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Almost isometries of balls

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ALMOST ISOMETRIES OF BALLS

EVA MATOUŠKOVÁ

ABSTRACT. Let f be a bi-Lipschitz mapping of the Euclidean ball $B_{\mathbb{R}^n}$ into ℓ_2 with both Lipschitz constants close to one. We investigate the shape of $f(B_{\mathbb{R}^n})$. We give examples of such a mapping f , which has the Lipschitz constants arbitrarily close to one and at the same time has in the supremum norm the distance at least one from every isometry of \mathbb{R}^n .

1. INTRODUCTION

By the classical theorem of Mazur and Ulam, every surjective isometry f of two Banach spaces X and Y is affine. There are various possibilities how to slightly relax the isometry condition on f and still ask if f can be well approximated by an affine mapping (see [BL] for an exposition and literature on this subject). Here we will consider the case when both X and Y are Euclidean spaces and $f : B_X \rightarrow Y$ is a bi-Lipschitz mapping with both Lipschitz constants $1 + \varepsilon$ for some $0 < \varepsilon < 1$ (for exact definitions of an ε -rigid mapping, or of an ε -quasi-isometry see Section 2). If $\dim X = \dim Y = n$, then by a result of F. John [J], there is an isometry $T : X \rightarrow Y$ so that $\|f(x) - T(x)\| \leq cn^{\frac{3}{2}}\varepsilon$ for $x \in B_X$, where c is an absolute constant. The estimation of the approximation error $\alpha(n, \varepsilon)$ was improved by Vestfrid [Ve] to $\alpha(n, \varepsilon) \leq cn^{\frac{1}{2}}\varepsilon$. He proved also that in the general case when $n = \dim X \leq \dim Y$ the approximation error is at most $cn^{\frac{1}{2}}\sqrt{\varepsilon}$. (If $\dim X < \dim Y$, the order of magnitude of the error has to be at least $\sqrt{\varepsilon}$. To see this, it is enough to take the mapping $f : [-1, 1] \rightarrow \mathbb{R}^2$ defined by $f(t) = (t, 0)$ if $t \in [-1, 0]$ and $f(t) = (t, t\sqrt{\varepsilon})$ if $t \in [0, 1]$. This mapping is ε -rigid and its distance from any affine mapping $T : \mathbb{R} \rightarrow \mathbb{R}^2$ is at least $\sqrt{\varepsilon}/8$.)

In Section 4 we give examples which show that the approximation error really does depend on the dimension of X , answering thus a

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question in [BL]. For example, for any $\varepsilon > 0$ we construct an ε -quasi-isometry $f : B_{\mathbb{R}^n} \rightarrow \mathbb{R}^n$ (n is about $\exp \frac{1}{\varepsilon}$) such that $f(B_{\mathbb{R}^n/2})$ contains an orthonormal basis of \mathbb{R}^n . This f has the distance at least $1/\sqrt{2}$ from every affine mapping of \mathbb{R}^n . Consequently, if we wish to write the approximation error in the form $\alpha(n, \varepsilon) = \varphi(n)\varepsilon$, then $\varphi(n) \geq c \log n$ for some constant $c > 0$.

This is very much unlike the situation when both X and Y are Banach spaces of continuous functions on some compact metric spaces. Here, by a result of Löfblom [Lo], an ε -rigid mapping of B_X into Y can be approximated on $(1 - 8\varepsilon)B_X$ by an isometry within an error of 8ε .

We also investigate the shape of $f(B_{\mathbb{R}^n})$, if f is an ε -rigid mapping. In Proposition 3.1 and Proposition 3.2 an easy application of the theorem of Borsuk and Ulam shows that f can not “squeeze” $B_{\mathbb{R}^n}$ close to a space of dimension less than n : if Y is an affine space with $\dim Y < n$ then $f(B_{\mathbb{R}^n})$ is *not* contained in $Y + B(0, 1 - 4\sqrt{\varepsilon})$. In Proposition 3.4 we show a counterpart to Proposition 3.1: the convex hull K of an ε -rigid image of $B_{\mathbb{R}^n}$ can not fill up too much of $B_{\mathbb{R}^m}$ if $n < m$.

If Z is a closed linear subspace of a Hilbert space H , we denote by P_Z the orthogonal projection on Z . By $B_X(x, r)$ we denote the closed ball with the center at x and radius r in the Banach space X ; $B_X^o(x, r)$ is the open ball. By $S_X(x, r)$ we denote the corresponding sphere. The unit ball with the center at zero is denoted by B_X . We reserve the notation $B_{\mathbb{R}^n}$ and $S_{\mathbb{R}^n}$ for the Euclidean ball and sphere. By e_1, \dots, e_n we denote the standard orthonormal basis of \mathbb{R}^n . By c, c_1, c_2, \dots we denote absolute constants, which may have different values even in the same formula.

2. PRELIMINARIES

Let f be a mapping from an open subset U of a Banach space X into a Banach space Y . The local distortion of distances by f can be measured by the functions

$$D^+ f(x) = \limsup_{y \rightarrow x} \frac{\|f(y) - f(x)\|}{\|y - x\|},$$

$$D^- f(x) = \liminf_{y \rightarrow x} \frac{\|f(y) - f(x)\|}{\|y - x\|}.$$

The following class of almost isometric mappings was introduced by F. John [J] (see [BL] for many of their properties).

Definition 2.1. Let $\varepsilon > 0$. A mapping f from an open subset U of a Banach space X into a Banach space Y is called an ε -quasi-isometry if it satisfies the following two conditions

- (i) f is a local homeomorphism; *i.e.* every point $x \in U$ has an open neighborhood V such that f is a homeomorphism of V onto an open subset of Y .
- (ii) f satisfies $(1 + \varepsilon)^{-1} \leq D^- f(x) \leq D^+ f(x) \leq 1 + \varepsilon$ for every $x \in U$.

We will mostly work simply with bi-Lipschitz mappings which have the Lipschitz constants close to one:

Definition 2.2. Let $\varepsilon > 0$. A mapping f from a subset A of a Banach space X into a Banach space Y is called ε -rigid if $(1 + \varepsilon)^{-1} \|x - y\| \leq \|f(x) - f(y)\| \leq (1 + \varepsilon) \|x - y\|$ for all $x, y \in A$.

We will usually assume that $0 \in A$ and $f(0) = 0$. Also, we will often use the trivial observation that $1 - \varepsilon \leq (1 + \varepsilon)^{-1} \leq 1 - \varepsilon/2$ for $0 < \varepsilon < 1$.

If $U \subset \mathbb{R}^n$ is open and $f : U \rightarrow \mathbb{R}^n$ is ε -rigid then by the invariance of domains f is an ε -quasi-isometry (the invariance of domains says that if $V \subset \mathbb{R}^n$ is homeomorphic to an open set $U \subset \mathbb{R}^n$, then V itself is open in \mathbb{R}^n). The other way round, if X, Y are Banach spaces and $f : B_X^o(x, r) \rightarrow Y$ is an ε -quasi-isometry then f is ε -rigid on $B_X(x, r/(1 + \varepsilon)^2)$ and $f(B_X(x, r)) \supset B_Y(f(x), r/(1 + \varepsilon))$ (see *e.g.* [BL], p. 345).

It is an elementary, but useful fact that ε -rigid mappings almost preserve angles (see *e.g.* [BL], p. 349).

