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FUNCTION SPACES ASSOCIATED WITH SCHRÖDINGER OPERATORS: THE PÖSCHL-TELLER POTENTIAL

GESTUR ÓLAFSSON AND SHIJUN ZHENG

Abstract. We address the function space theory associated with the Schrödinger operator $H = -d^2/dx^2 + V$. The discussion is featured with potential $V(x) = -n(n+1)\operatorname{sech}^2 x$, which is called in quantum physics Pöschl-Teller potential. Using a dyadic system, we introduce Triebel-Lizorkin spaces and Besov spaces associated with H. We then use interpolation method to identify these spaces with the classical ones for a certain range of $p, q > 1$. A physical implication is that the corresponding wave function $\psi(t, x) = e^{-itH} f(x)$ admits appropriate time decay in the Besov space scale.

1. Introduction

Let $H = -d^2/dx^2 + V$ be a Schrödinger operator on $\mathbb{R}$ with real-valued potential function V. In quantum physics, H is the energy operator of a particle having one degree of freedom with potential V. If the potential has certain decay at ∞, then one may expect that asymptotically, as time tends to infinity, the motion of the associated perturbed quantum system resembles the free evolution. Indeed, it is well-known that if $\int_{\mathbb{R}} (1 + |x|)|V(x)|dx < \infty$, then the absolute continuous spectrum of H is $[0, \infty)$, the singular continuous spectrum is empty, and there is only finitely many negative eigenvalues. Moreover, the wave operators $W_{\pm} = s - \lim_{t\to\pm\infty} e^{itH} e^{-itH_0}$ exists and are complete [C01, DT79, Z04a].

Recently, several authors have studied function spaces associated with Schrödinger operators [JN94, E95, E96, DZ98, DZ02, BZ05]. One of the goals has been to develop the associated Littlewood-Paley theory, in order to give a unified approach. Motivated by the treatment in [BZ05, E95] for the barrier and Hermite cases, we consider H with the negative potential

$$V_n(x) = -n(n+1)\operatorname{sech}^2 x, \quad n \in \mathbb{N}, \tag{1.1}$$

which is called the Pöschl-Teller potential [B99, G89]. The study of H with this potential is related to linearization of nonlinear wave and Schrödinger equations. In this paper,

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we are mainly concerned with characterization and identification of the Triebel-Lizorkin spaces and Besov spaces associated with H. Notice that in contrast to the potentials studied in [BZ05, E95, DZ98, DZ02], $H = H_0 + V_n$ is not a positive operator and it has a resonance at zero.

Suppose $\{\varphi_j\}_0^\infty \subset C_0^\infty(\mathbb{R})$ satisfy: (i) supp $\varphi_0 \subset \{|x| \le 1\}$, supp $\varphi_j \subset \{2^{j-2} \le |x| \le 2^j\}$, $j \ge 1$; (ii) $|\varphi_j^{(m)}(x)| \le c_m 2^{-mj}, \quad \forall j, m \in \mathbb{N}_0$; and (iii)

$$\sum_{j=0}^{\infty} \varphi_j(x) = 1, \qquad \forall\, x \in \mathbb{R}. \tag{1.2}$$

Let $\alpha \in \mathbb{R}$, $0 < p < \infty$ and $0 < q \le \infty$. The *Triebel-Lizorkin space associated with* H, denoted by $F_p^{\alpha,q}(H)$, is defined to be the completion of the subspace $L_0^2 := \{f \in L^2(\mathbb{R}) : \|f\|_{F_p^{\alpha,q}(H)} < \infty\}$, where the quasi-norm $\|\cdot\|_{F_p^{\alpha,q}(H)}$ is initially defined for $f \in L^2(\mathbb{R})$ as

$$\|f\|_{F_p^{\alpha,q}(H)} = \|\big(\sum_{j=0}^{\infty} 2^{j\alpha q} |\varphi_j(H) f|^q\big)^{1/q}\|_{L^p} \tag{1.3}$$

(with usual modification if $q = \infty$).

Similarly, the *Besov space associated with* H, denoted by $B_p^{\alpha,q}(H)$, is defined by the quasi-norm

$$\|f\|_{B_p^{\alpha,q}(H)} = \big(\sum_{j=0}^{\infty} 2^{j\alpha q} \|\varphi_j(H) f\|_{L^p}^q\big)^{1/q}. \tag{1.4}$$

In Section 3 we give a maximal function characterization of $F_p^{\alpha,q}(H)$. We show in Theorem 3.5 that

$$\|f\|_{F_p^{\alpha,q}(H)} \approx \|\big(\sum_{j=0}^{\infty} (2^{j\alpha} \varphi_{j,s}^* f)^q\big)^{1/q}\|_p\,, \tag{1.5}$$

where $\varphi_{j,s}^* f$ is the Peetre type maximal function with $s > 1/\min(p,q)$. Therefore the definition of the $F_p^{\alpha,q}(H)$-norm is independent of the choice of $\{\varphi\}_{j\ge 0}$.

The proof of (1.5) essentially depends on the decay estimates in Lemma 3.1 for the kernel of $\varphi_j(H)$, which can be expressed in terms of continuum and discrete eigenfunctions of H. In Section 2 we solve the eigenfunction equation (2.1) for $k \in \mathbb{R} \cup \{i, \dots, ni\}$ ($i = \sqrt{-1}$), based on a method suggested in [Lam80]. In Section 4, using the explicit kernel of $\varphi_j(H)$ we give a proof of Lemma 3.1 for high and local energies. It turns out that for the absolute continuous part of H, the high and local energy analysis is simpler than the barrier potential, although H has a nonempty pure point spectrum.

A natural question arises: What is the relation between the perturbed function spaces and the ordinary ones, namely, $F_p^{\alpha,q}(\mathbb{R})$ and $B_p^{\alpha,q}(\mathbb{R})$? In this regard, we show in Section 5 that $F_p^{0,2}(H)$ is identically the L^p space, $1 < p < \infty$. Futhermore, in Section 6 we obtain the following result (Theorem 6.1) by means of complex interpolation: If $\alpha > 0$, $1 < p < \infty$

and $2p/(p+1) < q < 2p$, then

$$F_p^{\alpha,q}(H) = F_p^{2\alpha,q}(\mathbb{R}) \tag{1.6}$$

and if $\alpha > 0$, $1 \le p < \infty$, $1 \le q \le \infty$, then

$$B_p^{\alpha,q}(H) = B_p^{2\alpha,q}(\mathbb{R}).$$

The method in proving $F_p^{0,2}(H) = L^p$ is similar to the Hermite case [E95]. However, the identification (1.6) seems new for $\alpha > 0$. It is not difficult to see that the analogue of (1.6) does not hold for the Hermite case, where the potential is x^2.

As an application of the function space method we obtain a global time decay result (Theorem 6.3) for the solution to the Schrödinger equation (6.1), namely,

$$\|e^{-itH} f\|_{L^{p'}} \lesssim \langle t\rangle^{-(\frac{1}{p}-\frac{1}{2})} \|f\|_{B_p^{4\beta,2}(\mathbb{R})}$$

for $1 < p \le 2$ and $\beta = |\frac{1}{p} - \frac{1}{2}|$ being the critical exponent, which is a consequence of the local and long time decay estimates from [JN94] and [GSch04]. Here the perturbed function spaces play an important role in the interpretation of the mapping properties of operators between the abstract and classical spaces. It provides a necessary tool in realizing the above inequality by means of embedding and interpolation.

Finally, we mention that the homogeneous F and B spaces seem to deserve special attention. The crucial reason is that, to our surprise somehow, the decay estimates for the *low energy* ($-\infty < j < 0$) that are required for the derivative of $\varphi_j(H)E_{ac}(x,y)$ does not hold, which leaves open the question on obtaining the homogeneous version of Theorem 3.5. In a sequal to this paper we will consider the homogeneous case and study the spectral multiplier problem on the F and B spaces.

2. The eigenfunctions of H

Let $V_n = -n(n+1)\,\mathrm{sech}^2 x$ and $H_0 = -d^2/dx^2$. In this section we derive a simple expression for the continuum eigenfunctions of $H = H_0 + V_n$, which are the scattering solutions to the Lippman-Schwinger equation (2.3). We also show that the bound state eigenfunctions are rapid decaying functions.

