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A criterion for convergence of Weak Greedy Algorithms¹

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1. INTRODUCTION

This paper completes the investigation of necessary and sufficient conditions on the "weakness" sequence $\tau := \{t_k\}_{k=1}^{\infty}$ for convergence of Weak Greedy Algorithm for all dictionaries \mathcal{D} and each function (vector) f in Hilbert space H . This paper is a follow up to the papers [T] and [LT]. The Weak Greedy Algorithms (WGA) were introduced in [T]. The paper [T] contains also historical remarks and some motivation of studying greedy and weak greedy algorithms. We will not repeat historical remarks from [T] here and refer the reader to [T] for prehistory of WGA. We discuss here results on WGA in detail.

We remind first some notations and definitions from the theory of greedy algorithms. Let H be a real Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ and the norm $\|x\| := \langle x, x \rangle^{1/2}$. We say a set \mathcal{D} of functions (elements) from H is a dictionary if each $g \in \mathcal{D}$ has norm one ($\|g\| = 1$) and $\overline{\text{span}}\mathcal{D} = H$. We give now the definition of WGA (see [T]). Let a weakness sequence $\tau = \{t_k\}_{k=1}^{\infty}$, $0 \leq t_k \leq 1$, be given.

Weak Greedy Algorithm. *We define $f_0^\tau := f$. Then for each $m \geq 1$, we inductively define:*

1). $\varphi_m^\tau \in \mathcal{D}$ is any satisfying

$$|\langle f_{m-1}^\tau, \varphi_m^\tau \rangle| \geq t_m \sup_{g \in \mathcal{D}} |\langle f_{m-1}^\tau, g \rangle|;$$

2).

$$f_m^\tau := f_{m-1}^\tau - \langle f_{m-1}^\tau, \varphi_m^\tau \rangle \varphi_m^\tau;$$

3).

$$G_m^\tau(f, \mathcal{D}) := \sum_{j=1}^m \langle f_{j-1}^\tau, \varphi_j^\tau \rangle \varphi_j^\tau.$$

In the case $t_k = 1$, $k = 1, 2, \dots$, we call WGA by Pure Greedy Algorithm (PGA). The convergence of PGA and WGA with $t_k = t$, $0 < t < 1$, was established in [J] and [RW]. The first sufficient condition on τ which includes sequences with $\lim_{k \rightarrow \infty} t_k = 0$ was obtained in [T].

Theorem A. *Assume*

$$(1.1) \quad \sum_{k=1}^{\infty} \frac{t_k}{k} = \infty.$$

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Then for any dictionary \mathcal{D} and any $f \in H$ we have

$$\lim_{m \rightarrow \infty} \|f - G_m^\tau(f, \mathcal{D})\| = 0.$$

In [T] we reduced the proof of convergence of WGA with weakness sequence τ to some properties of l_2 -sequences with regard to τ . Theorem A was derived from the following two statements proved in [T].

Proposition 1.1. *Let τ be such that for any $\{a_j\}_{j=1}^\infty \in l_2$, $a_j \geq 0$, $j = 1, 2, \dots$ we have*

$$\liminf_{n \rightarrow \infty} a_n \sum_{j=1}^n a_j / t_n = 0.$$

Then for any H , \mathcal{D} , and $f \in H$ we have

$$\lim_{m \rightarrow \infty} \|f_m^\tau\| = 0.$$

Proposition 1.2 (Lemma 2.3, [T]). *If τ satisfies the condition (1.1) then τ satisfies the assumption of Proposition 1.1.*

The following simple necessary condition

$$\sum_{k=1}^{\infty} t_k^2 = \infty$$

was mentioned in [T]. The first nontrivial necessary conditions were obtained in [LT]. We proved in [LT] the following theorem.

Theorem B. *In the class of monotone sequences $\tau = \{t_k\}_{k=1}^\infty$, $1 \geq t_1 \geq t_2 \geq \dots \geq 0$, the condition (1.1) is necessary and sufficient for convergence of Weak Greedy Algorithm for each f and all Hilbert spaces H and dictionaries \mathcal{D} .*

The proof of this theorem is based on a special procedure which we called Equalizer. The generalization of that procedure plays an important role in this paper also (see S.3). In [LT] we gave an example of a class of sequences τ for which the condition (1.1) is not a necessary condition for convergence. We also proved in [LT] a theorem which covers Theorem A.

