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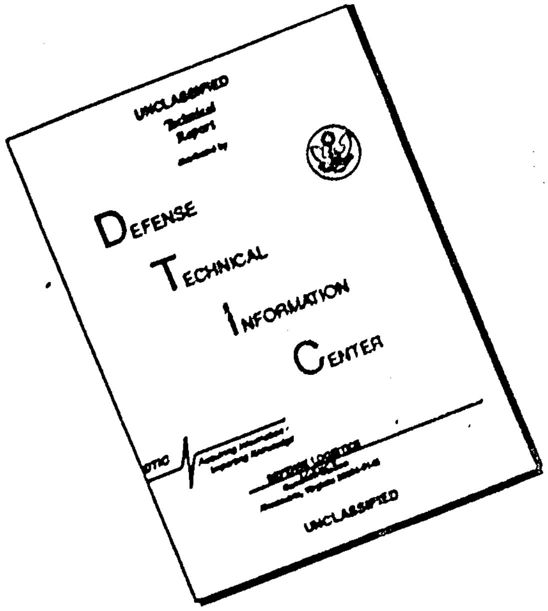

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THEORY OF THE ANOMALOUS PROPAGATION OF ACOUSTIC WAVES IN THE OCEAN

by

H. W. Marsh, Jr.

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U. S. NAVY UNDERWATER SOUND LABORATORY
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H. W. Marsh, Jr.

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ABSTRACT

This report contains a theoretical study of acoustic wave propagation in the ocean. The ordinary wave equation is used, and solutions in terms of normal modes, supplemented by certain integral terms, are developed. An alternative solution, which is useful at very low frequencies, is obtained by the use of integral equations.

FOREWORD

This report embodies the material of a thesis submitted by the author in partial fulfillment of the requirements for the Degree of Doctor of Philosophy in the Department of Physics at Brown. The subject matter was considered to be of interest to the Navy in connection with experimental studies of acoustic wave propagation being carried out at the U. S. Navy Underwater Sound Laboratory.

Much of the work represented in this report was carried out at this laboratory, and the gratitude of the author is due Dr. J. M. Ide, Chief Scientist, and Captain W. L. Pryor, Commanding Officer, for their encouragement and support of this work. For invaluable assistance in the numerical aspects of Section 7, the author is indebted to the Misses M. R. Powers, R. M. Scully, M. A. Gramberger, and E. R. Friedman, of the Mathematics Consultant Group at the Laboratory. Finally, his sincere thanks are due Professor R. B. Lindsay, Chairman of the Department of Physics at Brown University, and to his colleagues, Professors A. O. Williams and G. S. Heller, of Brown University, under whose guidance and stimulation this research was carried out.

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SECTION 1

INTRODUCTION

This thesis has to deal with certain portions of the theory of wave motion. Of especial interest are those phenomena associated with the propagation of waves in a medium whose physical properties are not uniform. When the propagating medium is not homogeneous, curious circumstances may exist which cannot always be understood from physical intuition alone. Consequently, the highest resources of applied mathematics must be called upon in order to produce a satisfactory theory.

Inhomogeneities in the medium may cause traveling waves to be focused, as if by a lens; the waves may mutually interfere so as to produce shadows; or the wave intensity at a given point may fluctuate greatly in time (fade). Waves may be scattered by local irregularities, as in the formation of the rainbow or the blue aspect of the sky.

There was much interest at the beginning of the present century in the theory of radio wave propagation around the earth, and this theory is formally equivalent to that treated in this text. The obstacles encountered by those theorists were such as to cause Nicholson to characterize the problem as the most difficult one confronting the mathematical physicists of that time.

The subject of this tract is an unusual point of focus for the interests of the mathematician, the physicist, and the engineer concerned with problems of wave propagation. This fact makes anything resembling a broad treatment of the topic an impossibility within the confines of a dissertation of moderate length. It is nevertheless to be hoped that something of interest to each profession will be found in subsequent pages, and that the rather diverse points of view presented will be tolerantly accepted as indigenous to the character of the problem.

Anomalous wave propagation has been of interest recently in the fields of microwave radio in the terrestrial atmosphere and of ultrasonics in submarine areas. These two fields coalesce from the mathematical viewpoint, as they also must do with the bulk of wave mechanics. In fact, the dichotomy of modern physics is clearly recognized in the procedures of practical analysis employed for engineering purposes. Traditionally, the appropriate methods of analysis for wave problems depend largely upon the relationship between the wave length and the physical dimensions in question. It is thus a stroke of good fortune that the parameters of many important phases of the above three fields fall within a rather narrow numerical range. This is due, no doubt, to the fact that investigators in this relatively new subject tend to adopt conventional methods where possible. One must nonetheless be grateful for the considerable wealth of powerful methodology which is available from the earlier efforts of other researchers.

The point of departure for an analytic discussion of wave propagation is the wave equation. Small motion acoustic waves and electromagnetic waves emanated from a suitably oriented dipole of angular frequency ω may be described from a knowledge of the wave potential φ, which satisfies the wave equation

$$\Delta\varphi + \omega^2/c^2\varphi = 0 \tag{1.1}$$

Inhomogeneities in the propagating medium express themselves in terms of spatial variations in the phase velocity c. The object in restricting c to spatial variations only is that the time dependent wave field may be obtained from φ by Fourier transformation only if the properties of the medium are stationary. We shall be concerned only with this situation and shall be further limited to the case where time dependency is manifested only through the factor $\exp(i\omega t)$.

