval $[0, \ell_2(p)p)$ into the union of disjoint intervals

$$I_i := \left[i(\ell_2(p)p)^{1/7}, (i+1)(\ell_2(p)p)^{1/7}\right) \qquad \text{for} \quad i = 0, 1, \ldots, K,$$

where $K := \lfloor(\ell_2(p)p)^{6/7}\rfloor$. Since $\nu_p > 2K + 2$, one of these intervals, say I_i, will contain three solutions $n_1 < n_2 < n_3$ to the congruence $w_{n_i} \equiv y \pmod{p}$. We write $n_2 = n_1 + x_1$, $n_3 = n_1 + x_2$ for some integers $1 \leq x_1 < x_2 \leq (\ell_2(p)p)^{1/7}$. Note that $(\ell_2(p)p)^{1/7} < x^{2/7}$. The above congruences can be written as

$$\begin{aligned}
2^{n_1} \cdot 1 + (-1)0 - (n_1 + y) \cdot 1 &\equiv 0 \pmod{p}; \\
2^{n_1} \cdot 2^{x_1} + (-1)x_1 - (n_1 + y) \cdot 1 &\equiv 0 \pmod{p}; \\
2^{n_1} \cdot 2^{x_2} + (-1)x_2 - (n_1 + y) \cdot 1 &\equiv 0 \pmod{p}.
\end{aligned}$$

In particular, the vector $\mathbf{x} = (2^{n_1}, -1, -(n_1 + y))^T$ is a solution to the homogeneous system $A\mathbf{x} \equiv \mathbf{0} \pmod{p}$, where

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 2^{x_1} & x_1 & 1 \\ 2^{x_2} & x_2 & 1 \end{pmatrix}.$$

Let $D(x_1, x_2)$ be determinant of the above matrix A. By Lemma 2.2, we have that $D(x_1, x_2) \neq 0$. Thus, p divides the nonzero integer $D(x_1, x_2)$. It then follows that

$$\prod_{p \in \mathcal{P}(x)} p \ \bigg| \prod_{1 \leq x_1 < x_2 \leq x^{2/7}} |D(x_1, x_2)|.$$

By Hadamard's inequality,

$$|D(x_1, x_2)| \leq 2^{1/2}(2^{2x_1} + x_1^2 + 1)^{1/2}(2^{2x_2} + x_2^2 + 1)^{1/2} < 2^{2x_2+1} < 3^{2x_2},$$

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where we used the fact that $2^{2m} \geq m^2 + 1$ holds for all positive integers m. Since $2 \notin \mathcal{P}$, we get that

$$3^{\#\mathcal{P}(x)} \leq \prod_{p \in \mathcal{P}(x)} p \leq \prod_{1 \leq x_1 < x_2 \leq x^{2/7}} 3^{2x_2} < 3^{2x^{6/7}},$$

giving that $\#\mathcal{P}(x) < 2x^{6/7}$. This proves (2.4). The claim about the finiteness of the series shown at (2.5) follows from (2.4) by Abel's summation formula. $\quad\square$

2.2. Final step in the proof of Theorem 1.1

We need to show that the series shown at (1.3) converges. For simplicity, we write $z(n)$ instead of $z_w(n)$. Let $P(n)$ be the largest prime factor of n. We split that series as follows:

$$\sum_{n \geq 1} \frac{\mu(n)^2}{nz(n)} := S_1 + S_2,$$

where

$$S_1 = \sum_{\substack{n \geq 1 \\ P(n) \text{ is bad}}} \frac{\mu(n)^2}{nz(n)} \quad \text{and} \quad S_2 = \sum_{\substack{n \geq 1 \\ P(n) \text{ is not bad}}} \frac{\mu(n)^2}{nz(n)}.$$

For S_1, we just use the fact that $z(n) \geq 1$, to get that S_1 is upper bounded by

$$
\begin{aligned}
S_1' &= \sum_{n, P(n) \in \mathcal{P}} \frac{\mu(n)^2}{n} = \sum_{p \in \mathcal{P}} \frac{1}{p} \sum_{m, P(m) < p} \frac{\mu(m)^2}{m} \\
&\leq \sum_{p \in \mathcal{P}} \frac{1}{p} \prod_{q < p} \left(1 + \frac{1}{q}\right) \ll \sum_{p \in \mathcal{P}} \frac{\log p}{p} = O(1),
\end{aligned}
$$

by (2.5). As for S_2, note that if $p \notin \mathcal{P}$, then $\nu_p \leq 2\lfloor (\ell_2(p)p)^{6/7} \rfloor + 2$, giving that

$$z(n) \geq z(p) = k(p)/\nu_p > 0.25(\ell_2(p)p)^{1/7} \geq 0.25p^{1/7}.$$

Thus, S_2 is upper bounded by

$$
\begin{aligned}
S_2' &= \sum_{n \geq 1} \frac{4\mu(n)^2}{nP(n)^{8/7}} = \sum_{p \geq 2} \frac{4}{p^{8/7}} \sum_{m, P(m) < p} \frac{\mu(m)^2}{m} \\
&= 4 \sum_{p \geq 2} \frac{1}{p^{8/7}} \prod_{q < p} \left(1 + \frac{1}{q}\right) \ll \sum_{p \geq 2} \frac{\log p}{p^{8/7}} = O(1).
\end{aligned}
$$

Hence, $S_1 + S_2 \leq S_1' + S_2' = O(1)$, and the theorem is proved.

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