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Under-Ice Roughness: Shot Noise Model

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NAVAL UNDERWATER SYSTEMS CENTER

New London Laboratory

New London, Connecticut

UNDER-ICE ROUGHNESS: SHOT NOISE MODEL

Date: 31 December 1984

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ABSTRACT

The one-dimensional roughness of an under-ice profile of elliptical bosses is modeled in the time domain by a shot noise process of elliptical pulses of random amplitude, duration, and time of occurrence. A sample realization of 8000 data points is generated and plotted for visual comparison with experimental under-ice data. Also, theoretical and simulation results for the power density spectrum, the autocorrelation function, the characteristic function, the cumulative distribution function, and the probability density function of the shot noise process are plotted and compared.

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INTRODUCTION

The under-ice profile has been observed to appear like a random collection of superposed elliptical bosses, each of random amplitude, length, and location. An analogous model in the time domain is shot noise composed of overlapping pulses of random amplitude, duration, and time of occurrence. Accordingly, we have generated a sample realization of a shot noise process for visual comparison with experimental under-ice data, and for possible corroboration of this model. The particular realization generated has 8000 data points, although the number of effectively-independent samples is far fewer, as will be demonstrated.

A number of analytical results for shot noise have been derived in the past [1]; however, they did not cover the case of random duration modulation. We have extended the analyses to include random durations (as well as random amplitudes and random time occurrences) and evaluated the spectrum of the shot noise process, as well as the autocorrelation function and the first-order characteristic function of the instantaneous amplitude. From the latter, the first-order probability density function and cumulative distribution function of shot noise have been evaluated via a generalized Laguerre expansion employing 32 cumulants or moments. Comparisons of all these theoretical results with the corresponding sample quantities, obtained from the 8000 data point realization above, reveal excellent agreement.

A REALIZATION OF A SHOT-NOISE PROCESS

Shot noise is characterized by a superposition of pulses, each located independently and uniformly on the time scale. A sample pulse is illustrated in figure 1. The time of occurrence t_k (center of symmetrical pulse, for

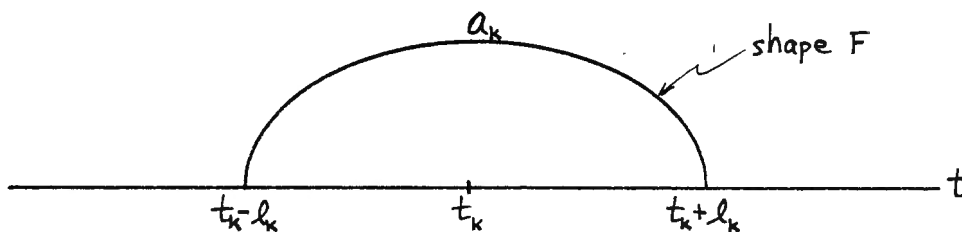


Figure 1. Sample Pulse of Shot Noise

example) is uniformly distributed in time t , with an average number of pulses per second, ν . The amplitude a_k and half-duration l_k of an individual pulse are also all independent and are each identically randomly-distributed with arbitrary probability density functions. Finally the fundamental pulse shape F in figure 1 is arbitrary.

A realization of shot noise is given by

$$I(t) = \sum_k a_k F\left(\frac{t-t_k}{l_k}\right), \quad (1)$$

where the summation extends over all k . The particular data we generate here employs the following example; unscaled pulse shape F is circular:

$$F(x) = \begin{cases} (1-x^2)^{1/2} & \text{for } |x| < 1 \\ 0 & \text{for } |x| > 1 \end{cases}. \quad (2)$$

This pulse is continuous; however, it has cusps (infinite slope) at $x = \pm 1$. The reason for this selection will become apparent when we discuss the spectrum of shot noise process (1).

The amplitude probability density function for random variable a_k is Rayleigh,

$$p(a) = \frac{a}{\sigma_a^2} \exp\left(-\frac{a^2}{2\sigma_a^2}\right) U(a), \quad (3)$$

and the duration probability density function for random variable ℓ_k is also Rayleigh,

$$p(\ell) = \frac{\ell}{\sigma_\ell^2} \exp\left(-\frac{\ell^2}{2\sigma_\ell^2}\right) U(\ell). \quad (4)$$

Here, step function

$$U(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x < 0 \end{cases}. \quad (5)$$

The mean values of random variables a_k and ℓ_k are given respectively by

$$\bar{a} = \bar{a}_k = \left(\frac{\pi}{2}\right)^{1/2} \sigma_a, \quad \bar{\ell} = \bar{\ell}_k = \left(\frac{\pi}{2}\right)^{1/2} \sigma_\ell, \quad (6)$$

in terms of the parameters σ_a and σ_ℓ of probability density functions (3) and (4). Alternatively, the mean square values are given by

$$\overline{a^2} = \overline{a_k^2} = 2\sigma_a^2, \quad \overline{\ell^2} = \overline{\ell_k^2} = 2\sigma_\ell^2. \quad (7)$$

Three typical component pulses are depicted in figure 2, and can range from circular through various elongated elliptical shapes. The total length of an individual pulse is $L_k = 2\ell_k$. An important parameter of this time-

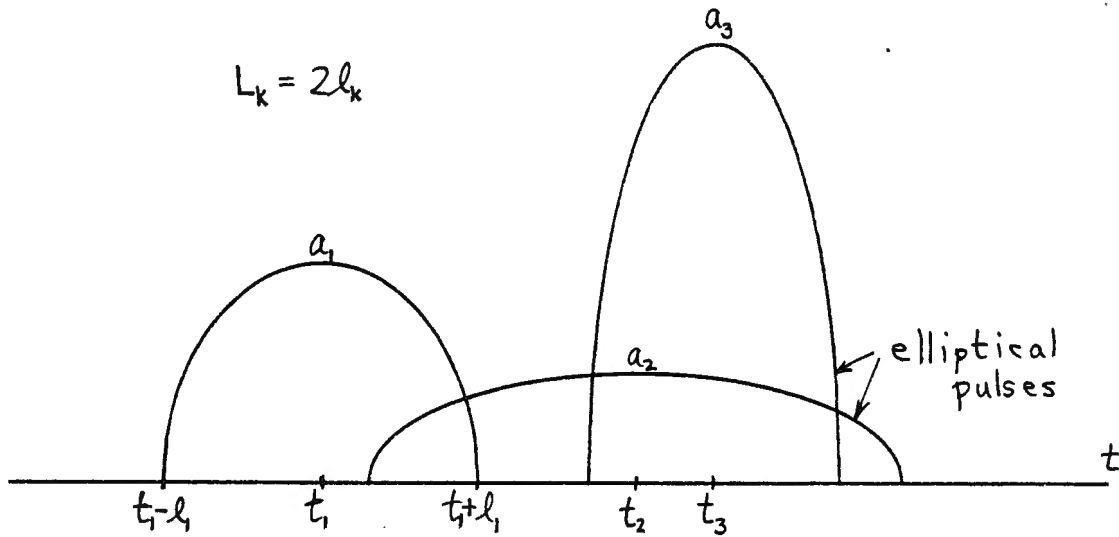


Figure 2. Three Component Pulses

limited pulse shape in figures 1 and 2 is the (dimensionless) overlap factor

$$\bar{L}_k v = 2\bar{\mathcal{L}}_k v = 2 \left(\frac{\pi}{2} \right)^{1/2} \sigma_{\mathcal{L}} v. \quad (8)$$

This is the average number of pulses that are overlapping at any one instant of time, and is a partial measure of the applicability of the central limit theorem. A more meaningful measure are the cumulants; for probability density function (3) and pulse shape (2), the normalized third and fourth cumulants are

$$\frac{1.017}{(\bar{\mathcal{L}}_k v)^{1/2}} \quad \text{and} \quad \frac{1.2}{\bar{\mathcal{L}}_k v}, \quad (9)$$

respectively. In the sample realization generated here, the overlap factor in (8) was 6.2, leading to normalized cumulant values in (9) of .58 and .39, respectively. Since a Gaussian probability density function would lead to zero cumulants above second-order, the shot noise realization dealt with here is distinctly non-Gaussian.

In the three parts of figure 3, a realization of shot noise model (1) is given for parameter values

$$\sigma_a = 1 \text{ sec}, \quad \sigma_p = 20 \text{ sec}, \quad v = .124 \text{ pulses/sec.} \quad (10)$$

The waveform (1) is sampled at unit time increments and connected by straight lines; thus the initial 100 data points illustrated in figure 3A have a jagged appearance for those component pulses with small λ_k , as for example at time instants 67-68. The larger duration pulses, like the one centered at $t = 29$, have a smoother appearance.

In figure 3B, the initial 1000 data points illustrate the very erratic character of shot noise; the waveform consists of some very sharp spiky pulses and other broader smooth components. The appearance of a downward trend in these 1000 data points is erased when the entire 8000 data point sequence is viewed in figure 3C. The possibility of shot noise process (1) reaching a zero value (when no pulses overlap) is confirmed by the waveform values near $t = 6400$.