For many purposes, an approximate but simpler differential equation than 1.1 is the Eikonal equation

$$\nabla\theta \cdot \nabla\theta = \left(\frac{c_0}{c}\right)^2 \qquad 1.2$$

c_0 denotes some standard (constant) value of c. φ may be obtained to a first approximation, in the limit of large k $(= \omega/c_0)$ from a knowledge of θ. The surfaces θ = const are the characteristic surfaces of the time dependent wave equation, and the orthogonal trajectories to these surfaces are the rays. Eq. 1.2 gives useful and accurate results in regions of space reached by the rays. Other portions of space may be studied by using the Airy analysis of diffraction from caustic surfaces (1).*

A more general application of Eq. 1.2 may be made by the use of "imaginary" rays. This procedure is equivalent to the W. K. B. or phase integral approximation to the three dimensional wave field specified by Eq. 1.1 and leads us to another restriction of the types of inhomogeneity to be considered. We shall discuss this question as it bears on submarine acoustics, that for the radio case being exhaustively treated in (17).

The acoustic phase velocity c may be calculated from the equation of state for the medium in question, provided that the approximate values of the state variables are known. Thus, a knowledge of temperature, pressure, and the concentration of foreign matter present (salinity) suffices to determine the velocity. The temperature is the most important single factor and serves for a gross estimate of the velocity. It is found that the primary temperature structure in the ocean is depth-dependent only. That is to say, at a given time the isotherms are roughly parallel to the ocean surface. Because of diurnal and seasonal effects, the position of the isotherms will shift from time to time, but the changes are slow in comparison with propagation times of any interest.

To the extent that the isotherms are not parallel to the surface and have a short-term variation, a description of the ocean in terms of the depth as the only parameter is unrealistic. It is realized that very many important practical problems will be completely begged. However, it is imperative that the number of parameters entering the problem be minimized. Accordingly, in the interest of general results which may be used for an approximate study of specific situations, we shall henceforth consider the velocity structure to be depth-dependent only.

In this case Eq. 1.1 is separable, yielding an equation for β, the Fourier-Bessel transform of φ:

$$\frac{d^2\beta}{dz^2} + [\omega^2/c^2(z) - \lambda^2]\beta = 0 \qquad 1.3$$

*The numbers in parentheses indicate the references listed in the bibliography at the end of this thesis.

The central problem then becomes that of determining the eigenvalues of the parameter λ corresponding to the boundary conditions on φ, and the subsequent calculation of β for these values of λ. The characteristic difference between Eq. 1.3 for radio and acoustic waves and Eq. 1.3 for the wave mechanical problem is that the eigenvalues for the former are, in general, complex. For an application of the W. K. B. method to this case of complex eigenvalues, c must be known as an analytic function of the complex variable z. Pekeris (20) has considered special cases where c(z) is given this analytic character. Other, more general, cases are considered in some detail in (17), Vol. I.

It will be realized that the problems of World War II provided a fertile field for the investigation of wave propagation. The studies of Sommerfeld (24), Watson (26), and several others laid the groundwork for research yet to come. (For a complete bibliography see (17), Vol. I.) During the period 1925-1940, the methods of wave mechanics were developed to a high degree and thus were available when military problems became of interest to the scientist. The entire body of work on this problem during the war was accomplished under the cognizance of the National Defense Research Committee, Committee on Propagation, of which C. R. Burrows was chairman. The summary reports of the latter committee (17), Vols. I, II, and III, provide a wealth of useful information on the radio aspects of the problem and include a most comprehensive bibliography. The study of related problems involving the diffraction of waves in other fields was made by Rayleigh (22) in the case of scattering by small particles and by Airy (1) in the question of the formation of the rainbow. The study of stratified media occurs in the theory of acoustic or electric filtration, as, for example, in a work of Lindsay (15). This dissertation is primarily concerned with the problem of acoustic wave propagation, and so will be supplementary to the great volume of other work done heretofore.

The ordinary wave equation will be shown in Section 2 to come from the equations of hydrodynamics if suitable approximations are made. In Section 3 the formal aspects of propagation in the vicinity of a reflecting surface are considered, and a series expansion of the wave potential is obtained in terms of cylindrical waves. An equivalent integral equation is developed in Section 4, and the use of a Liouville-Neumann series is considered for its solution. Finally, in the remaining sections, special cases with numerical illustrations are considered.

SECTION 2

HYDRODYNAMICAL EQUATIONS; DETERMINATION OF VELOCITY FROM THE EQUATION OF STATE

The fundamental physical laws governing the propagation of acoustic waves differ markedly from those controlling electromagnetic undulation in the practical sense that the former are approximated by the neglect of all non-linear terms in almost the entire field of acoustics. In addition, a knowledge of the equation of state of the acoustic medium permits an accurate calculation of the phase velocity, in contradistinction to the semi-empirical basis for the determination of electromagnetic velocity.

Although the science of oceanography lags behind that of meteorology in many respects, it is believed that present measuring techniques suffice for a satisfactory determination of acoustic velocity in the ocean. These measurements are introduced into the wave theory by means of the following considerations: If we make the abbreviations

ρ = fluid density

p = fluid pressure

μ = coefficient of viscosity

$\underline{v}$ = fluid velocity

$\underline{F}$ = body force per mass unit

then we have for the equation of motion the Navier-Stokes formula in the Eulerian form

$$\rho \frac{\partial \underline{v}}{\partial t} + \rho \underline{v} \cdot \nabla \underline{v} = \rho \underline{F} - \nabla p + \frac{1}{3} \mu \nabla \nabla \cdot \underline{v} + \mu \nabla \cdot \nabla \underline{v} \qquad 2.1$$

and for the conservation of mass the equation of continuity

$$\nabla \cdot \rho \underline{v} + \frac{\partial \rho}{\partial t} = 0 \qquad 2.2$$