Lemma 2.3. *Let X be a Hilbert space, $0 < \varepsilon < 1$, $0 \in A \subset X$, and let $f : A \rightarrow X$ be ε -rigid and such that $f(0) = 0$. Then*

$$|\langle f(x), f(y) \rangle - \langle x, y \rangle| \leq \frac{3}{2}\varepsilon(\|x - y\|^2 + \|x\|^2 + \|y\|^2)$$

for all $x, y \in A$.

Proof. Since f is ε -rigid, $|\|f(x) - f(y)\|^2 - \|x - y\|^2| \leq 3\varepsilon\|x - y\|^2$ for $x, y \in A$. Hence

$$\begin{aligned} & 2|\langle f(x), f(y) \rangle - \langle x, y \rangle| \\ & \leq |\|f(x) - f(y)\|^2 - \|x - y\|^2| + |\|f(x)\|^2 - \|x\|^2| + |\|f(y)\|^2 - \|y\|^2| \\ & \leq 3\varepsilon(\|x - y\|^2 + \|x\|^2 + \|y\|^2). \end{aligned}$$

□

The following lemma states that ε -rigid mappings almost preserve linearity for convex combinations. It is derived in [Ve] from a result of [Za].

Lemma 2.4. *Let X be a Hilbert space, $A \subset X$ be convex, $f : A \rightarrow X$ ε -rigid. Then for any $x_1, \dots, x_n \in A$, $\lambda_i \geq 0$, $\sum_{i=1}^n \lambda_i = 1$ it holds*

$$\|f(\sum_{i=1}^n \lambda_i x_i) - \sum_{i=1}^n \lambda_i f(x_i)\| \leq \sqrt{2} \cdot \sqrt{\varepsilon} \max \|x_i - x_j\|.$$

This means in particular, that ε -rigid mappings of convex sets almost preserve the mid-points of line segments: $\|f(\frac{1}{2}(x + y)) - \frac{1}{2}(f(x) + f(y))\| \leq \sqrt{2}\sqrt{\varepsilon}\|x - y\|$ for $x, y \in A$.

Assume now that f is an ε -rigid mapping of a convex symmetric set A and $f(0) = 0$. Then f is almost antipodal; that is, $\|f(x) + f(-x)\| \leq 4\sqrt{2}\sqrt{\varepsilon}\|x\|$ for $x \in A$. Consequently, if $\lambda_i \in \mathbb{R}$ are such that $\sum_{i=1}^n |\lambda_i| = 1$, then

$$\begin{aligned} & \|f(\sum_{i=1}^n \lambda_i x_i) - \sum_{i=1}^n \lambda_i f(x_i)\| \\ & \leq \|f(\sum_{i=1}^n |\lambda_i|(x_i \cdot \operatorname{sgn} \lambda_i)) - \sum_{i=1}^n |\lambda_i| f(x_i \cdot \operatorname{sgn} \lambda_i)\| \\ (1) \quad & + \|\sum_{i=1}^n |\lambda_i| f(x_i \cdot \operatorname{sgn} \lambda_i) - \sum_{i=1}^n \lambda_i f(x_i)\| \\ & \leq \sqrt{2} \cdot \sqrt{\varepsilon} \operatorname{diam} A + \sum_{i=1}^n |\lambda_i| \|f(x_i \cdot \operatorname{sgn} \lambda_i) - f(x_i) \cdot \operatorname{sgn} \lambda_i\| \\ & \leq \sqrt{2} \cdot \sqrt{\varepsilon} \operatorname{diam} A + 4\sqrt{2} \cdot \sqrt{\varepsilon} \max \|x_i\| \\ & \leq 3\sqrt{2}\sqrt{\varepsilon} \operatorname{diam} A. \end{aligned}$$

This means that the image of a convex symmetric set by an ε -rigid mapping is again almost convex and almost symmetric.

Quasi-isometries preserve the mid-points of line segments with a smaller error $c\varepsilon$, instead of $c\sqrt{\varepsilon}$ for the ε -rigid mappings. The following lemma appears in [Ve] in a more general setting (f is a quasi-isometry between two Banach spaces), and with a rather involved proof. As we will use it only for quasi-isometries of Hilbert spaces, we provide here an elementary proof of this case.

Lemma 2.5. *Let $0 < a < 1$. There exists $\varepsilon_a > 0$ with the following property. Let X be a Hilbert space, and let $f : B_X^o(0, 1+a) \rightarrow X$ be an*

$\|v - f(x)\| \leq \|x - y\|(\frac{a}{4} + (1 + \varepsilon))$. Since $\langle v, f(x) \rangle = \langle v, f(y) \rangle = 0$, by Lemma 2.3

$$|\langle f^{-1}(v), x \rangle| \leq \frac{3}{2}\varepsilon(\|f(x)\|^2 + \|v\|^2 + \|f(x) - v\|^2) \leq 6\varepsilon\|x - y\|^2,$$

and, similarly, $|\langle f^{-1}(v), y \rangle| \leq 6\varepsilon\|x - y\|^2$. Hence $|\langle f^{-1}(v), \frac{x+y}{2} \rangle| \leq 6\varepsilon\|x - y\|^2$, and again by Lemma 2.3

$$\begin{aligned} |\langle v, f(\frac{x+y}{2}) \rangle| &\leq |\langle f^{-1}(v), \frac{x+y}{2} \rangle| \\ &+ \frac{3}{2}\varepsilon(\|f^{-1}(v)\|^2 + \|\frac{x+y}{2}\|^2 + \|f^{-1}(v) - \frac{x+y}{2}\|^2) \\ &\leq 6\varepsilon\|x - y\|^2 + \frac{3}{2}\varepsilon \cdot 2(\|f^{-1}(v)\|^2 + \|\frac{x+y}{2}\|^2 + \|f^{-1}(v)\| \cdot \|\frac{x+y}{2}\|) \\ &\leq 6\varepsilon\|x - y\|^2 + 3\varepsilon\|x - y\|^2((1 + \varepsilon)^2 a^2/16 + (1 + \varepsilon)^4 + (1 + \varepsilon)^3 a/4) \\ &\leq 16\varepsilon\|x - y\|^2. \end{aligned}$$

By the definition of v

$$\|f(\frac{x+y}{2})\| = \langle v, f(\frac{x+y}{2}) \rangle \cdot \frac{4}{a} \cdot \frac{1}{\|x-y\|} \leq 16\varepsilon\|x-y\|^2 \cdot \frac{4}{a} \cdot \frac{1}{\|x-y\|} \leq \frac{64}{a} \cdot \varepsilon \cdot \|x-y\|,$$

and by (3)

$$\|f(\frac{x+y}{2}) - \frac{f(x)+f(y)}{2}\| \leq \|z - \frac{f(x)+f(y)}{2}\| + \|f(\frac{x+y}{2}) - z\| \leq \frac{65}{a}\varepsilon\|x - y\|.$$

□

Suppose that an ε -rigid mapping $f : B_{\mathbb{R}^n} \rightarrow \ell_2$ is well approximated by an affine mapping. Then f is well approximated by an isometry. This statement is used several times in [Ve]; for an easy reference we state it as a lemma.