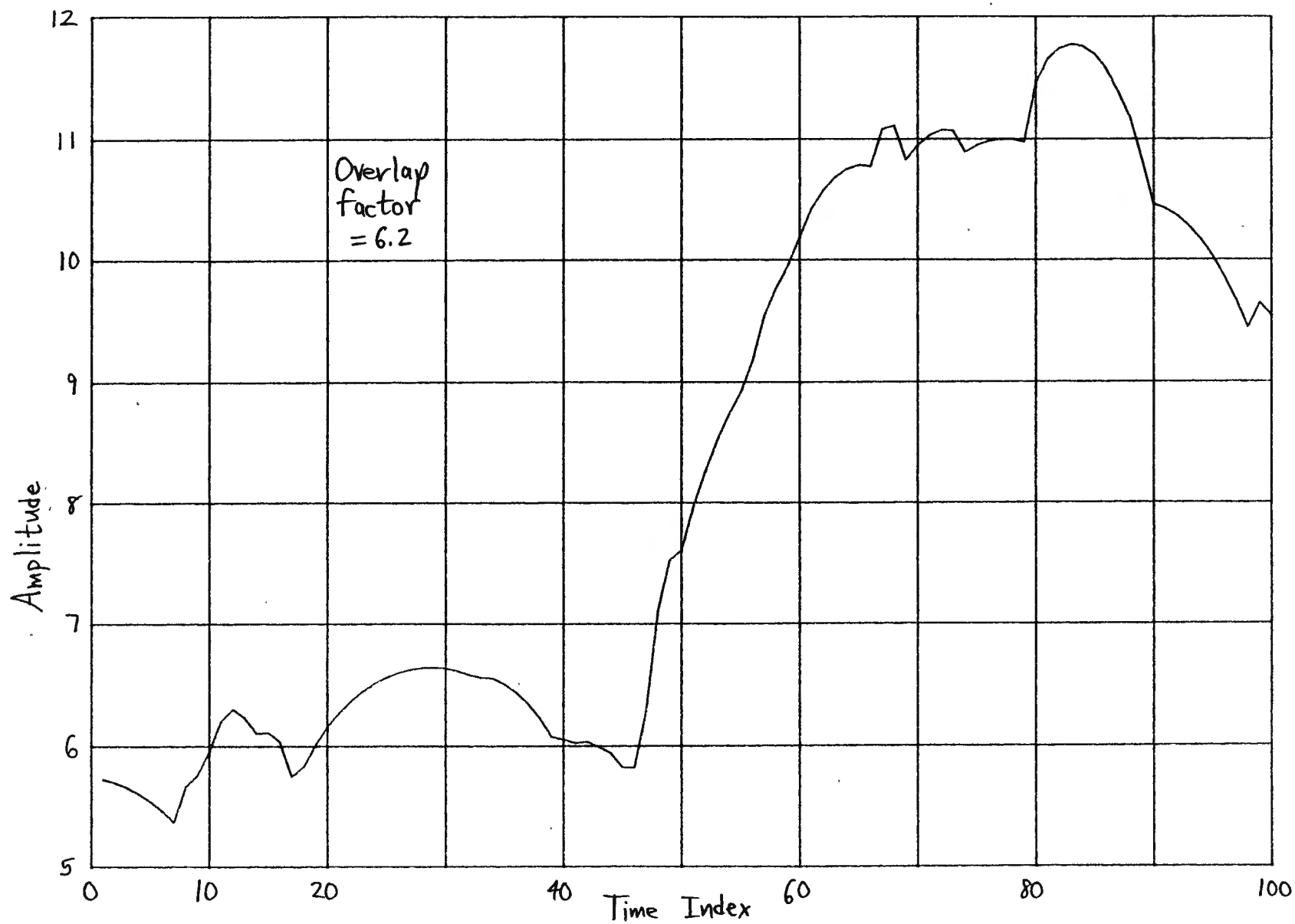


Figure 3A. Initial 100 Data Points

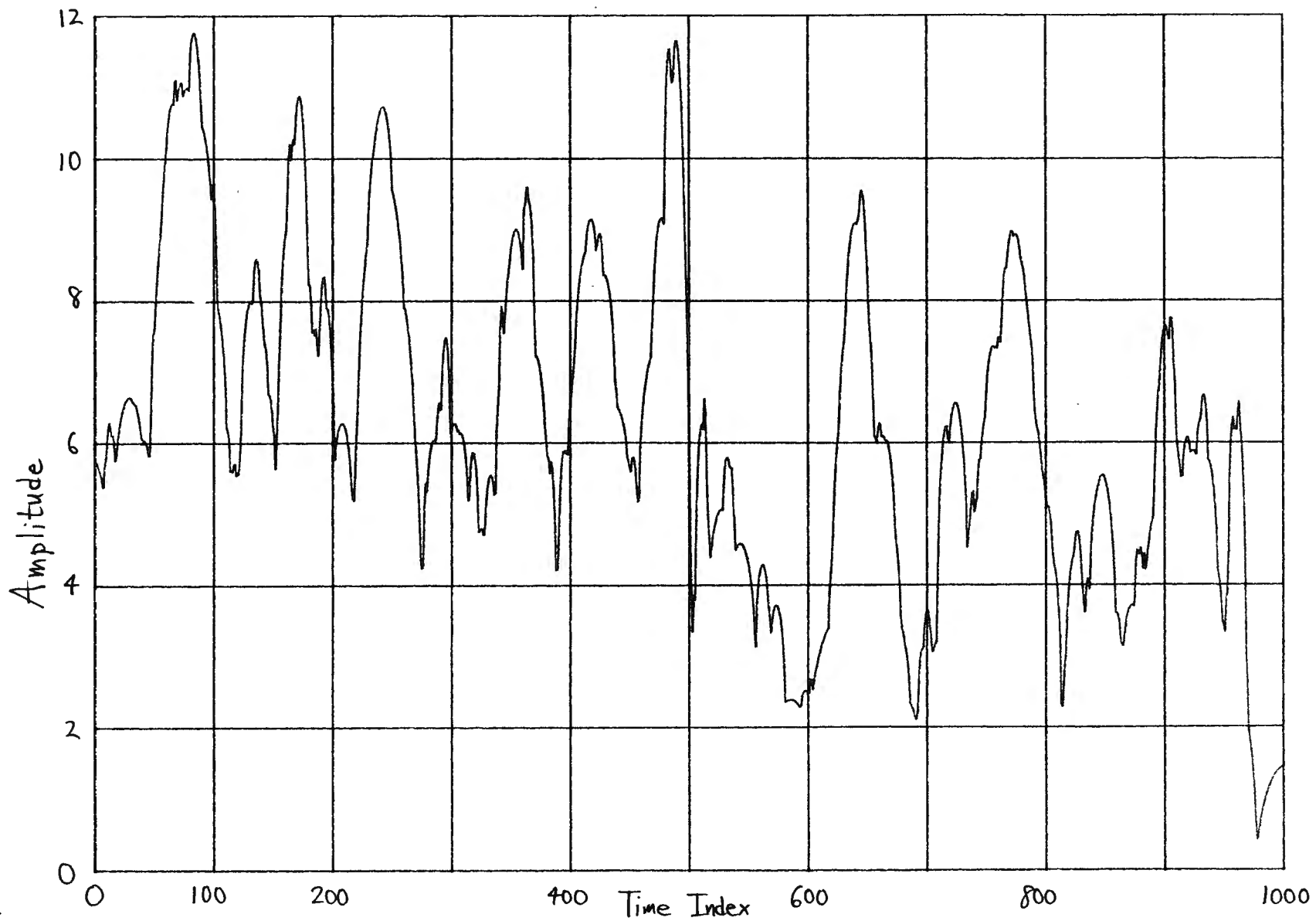


Figure 3B. Initial 1000 Data Points

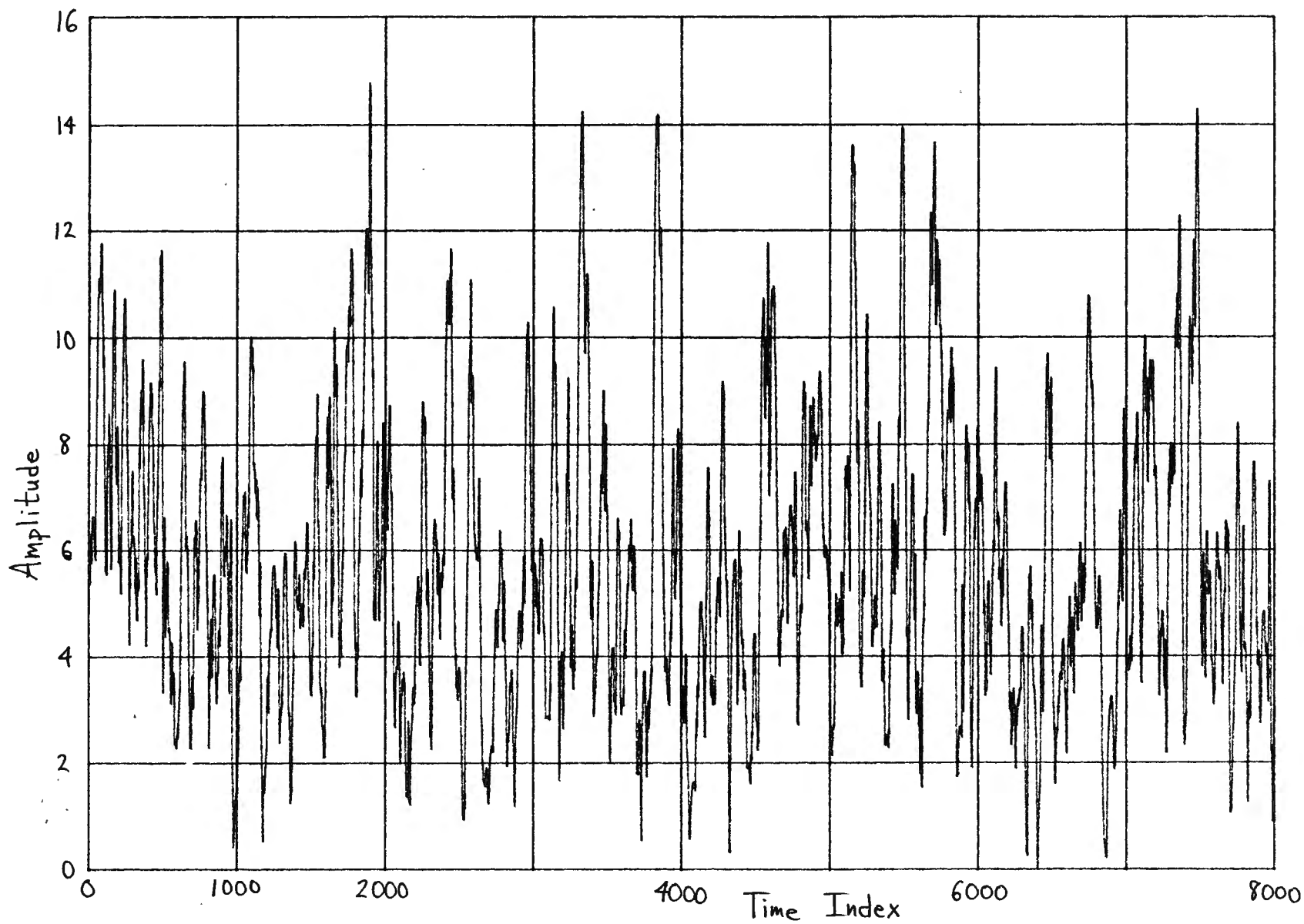


Figure 3C. Total Data Segment

CORRELATION AND SPECTRUM OF SHOT NOISE PROCESS

The derivations of the correlation and spectrum of the shot noise process (1) are given in appendix A; from (A-12), we have, in general, the correlation function at delay τ ,

$$R_I(\tau) = v \overline{a^2} \int d\ell p(\ell) \ell \phi(\tau/\ell) + I_{dc}^2, \quad (11)$$

where the dc component of $I(t)$ is, from (A-13),

$$I_{dc} = v \overline{a} \overline{\ell} \int dx F(x), \quad (12)$$

and

$$\phi(y) = \int dx F(x) F(x-y) \quad (13)$$

is the (aperiodic) correlation of an individual pulse F . (All integrals are over the range of non-zero integrand.)

Also, from (A-16), the general spectrum of process $I(t)$ is, at frequency f ,

$$G_I(f) = v \overline{a^2} \int d\ell p(\ell) \ell^2 |S(\ell f)|^2 + I_{dc}^2 \delta(f), \quad (14)$$

where

$$S(f) = \int dx \exp(-i2\pi f x) F(x) \quad (15)$$

is the voltage density spectrum (Fourier transform) of pulse F . Thus $|S(f)|^2$ is the energy density spectrum corresponding to pulse F .

It should be observed that the entire probability density function $p(\ell)$ of half-duration random variable ℓ_k is required in order to evaluate the correlation or spectrum of shot noise. However, only the first two moments, $\bar{a}$ and $\overline{a^2}$, are required known about probability density function $p(a)$ of amplitude random variable a_k . The only way that the dc term I_{dc} can be zero is if random variable a_k has zero mean ($\bar{a} = 0$), or if pulse F has zero area ($S(0) = 0$).

Example

The example of interest here was given earlier in (2) and (4), namely a circular pulse F and a Rayleigh probability density function for random variable ℓ_k . The spectrum $G_I(f)$ in (14) is evaluated in (A-17) through (A-22), with the results

$$S(f) = \frac{J_1(2\pi f)}{2f}, \quad S(0) = \frac{\pi}{2},$$

$$I_{dc} = \left(\frac{\pi}{2}\right)^{3/2} \nu \bar{a} \sigma_\ell^2,$$

$$G_I(f) = 2\pi^2 \nu \overline{a^2} \sigma_\ell^2 \frac{2 \exp(-z) I_1(z)}{z} + \left(\frac{\pi}{2}\right)^3 \nu^2 \frac{2}{\bar{a}} \sigma_\ell^2 \delta(f)$$

with $z = (2\pi\sigma_\ell f)^2$. (16)

The asymptotic behavior of spectrum (16) is [2, eq. 9.7.1]

$$G_I(f) \sim \frac{\nu \overline{a^2}}{(2\pi)^{3/2} \sigma_\ell} f^{-3} \quad \text{as } f \rightarrow +\infty. \quad (17)$$

That is, the spectrum decays at a -30 dB/decade rate at large frequencies; this is due to the square root singularities at $x = \pm 1$ of pulse F given in (2). This decay rate has been observed in some spectral analyses of under-ice profiles, and was one of the reasons for choosing the specific circular pulse in (2) for this investigation.

The spectrum in (16) is plotted in figure 4, for the choice of parameters earlier in (10), as a dashed line, normalized to 0 dB at $f = 0$. Superposed is a linear-predictive spectral analysis result with predictive order 10, for the 8000 data points of figure 3C. The two results are in excellent agreement, even at the -50 dB level, with the inevitable 3 dB aliasing effect at the Nyquist frequency, as indicated.

The correlation $R_I(\tau)$ in (11) is evaluated in (A-23) through (A-33), for the example (2) and (4), with the result

$$R_I(\tau) = \frac{8}{3}(2\pi)^{1/2} \sqrt{a^2 \sigma_L^2} s \exp(-s) [(1+4s) K_1(s) - (3+4s) K_0(s)] + \\ + \left(\frac{\pi}{2}\right)^3 \sqrt{a^2 \sigma_L^2}, \quad \text{with } s = \left(\frac{\tau}{4\sigma_L}\right)^2. \quad (18)$$

This quantity, exclusive of the I_{dc}^2 term, and normalized at the origin, is plotted in figure 5 as a dashed line, for delays (lags) τ up to 100. It is seen to decay monotonically to zero as τ increases, and reach its $1/e$ value at approximately $\tau = 30$.

The remaining solid curve on figure 5 is the normalized sample autocorrelation function of the 8000 data point sequence in figure 3C, where the sample mean was subtracted from the given data. The agreement with theoretical result (18) is excellent. The dotted horizontal lines at $\pm 2\sigma$ in figure 5 are the ± 2 sigma values of the correlation estimate at delays where the true correlation is presumed zero; the details of this analysis are given in appendix B.

This procedure is duplicated in figure 6, where the correlation function estimate out to lag $\tau = 1000$ is plotted. The drifting of the estimate outside the $\pm 2\sigma$ limits (at $\tau = 470$ and 820) is consistent with an occasional excursion of a random variable outside its $\pm 2\sigma$ range. The correlation estimate (used for figures 5 and 6) at time separation k is

$$R_k = \frac{1}{N} \sum_{n=k+1}^N x_n x_{n-k} \quad \text{for } k \geq 0, \quad (19)$$

where $\{x_n\}_1^N$ is the available data in figure 3C, with its sample mean removed.