We shall assume the existence of a velocity potential φ such that

$$\underline{v} = \nabla \varphi$$

and that the equilibrium density ρ_o and pressure p_o are not functions of the time. Then, if the following quantities are introduced,

s = condensation = $\frac{1}{\rho_o} (\rho - \rho_o)$

p_e = excess pressure = $p - p_o$

Eq. 2.1 becomes, after two differentiations,

$$\Delta\left\{\rho\ddot{\varphi} + \dot{p}_e - \frac{4}{3}\mu\Delta\dot{\varphi}\right\} = \frac{\partial}{\partial t}\left\{\dot{\varphi}\Delta\rho + \nabla p\cdot\underline{v} - \nabla\cdot(\rho\underline{v}\cdot\nabla\underline{v}) + \nabla\cdot(s\nabla p_o) - \frac{4}{3}\nabla\cdot\underline{v}\Delta\mu - \frac{4}{3}\nabla\mu\cdot\nabla\nabla\cdot\underline{v}\right\} \qquad 2.3$$

and Eq. 2.2 becomes

$$\rho\Delta\varphi + \rho_o\dot{s} = -\underline{v}\cdot\nabla\rho \qquad 2.4$$

The body force is eliminated through the equilibrium condition that $\rho_o\underline{F}$ be balanced by the buoyant force ∇p_o.

Acoustic wave propagation is believed to correspond to adiabatic changes of state. Then

$$p_e = \left(\frac{\partial p}{\partial \rho}\right)_S \rho_o s = c^2\rho_o s \qquad 2.5$$

where S is the entropy. Blokhintzev (4) may be consulted for a thorough consideration of the thermodynamics behind Eq. 2.5. If the equations be combined, there results

$$\Delta\left\{\rho\ddot{\varphi} - c^2\rho\Delta\varphi - \frac{4}{3}\mu\Delta\dot{\varphi}\right\} = G \qquad 2.6$$

where

$$G = \frac{\partial}{\partial t}\left\{\nabla\cdot(s\nabla p_o - \rho\underline{v}\cdot\nabla\underline{v}) + \dot{\varphi}\Delta\rho + \nabla\rho\cdot\dot{\underline{v}} - \frac{4}{3}\nabla\cdot\underline{v}\Delta\mu - \frac{4}{3}\nabla\mu\cdot\nabla\nabla\cdot\underline{v}\right\} + \Delta\left\{c^2\underline{v}\cdot\nabla\rho - \dot{\rho}\dot{\varphi}\right\}$$

The entire field of linear acoustics is based on the supposition that the quantity G is so small that it can be neglected. We shall give a numerical argument in favor of this presently, but shall assume for the moment that G is zero. We then obtain

$$c^2\Delta\varphi + \frac{4}{3}\frac{\mu}{\rho}\Delta\dot{\varphi} - \ddot{\varphi} = 0 \qquad 2.7$$

since only functions of time and space are of interest. For simple harmonic motion of angular frequency ω,

$$\Delta\varphi + \frac{\omega^2}{c^2\left(1 + i\frac{4\omega\mu}{3\rho_o c^2}\right)}\varphi = 0$$

Since the quantity $\frac{4}{3}\frac{\omega\mu}{\rho_o c^2}$ is very small in comparison with unity at all frequencies of any interest, we may write

$$\Delta\varphi + \frac{\omega^2}{c^2}\left(1 - i\frac{4\omega\mu}{3\rho_o c^2}\right)\varphi = 0 \qquad 2.8$$

In principle, then, one can calculate the phase velocity from the formula

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_S \qquad 2.9$$

and the attenuation coefficient from ω, ρ_o, and c^2. Thus, the plane wave attenuation coefficient is defined to be

$$a = \frac{2\mu}{3\rho_o c^3}\omega^2 \qquad 2.10$$

Formula 2.10 is fairly reliable at frequencies above a megacycle/sec., but is in error by about two orders of magnitude in the kilocycle range. Here one must adopt the results of measurements. Fig. 2.1 shows a compilation made by Dr. J. Warren Horton, of the U. S. Navy Underwater Sound Laboratory, and is reproduced with his kind permission.

For the determination of the phase velocity, it is necessary to use Eq. 2.5, since there are insufficient data to permit a direct calculation of the velocity from the values of other physical parameters. The most reliable source of information at present is the report by Susumu Kuwahara (11), which gives complete numerical information, based on thermodynamic measurements, for the estimation of the phase velocity from known values of temperature, pressure, and salinity. A condensation of the information contained in this report (see Fig. 2.2) was prepared at the Columbia Division of War Research, Fort Trumbull, New London, Conn., in 1945, from data prepared by Fleming and Revelle. From Fig. 2.2 a precision of about 1 part in 5×10^4 may be obtained with linear interpolation. Such precision is at times necessary for an analytic consideration of propagation conditions. It is therefore somewhat disquieting that the thermodynamic basis of this tables does not admit extension to this precision by at least an order of magnitude. Moreover, of the thermodynamic parameters, only pressure can be known with satisfactory precision. Practical measurements of temperature are at present reliable only within approximately 0.5° Fahrenheit, a fact which immediately reduces the ultimate velocity precision to 1 part in perhaps 5×10^3. It is probable, however, that these factors are not in themselves dominant, since other assumptions, such as that of horizontal stratification, are likely to be of more importance.