Lemma 2.6. *Let $\varepsilon > 0$, $a > 0$ be such that $a + \varepsilon < 1$. Let $f : B_{\mathbb{R}^n} \rightarrow \ell_2$ be ε -rigid and $T : \mathbb{R}^n \rightarrow \ell_2$ linear such that $\|f(x) - T(x)\| \leq a$ for all $x \in B_{\mathbb{R}^n}$. Then there exists an isometry $\mathbb{R}^n \rightarrow \ell_2$ so that $\|f(x) - S(x)\| \leq \varepsilon + 2a$ for all $x \in B_{\mathbb{R}^n}$.*

Proof. Let u_1, \dots, u_n be an orthonormal basis of \mathbb{R}^n , v_1, \dots, v_m an orthonormal basis of $T(\mathbb{R}^n)$ and $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m > 0$ so that $T(u_i) = \lambda_i v_i$ for $i = 1, \dots, m$ and $T(u_i) = 0$ and $\lambda_i = 0$ for $i > m$. Then

$$\lambda_i = \|T(u_i)\| \leq \|f(u_i)\| + a \leq 1 + \varepsilon + a \quad \text{and}$$

$$\lambda_i = \|T(u_i)\| \geq \|f(u_i)\| - a \geq 1 - \varepsilon - a,$$

hence $1 + \varepsilon + a \geq \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 1 - \varepsilon - a > 0$; in particular, $m = n$. Define S by $S(u_n) = v_n$. Then $\|S - T\| = \max_{i \in \{1, \dots, n\}} |1 - \lambda_i| \leq a + \varepsilon$, and the lemma follows from the triangle inequality. □

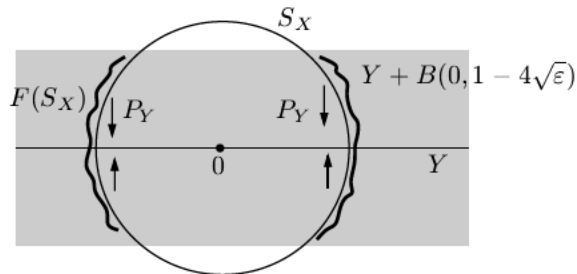


FIGURE 2. Illustration to the proof of Proposition 3.1.

3. ϵ -RIGID MAPPINGS AND LINEAR SUBSPACES

Let X be a Banach space and $A \subset X$; let $k \in \mathbb{N}$. Recall, that the Kolmogorov k -diameter $d_k(A, X)$ of A expresses how well can A be approximated by k -dimensional subspaces of X :

$$d_k(A, X) = \inf_{X_k} \sup_{x \in A} \inf_{y \in X_k} \|x - y\|,$$

the left-most infimum being taken over all k -dimensional subspaces X_k of X . The sum of a linear subspace and of a ball is a convex set, hence $d_k(A, X) = d_k(\text{conv } A, X)$ (for other properties of the Kolmogorov diameter see *e.g.* [Pi]).

First we observe that an ϵ -rigid mapping f can not squeeze the unit ball of a k -dimensional Hilbert space inside of a small neighborhood of a space with dimension $l < k$. We will actually show that the Kolmogorov l -diameter of $f(B_{\mathbb{R}^k})$ is almost one.

Proposition 3.1. *Let $0 < \epsilon < 1$ and $f : B_{\mathbb{R}^n} \rightarrow \ell_2$ be ϵ -rigid, $f(0) = 0$. Let $X \subset \mathbb{R}^n$, $Y \subset \ell_2$ with $\dim Y < \dim X$. Then $f(B_X)$ is not contained in $Y + B_{\ell_2}(0, 1 - 4\sqrt{\epsilon})$.*

Proof. Suppose that

$$f(B_X) \subset U := Y + B_{\ell_2}(0, 1 - 4\sqrt{\epsilon}).$$

Assuming this, we will construct a continuous antipodal mapping $\Phi : S_X \rightarrow Y$ such that $\Phi(x) \neq 0$ for all $x \in S_X$, which will contradict the Borsuk-Ulam theorem. For $x \in S_X$ put $F(x) = \frac{1}{2}(f(x) - f(-x))$ and define $\Phi = P_Y \circ F$. The mapping F is antipodal, as $F(-x) = \frac{1}{2}(f(-x) - f(x)) = -F(x)$, hence Φ is antipodal as well. By the remark after Lemma 2.4

$$(4) \quad \begin{aligned} \|F(x)\| &= \|f(x) - \frac{1}{2}(f(-x) + f(x))\| \geq \|f(x)\| - 2\sqrt{2}\sqrt{\epsilon} \\ &\geq 1 - \epsilon - 2\sqrt{2}\sqrt{\epsilon} > 1 - 4\sqrt{\epsilon}. \end{aligned}$$

Since U is convex and symmetric, $F(S_X) \subset U$, and

$$(5) \quad \|P_Y(F(x))\| \geq \|F(x)\| - (1 - 4\sqrt{\varepsilon}).$$

By (4) and (5)

$$\|\Phi(x)\| = \|P_Y(F(x))\| > 1 - 4\sqrt{\varepsilon} - (1 - 4\sqrt{\varepsilon}) = 0.$$

□

The midpoints of line segments are for ε -quasi-isometries by Lemma 2.5 preserved with the error $c\varepsilon$ instead just $c\sqrt{\varepsilon}$ as it was for ε -rigid mappings. This enables slightly improve Proposition 3.1; the proof is the same.

Proposition 3.2. *Let $0 < a < 1$. There exists $\varepsilon_a > 0$ with the following property. Let $f : B_{\mathbb{R}^n}^\circ(0, 1+a) \rightarrow \mathbb{R}^n$, $f(0) = 0$ be an ε -quasi isometry for some $0 < \varepsilon \leq \varepsilon_a$. Suppose $X, Y \subset \mathbb{R}^n$ with $\dim Y < \dim X$. Then $f(B_X)$ is not contained in $Y + B_{\mathbb{R}^n}(0, 1 - \frac{140}{a} \cdot \varepsilon)$.*

To prove a counterpart to Proposition 3.1, we will need the following version of the theorem of Bartle and Graves.

Theorem 3.3. *Let X, Y be Banach spaces, $T : X \rightarrow Y$ continuous, linear and surjective and $K \subset X$ closed and convex. Then there exists a continuous mapping $f : T(K) \rightarrow K$ so that $T(f(y)) = y$ for all $y \in T(K)$. Moreover, if K is symmetric, f can be chosen so that $f(y) = -f(-y)$ for all $y \in T(K)$.*

Proof. (Sketch) We can simply follow the proof in ([BP], p. 86). Let Φ be the inverse of T restricted to K . Then $\Phi : T(K) \rightarrow 2^X$ is a complete convex lower semi-continuous mapping. By Michael's theorem Φ admits a continuous selection f . If K is moreover symmetric, we replace $f(y)$ by $\frac{1}{2}(f(y) - f(-y))$. □

Next we prove that the convex hull K of an ε -rigid image of a k -dimensional unit ball can not fill up too much of an l -dimensional unit ball if $l > k$. Namely, the maximal inscribed ball of the projection $P_Y(K)$ onto any Y with $\dim Y = l > k$ has radius only $c\sqrt{\varepsilon}$. Notice however, that this does not mean that K is contained in a small neighborhood of a k -dimensional space. This follows from Example 4.1.