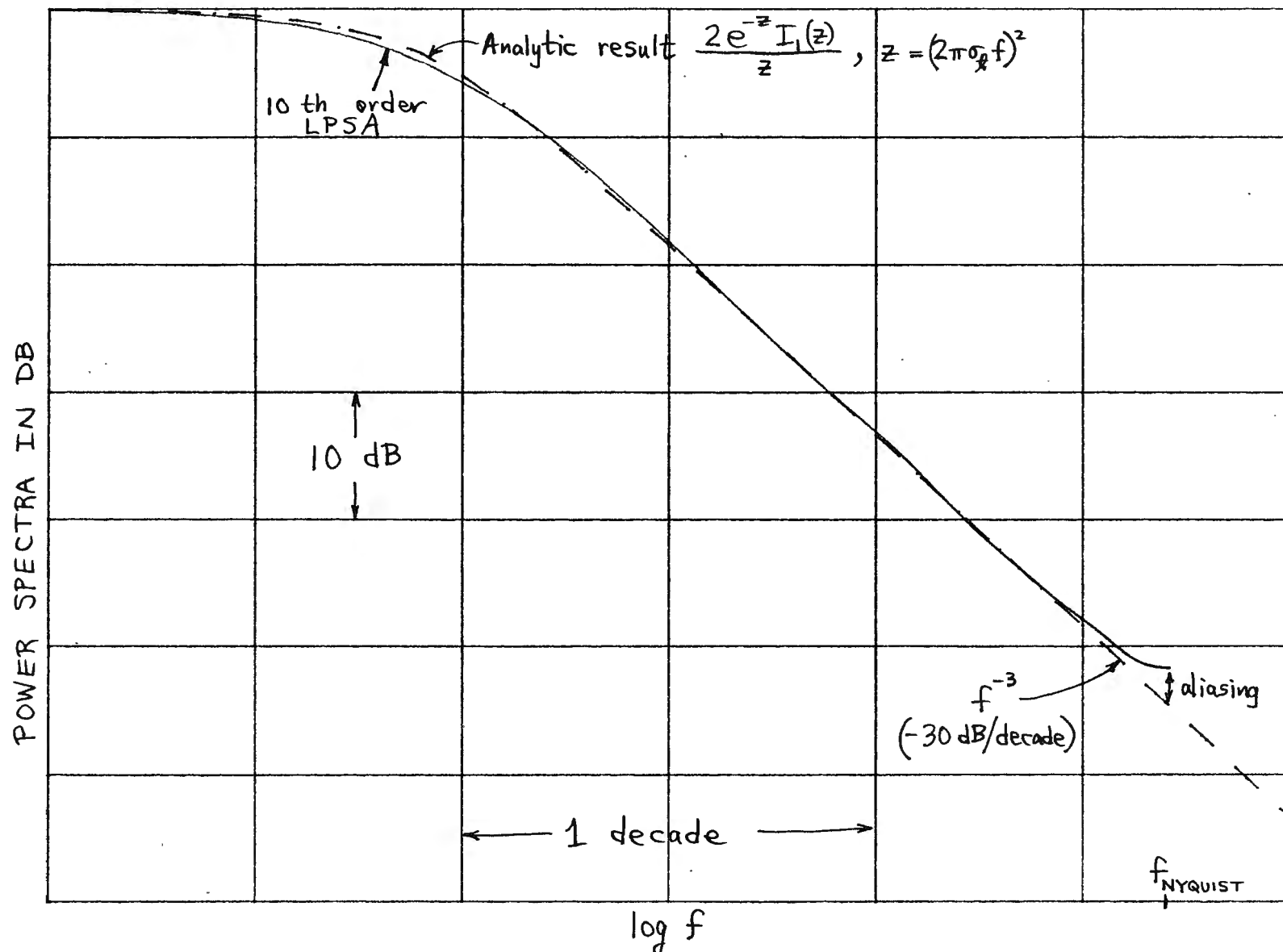


Figure 4. Comparison of Power Spectra

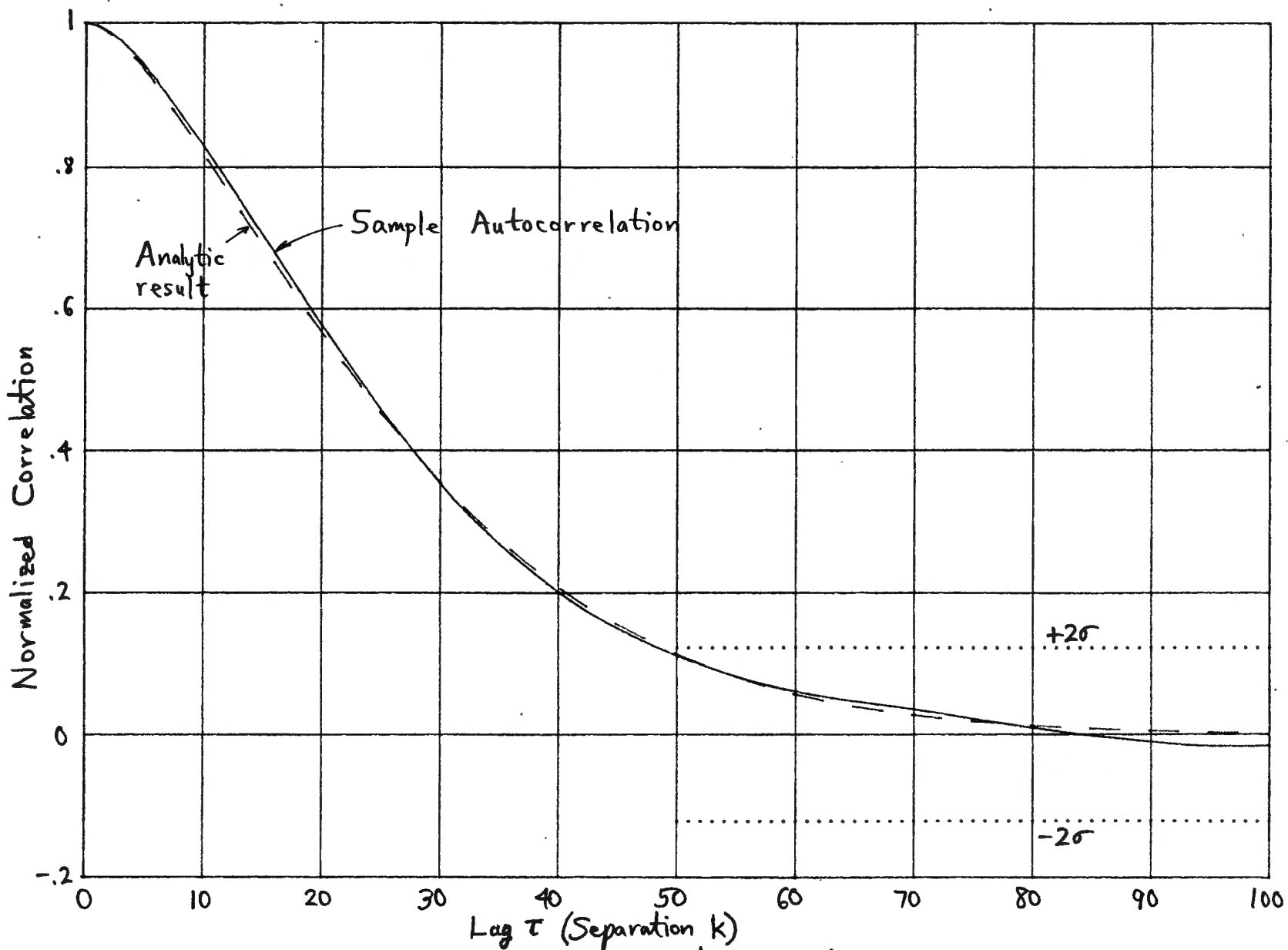


Figure 5. Estimate of Autocorrelation Function near origin

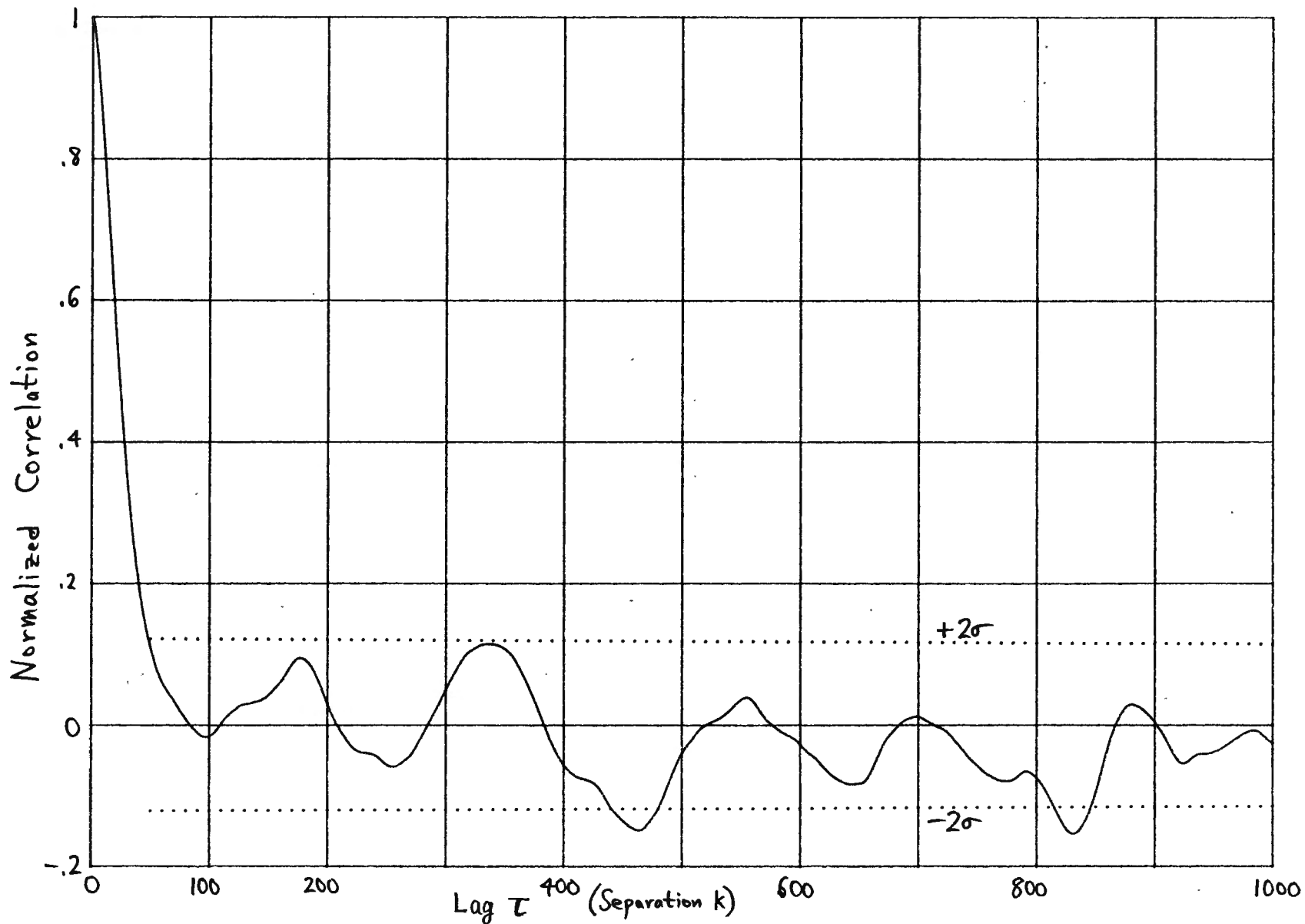


Figure 6. Estimate of Autocorrelation Function

AMPLITUDE STATISTICS OF SHOT NOISE

The first-order characteristic function of shot noise process $I(t)$ is derived in appendix C; it is given by (C-9) as

$$f_I(\xi) = \exp \left[v \bar{\ell} \int dx \left\{ f_a[\xi F(x)] - 1 \right\} \right]. \quad (20)$$

Here f_a is the first-order characteristic function of amplitude random variable a_k . Observe that the probability density function $p(\ell)$ of duration ℓ_k is irrelevant to characteristic function f_I , except for its mean $\bar{\ell}$; this is in contrast to the spectrum and correlation results in (11) and (14), where $p(a)$ was irrelevant except for parameters $\bar{a}$ and $\bar{a}^2$. (For $\ell_k = 1$ for all k , (20) reduces to a simplified version of [1, eq. 1.5-4].)

The characteristic function of the amplitude random variable a_k can be expanded in terms of its moments

$$\mu_a(n) = \overline{a^n} = \int da a^n p(a) \quad \text{for } n \geq 0, \quad (21)$$

according to

$$f_a(\xi) = \sum_{n=0}^{\infty} \mu_a(n) (i\xi)^n / n!. \quad (22)$$

This result is useful if the ℓ^n of (20) is expanded in a series in ξ ; namely

$$\ell^n f_I(\xi) = v \bar{\ell} \sum_{n=1}^{\infty} \mu_a(n) (i\xi)^n \int dx F^n(x) / n!, \quad (23)$$

giving immediately the cumulants of $I(t)$ as

$$\chi_I(n) = v \bar{\ell} \mu_a(n) \int dx F^n(x) \quad \text{for } n \geq 1. \quad (24)$$

That is, the n -th cumulant of $I(t)$ is proportional to the n -th moment of random variable a_k as well as the n -th "moment" of pulse F . (For $\ell_k = 1$ for all k , (24) reduces to [1, eq. 1.5-2].)

The normalized cumulant of $I(t)$ is

$$\gamma_I(n) = \frac{\chi_I(n)}{[\chi_I(2)]^{n/2}} = \frac{1}{(\nu \bar{\ell})^{\frac{n}{2}-1}} \frac{\mu_a(n) \int dx F^n(x)}{\left[\mu_a(2) \int dx F^2(x) \right]^{n/2}}. \quad (25)$$

In particular, the coefficients of skewness and excess [3, pp. 184 and 187] are

$$\gamma_I(3) = \frac{1}{(\nu \bar{\ell})^{1/2}} \frac{\mu_a(3) \int dx F^3(x)}{\left[\mu_a(2) \int dx F^2(x) \right]^{3/2}} \quad (26)$$

and

$$\gamma_I(4) = \frac{1}{\nu \bar{\ell}} \frac{\mu_a(4) \int dx F^4(x)}{\left[\mu_a(2) \int dx F^2(x) \right]^2}. \quad (27)$$

These quantities are very important measures of the approach of $I(t)$ to a Gaussian process; if $\nu \bar{\ell}$ is very large, the normalized cumulants $\gamma_I(n)$ are all substantially zero for $n \geq 3$, meaning that $I(t)$ is nearly Gaussian. Thus although probability density function $p(\ell)$ is not directly relevant to the probability density function or characteristic function (20) of $I(t)$, the exact probability density function of $I(t)$ is critically dependent on the mean $\bar{\ell}$ through the dimensionless parameter $\nu \bar{\ell}$. More precisely, (26) and (27) are the critical quantities; see also [1, eq. 1.6-3].

If either the third moment of random variable a_k is zero, or if the third moment of pulse F is zero, then $\gamma_I(3) = 0$. In that case, $\gamma_I(4)$ is the most important statistic measuring the applicability of the central limit theorem; $\gamma_I(4)$ can never be zero for shot noise, since neither the fourth moment of random variable a_k or pulse F can be zero (except in a trivial case).

The first moment of shot noise $I(t)$ is the mean

$$I_{dc} = \overline{I(t)} = \chi_I(1) = \nu \overline{a} \int dx F(x) \quad (28)$$

and has already been encountered in (12). It can be zero only if the first moment of random variable a_k or of pulse F is zero.

Example

Numerous cases have been considered in appendix C; in the main body here, we limit attention to example (2) and (3) presented earlier. We find

$$\begin{aligned} \mu_a(n) &= 2^{\frac{n}{2}} \Gamma\left(\frac{n}{2} + 1\right) \sigma_a^n \quad \text{for } n \geq 0, \\ \int dx F^n(x) &= \frac{2^{n+1} \Gamma^2\left(\frac{n}{2} + 1\right)}{\Gamma(n+2)} \quad \text{for } n \geq 0. \end{aligned} \quad (29)$$

Then (26) and (27) yield result (9) quoted earlier.