2.1. Scattering equation.

Consider the eigenvalue problem for $(1+|x|)V \in L^1$,

$$He(x,k) = k^2 e(x,k), \quad k \in \mathbb{R}, \tag{2.1}$$

with asymptotics

$$e_\pm(x,k) \sim \begin{cases} T_\pm(k)e^{ikx} & \text{if } x \to \pm\infty \\ e^{ikx} + R_\pm(k)e^{-ikx} & \text{if } x \to \mp\infty, \end{cases} \tag{2.2}$$

where ± indicate the sign of k. We will use the notation

$$e(x,k) = \begin{cases} e_+(x,k) & \text{if } k > 0 \\ e_-(x,k) & \text{if } k < 0. \end{cases}$$

The coefficients $T_\pm(k)$ and $R_\pm(k)$ in (2.2) are called the *transmission coefficients* and *reflection coefficients*, resp. They satisfy the conservation law $|T_\pm(k)|^2 + |R_\pm(k)|^2 = 1$. It is easy to see that (2.1) together with (2.2) is equivalent to the Lippman-Schwinger equation

$$e_\pm(x,k) = e^{ikx} + \frac{1}{2i|k|}\int e^{i|k|\,|x-y|}V(y)e_\pm(y,k)\,dy. \tag{2.3}$$

2.2. **Inductive construction of the solution.** Let y_n be the general solution of

$$y_n'' + n(n+1)\operatorname{sech}^2 x\, y_n = -k^2 y_n\,.$$

If $n = 0$, $y_0 = Ae^{ikx} + Be^{-ikx}$. If $n \geq 1$, according to [Lam80, Section 2.6] we have by induction

$$y_n(x) = A(k)D_n\cdots D_1(e^{ikx}) + B(k)D_n\cdots D_1(e^{-ikx}),$$

where D_n denotes the differential operator

$$D_n = \frac{d}{dx} - n\tanh x, \qquad n \in \mathbb{N}. \tag{2.4}$$

Here we observe that since $\frac{d}{dx}(\tanh x) = 1 - \tanh^2 x$,

$$\begin{aligned} D_n\cdots D_1(e^{ikx}) &= p_n(\tanh x, ik)e^{ikx},\\ D_n\cdots D_1(e^{-ikx}) &= q_n(\tanh x, ik)e^{-ikx}, \end{aligned} \tag{2.5}$$

where $p_n(x,k)$ and $q_n(x,k)$ are polynomials of degree n in x, k and have real coefficients.

Let $e_n(x,k)$ denote the particular solution of (2.3) with $V = V_n$. Using the asymptotics (2.2) we solve $e_n(x,k)$ as in the following lemma.

Lemma 2.3. *Let $n \in \mathbb{N}$. There exists a polynomial $p_n(x,k)$ of degree n in x, k such that*

$$e_{n,\pm}(x,k) = A_n^\pm(k)p_n(\tanh x, ik)e^{ikx}\,,$$

Furthermore the following hold.

(a) *The constants $A_n^\pm(k)$ are given by*

$$A_n^+(k) = \prod_{j=1}^n \frac{1}{j+ik} \qquad \text{and} \qquad A_n^-(k) = (-1)^n\prod_{j=1}^n \frac{1}{j-ik}\,.$$

(b) *The transmission coefficients $T_{n,\pm}(k)$ are*

$$T_{n,+}(k) = (-1)^n\prod_{j=1}^n \frac{j-ik}{j+ik} \qquad \text{and} \qquad T_{n,-}(k) = (-1)^n\prod_{j=1}^n \frac{j+ik}{j-ik}\,.$$

(c) *The reflection coefficients $R_{n,\pm}(k)$ are all zero.*

Proof. In light of the above discussion we write

$$e_{n,\pm}(x,k) = A_n^{\pm}(k)p_n(\tanh x, ik)e^{ikx} + B_n^{\pm}(k)q_n(\tanh x, ik)e^{-ikx}. \tag{2.6}$$

First we assume $k > 0$. Substituting (2.6) into the (2.2), we obtain that $B_n^+(k) = 0 = R_{n,+}(k)$,

$$A_n^+(k)p_n(-1, ik) = 1 \tag{2.7}$$

and

$$T_{n,+}(k) = A_n^+(k)p_n(1, ik) = \frac{p_n(1, ik)}{p_n(-1, ik)}\,. \tag{2.8}$$

Thus (2.6) becomes

$$e_{n,+}(x,k) = A_n^+(k)p_n(\tanh x, ik)e^{ikx}.$$

From (2.5) we easily derive the recurrence formula

$$p_n(\tanh x, ik) = \operatorname{sech}^2 x\, p'_{n-1}(\tanh x, ik) + (ik - n\tanh x)p_{n-1}(\tanh x, ik). \tag{2.9}$$

Since $p'_{n-1}(x,k)$ is a polynomial in x, it follows that

$$\lim_{x\to\pm\infty} p'_{n-1}(\tanh x, ik) = p'_{n-1}(\pm 1, ik)$$

is bounded. Taking the limit in (2.9) as $x \to \pm\infty$ we find

$$p_n(\pm 1, ik) = (ik \mp n)p_{n-1}(\pm 1, ik).$$

Since $e_0(x,k) = e^{ikx}$, i.e., $p_0 = 1$, $A_0^+ = 1$, we obtain

$$p_n(1, ik) = (-1)^n \prod_{j=1}^{n} (j - ik)$$

and

$$p_n(-1, ik) = \prod_{j=1}^{n} (j + ik) = (-1)^n \overline{p_n(1, k)}.$$

Now for $k > 0$, (a), (b) in the lemma follow from (2.7), (2.8).

For k negative, similarly it holds that $B_n^-(k) = 0 = R_{n,-}(k)$ and instead of (2.7), (2.8), we have

$$A_n^-(k)p_n(1, ik) = 1$$

and

$$T_{n,-}(k) = A_n^-(k)p_n(-1, ik)\,.$$

Then the results for A_n^-, $T_{n,-}$ and $e_{n,-}(x,k)$ follow. □

From (2.5) we can also see

$$p_n(\tanh x, -ik) = (-1)^n p_n(-\tanh x, ik) \tag{2.10}$$

by simple induction. Thus we obtain the following formula for the continuum eigenfunctions.

Theorem 2.4. *Assume* $k \in \mathbb{R} \setminus \{0\}$. *Then*

$$e_n(x,k) = (\operatorname{sign}(k))^n \left(\prod_{j=1}^{n} \frac{1}{j + i|k|} \right) P_n(x,k) e^{ikx},$$

where $P_n(x,k) = p_n(\tanh x, ik)$ *is defined by the recursion formula*

$$p_n(\tanh x, ik) = \frac{d}{dx}\big(p_{n-1}(\tanh x, ik)\big) + (ik - n \tanh x) p_{n-1}(\tanh x, ik)\,.$$

In particular, the function

$$\mathbb{R} \times (\mathbb{R} \setminus \{0\}) \ni (x,k) \mapsto e_n(x,k) \in \mathbb{C}$$

is analytic with $e_n(x,-k) = e_n(-x,k)$. *Moreover, the function*

$$(x,y,k) \mapsto e_n(x,k)\overline{e_n(y,k)} = \left(\prod_{j=1}^{n} \frac{1}{j^2 + k^2} \right) P_n(x,k) P_n(y,-k) e^{ik(x-y)}$$

is real analytic on $\mathbb{R}^3$.

2.5. **The point spectrum.** For $(1 + |x|)V \in L^1$, we know that the point spectrum of $H_0 + V$ is given by the simple eigenvalues $-\mu^2$ such that $T_+(k)$ has a (simple) pole at $i\mu$; see e.g., [DT79, p.146]. Therefore we have

Lemma 2.6. *The point spectrum of* $H = H_0 + V_n$ *consists of*

$$\sigma_{pp} = \{-1, -4, \ldots, -n^2\}\,.$$

The corresponding eigenfunctions are Schwartz functions that are linear combinations of $\operatorname{sech}^m x \tanh^k x$, $m \in \mathbb{N}$, $k \in \mathbb{N}_0$.