Proposition 3.4. *Let $0 < \varepsilon < \frac{1}{2}$ and $f : B_{\mathbb{R}^n} \rightarrow \ell_2$ be ε -rigid, $f(0) = 0$. Let $X \subset \mathbb{R}^n$, $Y \subset \ell_2$ with $\dim Y = \dim X + 1$, and $K = \text{sym conv } f(B_X)$. Then $\max\{r : B_Y(0, r) \subset P_Y(K)\} < 30\sqrt{\varepsilon}$.*

Proof. Assume that $B_Y(0, 30\sqrt{\varepsilon}) \subset P_Y(K)$. As in the proof of Proposition 3.1, we will construct under this assumption a continuous antipodal mapping $\Phi : S_Y(0, 30\sqrt{\varepsilon}) \rightarrow X$ such that $\Phi(y) \neq 0$ for all

$y \in S_Y(0, 30\sqrt{\varepsilon})$. This will contradict the Borsuk-Ulam theorem. The mapping $f^{-1} : f(B_X) \rightarrow X$ is $(1 + \varepsilon)$ -Lipschitz; by the theorem of Kirszbraun (see *e.g.* [BL], p. 19) it can be extended to a $(1 + \varepsilon)$ -Lipschitz mapping $\varphi : \ell_2 \rightarrow X$. For $v \in \ell_2$ put $F(v) = \frac{1}{2}(\varphi(v) - \varphi(-v))$; clearly, F is antipodal. Let $v \in K$. By (1), there exists $x \in B_X$ so that $\|f(x) - v\| \leq 6\sqrt{2}\sqrt{\varepsilon}$. Since $\|f(x)\| \leq (1 + \varepsilon)\|x\|$,

$$\|x\| \geq (\|v\| - 6\sqrt{2}\sqrt{\varepsilon})(1 + \varepsilon)^{-1} \geq \frac{2}{3}\|v\| - 4\sqrt{2}\sqrt{\varepsilon}.$$

By the definition of φ we have $\|x - \varphi(v)\| \leq (1 + \varepsilon)\|f(x) - v\|$, hence

$$(6) \quad \begin{aligned} \|\varphi(v)\| &\geq \|x\| - (1 + \varepsilon)\|v - f(x)\| \\ &\geq \frac{2}{3}\|v\| - 13\sqrt{2}\sqrt{\varepsilon}. \end{aligned}$$

By Theorem 3.3, there exists a continuous mapping $\psi : P_Y(K) \rightarrow K$ such that $P_Y(\psi(y)) = y$ and $\psi(y) = -\psi(-y)$ for all $y \in P_Y(K)$. As ψ is a selection from the inverse of an orthogonal projection, it is also $\|\psi(y)\| \geq \|y\|$. Define $\Phi : P_Y(K) \rightarrow X$ by $\Phi = \varphi \circ \psi$. Let $y \in S_Y(0, 30\sqrt{\varepsilon}) \subset P_Y(K)$. Then by (6)

$$(7) \quad \begin{aligned} \|\Phi(y)\| &= \|\varphi(\psi(y))\| \geq \frac{2}{3}\|\psi(y)\| - 13\sqrt{2}\sqrt{\varepsilon} \\ &\geq \frac{2}{3}\|y\| - 13\sqrt{2}\sqrt{\varepsilon} > 0, \end{aligned}$$

and this contradicts the Borsuk-Ulam theorem. \square

If $T : \mathbb{R}^n \rightarrow \ell_2$ is affine, then, clearly, the graph of T is contained in an n -dimensional affine subspace of $\mathbb{R}^n \oplus \ell_2$. If some $f : B_{\mathbb{R}^n} \rightarrow \ell_2$ is well approximated by an affine mapping, then the graph of f is contained in a small neighborhood of an n -dimensional affine subspace of $\mathbb{R}^n \oplus \ell_2$. In Lemma 3.5 we observe that the converse also holds. If the graph of a mapping $f : B_{\mathbb{R}^n} \rightarrow 2B_{\ell_2}$ is contained in a small neighborhood of an n -dimensional affine subspace of $\mathbb{R}^n \oplus \ell_2$, then f is well approximated by an affine mapping.

Lemma 3.5. *Let $f : B_{\mathbb{R}^n} \rightarrow \ell_2$ be a mapping with $\|f(x)\| \leq 2$ for $x \in B_{\mathbb{R}^n}$. Suppose there is an n -dimensional subspace $Z \subset \mathbb{R}^n \oplus \ell_2$ and $0 < \delta < \frac{1}{2}$ such that the graph of f is contained in $Z + B_{\mathbb{R}^n \oplus \ell_2}(0, \delta)$. Then there is a linear mapping $T : \mathbb{R}^n \rightarrow \ell_2$ so that $\|T(x) - f(x)\| \leq 7\delta$ for all $x \in B_{\mathbb{R}^n}$.*

Proof. Let $P = P_{\mathbb{R}^n}$ be the orthogonal projection on \mathbb{R}^n . We can assume that $P : Z \rightarrow \mathbb{R}^n$ is a bijection; this can be achieved by an arbitrarily small perturbation of Z . Put $S = P^{-1}$; S has necessarily the form $S(x) = (x, T(x))$ with T linear. Choose orthonormal bases

$\{u_1, \dots, u_n\}$ of \mathbb{R}^n and $\{v_1, \dots, v_n\}$ of Z , so that $T(u_i) = \lambda_i v_i$ for some $\lambda_1 \geq \dots \geq \lambda_n \geq 0$. Choose $y \in \mathbb{R}^n$ so that

$$\frac{1}{2} > \delta \geq \text{dist}((u_1, f(u_1)), Z) = (\|u_1 - y\|^2 + \|T(y) - f(u_1)\|^2)^{\frac{1}{2}}.$$

If $\langle z, u_1 \rangle < \frac{1}{2}$ for some $z \in \mathbb{R}^n$, then $\|z - u_1\| \geq \frac{1}{2}$. Hence $\langle y, u_1 \rangle \geq \frac{1}{2}$, and

$$\frac{1}{2} \geq \|T(y) - f(u_1)\| \geq \|T(y)\| - \|f(u_1)\| \geq \lambda_1 \langle y, u_1 \rangle - 2.$$

This implies that $\|T\| = \lambda_1 \leq 5$, and $\|S\| \leq 6$.

Let $x \in B_{\mathbb{R}^n}$; denote $F(x) = (x, f(x))$. For $y = P(P_Z(F(x)))$ it holds

$$\begin{aligned} \|x - y\| &= \|P(F(x)) - P(S(y))\| \leq \|P\| \cdot \|F(x) - S(y)\| \\ &= \|F(x) - P_Z(F(x))\| \leq \delta. \end{aligned}$$

Hence

$$\begin{aligned} \|T(x) - f(x)\| &= \|S(x) - F(x)\| \leq \|S(x) - S(y)\| + \|S(y) - F(x)\| \\ &\leq \|S\|\delta + \delta \leq 7\delta. \end{aligned}$$

□

If f is an ε rigid mapping, then by an elementary computation (which we perform below) the mapping $F(x) = \frac{1}{\sqrt{2}}(x, f(x))$ is 2ε -rigid. Suppose f is not well approximated by affine mappings, for example, $f(0) = 0$ and $\sup_{x \in B_{\mathbb{R}^n}} \|f(x) - T(x)\| \geq \delta > 0$ for all linear mappings T . Then by Lemma 3.5, the Kolmogorov n -diameter of $F(B_{\mathbb{R}^n})$ is large, namely $d_n(F(B_{\mathbb{R}^n}), \ell_2) \geq \delta/7$.