The realization of shot noise process $I(t)$ in figure 3C employed the parameters in (10). The sample cumulative distribution function of these 8000 data points is depicted in figure 7, on a normal probability ordinate; thus a

truly Gaussian random variable would have the straight line character indicated. The significant deviation of the sample cumulative distribution function from the Gaussian line is due to the small value of the overlap factor in (8), namely

$$\Gamma_k \nu = 2 \quad \mathcal{I}_k \nu = 6.2 . \quad (30)$$

The moments in (29) are all positive and are easily numerically evaluated via recursion; hence the cumulants in (24) can be accurately evaluated for high-order n . When these cumulants are employed in a generalized Laguerre expansion of the cumulative distribution function of $I(t)$, using 32 moments of (29), the solid curve in figure 8 is obtained. The sample cumulative distribution function of figure 7 is duplicated here, although the abscissa is scaled differently. The agreement between theory and experiment in figure 8 is excellent, considering the fact that we only have about $8000/30 = 270$ effectively independent samples of $I(t)$ in figure 3C; the denominator factor of 30 here is the effective correlation duration, previously identified in figure 5 at the $1/e$ point.

Finally, when the same 32 moments are used in a generalized Laguerre expansion of the probability density function of $I(t)$, the result in figure 9 is obtained. The small bump near the origin is real and accurate; it and the non-symmetric tails of the probability density function confirm the distinctly non-Gaussian character of $I(t)$. The method for the determination of the cumulative distribution function and probability density function in figures 8 and 9 will be presented in a NUSC Technical Report [4] by the author; the programs are listed here in appendix D, along with an example of the sequence of Laguerre coefficients.

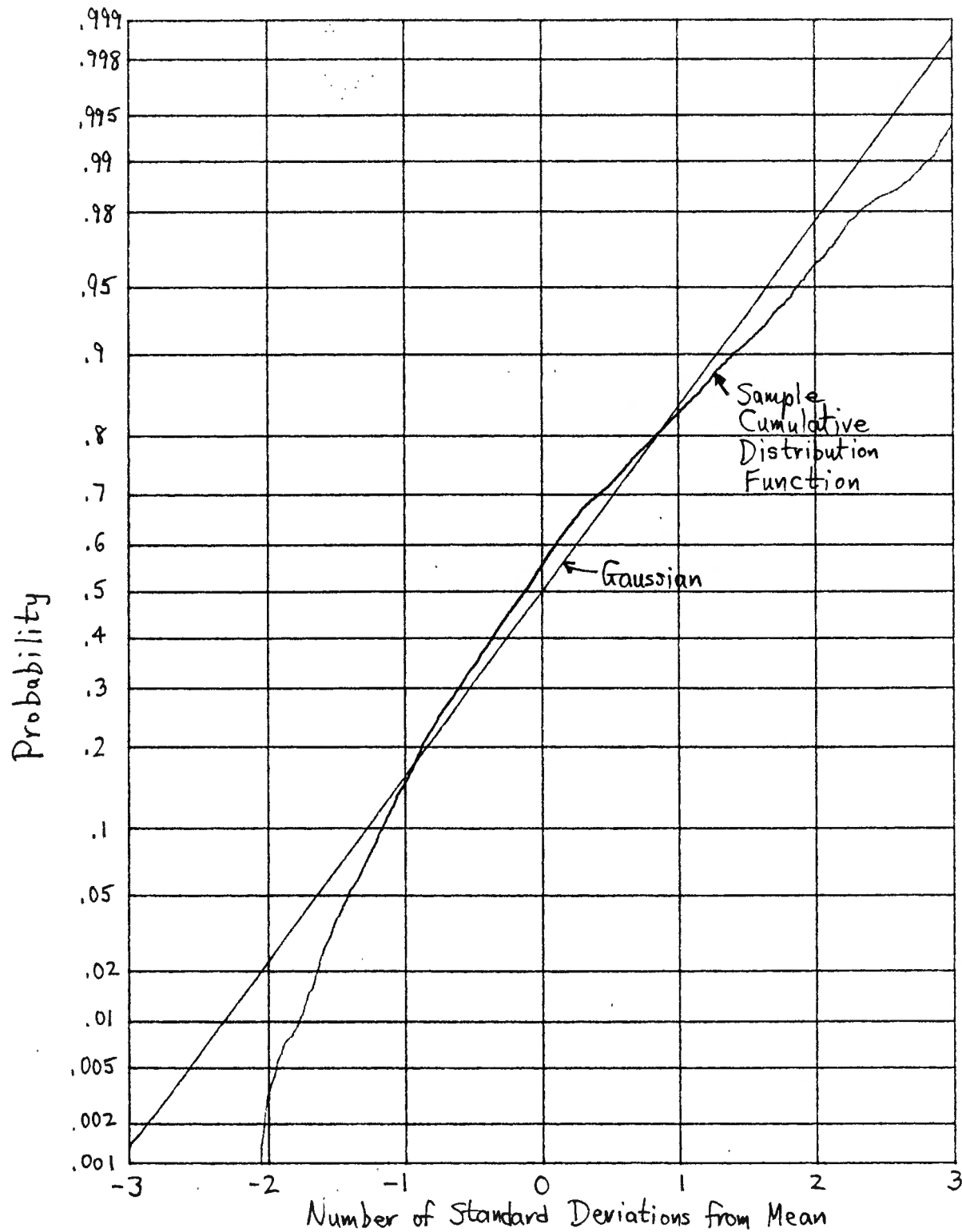


Figure 7. Estimate of Cumulative Distribution Function

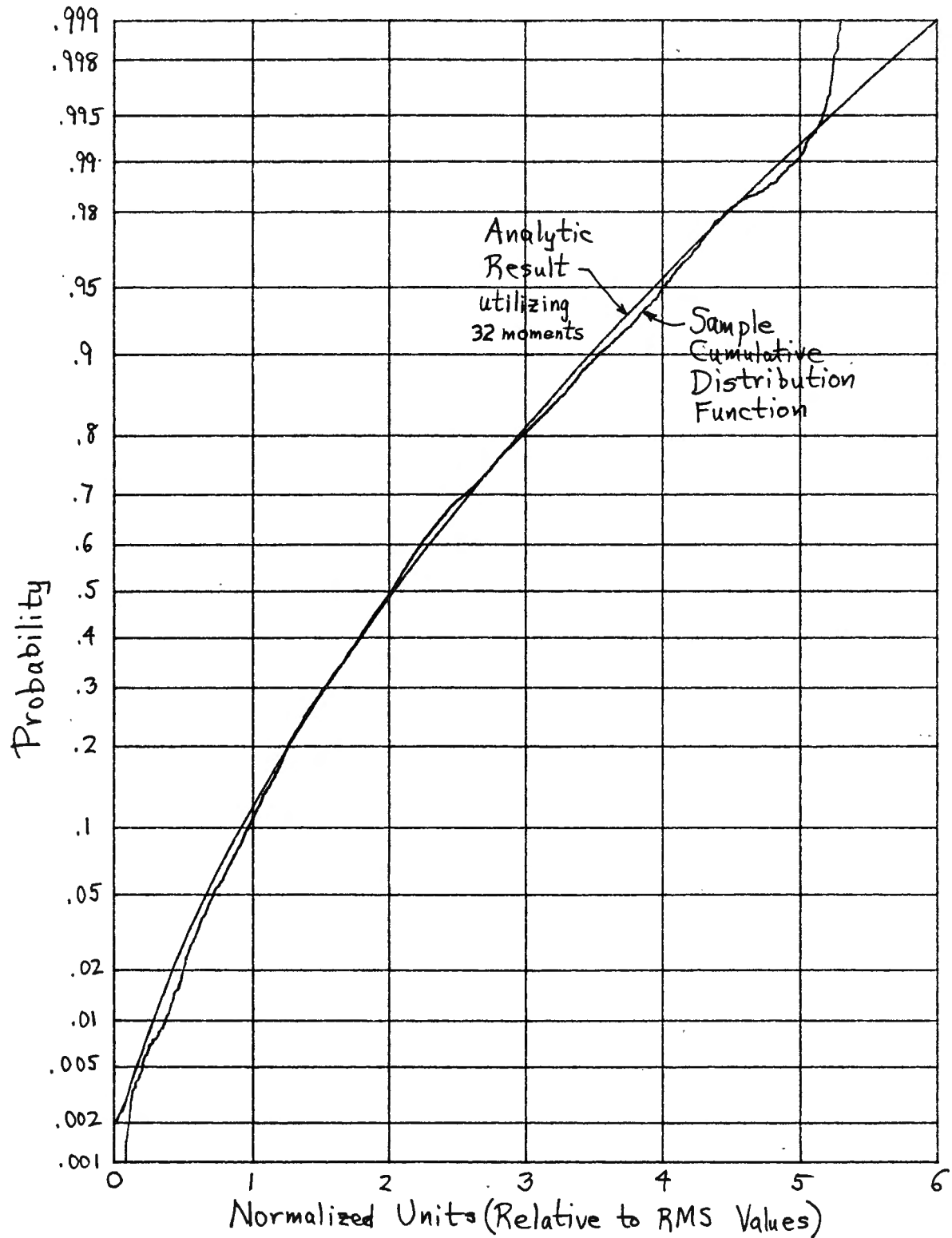


Figure 8. Comparison of Cumulative Distribution Functions

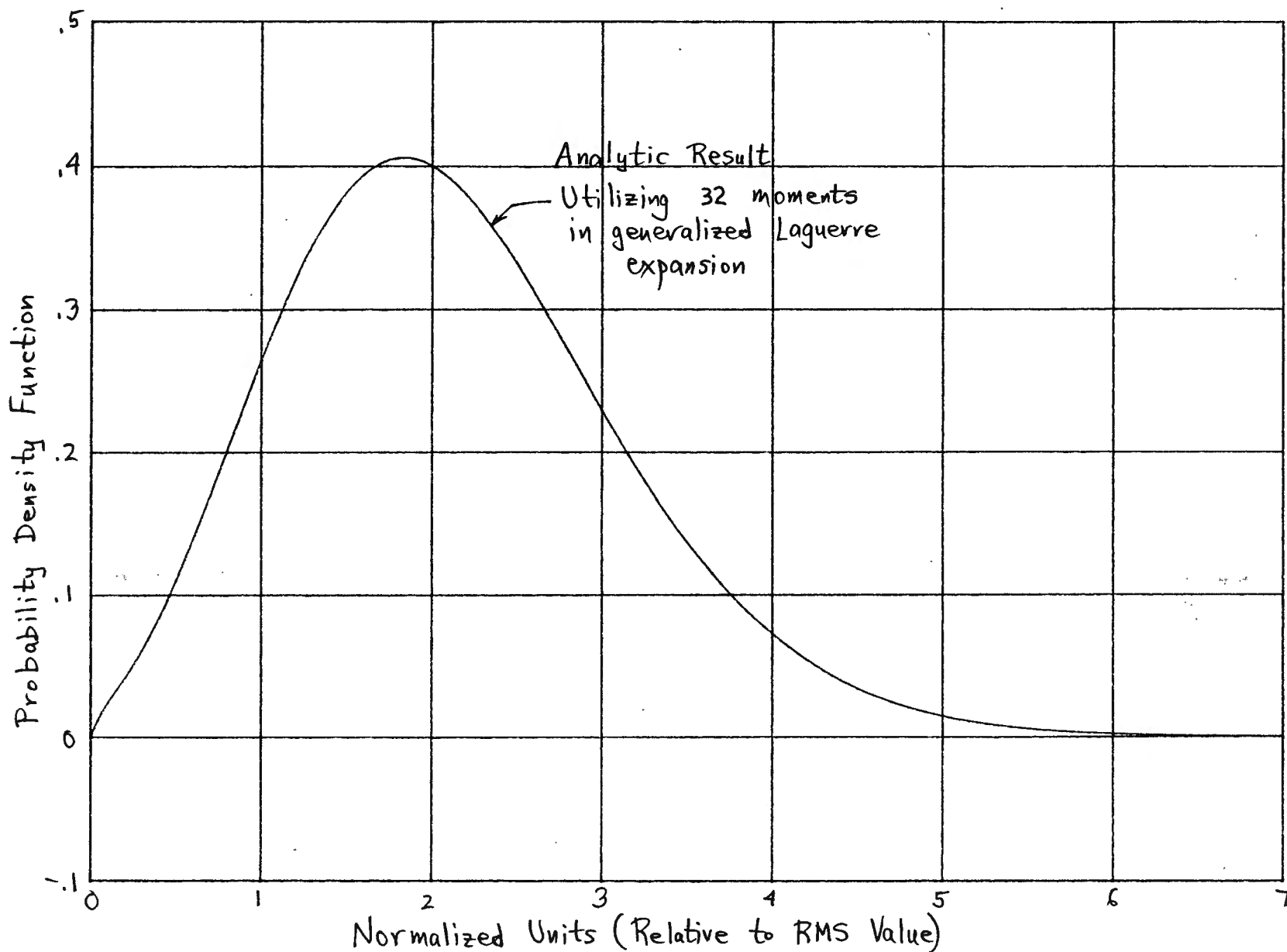


Figure 9. Theoretical Probability Density Function (Continuous Portion)

APPENDIX A. DERIVATION OF SPECTRUM AND CORRELATION

The method employed below follows that given by Rice [1, sections 1.4 and 1.5] rather closely. We generalize [1, eq. 1.3-1] to the current form introduced in (1):

$$I_K(t) = \sum_{k=1}^K a_k F\left(\frac{t-t_k}{\ell_k}\right), \quad (A-1)$$

where $\{a_k\}$, $\{t_k\}$, $\{\ell_k\}$ are all independent random variables. K is the presumed number of pulses to occur in a large time interval T , and a_k is a random amplitude as in [1, eq. 1.5-1]; but random duration ℓ_k is new. Then product

$$\begin{aligned} I_K(t) I_K(t-\tau) &= \sum_{k=1}^K a_k^2 F\left(\frac{t-t_k}{\ell_k}\right) F\left(\frac{t-\tau-t_k}{\ell_k}\right) + \\ &+ \sum_{k=1}^K \sum_{\substack{m=1 \\ k \neq m}}^K a_k a_m F\left(\frac{t-t_k}{\ell_k}\right) F\left(\frac{t-\tau-t_m}{\ell_m}\right). \end{aligned} \quad (A-2)$$

Holding random variables $\{a_k\}$ and $\{\ell_k\}$ fixed for now, the statistical average of (A-2) over $\{t_k\}$ is

$$\begin{aligned} &\sum_{k=1}^K a_k^2 \frac{1}{T} \int_0^T dt_k F\left(\frac{t-t_k}{\ell_k}\right) F\left(\frac{t-\tau-t_k}{\ell_k}\right) + \\ &+ \sum_{k=1}^K \sum_{\substack{m=1 \\ k \neq m}}^K a_k a_m \frac{1}{T} \int_0^T dt_k F\left(\frac{t-t_k}{\ell_k}\right) \frac{1}{T} \int_0^T dt_m F\left(\frac{t-\tau-t_m}{\ell_m}\right) = \\ &\cong \frac{1}{T} \sum_{k=1}^K a_k^2 \ell_k \phi(\tau/\ell_k) + \frac{1}{T^2} \sum_{k=1}^K \sum_{\substack{m=1 \\ k \neq m}}^K a_k a_m \ell_k \ell_m S^2(0), \end{aligned} \quad (A-3)$$

where T is an arbitrary large (but finite) time interval, and

$$\phi(y) = \int dx F(x) F(x-y) \quad (A-4)$$

is the aperiodic autocorrelation of pulse F, while

$$S(f) = \int dx \exp(-i2\pi fx) F(x) \quad (A-5)$$

is the voltage density spectrum of F.