Theorem C. *Assume*

$$\sum_{s=0}^{\infty} (2^{-s} \sum_{k=2^s}^{2^{s+1}-1} t_k^2)^{1/2} = \infty.$$

Then for any dictionary \mathcal{D} and any $f \in H$ we have

$$\lim_{m \rightarrow \infty} \|f - G_m^\tau(f, \mathcal{D})\| = 0.$$

We prove in this paper a criterion on τ for convergence of WGA. Let us introduce some notation.

We define by \mathcal{V} the class of sequences $x = \{x_k\}_{k=1}^\infty$, $x_k \geq 0$, $k = 1, 2, \dots$, with the following property: there exists a sequence $0 = q_0 < q_1 < \dots$ such that

$$(1.2) \quad \sum_{s=1}^{\infty} \frac{2^s}{\Delta q_s} < \infty;$$

and

$$(1.3) \quad \sum_{s=1}^{\infty} 2^{-s} \sum_{k=1}^{q_s} x_k^2 < \infty,$$

where $\Delta q_s := q_s - q_{s-1}$.

Remark 1.1. It is clear from this definition that if $x \in \mathcal{V}$ and for some N and c we have $0 \leq y_k \leq cx_k$, $k \geq N$, then $y \in \mathcal{V}$.

Theorem 1.1. *The condition $\tau \notin \mathcal{V}$ is necessary and sufficient for convergence of Weak Greedy Algorithm with weakness sequence τ for each f and all Hilbert spaces H and dictionaries \mathcal{D} .*

Sufficient part is proved in Section 2 and necessary part is proved in Section 3.

2. PROOF OF CONVERGENCE

We begin this section with the following lemma.

Lemma 2.1. *Let $\{a_j\}_{j=1}^{\infty} \in l_2$, $a_j \geq 0$, $j = 1, 2, \dots$. Then $\{a_n \sum_{j=1}^n a_j\}_{n=1}^{\infty} \in \mathcal{V}$.*

Proof. Assume $\{a_j\}_{j=1}^{\infty}$ contains infinitely many nonzero terms (if not the statement is trivial). Denote $y_n := a_n \sum_{j=1}^n a_j$ and define $q_s := q_s(y)$ inductively: $q_0 := 0$ and for q_0, \dots, q_{s-1} defined we choose q_s as the smallest q such that

$$(2.1) \quad (q - q_{s-1}) \sum_{n=q_{s-1}+1}^q y_n^2 \geq 2^{2s}.$$

Denote $Q_s := (q_{s-1}, q_s]$. Then (2.1) implies

$$\frac{2^s}{\Delta q_s} \leq 2^{-s} \sum_{n \in Q_s} y_n^2 \leq 2^{-s} \sum_{n=1}^{q_s} y_n^2.$$

Thus it is sufficient to check only (1.3)

$$\sum_s 2^{-s} \sum_{n=1}^{q_s} y_n^2 < \infty.$$

From the definition of q_s we have

$$(2.2) \quad \sum_{n=q_{s-1}+1}^{q_s-1} y_n \leq (\Delta q_s - 1)^{1/2} \left(\sum_{n=q_{s-1}+1}^{q_s-1} y_n^2 \right)^{1/2} < 2^s.$$

Next for any $N \leq M$ we have

$$\sum_{n=N}^M a_n \sum_{j=1}^n a_j \geq \sum_{N \leq j \leq n \leq M} a_n a_j =$$

$$(2.3) \quad = 1/2 \left(\sum_{j=N}^M a_j^2 + \left(\sum_{j=N}^M a_j \right)^2 \right) \geq \left(\sum_{j=N}^M a_j \right)^2 / 2.$$

Combining (2.2) and (2.3) we get

$$\sum_{j \in Q_s} a_j = \sum_{j=q_{s-1}+1}^{q_s-1} a_j + a_{q_s} \leq 2^{(s+1)/2} + \|a\|_\infty.$$

This implies

$$(2.4) \quad \sum_{j=1}^{q_s} a_j \leq C(a) 2^{s/2}.$$

We have now

$$\sum_s 2^{-s} \sum_{n=1}^{q_s} y_n^2 = \sum_s 2^{-s} \sum_{v=1}^s \sum_{n \in Q_v} y_n^2 \leq 2 \sum_s 2^{-s} \sum_{n \in Q_s} y_n^2 \leq$$

$$2 \sum_s 2^{-s} \left(\sum_{j=1}^{q_s} a_j \right)^2 \sum_{n \in Q_s} a_n^2 \leq C(a) \sum_n a_n^2 < \infty.$$

Lemma 2.1 is proved now.