Lemma 3.6. *Let $A \subset \ell_2$ and $f : A \rightarrow \ell_2$ ε -rigid for some $\varepsilon > 0$. Let $K > 0$ and $F : A \rightarrow \ell_2$ be the mapping which gives each $x \in A$ its image in the graph of $K \cdot f$; that is, $F(x) = (x, Kf(x))$ (here we write $\ell_2 = \ell_2 \oplus \ell_2$). Then for all $x, y \in A$*

$$(\sqrt{1 + K^2} - \varepsilon K)\|x - y\| \leq \|F(x) - F(y)\| \leq (\sqrt{1 + K^2} + \varepsilon K)\|x - y\|.$$

Proof. If $x \neq y$, then

$$\frac{\|F(x) - F(y)\|^2}{\|x - y\|^2} = 1 + K^2 \frac{\|f(x) - f(y)\|^2}{\|x - y\|^2},$$

and

$$(1 - \varepsilon)^2 \leq \frac{\|f(x) - f(y)\|^2}{\|x - y\|^2} \leq (1 + \varepsilon)^2.$$

Moreover

$$\sqrt{1 + K^2} - \varepsilon K \leq \sqrt{1 + K^2(1 - \varepsilon)^2} \text{ and } \sqrt{1 + K^2(1 + \varepsilon)^2} \leq \sqrt{1 + K^2} + \varepsilon K.$$

□

4. A QUASI-ISOMETRY CAN BE FAR FROM ALL ISOMETRIES

Consider the following example by F. John [J] (see also [BL], p. 352). Let $0 < \varepsilon < 1$. The mapping h of the unit disc $B_{\mathbb{R}^2}$ onto itself defined in the polar coordinates by $h(r, \varphi) = (r, \varphi + \varepsilon \log r)$ for $r > 0$ and by $h(0) = 0$ is an ε -quasi-isometry; it actually satisfies $(1 + \varepsilon)^{-1} \|x - y\| \leq \|h(x) - h(y)\| \leq (1 + \varepsilon) \|x - y\|$ for all $x, y \in B_{\mathbb{R}^2}$. If we define h outside of the unit disc by $h(x) = x$, the above inequality holds for all $x, y \in \mathbb{R}^2$. This can be seen by direct checking; also, it follows immediately from Lemma 2 of [IP] applied to both h and the inverse of h . In the supremum norm, h can be well approximated by the identity. It rotates each $x \in B_{\mathbb{R}^2}$ around the origin by an angle $\varepsilon \log(\|x\|)$; close to the origin this changes a lot.

We will use h to construct an ε -quasi-isometry f of $B_{\mathbb{R}^{2n}}$ onto itself (n is about $\exp \frac{1}{\varepsilon}$) so that the image of $B_{\mathbb{R}^n}$ nearly contains the unit ball $B_{\ell_1^{2n}}$. As any affine mapping carries \mathbb{R}^n to an affine subspace of dimension at most n , the mapping f can not be well approximated by an isometry.

Theorem 4.1. *Let $0 < \varepsilon < 1$ be given. There exists $n \in \mathbb{N}$ and a norm preserving ε -quasi-isometry f of \mathbb{R}^{2n} onto itself so that $f(x) = -f(-x)$ for $x \in \mathbb{R}^{2n}$, and $f(B_{\mathbb{R}^n})$ contains an orthonormal basis of \mathbb{R}^{2n} . Consequently,*

- (i) $d_k(f(B_{\mathbb{R}^n}), \ell_2^{2n}) \geq \sqrt{1 - \frac{k}{2n}}$ for $1 \leq k \leq 2n$;
- (ii) if $T : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ is affine, then $\sup_{x \in B_{\mathbb{R}^{2n}}} \|T(x) - f(x)\| \geq \frac{1}{\sqrt{2}}$,
and
- (iii) $B_{\ell_1^{2n}} \subset f(B_{\mathbb{R}^n}) + B_{\mathbb{R}^{2n}}(0, 2\sqrt{\varepsilon})$.

Proof. We can assume that ε is of the form $\varepsilon = \frac{\pi}{\log 2} \cdot \frac{1}{K}$, where $K \in \mathbb{N}$ is large enough, and put $n = 2^K$. We write \mathbb{R}^{2n} as $\mathbb{R}^n \oplus \mathbb{R}^n$. Let e_1, \dots, e_n be the standard orthonormal basis of the first copy of \mathbb{R}^n , and let $e_{n+1}, e_{n+2}, \dots, e_{2n}$ be the standard orthonormal basis of the second copy of \mathbb{R}^n . Let u_1, \dots, u_n be the orthonormal basis of the first \mathbb{R}^n which corresponds to the columns of the Hadamard matrix; that is, each u_j is of the form $u_j = \frac{1}{\sqrt{n}} \sum_{i=1}^n \varepsilon_{i,j} e_i$, where $\varepsilon_{i,j} \in \{1, -1\}$ are suitably chosen. Similarly, let v_1, \dots, v_n be an orthonormal basis of the second \mathbb{R}^n for which $v_j = \frac{1}{\sqrt{n}} \sum_{i=1}^n \varepsilon_{i,j} e_{n+i}$.

Let $h : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the mapping defined above; let $g = e^{\pi i/2} h$ be h composed with the rotation by $\pi/2$ around the origin. Then g rotates by $\pi/2$ all $z \in \mathbb{R}^2$ with $\|z\| \geq 1$ and $g(z) = z$ for all $z \in \mathbb{R}^2$ with

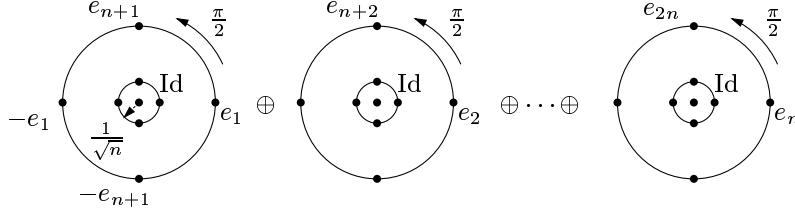


FIGURE 3. Illustration to the proof of Theorem 4.1.

$\|z\| = 1/\sqrt{n}$, as

$$\frac{\pi}{2} + \varepsilon \log \frac{1}{\sqrt{n}} = \frac{\pi}{2} + \frac{\pi}{\log 2} \cdot \frac{1}{K} \cdot \log \frac{1}{\sqrt{2^K}} = 0.$$

Below we will consider g written in the Cartesian coordinates. Now we will write \mathbb{R}^{2n} as the ℓ_2 -sum of n copies of \mathbb{R}^2 :

$$\mathbb{R}^{2n} = \mathbb{R}^2 \oplus \cdots \oplus \mathbb{R}^2 = \text{span}\{e_1, e_{n+1}\} \oplus \text{span}\{e_2, e_{n+2}\} \oplus \cdots \oplus \text{span}\{e_n, e_{2n}\}.$$

We define f “coordinate-wise”: if $x = \sum_{i=1}^n (x_i e_i + x_{n+i} e_{n+i})$ then

$$\begin{aligned} f(x) &= f((x_1, x_{n+1}), (x_2, x_{n+2}), \dots, (x_n, x_{2n})) \\ &= (g(x_1, x_{n+1}), g(x_2, x_{n+2}), \dots, g(x_n, x_{2n})). \end{aligned}$$