The remaining averages over independent random variables $\{a_k\}$ and $\{\ell_k\}$ in (A-3) now yield

$$\frac{1}{T} K \overline{a^2} \int d\ell p(\ell) \ell \phi(\tau/\ell) + \frac{1}{T^2} (K^2 - K) [\overline{a} \overline{\ell} S(0)]^2, \quad (A-6)$$

where $p(\ell)$ is the probability density function of random variable ℓ_k .

Now K is itself a random variable, with discrete probability (in an interval T) of [1, eq. 1.1-3]

$$\frac{(vT)^K}{K!} \exp(-vT) \quad \text{for } K = 0, 1, 2, \dots \quad (A-7)$$

There then follows the characteristic function of random variable K as

$$f_K(\xi) = \exp(vT[\exp(i\xi) - 1]), \quad (A-8)$$

with series expansion

$$\ln f_K(\xi) = vT [\exp(i\xi) - 1] = vT \sum_{n=1}^{\infty} (i\xi)^n / n! \quad (A-9)$$

Thus the cumulants of random variable K are all equal,

$$\chi_K(n) = vT \quad \text{for } n \geq 1, \quad (A-10)$$

giving in particular the first two moments

$$\overline{K} = vT, \quad \overline{K^2} = vT(vT + 1) \quad (A-11)$$

The use of (A-11), to perform the remaining average of (A-6) with respect to random variable K , then yields the correlation function of the shot noise process $I(t)$:

$$R_I(\tau) = v \bar{a}^2 \int d\ell p(\ell) \ell \phi(\tau/\ell) + [v \bar{a} \bar{\ell} S(0)]^2. \quad (A-12)$$

The dc component of $I(t)$ is

$$I_{dc} = v \bar{a} \bar{\ell} S(0) = v \bar{a} \bar{\ell} \int dx F(x). \quad (A-13)$$

The spectrum of $I(t)$ is the Fourier transform of (A-12):

$$G_I(f) = v \bar{a}^2 \int d\ell p(\ell) \ell^2 \Phi(\ell f) + I_{dc}^2 \delta(f), \quad (A-14)$$

where

$$\begin{aligned} \Phi(f) &= \int d\tau \exp(-i2\pi f\tau) \phi(\tau) = \\ &= \int d\tau \exp(-i2\pi f\tau) \int dx F(x) F(x-\tau) = |S(f)|^2, \end{aligned} \quad (A-15)$$

by use of (A-4) and (A-5). Thus (A-14) can be expressed as

$$G_I(f) = v \bar{a}^2 \int d\ell p(\ell) \ell^2 |S(\ell f)|^2 + I_{dc}^2 \delta(f). \quad (A-16)$$

$|S(f)|^2$ is the energy density spectrum of pulse F .

Example

The example of interest here is given in (2) and (4):

$$\begin{aligned} F(x) &= \begin{cases} (1-x^2)^{1/2} & \text{for } |x| < 1 \\ 0 & \text{for } |x| > 1 \end{cases}, \\ p(\ell) &= \frac{\ell}{\sigma_\ell^2} \exp\left(-\frac{\ell^2}{2\sigma_\ell^2}\right) u(\ell). \end{aligned} \quad (A-17)$$

Then from (A-5) and [5, eq. 3.752 2],

$$S(f) = \int_{-1}^1 dx \exp(-i2\pi f x) (1-x^2)^{1/2} = \frac{J_1(2\pi f)}{2f}. \quad (A-18)$$

Substitution of (A-17) and (A-18) in the integral in (A-16) yields, by use of [5, eq. 6.633 2],

$$\int d\ell \frac{\ell}{\sigma_\ell^2} \exp\left(\frac{-\ell^2}{2\sigma_\ell^2}\right) \frac{J_1^2(2\pi f)}{4f^2} = 2\pi^2 \sigma_\ell^2 \frac{2 \exp(-z) I_1(z)}{z}, \quad (A-19)$$

where

$$z = (2\pi\sigma_\ell f)^2. \quad (A-20)$$

Then the spectrum (A-16) is given by

$$G_I(f) = 2\pi^2 \nu \overline{a^2} \sigma_\ell^2 \frac{2 \exp(-z) I_1(z)}{z} + I_{dc}^2 \delta(f), \quad (A-21)$$

where

$$I_{dc} = \nu \overline{a} \overline{\ell} S(0) = \left(\frac{\pi}{2}\right)^{3/2} \nu \overline{a} \sigma_\ell \quad (A-22)$$

by means of (A-13), (A-18), and (6).

To determine the correlation of shot noise process $I(t)$, we consider first the continuous portion of the spectrum in (A-21):

$$G_C(f) = 2\pi^2 \nu \overline{a^2} \sigma_\ell^2 \frac{2 \exp(-4\pi^2 \sigma_\ell^2 f^2) I_1(4\pi^2 \sigma_\ell^2 f^2)}{4\pi^2 \sigma_\ell^2 f^2}. \quad (A-23)$$

The corresponding correlation is

$$\begin{aligned}
 R_c(\tau) &= \int_{-\infty}^{+\infty} df \exp(i2\pi f\tau) G_c(f) = \\
 &= 4\pi^2 \sqrt{a^2} \sigma_\ell^2 \int_0^\infty df 2 \cos(2\pi f\tau) \frac{\exp(-4\pi^2 \sigma_\ell^2 f^2) I_1(4\pi^2 \sigma_\ell^2 f^2)}{4\pi^2 \sigma_\ell^2 f^2} = \\
 &= 2\pi \sqrt{a^2} \sigma_\ell \int_0^\infty dz \cos\left(\frac{\tau}{\sigma_\ell} z^{\frac{1}{2}}\right) \frac{\exp(-z) I_1(z)}{z^{3/2}} = \\
 &= 2^{\frac{3}{2}} \pi \sqrt{a^2} \sigma_\ell \exp(-s) W_{-\frac{3}{2}, \frac{1}{2}}(2s), \tag{A-24}
 \end{aligned}$$

where we employed (A-20) and [5, eq. 6.755 2], and defined

$$s = \left(\frac{\tau}{4\sigma_\ell}\right)^2. \tag{A-25}$$

The W-function in (A-24) is the Whittaker function [2, p. 505].

Now by [5, eqs. 9.232 1 and 9.222 1], we have

$$\begin{aligned}
 W_{-\frac{3}{2}, \frac{1}{2}}(2s) &= W_{-\frac{3}{2}, -\frac{1}{2}}(2s) = \frac{\exp(-s)}{\Gamma(3/2)} \int_0^\infty dt \exp(-2st) t^{\frac{1}{2}} (1+t)^{-5/2} = \\
 &= \frac{2}{\pi^{\frac{1}{2}}} \exp(-s) \int_0^\infty dt \exp(-2st) \left[\frac{1}{t^{\frac{1}{2}} (1+t)^{3/2}} - \frac{1}{t^{\frac{1}{2}} (1+t)^{5/2}} \right]. \tag{A-26}
 \end{aligned}$$

But according to [5, eq. 3.364 3],

$$\int_0^\infty dt \frac{\exp(-2st)}{t^{\frac{1}{2}} (a+t)^{\frac{1}{2}}} = \exp(as) K_0(as). \tag{A-27}$$

Partial differentiation with respect to a then yields:

$$-\frac{1}{2} \int_0^{\infty} dt \frac{\exp(-2st)}{t^{1/2} (a+t)^{3/2}} = s \exp(as) [K_0(as) - K_1(as)] \quad (A-28)$$

and (repeated)

$$\frac{3}{4} \int_0^{\infty} dt \frac{\exp(-2st)}{t^{1/2} (a+t)^{5/2}} = s^2 \exp(as) \left[2K_0(as) - 2K_1(as) + \frac{K_1(as)}{as} \right]. \quad (A-29)$$

Here we used [2, eq. 9.6.28] in the forms

$$K'_0(z) = -K_1(z), \quad K'_1(z) = -K_0(z) - \frac{K_1(z)}{z}. \quad (A-30)$$

If we now set $a = 1$ in (A-28) and (A-29), and then employ these results in (A-26), we obtain

$$W_{-\frac{3}{2}, \frac{1}{2}}(2s) = \frac{2}{\pi^{1/2}} \frac{2s}{3} \left[(1+4s)K_1(s) - (3+4s)K_0(s) \right]. \quad (A-31)$$

Finally, the use of (A-31) in (A-24) yields

$$R_C(\tau) = \frac{8}{3}(2\pi)^{1/2} \sqrt{a^2} \sigma_L^2 s \exp(-s) \left[(1+4s)K_1(s) - (3+4s)K_0(s) \right]. \quad (A-32)$$

The Fourier transform of the impulsive part of the spectrum in (A-21) is simply the constant

$$I_{dc}^2 = \left(\frac{\pi}{2} \right)^3 \sqrt{a^2} \sigma_L^2, \quad (A-33)$$

which must be added to $R_C(\tau)$ in (A-32) to obtain $R_I(\tau)$. Here s is given by (A-25).

As $\tau \rightarrow 0+$, there follows from (A-32),

$$\lim_{\tau \rightarrow 0+} R_C(\tau) = \frac{8}{3}(2\pi)^{1/2} \sqrt{a^2} \sigma_L^2. \quad (A-34)$$

APPENDIX B. VARIANCE OF CORRELATION ESTIMATE

Let the available data be $\{x_n\}_1^N$, with zero mean and variance σ^2 :

$$\bar{x}_n = 0, \quad \overline{x_n^2} = \sigma^2 \quad \text{for } 1 \leq n \leq N. \quad (\text{B-1})$$

The autocorrelation estimate at delay k is defined here as

$$R_k = \frac{1}{N} \sum_{n=k+1}^N x_n x_{n-k} \quad \text{for } k \geq 0. \quad (\text{B-2})$$

At delay 0, the mean value of estimate R_0 is

$$\overline{R_0} = \frac{1}{N} \sum_{n=1}^N \overline{x_n^2} = \sigma^2. \quad (\text{B-3})$$

We now want to evaluate the standard deviation of estimate R_k at delays k large enough that x_n and x_{n-k} are statistically independent. We have mean value

$$\overline{R_k} = \frac{1}{N} \sum_{n=k+1}^N \bar{x}_n \bar{x}_{n-k} = 0, \quad (\text{B-4})$$

using the independence at separation k . The mean square value of estimate R_k is

$$\overline{R_k^2} = \frac{1}{N^2} \sum_{m,n=k+1}^N \overline{x_m x_n x_{m-k} x_{n-k}}. \quad (\text{B-5})$$

For the large separation values k of interest here, the only statistical dependence that contributes non-trivially to the double sum is the following:

$$\begin{aligned}
\overline{R_k^2} &= \frac{1}{N^2} \sum_{m,n=k+1}^N \overline{x_m x_n} \overline{x_{m-k} x_{n-k}} = \frac{\sigma^4}{N^2} \sum_{m,n=k+1}^N \rho^2(m-n) = \\
&= \frac{\sigma^4}{N^2} \sum_{|n| < N-k} (N-k-|n|) \rho^2(n) \cong \frac{\sigma^4}{N^2} (N-k) \sum_n \rho^2(n) . \quad (B-6)
\end{aligned}$$

Here ρ is the correlation coefficient of data $\{x_n\}$, and we have assumed that N is moderately larger than the effective correlation length of ρ . The ratio of the standard deviation of estimate R_k to the mean value at $k=0$ is then the normalized standard deviation at separation k :

$$\sigma'_k \cong \frac{1}{N} \left[(N-k) \sum_n \rho^2(n) \right]^{1/2}. \quad (B-7)$$

Notice that no Gaussian assumptions on data $\{x_n\}_1^N$ have been employed in this analysis; however, $\rho(k)$ is essentially zero at the k values of interest.

As an example, for an exponential correlation of effective length K_e , there follows

$$\sum_n \rho^2(n) = \sum_n \exp\left(-2\frac{|n|}{K_e}\right) \cong \int dx \exp\left(-2\frac{|x|}{K_e}\right) = K_e, \quad (B-8)$$

where we assume that K_e is moderately larger than unity. Then (B-7) yields

$$\sigma'_k \cong \frac{1}{N} [(N-k)K_e]^{1/2}. \quad (B-9)$$

These results hold only for those values of k where $\rho(k)$ has substantially gone to zero. Larger values of K_e lead to larger relative standard deviations; this is consistent with the fact that there are then a lesser number of effectively-independent samples in the limited data set of length N .