Proof. The statement about σ_{pp} follows from the fact that $k = ij$, $j = 1, \ldots, n$, are the poles of $T_{n,+}(k) = (-1)^n \prod_{j=1}^{n} (j - ik)(j + ik)^{-1}$. For $k^2 = -j^2$, let $y_{n,j}$ be the corresponding eigenfunction. By induction we find that

$$\begin{aligned} y_{j,j} &= \operatorname{sech}^j x \\ y_{j+1,j} &= D_{j+1} \operatorname{sech}^j x \\ y_{j+m,j} &= D_{j+m} y_{j+m-1,j}, \qquad m \in \mathbb{N}. \end{aligned}$$

Hence the bound states are given by

$$y_{n,j}(x) = D_n \cdots D_{j+1} \operatorname{sech}^j x, \quad j = 1, \ldots, n-1,$$

and

$$y_{n,n}(x) = \operatorname{sech}^n x.$$

□

Remark 2.7. *There is a continuous extension of V_n when n is replaced by a continuous parameter in $\mathbb{R}$. We can find the scattering solutions of (2.3) by using the two real fundamental solutions given in* [Flu74]*. However we do not intend to include them here since the expression (which involves hypergeometric functions) seems quite complicated.*

2.8. **Projection of the spectral operator $\phi(H)$.** Given $V \in L^1 \cap L^2$, it is known that $H = H_0 + V$ is selfadjoint on the domain $D(H) = D(H_0) = W_2^2(\mathbb{R})$, the usual Sobolev space of order 2 in L^2. We decompose $L^2 = \mathcal{H}_{ac} \oplus \mathcal{H}_{pp}$, where $\mathcal{H}_{ac}$ denotes the absolute continuous subspace and $\mathcal{H}_{pp}$ the pure point subspace. Let E_{ac}, E_{pp} be the corresponding orthogonal projections, respectively. For a measurable function ϕ we define $\phi(H)$ by functional calculus as usual. Then it follows that

$$\phi(H)f = \phi(H)E_{ac}f + \phi(H)E_{pp}f = \phi(H)\big|_{\mathcal{H}_{ac}} f + \phi(H)\big|_{\mathcal{H}_{pp}} f.$$

Let $e(x,k)$ be the scattering solution of (2.3) and $e_j(x)$ the eigenfunction of H with (negative) eigenvalue λ_j. If ϕ is continuous and compactly supported, we have the following expression [Z04a]

$$\phi(H)f(x) = \int K_{ac}(x,y)f(y)dy + \sum_{\lambda_j \in \sigma_{pp}} \phi(\lambda_j)(f,e_j)e_j, \quad f \in L^1 \cap L^2, \tag{2.11}$$

where

$$K_{ac}(x,y) = (2\pi)^{-1} \int \phi(k^2) e(x,k)\bar{e}(y,k)dk \tag{2.12}$$

is the kernel of $\phi(H)E_{ac}$. Note that if $e(x,k)$ is smooth in x, then $K_{ac}(x,y)$ is smooth in x, y.

If letting $K_{pp}(x,y) = \sum_j \phi(\lambda_j)e_j(x)e_j(y)$, we can write (2.11) in a more compact form

$$\phi(H)f(x) = \int K(x,y)f(y)dy, \tag{2.13}$$

where $K = K_{ac} + K_{pp}$. We mention that in the case $(1+|x|)V \in L^1$ the kernel formula (2.12) agrees with the usual one using the Jost functions [GSch04, DT79].

3. Maximal function characterization

Let $H = H_0 + V_n$. This section is mainly to give a quasi-norm characterization of $F_p^{\alpha,q}(H)$ and $B_p^{\alpha,q}(H)$ using Peetre type maximal function. Consequently, the $F(H)$ and $B(H)$ spaces are well-defined in the sense that different dyadic systems give rise to equivalent quasi-norms.

Let $\{\varphi_j\}_0^\infty$ be a system satisfying conditions (i), (ii) as in Section 1, i.e.,

(i) supp $\varphi_0 \subset [-1,1]$, supp $\varphi_j \subset [-2^j, -2^{j-2}] \cup [2^{j-2}, 2^j]$, $j \geq 1$;

(ii) $|\varphi_j^{(m)}(x)| \leq c_m 2^{-mj}, \quad \forall j, m \in \mathbb{N}_0$.

Denote $K_j(x,y) = \varphi_j(H)(x,y)$ the kernel of $\varphi_j(H)$ as given by the formula (2.13). To simplify notation we let

$$w_j(x) := 1 + 2^{j/2}|x|\,. \tag{3.1}$$

Lemma 3.1. *Let $j \geq 0$. Then for each $m \in \mathbb{N}_0$ there exist constants $C_m, C'_m > 0$ such that*

(*a*) $|K_j(x,y)| \leq C_m 2^{j/2} w_j(x-y)^{-m}$

(*b*) $|\frac{\partial}{\partial x} K_j(x,y)| \leq C'_m 2^j w_j(x-y)^{-m}\,.$

We postpone the proof till Section 4.

For $s > 0$ define the analogue of Peetre maximal function:

$$\varphi^*_{j,s} f(x) = \sup_{t\in\mathbb{R}} \frac{|\varphi_j(H)f(t)|}{w_j(x-t)^s} \tag{3.2}$$

and

$$\varphi^{**}_{j,s} f(x) = \sup_{t\in\mathbb{R}} \frac{|(\varphi_j(H)f)'(t)|}{w_j(x-t)^s}.$$

Lemma 3.2. *Let $s > 0$ and $j \in \mathbb{N}_0$. Then there exists a constant $C = C_s > 0$ such that*

$$\varphi^{**}_{j,s} f(x) \leq C 2^{j/2} \varphi^*_{j,s} f(x).$$

Before the proof we note the following identity that will be used often later on. Suppose $\{\psi_j\}$ be a dyadic system as in Section 1. Then

$$\varphi_j(H)f = \sum_{\nu=-1}^{1} \psi_{j+\nu}(H)\varphi_j(H)f, \qquad f \in L^2, \tag{3.3}$$

with the convention $\psi_{-1} \equiv 0$, which follows from the equality $\varphi_j(x) = \sum_{\nu=-1}^{1} \psi_{j+\nu}(x)\varphi_j(x)$ for all x.

Proof. By (3.3) we have

$$\frac{d}{dt}(\varphi_j(H)f)(t) = \sum_{\nu=-1}^{1} \int_{\mathbb{R}} \frac{\partial}{\partial t}(\psi_{j+\nu}(H)(t,y))\varphi_j(H)f(y)\,dy\,.$$

Apply Lemma 3.1 to obtain

$$\frac{|\frac{d}{dt}(\varphi_j(H)f)(t)|}{w_j(x-t)^s} \leq C_m \sum_{\nu=-1}^{1} 2^{j+\nu} \int_{\mathbb{R}} \frac{|\varphi_j(H)f(y)|}{w_{j+\nu}(t-y)^m w_j(x-t)^s}\,dy\,.$$

It follows from the definition of $\varphi^*_{j,s}f$ that

$$\begin{aligned}\frac{|\frac{d}{dt}(\varphi_j(H)f)(t)|}{w_j(x-t)^s} \leq & C_m \sum_{\nu=-1}^{1} 2^{j+\nu}\varphi^*_{j,s}f(x)\int_{\mathbb{R}} \frac{w_j(t-y)^s}{w_{j+\nu}(t-y)^m}\,dy \\ \leq & C_s 2^{j/2}\varphi^*_{j,s}f(x)\,,\end{aligned}$$

provided $m - s > 1$. This proves Lemma 3.2. □

The next lemma (Peetre maximal inequality) follows from Lemma 3.2 by a standard argument; see [Tr83, p.16] or [BZ05]. Let M be the Hardy-Littlewood maximal function

$$Mf(x) := \sup_I \frac{1}{|I|}\int_I |f(x+y)|dy,$$

where the supremum runs over all intervals in $(-\infty, \infty)$.

Lemma 3.3. *Let $s > 0$ and $j \in \mathbb{N}_0$. There exists a constant $C_s > 0$ such that*

$$\varphi^*_{j,s}f(x) \leq C_s[M(|\varphi_j(H)f|^{1/s})]^s(x)\,.$$

Remark 3.4. *It is well known that M is bounded on $L^p, 1 < p < \infty$, i.e.,*

$$\|Mf\|_p \leq C\|f\|_p. \tag{3.4}$$

Moreover, if $1 < p < \infty$, $1 < q \leq \infty$ and $\{f_j\}$ is a sequence of functions, then

$$\|(\sum_j |Mf_j|^q)^{1/q}\|_{L^p} \leq C_{p,q}\,\|(\sum_j |f_j|^q)^{1/q}\|_{L^p}, \tag{3.5}$$

(usual modification if $q = \infty$) by the Fefferman-Stein vector-valued maximal inequality.

We now state the following theorem on maximal function characterization of $F^{\alpha,q}_p(H)$.

Theorem 3.5. *Let $\alpha \in \mathbb{R}$, $0 < p < \infty$ and $0 < q \leq \infty$. Let $\{\varphi_j\}_{j\geq 0}$ be a system satisfying (i), (ii) and (iii) as given in Section 1. If $s > 1/\min(p, q)$, then we have for $f \in L^2$*

$$\|f\|_{F^{\alpha,q}_p(H)} \approx \|\big(\sum_{j=0}^{\infty}(2^{j\alpha}\varphi^*_{j,s}f)^q\big)^{1/q}\|_p. \tag{3.6}$$

Furthermore, $F^{\alpha,q}_p(H)$ is a quasi-Banach space (Banach space if $p \geq 1$, $q \geq 1$) and it is independent of the choice of $\{\varphi_j\}_{j\geq 0}$.