Since g preserves the norm and $g(z) = -g(-z)$ for $z \in \mathbb{R}^2$, it holds $\|f(x)\| = \|x\|$ and $f(x) = -f(-x)$ for $x \in \mathbb{R}^{2n}$. Since g is a bi-Lipschitz mapping of \mathbb{R}^2 onto itself, f is a bi-Lipschitz mapping of \mathbb{R}^{2n} onto itself and the Lipschitz constants are the same; that is, $(1 + \varepsilon)^{-1} \|x - y\| \leq \|f(x) - f(y)\| \leq (1 + \varepsilon) \|x - y\|$ for all $x, y \in \mathbb{R}^{2n}$. The projection of e_j , $j \in \{1, \dots, n\}$ on each of the 2-dimensional blocks spanned by $\{e_k, e_{n+k}\}$ is either e_j itself (if $j = k$), or zero. As $g(0) = 0$ and g rotates by $\pi/2$ on the unit circle, we have $f(e_j) = e_{n+j}$ for $j = 1, \dots, n$. The projection $p_k(u_j)$ of u_j on each of the 2-dimensional blocks spanned by $\{e_k, e_{n+k}\}$ is $p_k(u_j) = \frac{1}{\sqrt{n}} \varepsilon_{k,j} e_k$, hence $\|p_k(u_j)\| = \frac{1}{\sqrt{n}}$. Therefore $g(p_k(u_j)) = p_k(u_j)$ for $k = 1, \dots, n$ and $f(u_j) = u_j$ for each $j = 1, \dots, n$. Consequently, as $f(x) = -f(-x)$, the image of the first copy of \mathbb{R}^n contains (both plus and minus) the orthonormal basis $Q = \{u_1, u_2, \dots, u_n, e_{n+1}, e_{n+2}, \dots, e_{2n}\}$ of \mathbb{R}^{2n} . For completeness, let us mention, that this way we also obtain that $f(e_{n+j}) = -e_j$ and $f(v_j) = v_j$ for $n = 1, \dots, n$.

Since $\pm Q \subset f(B_{\mathbb{R}^n})$, and $B_{\ell_1^{2n}} = \text{conv } \pm Q$, the statement (i) follows from the estimate $d_k(B_{\ell_1^{2n}}, \ell_2^{2n}) = \sqrt{1 - \frac{k}{2n}}$, $k \in \{1, \dots, 2n\}$ for the Kolmogorov diameter of the ball of ℓ_1^{2n} (see e.g. [T], p. 237). In particular, since $\pm Q$ is symmetric, if Z is an n -dimensional affine subspace of

\mathbb{R}^{2n} , then there exists $q \in \pm Q$ so that $\text{dist}(Z, q) \geq 1/\sqrt{2}$. This implies (ii), as $Z = T(\mathbb{R}^n)$ is an at most n -dimensional affine subspace of \mathbb{R}^n . The statement (iii) follows from Lemma 2.4, since $\pm Q \subset f(B_{\mathbb{R}^n})$. \square

Let $f : B_{\mathbb{R}^n} \rightarrow \mathbb{R}^n$ be an ε -quasi-isometry for some $0 < \varepsilon < 1$. Denote by $\alpha(f) = \inf_T \sup_{x \in B_{\mathbb{R}^n}} \|T(x) - f(x)\|$, where the infimum is taken over all affine mappings $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$. Let $\alpha(n, \varepsilon) = \sup_f \alpha(f)$, the supremum being taken over all f as above. By [J] and [Ve], $\alpha(n, \varepsilon) \leq c\sqrt{n}\varepsilon$. If we similarly define $\beta(n, \varepsilon)$ for ε -rigid mappings, then by [Ve], $\beta(n, \varepsilon) \leq c\sqrt{n}\sqrt{\varepsilon}$. Theorem 4.1 implies, that if we wish to write $\alpha(n, \varepsilon)$ in the form $\alpha(n, \varepsilon) = \varphi(n)\varepsilon$, then it holds $\varphi(n) \geq c \log n$, where $c > 0$ is a suitable constant. Indeed, if $n \in \mathbb{N}$, choose $K \in \mathbb{N}$ so that $2^{K+1} \leq n < 2^{K+2}$, that is, $K = \lfloor \log n / \log 2 \rfloor - 1$. In the proof of Theorem 4.1 we constructed an ε -quasi-isometry $f : \mathbb{R}^{2^{K+1}} \rightarrow \mathbb{R}^{2^{K+1}}$, with $\varepsilon = \frac{\pi}{\log 2} \cdot \frac{1}{K}$, so that $\alpha(f) = 1/\sqrt{2}$. If we write $\mathbb{R}^n = \mathbb{R}^{2^{K+1}} \oplus \mathbb{R}^{n-2^{K+1}}$ and define $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ by $F(x, y) = (f(x), y)$, then F is also an ε -quasi-isometry with $\alpha(f) = 1/\sqrt{2}$. Hence

$$\frac{1}{\sqrt{2}} \leq \varphi(n)\varepsilon = \varphi(n) \cdot \frac{\pi}{\log 2} \cdot \frac{1}{\lfloor \log n / \log 2 \rfloor - 1},$$

and $\varphi(n) \geq c \log n$ for a suitable $c > 0$. Similarly, if we wish to write $\beta(n, \varepsilon)$ in the form $\beta(n, \varepsilon) = \psi(n)\sqrt{\varepsilon}$, then it holds $\psi(n) \geq c \log^{\frac{1}{2}} n$, where $c > 0$ is a suitable constant. This shows that the approximation error for near-isometries which was estimated in [ATV] also does depend on the dimension.

A natural approach how to try to approximate an ε -quasi-isometry f defined on $B_{\mathbb{R}^n}$ by a linear mapping T is to fix an orthonormal basis of \mathbb{R}^n (for example $\{e_1, \dots, e_n\}$), and put $T(e_i) = \frac{1}{2}(f(e_i) - f(-e_i))$ for $i = 1, \dots, n$. This is basically used in both [J] and [Ve]. Again, if we wish the approximation error to be of the form $\alpha(n, \varepsilon) = \varphi(n)\varepsilon$ with $\varphi(n)$ as small as possible, the best this approach can give to us is $\varphi(n) = c\sqrt{n}$, as was achieved in [Ve].

Lemma 4.2. *Let n be large enough. There exists an isometry S of \mathbb{R}^n with $\|S - \text{Id}\| = 2$ and $\frac{8}{\sqrt{n}}$ -quasi-isometry $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ with $f(0) = 0$, so that $\|S(x) - f(x)\| \leq \frac{2}{\sqrt{n}}$ for $x \in B_{\mathbb{R}^n}$ and at the same time $f(\pm e_i) = \pm e_i$ for $i = 1, \dots, n$.*

Moreover, if $n = 2^k$ for some $k \in \mathbb{N}$, and u_1, \dots, u_n is the orthonormal basis of \mathbb{R}^n which corresponds to the columns of the Hadamard matrix then $f(\pm u_1) = \mp u_1$, and $f(\pm u_i) = \pm u_i$ for $i = 2, \dots, n$.

Proof. Let $v = \frac{1}{\sqrt{n}} \sum_{i=1}^n e_i$. Then $\|v\| = 1$ and v is “almost orthogonal” to all e_i ’s; that is, $\langle v, e_i \rangle = \frac{1}{\sqrt{n}}$ for all $i = 1, \dots, n$. Let $S : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be the isometry which coincides with the identity on $\text{Ker } v$ and $S(v) = -v$; that is, $S(x) = x - 2\langle x, v \rangle v$. Let φ be the function supported on the interval $[-\frac{1}{2}, \frac{1}{2}]$, for which $\varphi(0) = \frac{2}{\sqrt{n}}$ and φ is linear on $[-\frac{1}{2}, 0]$ and on $[0, \frac{1}{2}]$. Define $\varphi_i^+ : \mathbb{R}^n \rightarrow \mathbb{R}$ by $\varphi_i^+(x) = \varphi(\|x - e_i\|)$, and, similarly, $\varphi_i^-(x) = -\varphi(\|x + e_i\|)$. As the distances of different $\pm e_i$ ’s are at least $\sqrt{2}$, the functions φ_i are disjointly supported. Consequently, as the function φ is $\frac{4}{\sqrt{n}}$ -Lipschitz, the function $\Phi = \sum_{i=1}^n (\varphi_i^+ + \varphi_i^-)$ is $\frac{4}{\sqrt{n}}$ -Lipschitz as well, with $|\Phi| \leq \frac{2}{\sqrt{n}}$. For $x \in B_{\mathbb{R}^n}$ put $f(x) = S(x) + \Phi(x)v$. Then f is $\frac{8}{\sqrt{n}}$ -rigid and $\|S(x) - f(x)\| \leq |\Phi(x)| \leq \frac{2}{\sqrt{n}}$ for $x \in \mathbb{R}^n$. Moreover, $f(0) = 0$ and $f(\pm e_i) = \pm e_i - 2\langle \pm e_i, v \rangle v + \varphi_i^\pm(e_i)v = \pm e_i$.