For the 8000 data point example of interest here, inspection of figure 5 reveals that $K_e \cong 30$. Thus

$$\pm 2\sigma'_k = \pm \frac{(8000-k)^{1/2}}{730}. \quad (B-10)$$

These confidence limits are superposed as dotted lines on figures 5 and 6.

APPENDIX C. PROPERTIES OF CHARACTERISTIC FUNCTION OF SHOT NOISE

Derivation of Characteristic Function

The method of derivation of the characteristic function of $I(t)$ presented here parallels that of Rice [1, sections 1.4 and 1.5] very closely. We generalize [1, eq. 1.3-1] to

$$I_K(t) = \sum_{k=1}^K a_k F\left(\frac{t-t_k}{\lambda_k}\right) \quad (C-1)$$

where $\{a_k\}$, $\{t_k\}$, $\{\lambda_k\}$ are all independent random variables; see (A-1) and the ensuing discussion. The characteristic function of an individual component in (C-1) is

$$f_1(\xi) = \overline{\exp\left[i\xi a_k F\left(\frac{t-t_k}{\lambda_k}\right)\right]}, \quad (C-2)$$

where the statistical average is over a_k , t_k , λ_k . The average over t_k (for fixed a_k , λ_k) is, for T a large but finite time interval [1, p. 152],

$$\begin{aligned} & \frac{1}{T} \int_0^T dt_k \exp\left[i\xi a_k F\left(\frac{t-t_k}{\lambda_k}\right)\right] = \\ & = \frac{1}{T} \int_0^T dt_k \left\{ \exp\left[i\xi a_k F\left(\frac{t-t_k}{\lambda_k}\right)\right] - 1 \right\} + 1 = \\ & \cong \frac{1}{T} \int d\tau \left\{ \exp\left[i\xi a_k F\left(\frac{t-\tau}{\lambda_k}\right)\right] - 1 \right\} + 1, \end{aligned} \quad (C-3)$$

for large T , where we have used the fact that

$$F(x) \rightarrow 0 \text{ as } x \rightarrow \pm\infty. \quad (C-4)$$

Let $x = (t - \tau_k)/\lambda_k$ in (C-3) to get

$$\frac{1}{T} \lambda_k \int dx \{ \exp[i \xi a_k F(x)] - 1 \} + 1. \quad (C-5)$$

Now performing the averages on random variables λ_k and a_k , we have, for the characteristic function of an individual component of (C-1),

$$f_1(\xi) = \frac{1}{T} \bar{\lambda} \int da \, p(a) \int dx \{ \exp[i \xi a F(x)] - 1 \} + 1, \quad (C-6)$$

where $p(a)$ is the probability density function of random variable a_k .

Interchanging integrals, (C-6) becomes

$$f_1(\xi) = \frac{1}{T} \bar{\lambda} \int dx \{ f_a[\xi F(x)] - 1 \} + 1, \quad (C-7)$$

where f_a is the characteristic function of amplitude a_k . Then from (C-1), since all the individual random variables are independent, the characteristic function of $I_K(t)$ is

$$f_{I_K}(\xi) = [f_1(\xi)]^K. \quad (C-8)$$

Finally, the characteristic function of total shot noise process (1) is, by use of discrete probability distribution (A-7) for random variable K , given by the average

$$\begin{aligned}
f_I(\xi) &= \sum_{K=0}^{\infty} \frac{(vT)^K}{K!} \exp(-vT) f_{I_K}(\xi) = \\
&= \exp[-vT + vT f_1(\xi)] = \\
&= \exp[v\bar{\lambda} \int dx \{f_a[\xi F(x)] - 1\}] . \quad (C-9)
\end{aligned}$$

The (imprecise) large time interval T has dropped out of the general result (C-9). Also, the only parameter required about the duration random variable λ_k is its mean. The exact characteristic function f_a of amplitude a_k and the exact pulse shape F directly affect the characteristic function of $I(t)$. For $\lambda_k = 1$ for all k , (C-9) reduces to a simplified version of [1, eq. 1.5-4].

Cumulants of $I(t)$

The characteristic function of random amplitude a_k can be expanded in a power series

$$f_a(\xi) = \sum_{n=0}^{\infty} \mu_a(n) \frac{(i\xi)^n}{n!} , \quad (C-10)$$

where $\mu_a(n)$ is the n -th moment of a_k :

$$\mu_a(n) = \overline{a^n} = \int da a^n p(a) . \quad (C-11)$$

Then from (C-9), we develop

$$\begin{aligned}
\ln f_I(\xi) &= \nu \bar{\ell} \int dx \{f_a[\xi F(x)] - 1\} = \\
&= \nu \bar{\ell} \sum_{n=1}^{\infty} \mu_a(n) \frac{(i\xi)^n}{n!} \int dx F^n(x), \quad (C-12)
\end{aligned}$$

allowing for immediate identification of the cumulants of $I(t)$ as

$$\chi_I(n) = \nu \bar{\ell} \mu_a(n) \int dx F^n(x) \quad \text{for } n \geq 1; \chi_I(0) = 0. \quad (C-13)$$

For $\ell_k = 1$ for all k , this reduces to [1, eq. 1.5-2].

The normalized cumulants of $I(t)$ are

$$r_I(n) = \frac{\chi_I(n)}{[\chi_I(2)]^{n/2}} = \frac{1}{(\nu \bar{\ell})^{n/2-1}} \frac{\mu_a(n) \int dx F^n(x)}{\left[\mu_a(2) \int dx F^2(x) \right]^{n/2}}. \quad (C-14)$$

These quantities tend to zero rapidly for $\nu \bar{\ell} \gg 1$; see also [1, eq. 1.6-3].

Thus $\nu \bar{\ell}$ has a pronounced effect on how Gaussian $I(t)$ is.

Behavior of characteristic function $f_I(\xi)$ at $\xi = \pm\infty$

If pulse $F(x)$ is non-zero only over (x_1, x_2) , we have

$$\int dx \{f_a[\xi F(x)] - 1\} = \int_{x_1}^{x_2} dx \{f_a[\xi F(x)] - 1\}. \quad (C-15)$$

Now if random variable a_k has a characteristic function f_a with the property that

$$f_a(\pm\infty) = 0, \quad (C-16)$$

then

$$(C-15) \rightarrow \int_{x_1}^{x_2} dx \{0-1\} = -(x_2 - x_1) \text{ as } \xi \rightarrow \pm\infty, \quad (C-17)$$

in which case (C-9) yields

$$f_I(\pm\infty) = \exp[-\sqrt{2}(x_2 - x_1)] . \quad (C-18)$$

If pulse extent $x_2 - x_1$ is infinite, as for the Gaussian or exponential pulses,

$$F(x) = \exp(-x^2) \text{ or } \exp(-x)U(x) , \quad (C-19)$$

then (C-18) is zero. On the other hand, if $x_2 - x_1$ is finite, as for circular pulse

$$F(x) = \begin{cases} (1-x^2)^{\frac{1}{2}} & \text{for } |x| < 1 \\ 0 & \text{for } |x| > 1 \end{cases} , \quad (C-20)$$

then

$$f_I(\pm\infty) = \exp[-\sqrt{2}] > 0 \quad \text{for circular pulse.} \quad (C-21)$$

This non-zero characteristic function value corresponds to an impulse at the origin of probability density function p_I , with area (C-21). Physically, this means that there are occasionally regions of the t -scale where no pulses overlap, and there $I(t) = 0$. The probability of this happening is, generally,

$$P_0 = \text{Prob} \{I(t) = 0\} = f_I(\pm\infty) = \exp[-\sqrt{2}(x_2 - x_1)] . \quad (C-22)$$

On the other hand, for the Gaussian or exponential pulses cited in (C-19), $x_2 - x_1 = +\infty$, and $f_I(\pm\infty) = 0$, meaning that there is no impulse at the origin of probability density function p_I . Physically, the infinite tails (even if single-sided, as for the exponential pulse) disallow $I(t)$ ever from becoming zero.

Cumulants of Continuous Portion of p_I .

The impulse at the origin means that probability density function p_I and cumulative distribution function P_I can be expressed respectively as

$$p_I(u) = p_0 \delta(u) + p_c(u) ,$$

$$P_I(u) = p_0 + \int_0^u dt p_c(t) \quad \text{for } u > 0, \quad (C-23)$$

where $p_c(u)$ is a continuous function of u , with area $1-p_0$. The characteristic function relation corresponding to (C-23) is

$$f_I(\xi) = p_0 + f_c(\xi) , \quad (C-24)$$

and the moments are related according to

$$\mu_c(n) = \begin{cases} \mu_I(0) - p_0 & \text{for } n=0 \\ \mu_I(n) & \text{for } n \geq 1 \end{cases} . \quad (C-25)$$

The cumulants of f_c or p_c can then be found from these moments (C-25), by recursive relations; see [4] or [6]. This procedure is necessary to get accurate series expansions for the probability density function p_c and its cumulative distribution function, without having to approximate a delta function.

Overlap Factor

In the case where pulse extent $x_2 - x_1$ is finite, it is possible to find the average number of overlapping pulses at any one time instant; this statistic, denoted by $\overline{K}_1$, is called the overlap factor. In order to determine it in a simple fashion, we concoct a very special shot noise process: let

$$a_k = 1 \text{ for all } k ,$$

$$F(x) = 1 \text{ for } x_1 < x < x_2 . \quad (C-26)$$

Then $I(t)$ is a step function with amplitudes limited to the values 0, 1, 2, Then obviously, the average number of overlapping pulses at one time instant is just

$$\overline{K}_1 = \overline{I(t)} = v \overline{\lambda} \mu_a(1) \int dx F(x) = v \overline{\lambda} (x_2 - x_1) , \quad (C-27)$$

upon use of (C-13) with $n=1$ and (C-26). If we let

$$\overline{\tau} = \overline{\lambda} (x_2 - x_1) \quad (C-28)$$

denote the average pulse duration, we have the overlap factor in the form

$$\overline{K}_1 = v \overline{\tau} . \quad (C-29)$$

For the Gaussian or exponential pulses in (C-19), we have $x_2 - x_1 = +\infty$, giving $\overline{\tau} = +\infty$, $\overline{K}_1 = +\infty$. This is in fact true, since all the infinite tails overlap; however, it is not then an informative statistic.

Closed Form Characteristic Function Examples

There are a couple of examples of the circular pulse shape F and amplitude characteristic function f_a , where (C-9) can be evaluated in closed form. This furnishes an alternative to the moment approach [4] used here.

Consider the circular pulse in (C-20); then the integral in (C-9) is (using (C-15))

$$\int_{-1}^1 dx \left\{ f_a \left[\xi (1-x^2)^{1/2} \right] \right\} = 2 \int_0^{\pi/2} d\theta \cos \theta f_a [\xi \cos \theta] \quad (C-30)$$

which holds for any characteristic function f_a . Now first let the probability density function of a_k be exponential:

$$p(a) = \frac{1}{\mu_a} \exp\left(-\frac{a}{\mu_a}\right) U(a), \quad f_a(\xi) = (1 - i\xi\mu_a)^{-1}. \quad (C-31)$$

Substitution in (C-30) yields

$$2 \int_0^{\pi/2} \frac{d\theta \cos \theta}{1 - i\xi\mu_a \cos \theta} = 2. \quad (C-32)$$

But we know that

$$\begin{aligned} 2 \int_0^{\pi/2} \frac{d\theta \cos \theta}{1 - z \cos \theta} &= -\frac{\pi}{z} + \frac{4}{z(1-z^2)^{1/2}} \arctan \left[\left(\frac{1+z}{1-z} \right)^{1/2} \right] = \\ &= -\frac{\pi}{z} + \frac{2}{z(1-z^2)^{1/2}} \arccos(-z), \end{aligned} \quad (C-33)$$

via [5, eqs. 2.554 2 and 2.553 3]. Then letting $z = i\xi\mu_a$ and using [2, eqs. 4.4.2 with 4.4.26], (C-32) becomes

$$\frac{-2 \ln(s - \mu_a \xi) + i\pi(s-1)}{\mu_a \xi s} - 2, \quad \text{with } s = \left(1 + \mu_a^2 \xi^2\right)^{1/2}. \quad (\text{C-34})$$

Combining these results in (C-9), the closed form characteristic function is

$$f_I(\xi) = \exp \left[-\frac{2\sqrt{2}}{\mu_a \xi} \left\{ \mu_a \xi s + \ln(s - \mu_a \xi) - i \frac{\pi}{2}(s-1) \right\} \right], \quad (\text{C-35})$$

which holds for a circular pulse F and an exponential probability density function $p(a)$.