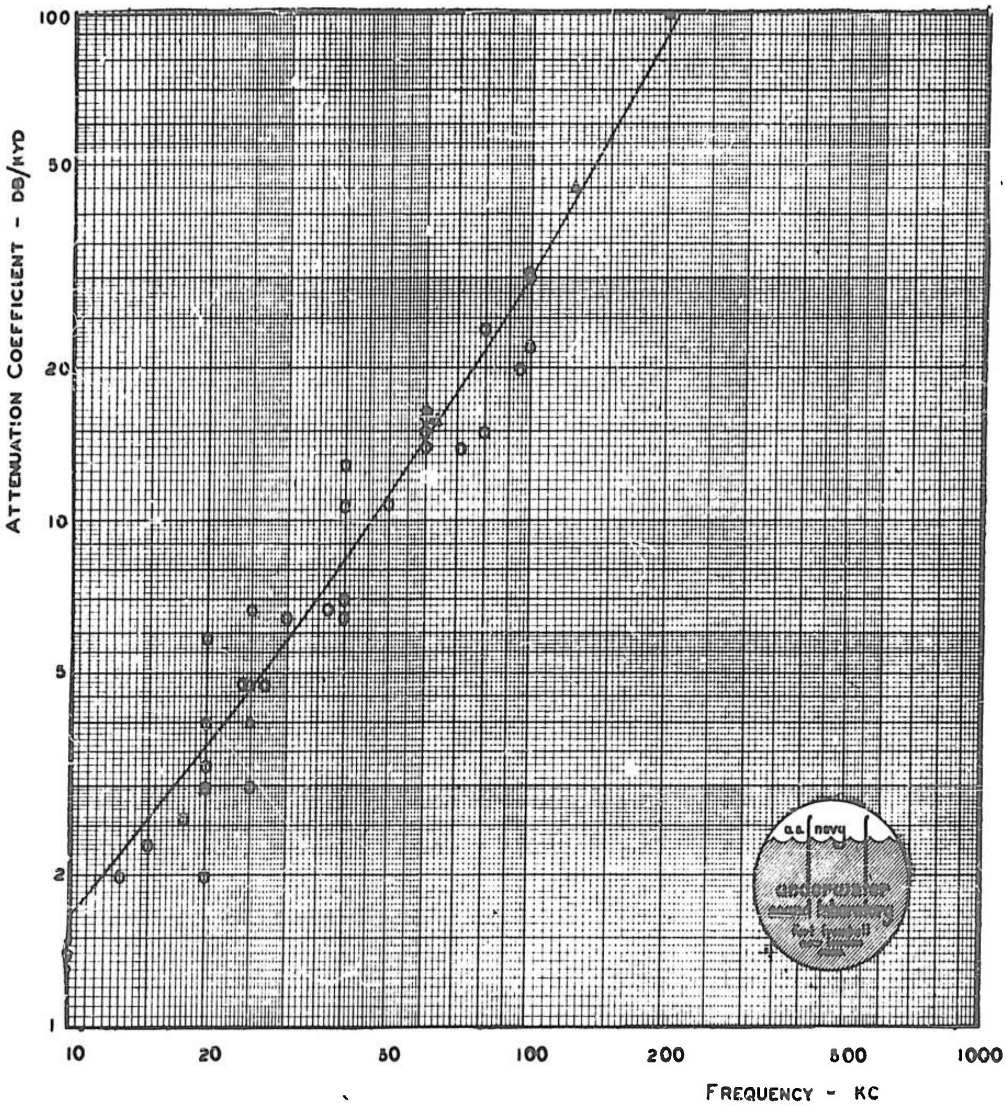
We have now to give a numerical argument justifying the neglect of the quantity G in Eq. 2.6. This argument will be based on a consideration of plane wave solutions to Eq. 2.8, from which the order of magnitude of all terms on both sides of Eq. 2.6 may be estimated. Let I_o be the intensity of plane waves traveling in the x direction, and let $\frac{\mu}{\rho_o c^2}$, c^2 assume their mean values for sea water. We then have

$$p_e = \sqrt{2\rho_o c I_o}$$

$$s = \frac{p_e}{\rho_o c^2}$$

$$|\underline{v}| = \frac{p_e}{\rho_o c}$$

$$\varphi = \frac{p_e}{\omega\rho}$$



Values of the attenuation coefficient applying to the exponential decrease, resulting from scattering and absorption, of the intensity of acoustical energy propagated through sea water

○ Miscellaneous measurements made in the ocean between 1938 and 1945

△ Values reported by Liebermann in University of California Marine Physical Laboratory Quarterly Report for July to September, 1947

——— Values computed by the relation $a = 0.15\ f + 0.0015\ f^2$

Fig. 2.1 - Attenuation Coefficient

SOUND VELOCITY ANOMALY (V^*) IN SEA WATER AT VARIOUS TEMPERATURES
($V = V^* + 4742.4$)