Suppose that $n = 2^k$. We can assume that $u_1 = v$. As $S = \text{Id}$ on $\text{Ker } v$ and $\|u_i \pm e_j\| \geq \sqrt{(n-1)/n} > 1/2$, $f(\pm u_i) = \pm u_i$ for $i = 2, \dots, n$. \square

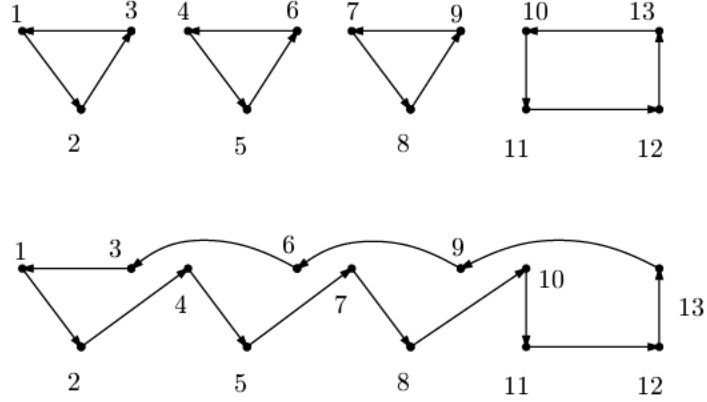
To present a modification of Theorem 4.1 we recall a few basic facts about permutations. A *permutation* p on a finite set Ω is a bijection of Ω onto itself. Each permutation can be decomposed uniquely, except for order, into disjoint cycles. For example,

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 1 & 5 & 4 & 6 & 7 \end{pmatrix} = (1, 2, 3)(4, 5)(6)(7) = (1, 2, 3)(4, 5);$$

in the last expression the single point cycles (that is, the fixed points of the permutation) are omitted. A transposition is a permutation which consists of one cycle of length two; all the other cycles have length one. Every permutation can be written as a composition of (enough many) transpositions. Each permutation can be composed from four permutations each of which consist only of *disjoint* transpositions (and of single point cycles). For convenience we include a simple proof of this.

Lemma 4.3. *Let p be permutation on a finite set Ω . Then $p = p_4 \circ p_3 \circ p_2 \circ p_1$, where each of the permutations p_1, \dots, p_4 consists of disjoint cycles of length at most two.*

Proof. We can assume that p is a cycle. For cycles of length up to five it holds: $(1, 2, 3) = (2, 3) \circ (1, 2)$; $(1, 2, 3, 4) = (2, 4) \circ [(1, 2)(3, 4)]$; and $(1, 2, 3, 4, 5) = (4, 5) \circ (2, 4) \circ [(1, 2)(3, 4)]$.

FIGURE 4. The permutations p' and p for $n = 13$.

If $|\Omega| = n > 5$, we write $n = 3k + l$, where $k \in \mathbb{N}$ and $l \in \{3, 4, 5\}$, and put

$$p' = (1, 2, 3)(4, 5, 6)(7, 8, 9) \dots (3k - 2, 3k - 1, 3k)(3k + 1, \dots, 3k + l),$$

$$p_4 = (3, 4)(6, 7) \dots (3k, 3k + 1).$$

By the special cases mentioned above, $p' = p_3 \circ p_2 \circ p_1$, where each of the permutations p_1, \dots, p_3 consists of disjoint cycles of length at most two. The permutation p_4 joins the k triangles and one l -gon of the permutation p' into a single cycle:

$$p_4 \circ p' = (1; 2, 4, 5, 7, \dots, 3k - 1, 3k + 1; 3k + 2, \dots, 3k + l; 3k, 3(k - 1), \dots, 3).$$

By denoting the elements of Ω successively (according to the cycle p) by $1, 2, 4, 5, 7, \dots, 3$ we get that $p = p_4 \circ p_3 \circ p_2 \circ p_1$. \square

Theorem 4.4. *Let $0 < \varepsilon < 1$ be given. There exist $n \in \mathbb{N}$ and two orthonormal bases $\{e_1, \dots, e_n\}$ and $\{u_1, \dots, u_n\}$ of \mathbb{R}^n with the following property. Let p be a permutation on $\{1, 2, \dots, n\}$, and let $\alpha_i \in \{-1, 1\}$. There exists an ε -quasi-isometry f of \mathbb{R}^n onto itself so that $f(\pm e_i) = \pm \alpha_i e_{p(i)}$ and $f = \text{Id}$ on $\{0, \pm u_1, \dots, \pm u_n\}$.*

Proof. Choose $K \in \mathbb{N}$ so that $(1 + \frac{2\pi}{\log 2} \cdot \frac{1}{K})^6 \leq 1 + \varepsilon$ and put $n = 2^{K+2}$. Let e_1, \dots, e_n be the standard orthonormal basis of \mathbb{R}^n . Let u_1, \dots, u_n be the orthonormal basis of \mathbb{R}^n which corresponds to the columns of the Hadamard matrix; that is, each u_j is of the form $u_j = \frac{1}{\sqrt{n}} \sum_{i=1}^n \varepsilon_{i,j} e_i$, where $\varepsilon_{i,j} \in \{1, -1\}$ are suitably chosen. We will prove two special cases of the theorem:

- (A) Suppose p consists of disjoint cycles of length at most two. Then there exists a norm preserving $(\frac{\pi}{\log 2} \cdot \frac{1}{K})$ -quasi-isometry f of \mathbb{R}^n

onto itself so that $f(x) = -f(-x)$ for $x \in \mathbb{R}^n$, $f(e_i) \in \{\pm e_{p(i)}\}$ and $f = \text{Id}$ on $\{u_1, \dots, u_n\}$.

- (B) Suppose that $|\{i : \alpha_i = -1\}|$ is even. Then there exists a norm preserving $(\frac{2\pi}{\log 2} \cdot \frac{1}{K})$ -quasi-isometry f of \mathbb{R}^n onto itself so that $f(x) = -f(-x)$ for $x \in \mathbb{R}^n$, $f(e_i) = \alpha_i e_i$ and $f = \text{Id}$ on $\{u_1, \dots, u_n\}$.