Proof. Because $\varphi^*_{j,s}f(x) \geq |\varphi_j(H)f(x)|$, we only need to show

$$\|(\sum_{j=0}^{\infty}(2^{j\alpha}\varphi^*_{j,s}f)^q)^{1/q}\|_p \leq C\|f\|_{F^{\alpha,q}_p(H)}, \tag{3.7}$$

but this follows from Lemma 3.3 and (3.5). Indeed, choosing $0 < r = 1/s < \min(p,q)$, we have

$$\begin{aligned}\|\{2^{j\alpha}\varphi^*_{j,s}f\}\|_{L^p(\ell^q)} &\leq C_s\|\{2^{j\alpha}[M(|\varphi_j(H)f|^r)]^{1/r}\}\|_{L^p(\ell^q)}\\ &= C_s\|\big(\sum_0^\infty [M(2^{j\alpha r}|\varphi_j(H)f|^r)]^{q/r}\big)^{r/q}\|^{1/r}_{L^{p/r}}\\ &\leq C_{s,p,q}\|\{2^{j\alpha}\varphi_j(H)f\}\|_{L^p(\ell^q)}\\ &= C_{s,p,q}\|f\|_{F_p^{\alpha,q}(H)}\,,\end{aligned}$$

which proves (3.7).

To show the second statement let $\psi = \{\psi_j\}$ be another system satisfying the same condtions as $\varphi = \{\varphi_j\}$. We use (3.3) and Lemma 3.1 (a) to estimate

$$\begin{aligned}|\varphi_j(H)f(x)| &\leq C2^{j/2}\sum_{\nu=-1}^{1}\int_{\mathbb{R}}\frac{|\psi_{j+\nu}(H)f(y)|}{w_j(x-y)^m}\,dy\\ &\leq C\sum_{\nu=-1}^{1} 2^{j/2}\psi^*_{j+\nu,s}f(x)\int_{\mathbb{R}}\frac{w_{j+\nu}(x-y)^s}{w_j(x-y)^m}\,dy\\ &\leq C\sum_{\nu=-1}^{1}\psi^*_{j+\nu,s}f(x),\end{aligned}$$

provided $m - s > 1$. Thus, for $f \in L^2$

$$\|f\|^{\varphi}_{F_p^{\alpha,q}(H)} \leq C_{s,p,q}\|\{2^{j\alpha}\psi^*_{j,s}f\}\|_{L^p(\ell^q)} \approx \|f\|^{\psi}_{F_p^{\alpha,q}(H)}\,. \tag{3.8}$$

This concludes the proof. □

Remark 3.6. *Note that the statement in Theorem 3.5 is true for the more general system* $\rho = \{\rho_j\}_0^\infty$ *satisfying conditions (i), (ii) and (iii')*

$$\sum_j \rho_j(x) \approx c > 0.$$

In fact, let us fix a system $\{\varphi_j\}_0^\infty$ *as given in Theorem 3.5. Then the same argument in the proof of (3.8) shows*

$$\|f\|^{\rho}_{F_p^{\alpha,q}(H)} \leq C\|f\|^{\varphi}_{F_p^{\alpha,q}(H)}.$$

To show the other direction, we define

$$\tilde\varphi_j(x) = \varphi_j(x)/(\sum_j \rho_j(x)).$$

Then it is easy to verify that $\{\tilde{\varphi}_j\}$ *satisfies (i), (ii), and so,* $\tilde{\varphi}_j(H)(x,y)$ *satisfies the nice decay in Lemma 3.1. Now the identity*

$$\varphi_j(x) = \sum_{\nu=-1}^{1} \tilde{\varphi}_j(x)\rho_{j+\nu}(x)$$

and the proof of (3.8) yield

$$\|f\|^{\varphi}_{F_p^{\alpha,q}(H)} \le C\|f\|^{\rho}_{F_p^{\alpha,q}(H)}.$$

3.7. **Besov spaces for H.** Let $\alpha \in \mathbb{R}$, $0 < p < \infty$, $0 < q \le \infty$. We define $B_p^{\alpha,q}(H)$, the *Besov space* associated with H to be the completion of the subspace $\{f \in L^2 : \|f\|_{B_p^{\alpha,q}(H)} < \infty\}$ with respect to the norm $\|\cdot\|_{B_p^{\alpha,q}(H)}$, which is given by (1.4). Then $B_p^{\alpha,q}(H)$ is a quasi-Banach space (Banach space if p, $q \ge 1$).

Theorem 3.8. *Let* $\alpha \in \mathbb{R}$, $0 < p < \infty$, $0 < q \le \infty$. *If* $s > 1/p$, *then for* $f \in L^2$

$$\|f\|_{B_p^{\alpha,q}(H)} \approx (\sum_{j=0}^{\infty} 2^{j\alpha q}\|\varphi^*_{j,s}f\|^q_{L^p})^{1/q}.$$

Furthermore, $B_p^{\alpha,q}(H)$ *is well defined and independent of the choice of* $\{\varphi_j\}_{j\ge 0}$.

The proof of Theorem 3.8 is analogous to that of Theorem 3.5 but we use (3.4) instead of (3.5).

There is an embedding relation between the $F(H)$ and $B(H)$ spaces that can be shown directly from the definitions, namely,

$$B_p^{s,\min(p,q)}(H) \hookrightarrow F_p^{s,q}(H) \hookrightarrow B_p^{s,\max(p,q)}(H), \tag{3.9}$$

$0 < p < \infty$, $0 < q \le \infty$, where $X \hookrightarrow Y$ means, as usual, continuous embedding in the sense that $X \subset Y$ and $\|f\|_Y \le C\|f\|_X$, $\forall f \in X$. The proof of (3.9) is the same as in the Fourier case; see [Tr78, 2.3.2].

3.9. **Lifting properties of $F(H)$ and $B(H)$ spaces.** Let $c_n > -\inf\sigma(H) = -\inf\sigma_{pp}(H) = n^2$. We need the following lemma in Section 6.

Lemma 3.10. *Let* $s \in \mathbb{R}$, $0 < p < \infty$ *and* $0 < q \le \infty$. *Then* $(H+c_n)^s$ *maps* $F_p^{\alpha,q}(H)$ *isomorphically and continuously onto* $F_p^{\alpha-s,q}(H)$. *Moreover,* $\|(H+c_n)^s f\|_{F_p^{\alpha-s,q}(H)} \approx \|f\|_{F_p^{\alpha,q}(H)}$. *The analogous statement holds for* $B_p^{\alpha,q}(H)$.

Proof. We only give the proof for $F(H)$. The proof for $B(H)$ is similar.

$$\|(H+c_n)^s f\|_{F_p^{\alpha-s,q}(H)} = \|2^{(\alpha-s)j}(H+c_n)^s\varphi_j(H)f\|_{L^p(\ell^q)} = \|2^{j\alpha}\psi_j(H)f\|_{L^p(\ell^q)},$$

where $\psi_j(x) = 2^{-sj}(x+c_n)^s\varphi_j(x)$. Since ψ_j satisfies condition (i), (ii) and (iii'), according to Remark 3.6 we have

$$\|(H+c_n)^s f\|_{F_p^{\alpha-s,q}(H)} \approx \|f\|_{F_p^{\alpha,q}(H)}.$$

Also, it is easy to see that the inverse of $(H+c_n)^s$ is $(H+c_n)^{-s}$. This proves that the mapping $(H+c_n)^s$: $F_p^{\alpha,q}(H) \to F_p^{\alpha-s,q}(H)$ is surjective. □

4. Proof of Lemma 3.1

From Section 2 we know $K_j = K_{j,ac} + K_{j,pp}$. We need to show that $K_{j,ac}, K_{j,pp}$ both satisfy the decay estimates (a), (b) in the lemma. For the pure point kernel, since $\sigma_{pp} = \{-1,-4,\dots,-n^2\}$ is finite, it amounts to showing for $0 \le j \le 2 + 2\log_2 n$

$$|\partial_x^\alpha K_{j,pp}(x,y)| \le C_{m,\alpha}(1+|x-y|)^{-m}, \quad \forall m \in \mathbb{N}_0,\ \alpha = 0,1. \tag{4.1}$$

For other $j's$, the p.p. kernel vanish because $\operatorname{supp}\varphi_j$ are disjoint from the set σ_{pp}. But (4.1) follows from the fact that the eigenfunctions $e_j(x)$ are all Schwartz functions according to Lemma 2.6. So the nontrivial part will be to prove the decay for the a.c. kernel.