Theorem 2.1. *The following two conditions are equivalent*

$$(C.1) \quad \tau \notin \mathcal{V},$$

$$(C.2) \quad \forall \{a_j\}_{j=1}^\infty \in l_2, \quad a_j \geq 0, \quad \liminf_{n \rightarrow \infty} a_n \sum_{j=1}^n a_j / t_n = 0.$$

Proof. We prove first that (C.1) \Rightarrow (C.2). Assume (C.2) is not satisfied: $\exists \{a_j\}_{j=1}^\infty \in l_2, a_j \geq 0$, such that

$$(2.5) \quad \liminf_{n \rightarrow \infty} a_n \sum_{j=1}^n a_j / t_n > 0.$$

Relation (2.5) implies that for some N and $c > 0$ we have for $n \geq N$ that

$$a_n \sum_{j=1}^n a_j / t_n \geq c$$

or

$$t_n \leq C a_n \sum_{j=1}^n a_j.$$

This inequality, Lemma 2.1, and Remark 1.1 imply that $\tau \in \mathcal{V}$. The first implication is proved now.

We proceed to the second implication (C.2) \Rightarrow (C.1). Let $\tau \in \mathcal{V}$. We construct a sequence $\{a_j\}_{j=1}^\infty \in l_2$ such that for all n

$$t_n \leq C a_n \sum_{j=1}^n a_j$$

with some C . This will imply that (C.2) is not satisfied. Let $\{q_s\} := \{q_s(\tau)\}$ be a sequence from the definition of \mathcal{V} . We define a sequence $\{a_j\}_{j=1}^\infty$ as follows. For $n \in Q_s$ we set

$$a_n := t_n 2^{-s/2} + 2^{s/2} / \Delta q_s.$$

Then

$$a_n^2 \leq 2(t_n^2 2^{-s} + 2^s (\Delta q_s)^{-2})$$

and

$$\sum_n a_n^2 \leq 2 \sum_s 2^{-s} \sum_{n \in Q_s} t_n^2 + 2 \sum_s \frac{2^s}{\Delta q_s} < \infty.$$

Next,

$$\sum_{n \in Q_s} a_n \geq 2^{s/2}.$$

Thus for $n \in Q_s$ we have

$$a_n \sum_{j=1}^n a_j \geq a_n \sum_{j \in Q_{s-1}} a_j \geq t_n 2^{-1/2}$$

and

$$t_n \leq \sqrt{2} a_n \sum_{j=1}^n a_j$$

for all n .

Theorem 2.1 is proved now.

The sufficient part of Theorem 1.1 follows from Theorem 2.1 and Proposition 1.1.

3. CONSTRUCTION OF A COUNTEREXAMPLE

The following procedure which is the generalization of Equalizer from [LT] plays an important role in the construction. Let H be a Hilbert space with an orthonormal basis $\{e_j\}_{j=1}^\infty$. We take two elements $e_i, e_j, i \neq j$, and define the following procedure.

Equalizer with schedule $\gamma := \{\gamma_k\}$. Let $\gamma_k \leq 1/5, f_0 := e_i$. Define:

$$(3.1) \quad g_1 := \alpha_1 e_i - (1 - \alpha_1^2)^{1/2} e_j; \quad \alpha_1 = \gamma_1; \quad \langle f_0, g_1 \rangle = \gamma_1;$$

$$(3.2) \quad f_n := f_{n-1} - \langle f_{n-1}, g_n \rangle g_n; \quad g_n := \alpha_n e_i - (1 - \alpha_n^2)^{1/2} e_j;$$

$$\langle f_n, g_{n+1} \rangle = \gamma_{n+1}; \quad f_n = a_n e_i + b_n e_j.$$

We check

$$a_{n-1} - b_{n-1} \geq 3\sqrt{2}\gamma_n$$

to continue. If

$$a_{n-1} - b_{n-1} < 3\sqrt{2}\gamma_n$$

then we take $g_n := 2^{-1/2}(e_i - e_j)$ and

$$f_n := f_{n-1} - \langle f_{n-1}, g_n \rangle g_n,$$

and stop after this step. We call this step "the final step" and all other steps "regular steps". At each regular step l we have

$$a_l - b_l = a_{l-1} - b_{l-1} - \gamma_l(\alpha_l + (1 - \alpha_l^2)^{1/2}) \geq a_{l-1} - b_{l-1} - 2^{1/2}\gamma_l > 0.$$