To get the general case, we write as in Lemma 4.3 $p = p_4 \circ p_3 \circ p_2 \circ p_1$, where each of the permutations p_1, \dots, p_4 consists of disjoint cycles of length at most two. For each $j \in \{1, \dots, 4\}$, let f_j be the quasi-isometry which exists by (A) for the permutation p_j . The quasi-isometry $\tilde{f} = f_4 \circ f_3 \circ f_2 \circ f_1$ satisfies $\tilde{f}(e_i) = \beta_i e_{p(i)}$ for some $\beta_i \in \{-1, 1\}$, and $\tilde{f} = \text{Id}$ on $\{u_1, \dots, u_n\}$. If $|\{i : \alpha_i \neq \beta_i\}|$ is even, there exists by (B) a quasi-isometry f_5 so that $f = f_5 \circ \tilde{f}$ satisfies the conclusion of the theorem. Suppose $|\{i : \alpha_i \neq \beta_i\}|$ is odd; we can assume that $\alpha_1 \neq \beta_1$. By (B) there exists a quasi-isometry f_5 so that $f_5 \circ \tilde{f}$ satisfies the conclusion of the theorem but for $(f_5 \circ \tilde{f})(e_1) = -\alpha_1 e_1$. By Lemma 4.2 (with the bases $\{e_1, \dots, e_n\}$ and $\{u_1, \dots, u_n\}$ interchanged), there exists a quasi-isometry f_6 , so that $f = f_6 \circ f_5 \circ \tilde{f}$ is as required.

Proof of (A). Let $p = (a_1, b_1) \dots (a_k, b_k)$. To keep the notation more transparent, we will treat the concrete case when $p = (1, 2)(3, 4) \dots (n-1, n)$; the generalization is obvious. Let $h : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the mapping defined above Theorem 4.1 with $\varepsilon = \frac{\pi}{\log 2} \cdot \frac{1}{K}$. Let $g = e^{\pi i/2} \cdot h$ be h composed with the rotation by $\pi/2$ around the origin. Then g rotates by $\pi/2$ all $z \in \mathbb{R}^2$ with $\|z\| \geq 1$ and $g(z) = z$ for all $z \in \mathbb{R}^2$ with $\|z\| = 2/\sqrt{n}$, as

$$\frac{\pi}{2} + \varepsilon \log \frac{2}{\sqrt{n}} = \frac{\pi}{2} + \frac{\pi}{\log 2} \cdot \frac{1}{K} \cdot \log \frac{2}{\sqrt{2^{K+2}}} = 0.$$

Below we will consider g written in the Cartesian coordinates. We write \mathbb{R}^n as the ℓ_2 -sum of $n/2$ copies of \mathbb{R}^2 and define f “coordinate-wise”: if $x = \sum_{i=1}^n x_i e_i$ then

$$f(x) = f(x_1, x_2, \dots, x_n) = (g(x_1, x_2), g(x_3, x_4), \dots, g(x_{n-1}, x_n)).$$

Since g is a bi-Lipschitz mapping of \mathbb{R}^2 onto itself, f is a bi-Lipschitz mapping of \mathbb{R}^n onto itself and the Lipschitz constants are the same, that is, f is a $(\frac{\pi}{\log 2} \cdot \frac{1}{K})$ -quasi-isometry. Since g preserves the norm and $g(z) = -g(-z)$ for $z \in \mathbb{R}^2$, f is also norm-preserving and $f(x) = -f(-x)$ for $x \in \mathbb{R}^n$. The projection of e_j on each of the 2-dimensional blocks spanned by $\{e_l, e_{l+1}\}$ is either e_j itself (if $j \in \{l, l+1\}$), or zero. As g rotates by $\pi/2$ on the unit circle and $g(0) = 0$, we have $f(e_{2k-1}) = e_{2k}$ and $f(e_{2k}) = -e_{2k-1}$ for $k \in \{1, \dots, \frac{n}{2}\}$. The projection

$p_l(u_j)$ of u_j on each of the 2-dimensional blocks spanned by $\{e_l, e_{l+1}\}$ is $p_l(u_j) = \frac{1}{\sqrt{n}}(\varepsilon_{l,j}e_l + \varepsilon_{l+1,j}e_{l+1})$, hence $\|p_l(u_j)\| = \frac{2}{\sqrt{n}}$. It follows that $g(p_l(u_j)) = p_l(u_j)$ and $f(u_j) = u_j$ for $j \in \{1, \dots, n\}$.

Proof of (B). Again, to keep the notation more transparent, we will treat a concrete case: assume that $\alpha_1 = \dots = \alpha_n = -1$. The generalization is obvious. Let $h : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the mapping defined above Theorem 4.1 with $\varepsilon = \frac{2\pi}{\log 2} \cdot \frac{1}{K}$. Let $g = e^{\pi i} \cdot h$ be h composed with the rotation by π around the origin. Then g rotates by π all $z \in \mathbb{R}^2$ with $\|z\| \geq 1$ and $g(z) = z$ for all $z \in \mathbb{R}^2$ with $\|z\| = 2/\sqrt{n}$, as

$$\pi + \varepsilon \log \frac{2}{\sqrt{n}} = \pi + \frac{2\pi}{\log 2} \cdot \frac{1}{K} \cdot \log \frac{2}{\sqrt{2^{K+2}}} = 0.$$

Below we will consider g written in the Cartesian coordinates. We write \mathbb{R}^n as the ℓ_2 -sum of $n/2$ copies of \mathbb{R}^2 and define f “coordinate-wise”: if $x = \sum_{i=1}^n x_i e_i$ then

$$f(x) = f(x_1, x_2, \dots, x_n) = (g(x_1, x_2), g(x_3, x_4), \dots, g(x_{n-1}, x_n)).$$

As in the proof of (A), f is a norm preserving $(\frac{2\pi}{\log 2} \cdot \frac{1}{K})$ -quasi-isometry of \mathbb{R}^n onto itself, and $f(x) = -f(-x)$ for $x \in \mathbb{R}^n$. The projection of e_j on each of the 2-dimensional blocks spanned by $\{e_l, e_{l+1}\}$ is either e_j itself (if $j \in \{l, l+1\}$), or zero. As g rotates by π on the unit circle and $g(0) = 0$, we have $f(e_j) = -e_j$ for $j \in \{1, \dots, n\}$. Exactly as in the proof of (A) we get that $f(u_j) = u_j$ for $j \in \{1, \dots, n\}$. \square

If f is the ε -quasi-isometry from Theorem 4.4 for which $f = -\text{Id}$ on the orthonormal basis $\{e_1, \dots, e_n\}$ and $f = \text{Id}$ on the orthonormal basis $\{u_1, \dots, u_n\}$ (we treated this particular case in the proof of the statement (B)), then $\sup_{x \in B_{\mathbb{R}^n}} \|f(x) - T(x)\| \geq 1$ for any linear $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$. Indeed, suppose that for some linear $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ we have $\|T(x) - f(x)\| < 1$ for each $x \in B_{\mathbb{R}^n}$. Then

$$\langle T(e_i), e_i \rangle = \langle T(e_i) - f(e_i), e_i \rangle + \langle f(e_i), e_i \rangle \leq -1 + \|T(e_i) - f(e_i)\| < 0.$$

Similarly,

$$\langle T(u_i), u_i \rangle = \langle T(u_i) - f(u_i), u_i \rangle + \langle f(u_i), u_i \rangle \geq 1 - \|T(u_i) - f(u_i)\| > 0.$$

Let A be the matrix of T with respect to the basis $\{e_1, \dots, e_n\}$, and B be the matrix of T with respect to the basis $\{u_1, \dots, u_n\}$. Then $\text{trace } A = \text{trace } B$. At the same time

$$\text{trace } A = \sum_{i=1}^n \langle T(e_i), e_i \rangle < 0,$$

and

$$\text{trace } B = \sum_{i=1}^n \langle T(u_i), u_i \rangle > 0,$$

which is a contradiction. As the mapping f in the proof of (B) satisfies moreover $f(x) = -f(-x)$ for $x \in \mathbb{R}^n$, it holds also that $\sup_{x \in B_{\mathbb{R}^n}} \|f(x) - T(x)\| \geq 1$ for any affine $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$.

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