$V_a = 4.27\ (S - 35.00\ ^o/oo)$
$V_p = 0.0182$/ft depth

Temp. °F	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
35	24.6	25.4	26.2	27.0	27.8	28.6	29.4	30.2	31.0	31.8
36	32.6	33.4	34.2	35.0	35.8	36.6	37.4	38.2	39.0	39.8
37	40.6	41.4	42.2	43.0	43.8	44.5	45.3	46.1	46.9	47.7
38	48.5	49.3	50.1	50.8	51.6	52.4	53.2	53.9	54.7	55.5
39	56.3	57.0	57.8	58.6	59.3	60.1	60.8	61.6	62.4	63.1
40	63.9	64.7	65.4	66.2	67.0	67.7	68.5	69.2	70.0	70.7
41	71.5	72.2	73.0	73.7	74.5	75.2	76.0	76.7	77.5	78.2
42	79.0	79.7	80.5	81.2	82.0	82.7	83.4	84.2	84.9	85.7
43	86.4	87.1	87.9	88.6	89.3	90.1	90.8	91.5	92.2	93.0
44	93.7	94.4	95.1	95.9	96.6	97.3	98.0	98.7	99.5	100.2
45	100.9	101.6	102.3	103.0	103.7	104.4	105.2	105.9	106.6	107.3
46	108.0	108.7	109.4	110.1	110.8	111.5	112.2	112.9	113.6	114.3
47	115.0	115.7	116.4	117.1	117.8	118.4	119.1	119.8	120.5	121.2
48	121.9	122.6	123.3	123.9	124.6	125.3	126.0	126.7	127.3	120.0
49	128.7	129.4	130.0	130.7	131.4	132.1	132.7	133.4	134.1	134.7
50	135.4	136.1	136.7	137.4	138.0	138.7	139.4	140.0	140.7	141.3
51	142.0	142.6	143.3	143.9	144.6	145.2	145.9	146.5	147.2	147.8
52	148.5	149.1	149.8	150.4	151.1	151.7	152.4	153.0	153.7	154.3
53	154.9	155.5	156.2	156.8	157.4	158.0	158.7	159.3	159.9	160.6
54	161.2	161.8	162.4	163.1	163.7	164.3	165.0	165.6	166.2	166.8
55	167.5	168.1	168.7	169.3	169.9	170.5	171.2	171.8	172.4	173.0
56	173.6	174.2	174.8	175.4	176.0	176.6	177.2	177.8	178.4	179.0
57	179.6	180.2	180.8	181.4	182.0	182.5	183.1	183.7	184.3	184.9
58	185.5	186.1	186.7	187.3	187.8	188.4	189.0	189.6	190.2	190.8
59	191.3	191.9	192.5	193.1	193.7	194.2	194.8	195.4	196.0	196.5
60	197.1	197.7	198.3	198.8	199.4	200.0	200.5	201.1	201.7	202.2
61	202.6	203.3	203.9	204.5	205.0	205.6	206.2	206.7	207.3	207.8
62	208.4	208.9	209.5	210.1	210.6	211.2	211.7	212.3	212.8	213.4
63	213.9	214.4	215.0	215.5	216.1	216.6	217.2	217.7	218.2	218.8
64	219.3	219.8	220.4	220.9	221.4	222.0	222.5	223.0	223.6	224.1
65	224.6	225.2	225.7	226.2	226.8	227.3	227.8	228.3	228.9	229.4
66	229.9	230.4	230.9	231.5	232.0	232.5	233.0	233.5	234.0	234.5
67	235.0	235.5	236.1	236.6	237.1	237.6	238.1	238.6	239.1	239.6
68	240.1	240.6	241.1	241.6	242.1	242.6	243.1	243.6	244.1	244.6
69	245.1	245.6	246.1	246.6	247.1	247.6	248.0	248.5	249.0	249.5
70	250.0	250.5	251.0	251.5	251.9	252.4	252.9	253.4	253.9	254.4
71	254.9	255.3	255.8	256.3	256.8	257.3	257.7	258.2	258.7	259.1
72	259.6	260.1	260.6	261.0	261.5	262.0	262.4	262.9	263.4	263.8
73	264.3	264.8	265.2	265.7	266.2	266.6	267.1	267.6	268.1	268.5
74	269.0	269.5	269.9	270.4	270.8	271.3	271.8	272.2	272.7	273.1
75	273.6	274.0	274.5	274.9	275.4	275.8	276.3	276.7	277.2	277.6
76	278.1	278.5	279.0	279.4	279.9	280.3	280.8	281.2	281.7	282.1
77	282.6	283.0	283.5	283.9	284.4	284.8	285.3	285.7	286.1	286.6
78	287.0	287.4	287.9	288.3	288.8	289.2	289.6	290.1	290.5	291.0
79	291.4	291.8	292.2	292.7	293.1	293.5	293.9	294.3	294.8	295.2
80	295.7	296.1	296.5	297.0	297.4	297.8	298.2	298.6	299.1	299.5
81	299.9	300.3	300.7	301.2	301.6	302.0	302.4	302.8	303.3	303.7
82	304.1	304.5	305.0	305.4	305.8	306.2	306.7	307.1	307.5	307.9
83	308.3	308.7	309.1	309.5	309.9	310.3	310.8	311.2	311.6	312.0
84	312.4	312.8	313.2	313.6	314.0	314.4	314.8	315.2	315.6	316.0
85	316.4	316.8	317.2	317.6	318.0	318.4	318.8	319.2	319.6	320.0

Fig. 2.2 - Sound Velocity Anomaly in Sea Water at Various Temperatures

The space derivative, to the first order, may be replaced by using the quantity

$$\frac{\partial}{\partial x} \sim k + \frac{\partial T}{\partial x} \cdot \frac{\partial}{\partial T}$$

except in the case of the equilibrium pressure, where

$$\frac{\partial}{\partial x} p_o \sim \rho_o g$$

The time derivatives are to be replaced by use of

$$\frac{\partial}{\partial t} = \omega$$

If we use cgs units throughout, then

$$\rho_o = 1$$

$$c = 1.5 \times 10^5$$

$$\mu = 1 \times 10^{-2}$$

$$\frac{1}{\rho}\frac{\partial \rho}{\partial T} = 10^{-3}/^oF$$

$$\frac{1}{\mu}\frac{\partial \mu}{\partial T} = .02/^oF$$

$$\frac{1}{c}\frac{\partial c}{\partial T} = 1.6 \times 10^{-3}/^oF$$

We then have, for the terms in G

$$\frac{\partial}{\partial t}(\nabla \cdot s\nabla p_o) = \frac{\omega^2 p_e g}{\rho_o c^3}$$

$$\frac{\partial}{\partial t}(\rho \underline{v} \cdot \nabla \underline{v}) = \frac{\omega^3 p_e{}^2}{\rho_o c^4}$$

$$\frac{\partial}{\partial t}(\dot{\phi}\Delta\rho) = \frac{\omega^3 p_e{}^2}{\rho_o c^4}$$

$$\frac{\partial}{\partial t}(\nabla\rho \cdot \dot{\underline{v}}) = \frac{\omega^2 p_e}{\rho_o c}\left(10^{-3}\gamma + \frac{\omega p_e}{c^3}\right)$$

$$\frac{\partial}{\partial t}(\nabla \cdot \underline{v}\Delta\mu) = 0$$

$$\frac{\partial}{\partial t}(\nabla\mu\cdot\nabla\nabla\cdot\underline{v}) = \frac{2\times10^{-2}\omega^3 p_e \gamma}{\rho_o c^3}$$

$$\Delta\,(c^2\underline{v}\cdot\nabla\rho) = \frac{\omega^2 p_e}{\rho_o c}\,(10^{-8}\gamma + \frac{\omega p_e}{c^3})$$

$$\Delta\,(\dot{\rho}\dot{\varphi}) = \frac{\omega^3 p_e^{\,2}}{\rho_o c^4}$$

where γ has been written for $\frac{\partial T}{\partial x}$.