4.1. **The kernel of $\varphi_j(H)E_{ac}$.**

Recall from Theorem 2.4 that

$$e_n(x,k) = A_n(k)\, P_n(x,k) e^{ikx},$$

where $A_n(k) = (\operatorname{sign}(k))^n \prod_{j=1}^n (j+i|k|)^{-1}$ and $P_n(x,k) = p_n(\tanh x, ik)$ is a polynomial of real coefficients and of oder n in $\tanh x$ and ik.

4.1.1. *High energy estimates ($j > 0$).* Let $\varphi_j \in C_0^\infty(\mathbb{R})$ be given as in the beginning of Section 3. By (2.12) the kernel of $\varphi_j(H)E_{ac}$ is given by

$$\begin{aligned} K_{j,ac}(x,y) =& \frac{1}{2\pi}\int \varphi_j(k^2) e_n(x,k)\overline{e_n(y,k)}\, dk \\ =& \int_0^\infty + \int_{-\infty}^0 \varphi_j(k^2) R(x,y,k)\, e^{ik(x-y)} dk := K^+(x,y) + K^-(x,y), \end{aligned}$$

where

$$R(x,y,k) = P(x,k)P(y,-k)/\prod_{j=1}^n (j^2+k^2). \tag{4.2}$$

We only need to deal with $K^+(x,y)$ because $K^-(x,y) = K^+(-x,-y)$ in light of the relation $e_n(x,-k) = e_n(-x,k)$. Let $\lambda = 2^{-j/2}$ throughout this section. We have by integration by parts

$$\begin{aligned} 2\pi|K^+(x,y)| =& \left| \frac{(-1)^m}{i^m(x-y)^m} \int_{2^{j/2-1}}^{2^{j/2}} \frac{d^m}{dk^m}[\varphi_j(k^2)R(x,y,k)] e^{ik(x-y)}\, dk \right| \\ \le& C_m \lambda^{m-1}/|x-y|^m, \quad m \ge 0, \end{aligned}$$

where we used for $k \sim \lambda^{-1} \to \infty$ as $j \to \infty$,

$$\begin{cases} \frac{d^i}{dk^i}[\varphi_j(k^2)] &= O(\lambda^i) \\ \frac{\partial^j}{\partial k^j} R(x,y,k) &= O(\lambda^j) \quad \text{uniformly in } x,\ y. \end{cases} \tag{4.3}$$

The same estimate also holds for $K^-(x,y)$. Hence we obtain

$$|K_{j,ac}(x,y)| \le C_m\lambda^{-1}/(1+\lambda^{-1}|x-y|)^m\,. \tag{4.4}$$

4.1.2. *Low energy estimates ($-\infty < j < 0$).* If we allow $j<0$ with φ_j satisfying conditions (i), (ii) in Section 3, then (4.4) also holds for $j<0$ by the same proof above, except that instead of (4.3) we use the following estimates: if $k \sim \lambda^{-1} \to 0$ as $j \to -\infty$,

$$\begin{cases} \frac{d^i}{dk^i}[\varphi_j(k^2)] & = & O(\lambda^i) \le O(\lambda^m) \quad \text{if } 0 \le i \le m \\ \frac{\partial^j}{\partial k^j}R(x,y,k) & = & O(1) \quad \text{uniformly in } x,\, y. \end{cases}$$

However, the low energy case will be needed only in the discussion of homogeneous spaces $\dot{F}_p^{\alpha,q}(H)$, $\dot{B}_p^{\alpha,q}(H)$.

4.1.3. *Local energy estimates.* Fix $\Phi := \varphi_0 \in C_0^\infty(\mathbb{R})$ with support $\subset [-1,1]$.

$$2\pi\Phi(H)E_{ac}(x,y) = \int_{-1}^{1} \Phi(k^2)R(x,y,k)e^{ik(x-y)}dk.$$

Using for $k \to 0$,

$$\begin{cases} \frac{d^i}{dk^i}[\Phi(k^2)] & = & O(1) \\ \frac{\partial^j}{\partial k^j}R(x,y,k) & = & O(1) \quad \text{uniformly in } x,\, y \end{cases}$$

and integrating by parts on $[-1,1]$, where we note that $k \mapsto R(x,y,k)$ is analytic at zero, we obtain for each m

$$|\Phi(H)E_{ac}(x,y)| \le C_m(1+|x-y|)^{-m}.$$

4.2. **The derivative of the kernel.** Using the notation in Subsection 4.1, we proceed

$$\begin{aligned} 2\pi\frac{\partial}{\partial x}K_{j,ac}(x,y) =& \frac{\partial}{\partial x}\int \varphi_j(k^2)R(x,y,k)e^{ik(x-y)}\,dk \\ =& \int \varphi_j(k^2)\frac{\partial}{\partial x}[R(x,y,k)e^{ik(x-y)}]\,dk \\ =& \int \varphi_j(k^2)|A(k)|^2[ikP(x,k)+\frac{\partial}{\partial x}P(x,k)]P(y,-k)e^{ik(x-y)}\,dk. \end{aligned}$$

The function $\frac{\partial}{\partial x}P(x,k)$ is a polynomial of $\tanh x$ and ik having degrees $n+1$ and $n-1$, resp. Note that if $|k| \sim \lambda^{-1} = 2^{j/2}$, $j>0$,

$$\Big|\frac{d^i}{dk^i}\big(k\varphi_j(k^2)\big)\Big| = O(\lambda^{i-1}),$$

and if $|k| \le 1$,

$$\Big|\frac{d^i}{dk^i}\big(k\Phi(k^2)\big)\Big| = O(1).$$

We obtain, by similar arguments as in Subsection 4.1, for each $m \ge 0$

$$\Big|\frac{\partial}{\partial x}K_{j,ac}(x,y)\Big| \le C_m\lambda^{-2}(1+\lambda^{-1}|x-y|)^{-m}, \qquad j>0$$

and

$$\left|\frac{\partial}{\partial x}\Phi(H)E_{ac}(x,y)\right| \le C_m(1+|x-y|)^{-m}.$$

This completes the proof of Lemma 3.1. □

Remark 4.3. *For $-\infty < j < 0$, the best estimate is, for each $m \ge 0$*

$$\begin{aligned} &\left|\frac{\partial}{\partial x}K_{j,ac}(x,y)\right| \\ &\lesssim \lambda^{-1}\mathrm{sech}^2 x \tanh y(1+\lambda^{-1}|x-y|)^{-m} + \lambda^{-2}(1+\lambda^{-1}|x-y|)^{-m}. \end{aligned} \tag{4.5}$$

We observe that the first term has only a factor of $\lambda^{-1} = O(2^{j/2})$ as $j \to -\infty$, which makes unavailable the Bernstein inequality and Peetre maximal inequality, namely low energy *cases of Lemma 3.2 and Lemma 3.3, resp. Nevertheless, if we work a little harder, using (4.4) and (4.5) we can obtain a weaker form of Peetre maximal inequality and prove the following: if $1 \le p < \infty$, $0 < q < \infty$, $\alpha \in \mathbb{R}$,*

$$\|f\|_{\dot{B}_p^{\alpha,q}(H)} \approx \|\{2^{j\alpha}\varphi_j^*(H)f\}_{j\in\mathbb{Z}}\|_{\ell^q(L^p)}$$

and if $1 < p < \infty$, $1 < q < \infty$, $\alpha \in \mathbb{R}$,

$$\|f\|_{\dot{F}_p^{\alpha,q}(H)} \approx \|\{2^{j\alpha}\varphi_j^*(H)f\}_{j\in\mathbb{Z}}\|_{L^p(\ell^q)}.$$

5. IDENTIFICATION OF $F_p^{0,2}(H) = L^p$, $1 < p < \infty$

Let $\{\varphi_j\}_0^\infty$ be as in Section 1. Then there exists $\{\psi_j\}_0^\infty$ satisfying the same conditions (i), (ii) therein such that

$$\sum_{j=0}^\infty \varphi_j(x)\psi_j(x) = 1$$

by taking $\psi_j(x) = \overline{\varphi_j(x)}/\sum|\varphi_j(x)|^2$. We may assume that $\|\varphi_j\|_\infty$, $\|\psi_j\|_\infty$ are all ≤ 1. Let $Q_j = \varphi_j(H)$ and $R_j = \psi_j(H)$. Define the operators $Q : L^2 \to L^2(\ell^2)$ and $R : L^2(\ell^2) \to L^2$ as follows.

$$Q:\ f \mapsto \{Q_j(H)f\}_0^\infty$$

and

$$R:\ \{g_j\}_0^\infty \mapsto \sum_{j=0}^\infty R_j g_j.$$

It follows from the definition that

$$\|f\|_{F_p^{0,2}(H)} = \|Qf\|_{L^p(\ell^2)} \tag{5.1}$$

and it is easy to see that $RQ = I : L^2 \to L^2$ and $QR \le 3I : L^2(\ell^2) \to L^2(\ell^2)$. We will use Q and R to identify $F_p^{0,2}(H)$ with L^p.