The second example is the one considered in detail here, namely the Rayleigh probability density function $p(a)$ given in (3). First substituting (C-30) in (C-9), we have characteristic function

$$f_I(\xi) = \exp[2\sqrt{2}(J(\xi)-1)], \quad (\text{C-36})$$

where integral $J(\xi)$ is defined as

$$J(\xi) = \int_0^{\pi/2} d\theta \cos \theta f_a[\xi \cos \theta]. \quad (\text{C-37})$$

For Rayleigh probability density function (3), (C-37) can be expressed as follows:

$$J(\xi) = \int_0^{\pi/2} d\theta \cos \theta \int_0^{\infty} da \exp(ia\xi \cos \theta) \frac{a}{\sigma_a^2} \exp\left(-\frac{a^2}{2\sigma_a^2}\right). \quad (\text{C-38})$$

Transform to rectangular coordinates according to $a \cos \theta = \sigma_a x$,
 $a \sin \theta = \sigma_a y$, and obtain

$$J(\xi) = \int_0^{\infty} \int_0^{\infty} dx dy \frac{x}{(x^2+y^2)^{1/2}} \exp\left(i\xi \sigma_a x - \frac{x^2+y^2}{2}\right). \quad (\text{C-39})$$

But the integral on y here is, via $y = x u^{1/2}$, equal to

$$\frac{1}{2} \int_0^{\infty} \frac{du}{u^{1/2}(1+u)^{1/2}} \exp\left(-\frac{1}{2} x^2 u\right) = \frac{1}{2} \exp\left(\frac{x^2}{4}\right) K_0\left(\frac{x^2}{4}\right), \quad (C-40)$$

the latter by means of [5, eq. 3.364 3]. Thus (C-39) becomes

$$\begin{aligned} J(\xi) &= \int_0^{\infty} dx \, x \exp\left(i\xi\sigma_a x - \frac{x^2}{2}\right) \frac{1}{2} \exp\left(\frac{x^2}{4}\right) K_0\left(\frac{x^2}{4}\right) = \\ &= \int_0^{\infty} du \exp(i \, 2^{1/2} \sigma_a \xi u) u \exp\left(-\frac{u^2}{2}\right) K_0\left(\frac{u^2}{2}\right). \end{aligned} \quad (C-41)$$

At this point, we have two alternatives. First, (C-41) could be efficiently evaluated for all ξ via an FFT; the decay of the integrand is according to $\exp(-u^2)$ for large u . Secondly, $J(\xi)$ can be expressed in a closed form in terms of a hypergeometric function; specifically

$$\begin{aligned} J(\xi) &= {}_2F_2\left(1, 1; \frac{1}{2}, \frac{3}{2}; -2b^2\right) + \\ &+ i\left(\frac{\pi}{2}\right)^{1/2} b \exp(-b^2) [I_0(b^2) - I_1(b^2)], \end{aligned} \quad (C-42)$$

where $b = \sigma_a \xi / 2$. The upper line follows from [5, eq. 6.755 6], while the lower line used [5, eq. 6.755 9] with an application of partial derivative $\partial/\partial a$ to both sides. The characteristic function f_I is finally obtained by employing (C-42) in (C-36).

Still another alternative is afforded by use of the closed form for the characteristic function of the Rayleigh probability density function, as given in [7, eq. 6].

Moments for Some Particular Pulse Shapes F

The moments of pulse shape F were encountered in evaluating the cumulants $\chi_I(n)$ of shot noise $I(t)$, according to (24) or (C-13) as

$$\mu_F(n) = \int dx F^n(x) \quad \text{for } n \geq 0. \quad (\text{C-43})$$

For circular pulse (C-20), [5, eq. 3.621 1] yields moments

$$\mu_F(n) = \int_{-1}^1 dx (1-x^2)^{n/2} = 2 \int_0^{\pi/2} d\theta (\cos\theta)^{n+1} = \frac{2^{n+1} \Gamma^2\left(\frac{n+1}{2}\right)}{\Gamma(n+2)}, \quad (\text{C-44})$$

a result already quoted in (29).

More generally, for

$$F(x) = \begin{cases} (1-x^2)^\alpha & \text{for } |x| < 1 \\ 0 & \text{for } |x| > 1 \end{cases}, \quad (\text{C-45})$$

[5, eq. 3.621 1] yields, with a trigonometric substitution,

$$\mu_F(n) = 2^{2n\alpha+1} \frac{\Gamma^2(n\alpha+1)}{\Gamma(2n\alpha+2)}. \quad (\text{C-46})$$

For

$$F(x) = \begin{cases} (\cos x)^\alpha & \text{for } |x| < \frac{\pi}{2} \\ 0 & \text{for } |x| > \frac{\pi}{2} \end{cases}, \quad (\text{C-47})$$

[5, eq. 3.621 1] yields directly

$$\mu_F(n) = 2^{n\alpha} \frac{\Gamma^2\left(\frac{n\alpha+1}{2}\right)}{\Gamma(n\alpha+1)}. \quad (\text{C-48})$$

For

$$F(x) = x^\alpha \exp(-x) U(x) ,$$

$$\mu_F(n) = \frac{\Gamma(n\alpha+1)}{n^{\alpha+1}} , \quad (C-49)$$

while for

$$F(x) = x^\alpha \exp(-x^2/2) U(x) ,$$

$$\mu_F(n) = \frac{2^{\frac{n\alpha-1}{2}} \Gamma\left(\frac{n\alpha+1}{2}\right)}{n^{\frac{n\alpha+1}{2}}} . \quad (C-50)$$

Both relations follow directly from the definition of the Γ function.

Some Probability Density Functions for Amplitude a_k

For probability density function

$$p(a) = \frac{(a/\alpha)^\gamma \exp(-a/\alpha)}{\alpha \Gamma(\gamma+1)} U(a) , \quad (C-51)$$

we have characteristic function

$$f_a(\xi) = (1-i\xi\alpha)^{-\gamma-1} \quad (C-52)$$

with moments

$$\mu_a(n) = (\gamma+1)_n \alpha^n \quad \text{for } n \geq 0 \quad (C-53)$$

and cumulants

$$\chi_a(n) = (n-1)! (\gamma+1) \alpha^n \quad \text{for } n \geq 1 . \quad (C-54)$$

This example subsumes the exponential probability density function, upon setting $\gamma = 0$.

For probability density function

$$p(a) = \frac{2(a/\alpha)^\gamma \exp(-a^2/\alpha^2)}{\alpha \Gamma(\frac{\gamma+1}{2})} U(a) , \quad (C-55)$$

we have moments

$$\mu_a(n) = \frac{\Gamma(\frac{n+\gamma+1}{2})}{\Gamma(\frac{\gamma+1}{2})} \alpha^n \quad \text{for } n \geq 0 . \quad (C-56)$$

This example subsumes respectively the one-sided Gaussian for $\gamma=0$, the Rayleigh for $\gamma=1$, and the Maxwell probability density functions for $\gamma=2$. The result in (29) follows immediately by setting $\gamma=1$, $\alpha = 2^{1/2} \sigma_a$.

Convergence of Series for $\ln f_I(\xi)$

A power series expansion for $\ln f_I(\xi)$ was developed in (C-12), namely,

$$\ln f_I(\xi) = \nu \bar{\lambda} \sum_{n=1}^{\infty} \mu_a(n) \mu_F(n) \frac{(i\xi)^n}{n!} ; \quad (C-57)$$

here we employed (C-43). Since the moments in (C-57) can be easily evaluated via recursion, according to results in the above two subsections, it might be thought that (C-57) could be employed to evaluate the characteristic function of $I(t)$ directly, without recourse to the more difficult approaches required in (C-9) or (C-35) or (C-36)-(C-42).

To see the drawbacks of this approach, consider first a circular pulse F and a generalized exponential probability density function $p(a)$ as in (C-51); then a combination of (C-44) and (C-53) yields, for the n -th term of the sum in (C-57),

$$T_n = \frac{2(\gamma+1)_n \Gamma^2\left(\frac{n+1}{2}\right)}{n!(n+1)!} (i\xi\alpha 2)^n. \quad (C-58)$$

Then ratio

$$\frac{T_n}{T_{n-2}} = - \frac{(n+\gamma)(n+\gamma-1)}{n^2-1} \xi^2 \alpha^2 \sim - \xi^2 \alpha^2 \quad \text{as } n \rightarrow +\infty, \quad (C-59)$$

regardless of the value of γ . Therefore

$$\left| \frac{T_n}{T_{n-1}} \right| \sim |\xi| \alpha \quad \text{as } n \rightarrow +\infty, \quad (C-60)$$

meaning that series (C-57) only converges for $|\xi| < 1/\alpha$. So (C-57) is not a viable approach for the calculation of the characteristic function in this case.

As a second example, we consider the circular pulse F with the generalized Rayleigh probability density function in (C-55). Combination of (C-44) with (C-56) yields for the n -th term of series (C-57),

$$T_n = \frac{2\Gamma^2\left(\frac{n+1}{2}\right) \Gamma\left(\frac{n+\gamma+1}{2}\right)}{n!(n+1)! \Gamma\left(\frac{\gamma+1}{2}\right)} (i\xi\alpha 2)^n. \quad (C-61)$$

Then ratio

$$\frac{T_n}{T_{n-2}} = - \frac{n+\gamma-1}{n^2-1} \frac{\xi^2 \alpha^2}{2} \sim - \frac{\xi^2 \alpha^2}{2n} \quad \text{as } n \rightarrow +\infty, \quad (C-62)$$

regardless of γ . Therefore

$$\left| \frac{T_n}{T_{n-1}} \right| \sim \frac{|\xi| \alpha}{(2n)^{1/2}} \quad \text{as } n \rightarrow +\infty, \quad (C-63)$$

meaning that series (C-57) converges for all ξ . However, direct numerical evaluation of (C-57) via (C-61) and (C-62) loses all its significant digits for large ξ , long before $\ln f_I(\xi)$ reaches its final value of $-2\bar{\alpha}_v + i0$, due to the alternating character of the series. So (C-57) is not a useful approach for evaluation of the characteristic function, except for small ξ . By contrast, the series expansion technique employed in [4] uses the moments to directly estimate the desired probability density function and cumulative distribution function of interest, for large arguments as well as small.

APPENDIX D. PROGRAMS FOR CUMULATIVE DISTRIBUTION FUNCTION
AND PROBABILITY DENSITY FUNCTION

The programs used here to evaluate the cumulative distribution function and probability density function of shot noise are listed below. The n -th coefficient in a generalized Laguerre expansion of orthonormal polynomials is denoted by b_n and is plotted in figure D-1 for $n = 0(1)70$. It is seen to oscillate and decay with n until $n = 32$, at which point round-off error becomes important; however, by this time, $|b_n|$ has decayed below the $1E-5$ level. The round-off error is so dominant beyond $n = 35$, that no useful results for b_n can be obtained then. The particular parameter values (α, β) used for the Laguerre weighting are indicated in the listings.