Then, to an order of magnitude, we have

$$G \sim \frac{\omega^3 p_e^{\,2}}{\rho_o c^4} + \frac{\omega^2 p_e}{\rho_o c}\,[\frac{g}{c^2} + 2\cdot10^{-3}\gamma(1 + \frac{10^{-2}\omega}{c^2})]$$

while the left-hand side of Eq. 2.6 is

$$\Delta(\rho\ddot{\varphi} - c^2\rho\Delta\varphi - \frac{4}{3}\mu\Delta\dot{\varphi}) \sim \frac{\omega^3 p_e}{c^2} + \frac{10^{-2}\omega^4 p_e}{c^4}$$

and if this quantity be called F, we have

$$\frac{G}{F} \sim \frac{\frac{p_e}{c^2} + \frac{c^2}{\omega}[\frac{g}{c^3} + 2\cdot10^{-3}\frac{\gamma}{c}(1 + 10^{-2}\frac{\omega}{c^2})]}{1 + \frac{10^{-2}\omega}{c^2}}$$

The largest values of p_e, γ likely to be encountered in acoustical work may be taken as

$p_e \sim 10^6$ (corresponding to an intensity of .3 watt/cm^2)

$\gamma \sim 10^{-2}$ (corresponding to a temperature gradient of 1°F/10')

Using these values, the quantity $\frac{G}{F}$ is plotted against frequency in Fig. 2.3. The neglect of G would thus appear to be proper, at frequencies above a few hundred cycles, and at much lower frequencies for lower field intensities. However, the attenuation term $\frac{10^{-2}\omega}{c^2}$ is insignificant at frequencies below about 100 megacycles for the chosen intensity, although it is of more importance for lower intensities. It is possible that this fact may explain part of the discrepancy between the attenuation calculated from Eq. 2.10 and the measured values.

On the basis of the foregoing considerations, we shall assume that an acoustic wave field is characterized by the scalar velocity potential φ, which satisfies the equation

$$\Delta\varphi + \frac{\omega^2}{c^2}\varphi = 0$$

in which c has a small imaginary part. The attenuation to be associated with this imaginary part is justified primarily on the basis of experience (see Fig. 2.1). On the other hand, the mathematical problem is simplified if c is complex, since convergence factors or "generalized integrals" need not be used.