After the final step we have

$$a_n = b_n.$$

At each regular step we have by definition that

$$\langle f_{l-1}, g_l \rangle = \gamma_l.$$

At the final step we have

$$\begin{aligned} \langle f_{n-1}, g_n \rangle &= 2^{-1/2}(a_{n-1} - b_{n-1}) \geq 2^{-1/2}(a_{n-2} - b_{n-2} - 2^{1/2}\gamma_{n-1}) \geq \\ &2^{-1/2}(2\sqrt{2}\gamma_{n-1}) = 2\gamma_{n-1}. \end{aligned}$$

Thus, if $2\gamma_{n-1} \geq \gamma_n$ then the above described Equalizer is a WGA with weakness sequence $\gamma_1, \dots, \gamma_n$.

At regular step l we reduce the $\|\cdot\|^2$ by γ_l^2 . At the final step we reduce the $\|\cdot\|^2$ by

$$\frac{1}{2}(a_{n-1} - b_{n-1})^2 < 9\gamma_n^2.$$

We also have

$$a_{n-1} - b_{n-1} < 3\sqrt{2}\gamma_n$$

and

$$a_{n-1} - b_{n-1} \geq 1 - \sqrt{2} \sum_{j=1}^{n-1} \gamma_j.$$

Thus,

$$\sqrt{2} \sum_{j=1}^{n-1} \gamma_j + 3\sqrt{2}\gamma_n > 1.$$

On the other hand

$$a_l - b_l \leq a_{l-1} - b_{l-1} - \gamma_l.$$

Therefore,

$$0 \leq a_{n-1} - b_{n-1} \leq 1 - \sum_{l=1}^{n-1} \gamma_l$$

and

$$(3.3) \quad \sum_{l=1}^{n-1} \gamma_l \leq 1.$$

In order to apply the above Equalizer we need to have the inequality $2\gamma_{n-1} \geq \gamma_n$ satisfied. Let us use the following regularization procedure.

Regularization. For a given $\tau = \{t_k\}_{k=1}^{\infty}$, $\tau \in l_{\infty}$, we define $\tau^R := \{t_k^R\}_{k=1}^{\infty}$ with

$$t_k^R := \sum_{m=0}^{\infty} 2^{-m} t_{k+m}.$$

Lemma 3.1. *If $\tau \in \mathcal{V} \cap l_{\infty}$ then $\tau^R \in \mathcal{V} \cap l_{\infty}$.*

Proof. Assumption $\tau \in \mathcal{V}$ implies

$$(3.4) \quad \sum_s 2^{-s} \sum_{k=1}^{q_s} t_k^2 < \infty.$$

We will prove that

$$(3.5) \quad \sum_s 2^{-s} \sum_{k=1}^{q_s} (t_k^R)^2 < \infty$$

with the same $q_s = q_s(\tau)$ as above. Thus (3.5) will imply $\tau^R \in \mathcal{V}$. Let us prove (3.5). We have for any N

$$\begin{aligned} \sum_{k=1}^N (t_k^R)^2 &= \sum_{k=1}^N \left(\sum_{m=0}^{\infty} 2^{-m} t_{k+m} \right)^2 = \sum_{k=1}^N \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} 2^{-m-n} t_{k+m} t_{k+n} = \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} 2^{-m-n} \sum_{k=1}^N t_{k+m} t_{k+n} \leq \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} 2^{-m-n} \left(\sum_{k=1}^N t_{k+m}^2 \right)^{1/2} \left(\sum_{k=1}^N t_{k+n}^2 \right)^{1/2} = \\ &= \left(\sum_{m=0}^{\infty} 2^{-m} \left(\sum_{k=1}^N t_{k+m}^2 \right)^{1/2} \right)^2 \leq \left(\sum_{m=0}^{\infty} 2^{-m} \right) \left(\sum_{m=0}^{\infty} 2^{-m} \sum_{k=1}^N t_{k+m}^2 \right). \end{aligned}$$