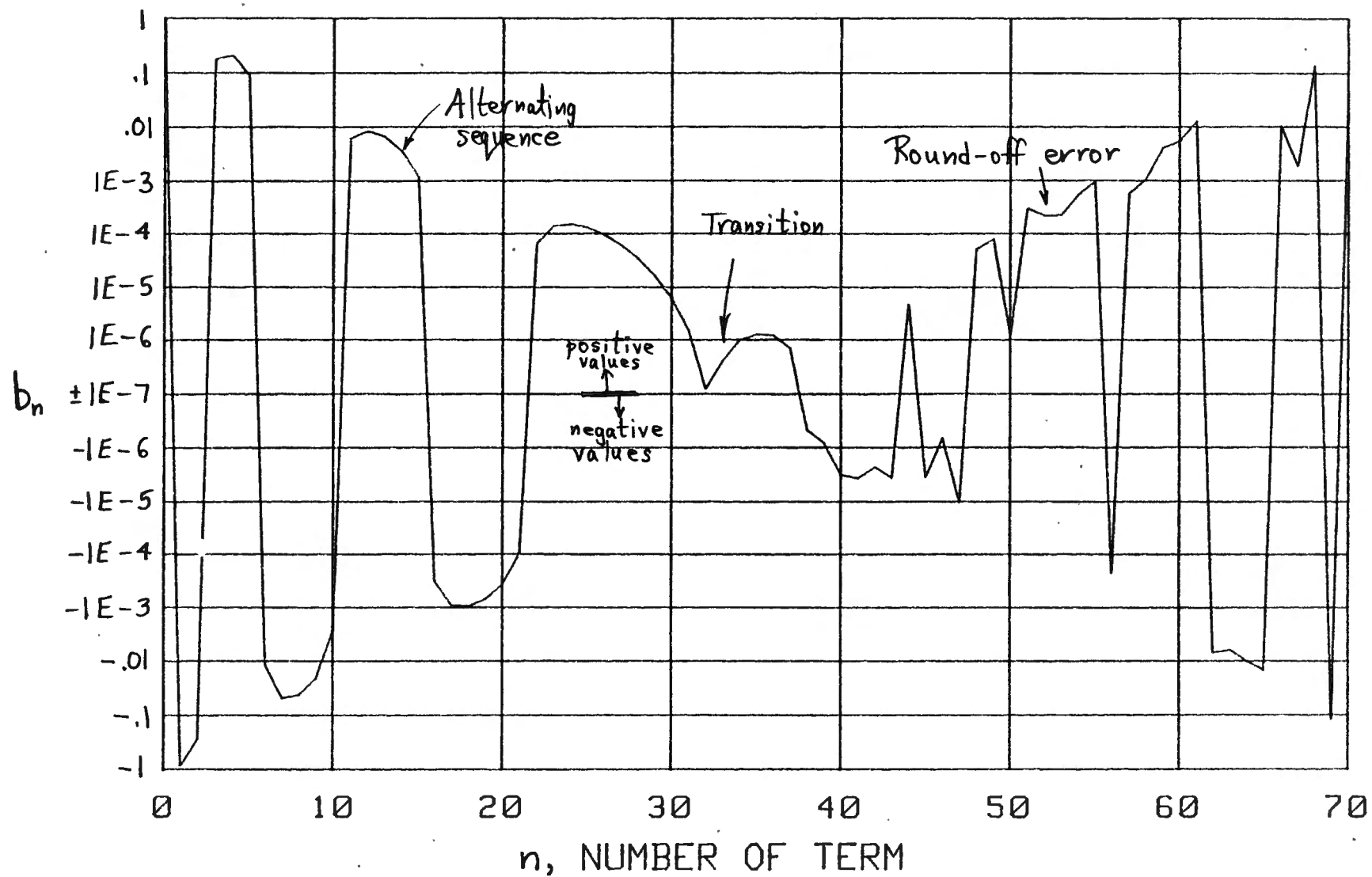


Figure D-1. Coefficient b_n in Laguerre Expansion

```

10 ! STEP PLUS CONTINUOUS PART OF SHOT NOISE CDF, Pc(u),
20 ! VIA GENERALIZED LAGUERRE EXPANSION AND MOMENTS
30 M=70 ! MAXIMUM ORDER OF APPROXIMATION; NUMBER OF MOMENTS REQUIRED
40 DOUBLE M,I,N,K ! INTEGERS < 2^31 = 2,147,483,648
50 REDIM Mom(0:M),A(0:M),L(0:M)
60 REAL Mom(0:100),A(0:100),L(0:100),Y(1:21)
70 CALL Moments(M,P0,Mom(*)) ! P0 IS STEP AT ORIGIN
80 Center=Mom(1)/Mom(0) ! CENTER OF pc(u)
90 R2=Mom(2)/Mom(0)-Center*Center ! MEAN SQUARE SPREAD OF pc(u)
100 Rms=SQR(R2) ! RMS SPREAD OF pc(u)
110 Alpha0=Center*Center/R2-1. ! THE CHOICES Alpha=Alpha0 AND
120 Beta0=R2/Center ! Beta=Beta0 WOULD MAKE A(1)=A(2)=0
130 Alpha=.74
140 Beta=2.1
150 CALL Coeffld_via_mom(M,Alpha,Beta,Mom(*),A(*)) ! DIRECT MOMENTS
160 ! CALL Coefflr_via_mom(M,Alpha,Beta,Mom(*),A(*)) ! RECURSIVE MOMENTS
170 PRINT "Center = ";Center
180 PRINT "Rms = ";Rms
190 A1=Alpha+1.
200 O1=1./A1
210 F1=1./FNGamma(A1)
220 DATA .001,.002,.005,.01,.02,.05,.1,.2,.3,.4,.5
230 DATA .6,.7,.8,.9,.95,.98,.99,.995,.998,.999
240 READ Y(*)
250 FOR I=1 TO 21
260 Y(I)=FNInvphi(Y(I))
270 NEXT I
280 Y1=Y(1)
290 Y2=Y(21)
300 INPUT "ORDER AND LIMITS:",N,U1,U2
310 PRINT "ORDER AND LIMITS:",N;U1;U2
320 Du=(U2-U1)/100.
330 PLOTTER IS "GRAPHICS"
340 GRAPHICS ON
350 WINDOW U1,U2,Y1,Y2
360 FOR U=U1 TO U2 STEP (U2-U1)*.1
370 MOVE U,Y1
380 DRAW U,Y2
390 NEXT U
400 FOR I=1 TO 21
410 MOVE U1,Y(I)
420 DRAW U2,Y(I)
430 NEXT I
440 PENUP
450 FOR I=1 TO 100
460 U=U1+Du*I
470 T=U/Beta
480 CALL Laguerre(N-1,A1,T,L(*))
490 Sum=A(0)*FNF11(A1,T)*O1
500 FOR K=1 TO N
510 Sum=Sum+A(K)*L(K-1)/K
520 NEXT K
530 P=P0+F1*EXP(-T+A1*LOG(T))*Sum ! PROBABILITY THAT RV < U
540 IF P>0. AND P<1. THEN 570
550 PENUP
560 GOTO 580
570 PLOT U,FNInvphi(P)
580 NEXT I
590 PENUP
600 GOTO 300
610 END
620 !

```

```

630 DEF FNInvphi(X) ! INVPHI(X) via 26.2.23 with modification
640 D=X-.5
650 IF ABS(D)>.01 THEN 680
660 P=2.50662827463*D*(1.+D*D*1.04719755120)
670 RETURN P
680 P=X
690 IF X>.5 THEN P=1.-X
700 P=SQR(-2.*LOG(P))
710 T=1.+P*(1.432788+P*(.189269+P*.001308))
720 P=P-(2.515517+P*(.802853+P*.010328))/T
730 IF X<.5 THEN P=-P
740 RETURN P
750 FNEND
760 !
770 DEF FNGamma(X) ! Gamma(X) via HART, page 282, #5243
780 DOUBLE N,K
790 N=INT(X)
800 R=X-N
810 IF N>0 OR R<>0. THEN 840
820 PRINT "FNGamma(X) IS NOT DEFINED FOR X = ";X
830 STOP
840 IF R>0. THEN 870
850 Gamma2=1.
860 GOTO 940
870 P=439.330444060025676+R*(50.1086937529709530+R*6.74495072459252899)
880 P=8762.71029785214896+R*(2008.52740130727912+R*P)
890 P=42353.6895097440896+R*(20886.8617892698874+R*P)
900 Q=499.028526621439048-R*(189.498234157028016-R*(23.081551524580125-R))
910 Q=9940.30741508277090-R*(1528.60727377952202+R*Q)
920 Q=42353.6895097440900+R*(2980.38533092566499-R*Q)
930 Gamma2=P/Q ! Gamma(2+R) for 0 < R < 1
940 IF N>2 THEN 980
950 IF N<2 THEN 1030
960 Gamma=Gamma2
970 RETURN Gamma
980 Gamma=Gamma2
990 FOR K=1 TO N-2
1000 Gamma=Gamma*(X-K)
1010 NEXT K
1020 RETURN Gamma
1030 R=1.
1040 FOR K=0 TO 1-N
1050 R=R*(X+K)
1060 NEXT K
1070 Gamma=Gamma2/R
1080 RETURN Gamma
1090 FNEND
1100 !

```

```

1110 DEF FNF11(A1,X) ! IF1<1;A1+1;X>
1120 DOUBLE K
1130 T=S=1.
1140 FOR K=1 TO 200
1150 T=T*X/(A1+K)
1160 S=S+T
1170 IF T<=1.E-17*S THEN RETURN S
1180 NEXT K
1190 PRINT "200 TERMS IN FNF11 AT";A1;X
1200 RETURN S
1210 FNEND
1220 !
1230 SUB Laguerre(DOUBLE N,REAL Alpha,X,L(*)) ! Ln\alpha(X)
1240 DOUBLE K
1250 A1=Alpha-1.
1260 L(0)=1.
1270 L(1)=Alpha+1.-X
1280 FOR K=2 TO N
1290 L(K)=((K+K+A1-X)*L(K-1)-(K+A1)*L(K-2))/K
1300 NEXT K
1310 SUBEND
1320 !
1330 SUB Momnt_via_cumnt(DOUBLE N,REAL Cum(*),Mom(*))
1340 DOUBLE K,J
1350 REAL Mom0
1360 Mom(0)=Mom0=EXP(Cum(0))
1370 FOR K=1 TO N
1380 T=1.
1390 S=Cum(K)*Mom0
1400 FOR J=1 TO K-1
1410 T=T*(K-J)/J
1420 S=S+T*Cum(K-J)*Mom(J)
1430 NEXT J
1440 Mom(K)=S
1450 NEXT K
1460 SUBEND
1470 !

```

```

1480 SUB Coeffld_via_mom(DOUBLE M,REAL Alpha,Beta,Mom(*),A(*))
1490 ALLOCATE B(0:M)
1500 DOUBLE K,K1,J,Mx
1510 T=1.
1520 FOR K=1 TO M
1530 T=T*(Alpha+K)*Beta
1540 Mom(K)=Mom(K)/T ! NORMALIZED MOMENTS, RELATIVE TO Alpha AND Beta
1550 NEXT K
1560 Q=1.
1570 A(0)=B(0)=Mom(0)
1580 FOR K=1 TO M
1590 K1=K+1
1600 T=1.
1610 S=Mom(0)
1620 FOR J=1 TO K
1630 T=T*(J-K1)/J
1640 S=S+T*Mom(J)
1650 NEXT J
1660 Q=Q*(Alpha+K)/K
1670 A(K)=S
1680 B(K)=S*SQR(Q)
1690 NEXT K
1700 Mx=Mx+10
1710 IF Mx<M THEN 1700
1720 Threshold=-7.
1730 T2=Threshold*2.
1740 V=10.^Threshold
1750 GINIT
1760 PLOTTER IS "GRAPHICS"
1770 GRAPHICS ON
1780 WINDOW 0.,FLT(Mx),T2,0.
1790 LINE TYPE 3
1800 FOR J=0 TO Mx STEP 10
1810 MOVE J,T2
1820 DRAW J,0.
1830 NEXT J
1840 FOR J=T2 TO 0
1850 MOVE 0.,J
1860 DRAW Mx,J
1870 NEXT J
1880 PENUP
1890 LINE TYPE 1
1900 IMAGE 4D,2(4X,M.17DE)
1910 PRINT " K B(K) Sum"
1920 Sum=0.
1930 FOR K=0 TO M
1940 B=B(K)
1950 Sum=Sum+B*B
1960 PRINT USING 1900;K,B,Sum
1970 IF B<V THEN 2000
1980 Y=LGT(B)
1990 GOTO 2040
2000 IF B>-V THEN 2030
2010 Y=T2-LGT(-B)
2020 GOTO 2040
2030 Y=Threshold
2040 PLOT K,Y
2050 NEXT K
2060 PENUP
2070 SUBEND
2080 !

```



```

2090 SUB Coefflr_via_mom(DOUBLE M,REAL Alpha,Beta,Mom(*),A(*))
2100 ALLOCATE B(0:M)
2110 DOUBLE K,K1,J,Mx
2120 T=1.
2130 FOR K=1 TO M
2140 T=T*(Alpha+K)*Beta
2150 Mom(K)=Mom(K)/T ! NORMALIZED MOMENTS, RELATIVE TO Alpha AND Beta
2160 NEXT K
2170 Q=1.
2180 A0=A(0)=B(0)=Mom(0)
2190 FOR K=1 TO M
2200 K1=K+1
2210 T=1.
2220 S=Mom(K)-A0
2230 FOR J=1 TO K-1
2240 T=T*(J-K1)/J
2250 S=S-T*A(J)
2260 NEXT J
2270 IF K MOD 2=1 THEN S=-S
2280 Q=Q*(Alpha+K)/K
2290 A(K)=S
2300 B(K)=S*SQR(Q)
2310 NEXT K
2320 Mx=Mx+10
2330 IF Mx<M THEN 2320
2340 Threshold=-7.
2350 T2=Threshold*2.
2360 V=10.^Threshold
2370 GINIT
2380 PLOTTER IS "GRAPHICS"
2390 GRAPHICS ON
2400 WINDOW 0.,FLT(Mx),T2,0.
2410 LINE TYPE 3
2420 FOR J=0 TO Mx STEP 10
2430 MOVE J,T2
2440 DRAW J,0.
2450 NEXT J
2460 FOR J=T2 TO 0
2470 MOVE 0.,J
2480 DRAW Mx,J
2490 NEXT J
2500 PENUP
2510 LINE TYPE 1
2520 IMAGE 4D,2(4X,M.17DE)
2530 PRINT " K B(K) Sum"
2540 Sum=0.
2550 FOR K=0 TO M
2560 B=B(K)
2570 Sum=Sum+B*B
2580 PRINT USING 2520;K,B,Sum
2590 IF B<V THEN 2620
2600 Y=LGT(B)
2610 GOTO 2660
2620 IF B>-V THEN 2650
2630 Y=T2-LGT(-B)
2640 GOTO 2660
2650 Y=Threshold
2660 PLOT K,Y
2670 NEXT K
2680 PENUP
2690 SUBEND
2700 !

```

```

2710 SUB Moments(DOUBLE M,REAL P0,Mom(*)) ! SHOT NOISE
2720 Overlap=6.2 ! AV. NO. PULSES/SEC * AVERAGE PULSE DURATION
2730 Sigmaa=1. ! PARAMETER OF RAYLEIGH AMPLITUDE PDF
2740 P0=EXP(-Overlap) ! PROBABILITY OF ZERO AMPLITUDE OF SHOT NOISE
2750 ALLOCATE Cum(0:M) ! ARRAY FOR CUMULANTS
2760 DOUBLE K
2770 S=Sigmaa*Sigmaa
2780 Cum(0)=0.
2790 Cum(1)=Overlap*Sigmaa*.25*PI*SQR(.5*PI)
2800 Cum(2)=Overlap*S*4./3.
2810 FOR K=3 TO M
2820 Cum(K)=Cum(K-2)*S*K*K/(K+1)
2830 NEXT K
2840 CALL Momnt_via_cumnt(M,Cum(*),Mom(*))
2850 Mom(0)=Mom(0)-P0
2860 SUBEND

```

```

10 ! CONTINUOUS PART OF SHOT NOISE PDF, pc(u), VIA
20 ! GENERALIZED LAGUERRE EXPANSION AND MOMENTS
30 M=90 ! MAXIMUM ORDER OF APPROXIMATION; NUMBER OF MOMENTS REQUIRED
40 DOUBLE M,I,N,K ! INTEGERS < 2^31 = 2,147,483,648
50 REDIM Mom(0:M),A(0:M),L(0:M)
60 REAL Mom(0:100),A(0:100),L(0:100)
70 CALL Moments(M,P0,Mom(*)) ! P0 IS STEP AT ORIGIN
80 Center=Mom(1)/Mom(0) ! CENTER OF pc(u)
90 R2=Mom(2)/Mom(0)-Center*Center ! MEAN SQUARE SPREAD OF pc(u)
100 Rms=SQR(R2) ! RMS SPREAD OF pc(u)
110 Alpha0=Center*Center/R2-1. ! THE CHOICES Alpha=Alpha0 AND
120 Beta0=R2/Center ! Beta=Beta0 WOULD MAKE A(1)=A(2)=0
130 Alpha=.74
140 Beta=2.1
150 CALL Coeffld_via_mom(M,Alpha,Beta,Mom(*),A(*)) ! DIRECT MOMENTS
160 ! CALL Coefflr_via_mom(M,Alpha,Beta,Mom(*),A(*)) ! RECURSIVE MOMENTS
170 PRINT "Center = ";Center
180 PRINT "Rms = ";Rms
190 F1=1./(Beta*FNGamma(Alpha+1.))
200 INPUT "ORDER AND LIMITS: ",N,U1,U2
210 PRINT "ORDER AND LIMITS: ",N;U1;U2
220 Du=(U2-U1)/100.
230 H=4./(U2-U1)
240 PLOTTER IS "GRAPHICS"
250 GRAPHICS ON
260 WINDOW U1,U2,-H*.1,H
270 GRID (U2-U1)*.1,H*.1
280 PLOT 0.,0.
290 FOR I=1 TO 100
300 U=U1+Du*I
310 T=U/Beta`a
320 CALL Laguerre(N,Alpha,T,L(*))
330 Sum=A(0)
340 FOR K=1 TO N
350 Sum=Sum+A(K)*L(K)
360 NEXT K
370 P=F1*EXP(-T+Alpha*LOG(T))*Sum ! PDF OF RV AT U
380 PLOT U,P
390 NEXT I
400 PENUP
410 GOTO 200
420 END
430 !

```

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