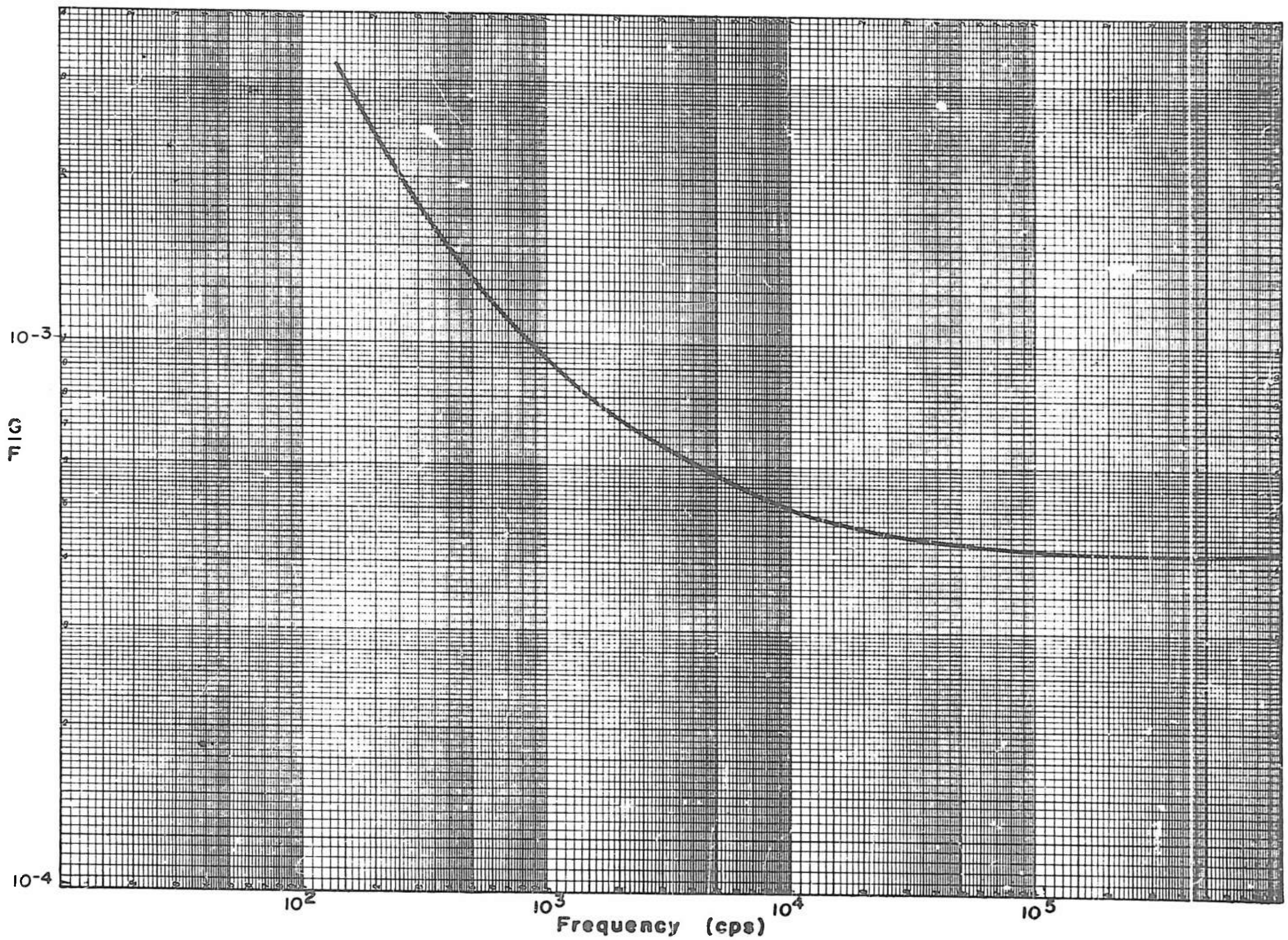


Fig. 2.3 - Neglected Terms in the Acoustic Wave Equation

SECTION 3

THE BOUNDARY VALUE PROBLEM. SOLUTION BY CONTOUR INTEGRAL. NORMAL MODES AND BRANCH LINE SOLUTION.

In this section, we shall concern ourselves with the formal aspects of the determination of the wave potential φ in the vicinity of a free surface on which φ vanishes.

Let c be the velocity of waves near the source, and c' the velocity at other points. The index of refraction then is

$$n = \frac{c}{c'}$$

Let the origin of cylindrical coordinates (r, θ, z) be taken on the free surface $z=0$, so that the coordinates of the source are $(0, 0, z_0)$, and let z be positive in the half space containing the source. We suppose that n is a function of z alone, so that the potential field possesses axial symmetry.

The problem may be formulated as follows:

$$\Delta\varphi + k^2 n^2 \varphi = 0 \qquad 3.1$$

$$\varphi(r,0) = 0 \qquad 3.2$$

$$\varphi \to \frac{e^{-ikR}}{R}, \quad R \to 0 \qquad 3.3$$

$$\varphi \to 0, \quad R \to \infty \qquad 3.4$$

wherein

$$R^2 = r^2 + (z - z_0)^2, \quad k = \frac{2\pi\nu}{c}(1 - i\sigma), \quad \sigma > 0$$

If we separate variables, a typical solution (3) is

$$U(r,z,\lambda) = J_0(\lambda r)\beta(\lambda, z) \qquad 3.5$$

provided

$$\frac{d^2\beta}{dz^2} + (k^2 n^2 - \lambda^2)\beta = 0 \qquad 3.6$$

and the parameter λ may take on general complex values. If the appropriate solution of Eq. 3.6 be chosen, we have the result that

$$\varphi(r,z) = \int_0^\infty U(r,z,\lambda)\, d\lambda \qquad 3.7$$

so that the function $\beta(\lambda, z)$ is the Fourier Bessel transform (of which fact we shall make use in Section 4) of the potential $\varphi(r, z)$.

If Eqs. 3.2, 3.3, and 3.4 be transformed into conditions on β, we have

$$\beta(\lambda, 0) = 0 \qquad 3.8$$

$$[\beta(\lambda, z_o)] = 0 \qquad 3.9$$

$$\left[\frac{d\beta}{dz}\right]_{z = z_o} = 2\lambda S \text{ , S being constant} \qquad 3.10$$

$$\beta(\lambda, \infty) = 0 \qquad 3.11$$

where the symbol [ ] denotes the discontinuity of the function at the point in question. These conditions may be understood most easily from the point of view of the equivalent integral equation, which automatically incorporates the boundary conditions. Consideration of this point will therefore be deferred until the discussion following Eq. 4.4. For an alternative demonstration, Furry (6) may be consulted. It may be mentioned that Eq. 3.10 has an heuristic appeal, since clearly the radial particle velocity must be oppositely directed on opposing sides of the source.

Now when $\beta(\lambda, z)$ is determined subject to the above conditions, we must evaluate Eq. 3.7 in order to arrive at the required potential. The appropriate method for carrying out this process is dependent upon the analytic character of $\beta(\lambda, z)$ qua function of the complex variable λ.

For cases of physical significance, it appears that the singularities of β may be classified as follows:

a. If $\frac{dn(z)}{dz} = 0$, say at $z = z_i$, then β will possess branch points satisfying $k^2n^2(z_i) - \lambda^2 = 0$. It may also have simple poles.

b. If $\frac{dn(z)}{dz}$ is everywhere (including infinity) bounded from zero, then β is an analytic function of λ, except for simple poles.

Although a proof of these conditions is wanting, they are plausible on physical grounds. The first condition is true for a homogeneous medium, where $\frac{dn}{dz}$ is identically zero, and it is to be expected that the condition will continue to be true for a locally homogeneous medium. This will be made clear from the following consideration.

Let independent solutions of Eq. 3.6 be $U(z, \lambda)$, $V(z, \lambda)$, U denoting that solution appropriate for upgoing waves and V that for downgoing. From Condition 3.11 $V(z, \lambda)$ is that solution which is bounded for large positive z. The function satisfying Conditions 3.8 through 3.11 may evidently be written

$$\left.\begin{aligned} \beta(\lambda, z, z_o) &= \frac{2\lambda S}{W} \frac{V(z_o, \lambda)}{V(0, \lambda)} [U(z,\lambda)V(0,\lambda) - V(z,\lambda)U(0,\lambda)] \qquad z \le z_o \\ &= \frac{2\lambda S}{W} \frac{V(z, \lambda)}{V(0, \lambda)} [U(z_o,\lambda)V(0,\lambda) - V(z_o,\lambda)U(0,\lambda)] \qquad z \ge z_o \end{aligned}\right\} 3.12$$