Next,

$$\sum_{k=1}^N t_{k+m}^2 \leq \sum_{k=1}^N t_k^2 + m \|\tau\|_{\infty}^2$$

and

$$\sum_{m=0}^{\infty} 2^{-m} \left(\sum_{j=1}^N t_j^2 + m \|\tau\|_{\infty}^2 \right) \leq 2 \sum_{j=1}^N t_j^2 + C(\tau).$$

Therefore we got

$$\sum_{j=1}^N (t_j^R)^2 \leq 2 \sum_{j=1}^N t_j^2 + C(\tau)$$

and

$$\sum_s 2^{-s} \sum_{k=1}^{q_s} (t_k^R)^2 \leq 2 \sum_s 2^{-s} \sum_{k=1}^{q_s} t_k^2 + C(\tau) < \infty.$$

It is easy to see that $\|\tau^R\|_{\infty} \leq 2\|\tau\|_{\infty}$.

Lemma 3.1 is proved now.

Thus for any $\tau \in \mathcal{V} \cap l_\infty$ we have $\tau^R \in \mathcal{V}$, $\|\tau^R\|_\infty \leq 2\|\tau\|_\infty$, and

$$2t_{n-1}^R \geq t_n^R, \quad n = 2, 3, \dots$$

Clearly, we also have for all n

$$t_n \leq t_n^R.$$

One more restriction in the Equalizer is $\gamma_n \leq 1/5$. Define a new sequence τ' by

$$t'_n := \min\{t_n^R, 1/5\}.$$

It is clear that $\tau' \in \mathcal{V}$ and also satisfies

$$2t'_{n-1} \geq t'_n.$$

Let $\{q_s\} := \{q_s(\tau')\}$ be the sequence for τ' from the definition of \mathcal{V} :

$$\sum_s \frac{2^s}{\Delta q_s} < \infty, \quad \sum_s 2^{-s} \sum_{n=1}^{q_s} (t'_n)^2 < \infty.$$

Let ϵ be a small number which we will specify later and s_0 be such that

$$\sum_{s \geq s_0} \frac{2^s}{\Delta q_s} < \epsilon, \quad \sum_{s \geq s_0} 2^{-s} \sum_{n=1}^{q_s} (t'_n)^2 < \epsilon.$$

Consider the function

$$f_{s_0} := 2^{-s_0/2}(e_1 + \dots + e_{2^{s_0}}).$$

We have $\|f_{s_0}\| = 1$. Define

$$t''_k := \max\{t'_k, 2^{s_0+2}(\Delta q_{s_0})^{-1}\}.$$

We apply a mixture of Equalizer with schedule $\{t''_k\}$ to vectors e_i , $i \leq 2^{s_0}$, and the PGA to the corresponding residual of f_{s_0} . We do this in the following way. If $t''_1 \geq 1/5$ we use PGA and throw away, say, $2^{-s_0/2}e_{2^{s_0}}$. If $t''_1 < 1/5$ we start using the Equalizer with schedule $\{t''_k\}$ to vectors e_1 and $e_{2^{s_0}+1}$. If at some step $t''_k \geq 1/5$ then we use PGA what means throwing away one term of the form $2^{-s_0/2}e_j$, $j \in [1, 2^{s_0}]$. Applying the Equalizer to the very last term of the form $2^{-s_0/2}e_m$ we may encounter with $t''_k \geq 1/5$. In such a case we apply PGA and stop. As a result we get