Theorem 5.1. *Let $1 < p < \infty$. Then $F_p^{0,2}(H)$ and L^p are isomorphic and have equivalent norms.*

To prove the theorem, we will show that $Q\colon\ L^p \to L^p(\ell^2)$ and $R\colon\ L^p(\ell^2) \to L^p$, $1 < p < \infty$, that is,

$$\|Qf\|_{L^p(\ell^2)} \lesssim \|f\|_p \quad \text{and} \quad \|Rg\|_p \lesssim \|g\|_{L^p(\ell^2)} \tag{5.2}$$

for $f \in L^2 \cap L^p$ and $g \in L^2(\ell^2) \cap L^p(\ell^2)$, resp. This means that, by a density argument,

$$\|f\|_{F_p^{0,2}(H)} \lesssim \|f\|_p \tag{5.3}$$

and

$$\|f\|_p \lesssim \|f\|_{F_p^{0,2}(H)}. \tag{5.4}$$

Here in view of (5.2), (5.3) follows from (5.1) and (5.4) follows, with $g = Qf$, from the identity $RQ = I$, i.e., $\sum \varphi_j(H)\psi_j(H) = I$. Thus (5.3) and (5.4) prove Theorem 5.1.

The remaining part of this section is devoted to showing the boundedness of Q and R in (5.2). In the following, Lemma 5.2 and Lemma 5.4 imply that Q is bounded from L^p to $L^p(\ell^2)$, and, Lemma 5.2 and Lemma 5.5 imply that R is bounded from $L^p(\ell^2)$ to L^p by interpolation and duality.

Lemma 5.2. *$Q : L^2 \to L^2(\ell^2)$ and $R : L^2(\ell^2) \to L^2$ are well-defined bounded operators.*

Proof. Let $\{g_j\} \in L^2(\ell^2)$. Note that R_j is bounded on L^2: $\|R_j g\|_2 \le \|\psi_j\|_\infty \|g\|_2 \le \|g\|_2$. Thus

$$\begin{aligned}(\sum_{j=0}^{\infty} R_j g_j, \sum_{j=0}^{\infty} R_j g_j) &= \sum_{\nu=-1}^{1} \sum_{j=0}^{\infty} (R_j g_j, R_{j+\nu} g_{j+\nu}) \\ &\le \sum_{\nu=-1}^{1} \sum_{j} \|R_j g_j\|_2 \|R_{j+\nu} g_{j+\nu}\|_2 \\ &\le 3 \sum_{j} \|g_j\|_2^2 = 3\|g_j\|_{L^2(\ell^2)}^2 .\end{aligned}$$

Similarly, we have $\|Qf\|_{L^2(\ell^2)} \le \sqrt{2}\|f\|_2$ because $\sum_j |\varphi_j(x)|^2 \le 2$ for all x. □

We now derive some necessary estimates for the kernel of $Q_j = \varphi_j(H)$, which is denoted by $Q_j(x,y)$. Define

$$\widetilde{Q}_j(x,y) = \begin{cases} Q_j(x,y) & if\ 2^{j/2}|I| \ge 1 \\ Q_j(x,y) - Q_j(x,\bar{y}) & if\ 2^{j/2}|I| < 1. \end{cases}$$

Lemma 5.3. *Let $I = (\bar{y} - \frac{t}{2}, \bar{y} + \frac{t}{2})$, $t = |I|$ and $I^* = (\bar{y} - t, \bar{y} + t)$. Then there exists a constant C independent of I such that (a) If $2^{j/2}|I| \ge 1$,*

$$\sup_{y \in I} \int_{\mathbb{R}\setminus I^*} |Q_j(x,y)| dx \le C(2^{j/2}|I|)^{-1}.$$

(b) If $2^{j/2}|I|<1$,

$$\sup_{y\in I}\int_{\mathbb{R}\setminus I^*}|Q_j(x,y)-Q_j(x,\bar{y})|dx\le C2^{j/2}|I|.$$

In particular, we have

$$\sum_j\int_{\mathbb{R}\setminus I^*}|\widetilde{Q}_j(x,y)|dx\le(2+\sqrt{2})C. \tag{5.5}$$

Proof. For (a), we let $2^{j/2}|I|\ge 1$ and $y\in I$. Then it follows from Lemma 3.1 (a) that

$$\begin{aligned}\int_{\mathbb{R}\setminus I^*}|Q_j(x,y)|\,dx\le&C_m\int_{|x-y|>t/2}\frac{2^{j/2}}{(1+2^{j/2}|x-y|)^m}\,dx\\ \le&C(2^{j/2}|I|)^{-1},\qquad(m=2).\end{aligned}$$

For (b) we let $2^{j/2}|I|<1$, $y\in I$ ($\bar{y}$ being the center of I) and apply Lemma 3.1 (b) to obtain

$$\begin{aligned}\int_{\mathbb{R}\setminus I^*}|Q_j(x,y)-Q_j(x,\bar{y})|\,dx=&\int_{\mathbb{R}\setminus I^*}|\int_{\bar{y}}^{y}\frac{\partial}{\partial z}Q_j(x,z)dz|\,dx\\ \le&C_m|y-\bar{y}|\int_{|x-\bar{y}|>t}\frac{2^j}{(1+2^{j/2-1}|x-\bar{y}|)^m}\,dx\\ \le&C2^{j/2}|I|,\qquad(m=2).\end{aligned}$$

□

Lemma 5.4. *Q is bounded from* L^1 *to weak-*$L^1(\ell^2)$*, i.e.,*

$$|\{x:(\sum_0^\infty|Q_jf(x)|^2)^{1/2}>\lambda\}|\le C\lambda^{-1}\|f\|_1,\quad\forall\lambda>0.$$

Proof. Let $f\in L^1$. By the Calderón-Zygmund decomposition, there exists a sequence of disjoint intervals $\{I_k\}$ and and functions $\{b_k\}$ with supp $b_k\subset I_k$ such that $f=g+b$ with $g\in L^2$ and $b=\sum_k b_k\in L^1$. Furthermore, for each $\lambda>0$ the following properties hold

(i) $|g(x)|\le C\lambda$ a.e.
(ii) $b_k(x)=f(x)-|I_k|^{-1}\int_{I_k}fdx,\ x\in I_k$
(iii) $\lambda\le|I_k|^{-1}\int_{I_k}|f|dx\le 2\lambda$
(iv) $\sum_k|I_k|\le\lambda^{-1}\|f\|_1$.

From Lemma 5.2 we know that $Q:L^2\to L^2(\ell^2)$ is bounded, i.e.,

$$\int\sum_0^\infty|Q_jg(x)|^2dx\le C\|g\|_2^2.$$

By Chebyshev inequality we have

$$|\{x : (\sum_0^\infty |Q_j g(x)|^2)^{1/2} > \lambda/2\}| \le C\lambda^{-2}\|g\|_2^2 \le C\lambda^{-1}\|f\|_1.$$

Now we only need to show

$$|\{x \notin \cup I_k^* : (\sum_j |Q_j b(x)|^2)^{1/2} > \lambda/2\}| \le C\lambda^{-1}\|f\|_1,$$

where $I_k^* = 2I_k$ means the interval of length $2|I_k|$ with the same center as I_k. Note that the left hand side of the above inequality is bounded by

$$\frac{2}{\lambda}\sum_k \int_{\mathbb{R}\setminus\cup I_k^*} (\sum_j |Q_j b_k(x)|^2)^{1/2} dx \le \frac{2}{\lambda}\sum_k \int_{\mathbb{R}\setminus\cup I_k^*} \sum_j |Q_j b_k(x)| dx. \tag{5.6}$$

For each k, since $\int b_k = 0$, we apply Lemma 5.3 with $I = I_k$ and estimate above the r.h.s. of (5.6) by

$$\begin{aligned} &\frac{2}{\lambda}\sum_k \int_{\mathbb{R}\setminus\cup I_k^*} \sum_j \int |\widetilde{Q}_j(x,y)||b_k(y)|\, dydx \\ \le&\frac{2}{\lambda}\sum_k \int_{y\in I_k} |b_k(y)|dy \int_{\mathbb{R}\setminus I_k^*} \sum_j |\widetilde{Q}_j(x,y)|dx \\ \le&\frac{C}{\lambda}\sum_k \int_{I_k} |b_k(y)|\, dy \le C\lambda^{-1}\|f\|_1\,. \end{aligned}$$

This completes the proof. □

Lemma 5.5. *Let $R_j = \psi_j(H)$. Then $R = \{R_j\}$ is bounded from $L^1(\ell^2)$ to weak-L^1.*

Proof. It suffices to show that there exists a constant C such that

$$|\{x : |\sum_0^N R_j f_j(x)| > \lambda\}| \le C\lambda^{-1}\|\{f_j\}\|_{L^1(\ell^2)} \tag{5.7}$$

for all $N \in \mathbb{N}$, $\{f_j\} \in L^1(\ell^2)$ and $\lambda > 0$. By passing to the limit we see that (5.7) also holds for $N = \infty$ and all $\{f_j\} \in L^1(\ell^2) \cap L^2(\ell^2)$. Then the lemma follows from the fact that $L^1(\ell^2) \cap L^2(\ell^2)$ is dense in $L^1(\ell^2)$.

Let $F(x) = (\sum_{j=0}^\infty |f_j(x)|^2)^{1/2} \in L^1$. By the Calderón-Zygmund decomposition there exists a sequence of disjoint open intervals $\{I_k\}$ such that

(i) $|F(x)| \le C\lambda$, a.e. $x \in \mathbb{R}\setminus \cup_k I_k$

(ii) $\lambda \le |I_k|^{-1}\int_{I_k} |F(x)|dx \le 2\lambda$, $\forall k$.

Define

$$g_j(x) = \begin{cases} |I_k|^{-1}\int_{I_k} f_j dy, & x \in I_k \\ f_j(x) & \text{otherwise,} \end{cases} \qquad b_j(x) = \begin{cases} f_j - g_j, & x \in I_k \\ 0 & \text{otherwise.} \end{cases}$$

Then, if $x \in \mathbb{R} \setminus \cup_k I_k$, $(\sum_{j=0}^{\infty} |g_j(x)|^2)^{1/2} = (\sum_{j=0}^{\infty} |f_j(x)|^2)^{1/2}$, and, if $x \in I_k$

$$\begin{aligned}(\sum_{j=0}^{\infty} |g_j(x)|^2)^{1/2} &= (\sum_{j=0}^{\infty} |I_k|^{-2} |\int_{I_k} f_j(y)\, dy|^2)^{1/2} \\ &\le |I_k|^{-1} \int_{I_k} (\sum_{j=0}^{\infty} |f_j(y)|^2)^{1/2}\, dy \le 2\lambda\end{aligned}$$

by Minkowski inequality. It follows that

$$\begin{aligned}\|\{g_j(x)\}\|_{L^2(\ell^2)}^2 &= \sum_k \int_{I_k} (\sum_j |g_j(x)|^2)\, dx + \int_{\mathbb{R}\setminus\cup I_k} (\sum_j |g_j(x)|^2)\, dx \\ &\le (2\lambda)^2 \sum_k |I_k| + 2\lambda \int_{\mathbb{R}\setminus\cup I_k} (\sum_j |f_j|^2)^{1/2}\, dx \\ &\le C\lambda \|F\|_1 .\end{aligned}$$

Now by Lemma 5.2 we obtain

$$\begin{aligned}|\{x : |\sum_0^N R_j g_j(x)| > \lambda/2\}| &\le C\lambda^{-2} \|\sum_0^N R_j g_j\|_2^2 \\ &\le C' \lambda^{-2} \|\{g_j\}\|_{L^2(\ell^2)}^2 \le C\lambda^{-1} \|F\|_1 .\end{aligned}$$

It remains to show

$$|\{x \notin \cup I_k^* : |\sum_0^N R_j b_j(x)| > \lambda/2\}| \le C\lambda^{-1} \|F\|_1 .$$

The left hand side is not exceeding $\frac{2}{\lambda} \sum_k \int_{\mathbb{R}\setminus\cup I_k^*} |\sum_{j=0}^N R_j b_{j,k}(x)| dx$, where $b_{j,k} = b_j \chi_{I_k}$, χ_{I_k} the characteristic function of I_k. For each k, define

$$\widetilde{R}_j^k(x,y) = \begin{cases} R_j(x,y) & \text{if } 2^{j/2}|I_k| \ge 1 \\ R_j(x,y) - R_j(x,\bar{y}_k) & \text{if } 2^{j/2}|I_k| < 1, \end{cases}$$

where $\bar{y}_k$ is the center of I_k. Then it follows from Lemma 5.3 with $I = I_k$ and Q_j replaced by R_j that

$$\int_{\mathbb{R}\setminus I_k^*} (\sum_{j=0}^N |\widetilde{R}_j^k(x,y)|^2)^{1/2} dx \le \int_{\mathbb{R}\setminus I_k^*} \sum_{j=0}^N |\widetilde{R}_j^k(x,y)| dx \le C, \ \ \forall y \in I_k,\ N.$$

Thus we obtain, using $\int b_{j,k}=0$,

$$\begin{aligned}\int_{\mathbb{R}\setminus I_k^*}|\sum_{j=0}^{N}R_jb_{j,k}(x)|\,dx &= \int_{\mathbb{R}\setminus I_k^*}|\sum_{j=0}^{N}\int_{I_k}\widetilde{R}_j^k(x,y)b_{j,k}(y)dy|\,dx\\ &\le \int_{I_k}(\sum_{j=0}^{N}|b_{j,k}|^2(y))^{1/2}\,dy\int_{\mathbb{R}\setminus I_k^*}(\sum_{j=0}^{N}|\widetilde{R}_j^k(x,y)|^2)^{1/2}\,dx\\ &\le C\int_{I_k}(\sum_{j=0}^{N}|b_{j,k}|^2)^{1/2}\,dy\\ &\le 2C\int_{I_k}(\sum_{j=0}^{\infty}|f_j|^2)^{1/2}\,dy.\end{aligned}$$

Hence

$$|\{x\notin\cup I_k^*:|\sum_{0}^{N}R_jb_j(x)|>\lambda/2\}|\le\frac{4C}{\lambda}\sum_k\int_{I_k}(\sum_j|f_j|^2)^{1/2}\,dy\le\frac{4C}{\lambda}\|(\sum_j|f_j|^2)^{1/2}\|_1\,,$$

as desired. This completes the proof.

$\square$

6. Remarks on boundedness of the wave function

We conclude the paper with a boundedness result on the wave function $\psi(t,x)=e^{-itH}f$ which is the solution to the Schrödinger equation

$$i\,\partial_t\psi=H\psi,\qquad \psi(0,x)=f(x). \tag{6.1}$$

We will see that using the $B(H)$ and $F(H)$ space one can obtain a global time decay for $\psi(t,x)$ (Theorem 6.3). The perturbed Besov space method has been considered in [JN94, Y95, Cu00, CuS01] and more recently, [BZ05, DP05, DF05] involving Schrödinger and wave equations.

By [BZ05, Theorem 7.1] or [JN94, Theorem 5.1] we know that if V is in the Kato class $\mathcal{K}_d$ and if $\mathcal{D}(H^m)=W_p^{2m}(\mathbb{R}^d)$ for some $m\in\mathbb{N}$, $1\le p<\infty$, then for $1\le q\le\infty, 0<\alpha<m, B_p^{\alpha,q}(H)=B_p^{2\alpha,q}(\mathbb{R}^d)$. It is easy to see that if V is C^∞ with all derivatives bounded, then the domain condition on H is verified for all $m\in\mathbb{N}$.

In the following we assume $H=-d^2/dx^2+V_n$ and restrict our discussion to the P-T potential, although results here have extensions to general potentials on $\mathbb{R}^d$.

Since $V_n\sim\operatorname{sech}^2 x$ is in the Schwartz class, we have

$$B_p^{\alpha,q}(H)=B_p^{2\alpha,q}(\mathbb{R})$$

for all $\alpha>0$. In particular, $F_p^{\alpha,p}(H)=F_p^{2\alpha,p}(\mathbb{R})$ since it always holds that $F_p^{\alpha,p}=B_p^{\alpha,p}$ by the definitions (see (1.3), (1.4)). On the other hand, by Theorem 5.1, $F_p^{0,2}(H)=L^p=$

$F_p^{0,2}(\mathbb{R})$. Thus we obtain the following theorem using complex interpolation method; consult [Tr78, Tr83] or [BL76] for details.