$$f_{s_0+1} := \sum_{k \in F_{s_0+1}} c_k^{s_0+1} e_k.$$

It is clear that for all $k \in F_{s_0+1}$ we have

$$(c_k^{s_0+1})^2 \leq 2^{-s_0-1}$$

and also

$$|F_{s_0+1}| \leq 2^{s_0+1}.$$

Assume that $\epsilon < 1/20$. Then $2^{s_0+2}(\Delta q_{s_0})^{-1} < 1/5$ and $t''_k \geq 1/5$ is equivalent to $t''_k = t'_k = 1/5$. If $t''_k < 1/5$ then $t'_k < 1/5$ and $t_k \leq t'_k$. Therefore, at all Equalizer steps we have a WGA with weakness parameters $\{t_k\}$. If $t''_k = 1/5$ we apply PGA what is a WGA with any t_k at this step. During this procedure which we call "working on s_0 -level" we perform $M_{s_0}^w$ steps of Equalizer and $M_{s_0}^G$ steps of PGA. Let us estimate $M_{s_0}^w$ and $M_{s_0}^G$. It is clear that $M_{s_0}^G \leq 2^{s_0}$. We have applied the Equalizer to terms of the form $2^{-s_0/2}e_j$ at most 2^{s_0} times. For each Equalizer application we have $\sum \gamma_j \leq 2$ (see (3.3)). Thus denoting $E(s_0) := \{k : t''_k < 1/5\}$ we get

$$\sum_{k \in E(s_0)} t''_k \leq 2^{s_0+1}.$$

On the other hand we have

$$\sum_{k \in E(s_0)} t''_k \geq M_{s_0}^w 2^{s_0+2} (\Delta q_{s_0})^{-1}$$

and

$$M_{s_0}^w \leq \Delta q_{s_0} / 2.$$

Therefore,

$$N_{s_0} := M_{s_0}^w + M_{s_0}^G \leq \Delta q_{s_0} / 2 + 2^{s_0} \leq \Delta q_{s_0}.$$

At each Equalizer step we reduced the $\|\cdot\|^2$ by at most $9(t''_k)^2 2^{-s_0}$ and at each PGA by at most $25(t'_k)^2 2^{-s_0}$. Thus the total reduction δ_{s_0} for the s_0 -level does not exceed

$$25(2^{-s_0}) \sum_{k=1}^{q_{s_0}} (t'_k)^2 + 9(2^{s_0+4})(\Delta q_{s_0})^{-1}.$$

We are on the $(s_0 + 1)$ -level now and perform the similar procedure. We describe it for the general case of an s -level. Assume we have after $N_{s-1} \leq q_{s-1}$ steps of our WGA the function

$$f_s = \sum_{k \in F_s} c_k^s e_k$$

with

$$(c_k^s)^2 \leq 2^{-s}, \quad |F_s| \leq 2^s.$$

Define now

$$t''_k := \max\{t'_k, 2^{s+2}(\Delta q_s)^{-1}\}, \quad k > N_{s-1}.$$

We pick $c_k^s e_k$ with the biggest c_k^s out of $\{c_k^s, k \in F_s\}$ and throw it away if $t''_{N_{s-1}+1} = 1/5$ (we remind that assumption $\epsilon < 1/20$ implies $2^{s+2}(\Delta q_s)^{-1} < 1/5$) and apply the Equalizer with schedule $\{t''_n\}$ otherwise. We continue to perform the above described procedure (the mixture of Equalizer and PGA steps) until we get

$$f_{s+1} = \sum_{k \in F_{s+1}} c_k^{s+1} e_k$$

with

$$(c_k^{s+1})^2 \leq 2^{-s-1}.$$

It is clear that then $|F_{s+1}| \leq 2^{s+1}$. Similarly to the above estimates of $M_{s_0}^w$ and $M_{s_0}^G$ we get

$$M_s^G \leq 2^s$$

and

$$M_s^w 2^{s+2} (\Delta q_s)^{-1} \leq \sum_{k \in E(s)} t_k'' \leq 2^{s+1}.$$

Thus

$$M_s^w + M_s^G \leq \Delta q_s / 2 + 2^s \leq \Delta q_s$$

and

$$N_s := N_{s-1} + M_s^w + M_s^G \leq q_s.$$

The total reduction δ_s of the $\|\cdot\|^2$ from working on the s -level does not exceed

$$25(2^{-s}) \sum_{k=1}^{q_s} (t_k')^2 + 9(2^{s+4})(\Delta q_s)^{-1}.$$

We continue this process and get that the $\|\cdot\|^2$ will be reduced by at most

$$\sum_{s=s_0}^{\infty} \delta_s \leq 25 \left(\sum_{s=s_0}^{\infty} 2^{-s} \sum_{k=1}^{q_s} (t_k')^2 \right) + 144 \sum_{s=s_0}^{\infty} 2^s (\Delta q_s)^{-1} \leq 169\epsilon.$$

Choosing ϵ small enough, say, $\epsilon = 0.005$ we get divergent WGA with the weakness sequence τ . This completes the construction of the counterexample.

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