Theorem 6.1. *If $\alpha > 0$, $1 < p < \infty$ and $2p/(p+1) < q < 2p$, then*

$$F_p^{\alpha,q}(H) = F_p^{2\alpha,q}(\mathbb{R}).$$

If $\alpha > 0$, $1 \le p < \infty$ and $1 \le q \le \infty$, then

$$B_p^{\alpha,q}(H) = B_p^{2\alpha,q}(\mathbb{R}).$$

From Theorem 6.1 and [JN94, Theorem 4.6, Remark 4.7] we obtain the boundedness of $\psi(t,x)$ on ordinary Besov spaces. Let $\langle t\rangle = (1+t^2)^{1/2}$ and let $\beta = \beta(p) = |\frac{1}{2} - \frac{1}{p}|$ be the critical exponent.

Proposition 6.2. *Let $\alpha > 0$, $1 \le p < \infty$, $1 \le q \le \infty$. Then*

$$\|e^{-itH} f\|_{B_p^{\alpha,q}(\mathbb{R})} \lesssim \langle t\rangle^{|\frac{1}{p}-\frac{1}{2}|} \|f\|_{B_p^{\alpha+2\beta,q}(\mathbb{R})}. \tag{6.2}$$

Moreover, if $2 \le p < \infty$,

$$\|e^{-itH} f\|_{L^p} \lesssim \langle t\rangle^{|\frac{1}{p}-\frac{1}{2}|} \|f\|_{B_p^{2\beta,2}(\mathbb{R})}$$

and if $1 \le p < 2$,

$$\|e^{-itH} f\|_{L^p} \lesssim \langle t\rangle^{|\frac{1}{p}-\frac{1}{2}|} \|f\|_{B_p^{2\beta,1}(\mathbb{R})}. \tag{6.3}$$

Proof. Let $\{\varphi_j\}_0^\infty$ be a smooth dyadic system. From the proof of [JN94, Theorem 4.6] we see that

$$\|e^{-itH}\varphi_j(H)f\|_p \lesssim 2^{j\beta}\langle t\rangle^{|\frac{1}{2}-\frac{1}{p}|}\|\varphi_j(H)f\|_p, \qquad j \ge 0.$$

This implies (6.2) by Theorem 6.1 and

$$\|e^{-itH} f\|_{B_p^{0,q}(H)} \lesssim \langle t\rangle^{|\frac{1}{2}-\frac{1}{p}|} \|f\|_{B_p^{\beta,q}(H)}. \tag{6.4}$$

Now if $p \ge 2$, then $B_p^{0,2}(H) \hookrightarrow F_p^{0,2}(H)$ according to (3.9). We have

$$\|e^{-itH} f\|_{L^p} \approx \|e^{-itH} f\|_{F_p^{0,2}(H)} \lesssim \langle t\rangle^{|\frac{1}{2}-\frac{1}{p}|} \|f\|_{B_p^{\beta,2}(H)}.$$

For $1 \le p < 2$, because

$$\|f\|_p \le \sum_{j=0}^{\infty} \|\varphi_j(H)f\|_p = \|f\|_{B_p^{0,1}(H)}\,,$$

we see $B_p^{0,1}(H) \hookrightarrow L^p$, which implies (6.3) in light of (6.4). □

One is also interested in understanding the long time behavior of $\psi(t,x)$. From [GSch04] and [DF05] we know that if $(1+x^2)V \in L^1(\mathbb{R})$, then

$$\|e^{-itH}E_{ac}f\|_{L^{p'}} \lesssim t^{-(\frac{1}{p}-\frac{1}{2})}\|f\|_{L^p}\,, \qquad \forall t > 0,\ 1 \le p \le 2, \tag{6.5}$$

where $\frac{1}{p}+\frac{1}{p'}=1$. So Proposition 6.2 and (6.5) yield

$$\|e^{-itH}E_{ac}f\|_{L^{p'}} \lesssim \langle t\rangle^{-(\frac{1}{p}-\frac{1}{2})}\|f\|_{B^{2\beta,2}_{p'}(\mathbb{R})\cap L^p}\,, \qquad 1<p\le 2, \tag{6.6}$$

where we note that E_{ac} is bounded on L^p because E_{pp}, which has the kernel $\sum_{j=1}^n e_j(x)e_j(y)$, is bounded on L^p (see the discussion at the beginning of Section 4).

Theorem 6.3. *Let $1<p\le 2$. Then*

$$\|e^{-itH}E_{ac}f\|_{L^{p'}} \lesssim \langle t\rangle^{-(\frac{1}{p}-\frac{1}{2})}\|f\|_{B^{4\beta,2}_{p}(\mathbb{R})}\,. \tag{6.7}$$

$$\|e^{-itH}E_{ac}f\|_{L^{p'}} \lesssim \langle t\rangle^{-(\frac{1}{p}-\frac{1}{2})}\|f\|_{F^{4\beta,2}_{p}(\mathbb{R})}\,. \tag{6.8}$$

Proof. Since $B^{4\beta,2}_p(\mathbb{R}) \hookrightarrow B^{2\beta,2}_{p'}(\mathbb{R})$ (Besov embedding; see e.g. [Tr83, 2.7.1]) and $B^{\epsilon,2}_p(\mathbb{R}) \hookrightarrow L^p$ if $\epsilon>0$, it follows from (6.6) that

$$\|e^{-itH}E_{ac}f\|_{L^{p'}} \lesssim \langle t\rangle^{-(\frac{1}{p}-\frac{1}{2})}\|f\|_{B^{4\beta,2}_{p}(\mathbb{R})}$$

provided $1<p\le 2$. The second inequality follows from (6.7) and the embedding $F^{s,2}_p(\mathbb{R}) \hookrightarrow B^{s,2}_p(\mathbb{R})$ in light of (3.9). □

Remark 6.4. *For (6.8), if alternatively starting with (6.6) (rather than (6.7)) and using an embedding of Jawerth [Tr83; 2.7.1], we can obtain an improved result: if $1<p<2$, $0<q\le\infty$, then*

$$\|e^{-itH}E_{ac}f\|_{L^{p'}} \lesssim \langle t\rangle^{-(\frac{1}{p}-\frac{1}{2})}\|f\|_{F^{4\beta,q}_{p}(\mathbb{R})}\,.$$

As a consequence we also obtain the following regularity result by the identification in Theorem 6.1.

Corollary 6.5. *Let $\alpha>0$. If $1<p\le 2$, $1\le q\le\infty$, then*

$$\|e^{-itH}E_{ac}f\|_{B^{\alpha,q}_{p'}(\mathbb{R})} \lesssim \langle t\rangle^{-(\frac{1}{p}-\frac{1}{2})}\|f\|_{B^{\alpha+4\beta,q}_{p}(\mathbb{R})}\,. \tag{6.9}$$

If $1<p\le 2$, $p\le q\le 2$, then

$$\|e^{-itH}E_{ac}f\|_{F^{\alpha,q}_{p'}(\mathbb{R})} \lesssim \langle t\rangle^{-(\frac{1}{p}-\frac{1}{2})}\|f\|_{F^{\alpha+4\beta,q}_{p}(\mathbb{R})}\,. \tag{6.10}$$

Proof. Since $B^{2\beta,2}_p(H)=B^{4\beta,2}_p(\mathbb{R})$ by Theorem 6.1, we can write (6.7) as

$$\|e^{-itH}E_{ac}f\|_{L^{p'}} \lesssim \langle t\rangle^{-(\frac{1}{p}-\frac{1}{2})}\|f\|_{B^{2\beta,2}_{p}(H)}. \tag{6.11}$$

Replace f with $\varphi_j(H)f$ in (6.11). Then the B-inequality (6.9) follows from the simple observation that

$$\Big(\sum_j 2^{j\alpha q}\|\varphi_j(H)f\|^q_{B^{\gamma,2}_p(H)}\Big)^{1/q} \approx \|f\|_{B^{\alpha+\gamma,q}_p(H)}\,.$$

To show the F-inequality, substitute $f = (H + c_n)^{-\alpha} f$ into (6.8) but use the $F_p^{2\beta,2}(H)$-norm instead. Then by the lifting property in Lemma 3.10 and Theorem 6.1, we have

$$\|e^{-itH} E_{ac} f\|_{F_{p'}^{\alpha,2}(\mathbb{R})} \lesssim \langle t\rangle^{-(\frac{1}{p}-\frac{1}{2})} \|f\|_{F_p^{\alpha+4\beta,2}(\mathbb{R})} . \tag{6.12}$$

Now (6.10) follows from the interpolation between (6.12) and (6.9) with $p = q$, where we note that $B_p^{\alpha,p}(\mathbb{R}) = F_p^{\alpha,p}(\mathbb{R})$. □

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