where

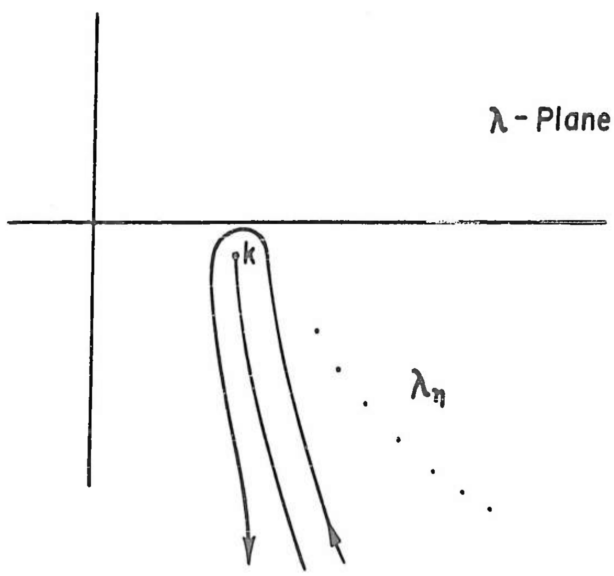
$$W = U'(z,\lambda)V(z,\lambda) - U(z,\lambda)V'(z,\lambda)$$

denotes the (constant) Wronskian of the system, and the primes signify differentiation with respect to z. The poles of β occur at the zeros of $V(0, \lambda)$. Let these zeros be λ_n. Then the residue of β at $\lambda = \lambda_n$ is

$$R_n(z, z_0) = \frac{\beta(\lambda_n)V(0,\lambda_n)}{\left[\frac{d}{d\lambda}V(0,\lambda)\right]_{\lambda = \lambda_n}} \qquad 3.13$$

According to Pekeris (20), there are no λ_n located in the first quadrant of the complex λ-plane, and further the contour of integration in Eq. 3. 7 may be taken to lie above the branch points of β. Let us consider, for simplicity, a single branch point $\lambda = k$.

If the λ-plane be cut along a contour L (see figure below), then for a particular determination of $\sqrt{\lambda^2 - k^2}$, β is analytic in the cut plane, except at $\lambda = \lambda_n$.



By a well known method devised by Sommerfeld, we may then write

$$\varphi(r, z, z_0) = \tfrac{1}{2}\Sigma R_n(z, z_0)H_0^{(2)}(\lambda_n r) + \tfrac{1}{2}\int_L H_0^{(2)}(\lambda r)\beta(\lambda, z, z_0)\,d\lambda \qquad 3.14$$

Methods of evaluating the integral in Eq. 3. 14 will be considered in Section 4, where it will appear that this integral may be interpreted as representing the field due to a succession of multipole sources. Now in a homogeneous portion of space, the rays are straight lines, so that normal spreading is associated with propagation through this space. However, there will be anomalous interference between the rays. It is on this basis that the existence of branch points is to be expected.

Let us now consider that β is a single valued function, in which case the integral in Eq. 3.14 vanishes. Let those solutions $V(z, \lambda)$ which satisfy the condition $V(0, \lambda) = 0$ be denoted $V_n(z)$, and let the associated values of λ be λ_n^*, so that

$$\begin{aligned} V(z, \lambda_n^*) &= V_n(z) \\ V_n(0) &= 0 \end{aligned} \qquad 3.15$$

$\beta(\lambda, z, z_0)$ will satisfy Eqs. 3.6 and 3.8 through 3.11 for all values of λ except λ_n^*, and $V_n(z)$ will satisfy these equations with $S = 0$, for $\lambda = \lambda_n^*$. We have, if $\lambda \neq \lambda_n^*$

$$\int_0^\infty [\beta''(\lambda, z, z_0) V_n(z) - \beta(\lambda, z, z_0) V_n''(z)] dz = (\lambda^2 - \lambda_n^{*2}) \int_0^\infty \beta(\lambda, z, z_0) V_n(z) dz$$

and if we integrate by parts, and use Eq. 3.10 ,

$$\int_0^\infty \beta(\lambda, z, z_0) V_n(z) dz = \frac{V_n(z_0) \cdot 2\lambda}{\lambda^2 - \lambda_n^{*2}} S \qquad 3.16$$

Similarly,

$$\int_0^\infty V_n(z) V_m(z) dz = 0 \; , \; m \neq n \qquad 3.17$$

Hence the V_n form an orthogonal set, which we may expect to be complete. If we write

$$\beta(\lambda, z, z_0) = \Sigma\Sigma \, a_{mn} V_m(z) V_n(z_0),$$

then

$$a_{mn} = \int_0^\infty \int_0^\infty \beta(\lambda, z, z_0) V_m(z) V_n(z_0) dz dz_0, \; \lambda \begin{matrix} \neq \lambda_n^* \\ \neq \lambda_m^* \end{matrix}$$

and it follows from Eqs. 3.16 and 3.17 that

$$a_{mn} = 0 \; , \; m \neq n$$

whence

$$\beta(\lambda, z, z_0) = \Sigma \frac{a_n V_n(z) V_n(z_0) \cdot 2\lambda}{\lambda^2 - \lambda_n^{*2}} S, \qquad 3.18$$

where

$$a_n^{-1} = \int_0^\infty [V_n(z)]^2 dz$$

Now the singularities of β have been shown to occur at the zeros λ_n of $V(0, \lambda)$. Hence $\lambda_n = \lambda_n^*$ and further, if we use Eq. 3.13,

$$a_n^{-1} = 2\left[\frac{1}{\lambda} \frac{\partial}{\partial z} V(z,\lambda) \frac{\partial}{\partial \lambda} V(z,\lambda)\right]_{z = 0, \; \lambda = \lambda_n} \qquad 3.19$$

If we evaluate Eq. 3.14 for this case, we get

$$\varphi(r, z, z_0) = S \Sigma H_0^{(2)} (\lambda_n r) V_n(z) V_n(z_0) \tag{3.20}$$

Where the $V_n(z)$ form a complete orthonormal set, and

$$\left[\frac{2}{\lambda} \frac{\partial}{\partial z} V_n(z) \frac{\partial}{\partial \lambda} V_n(z)\right]_{z = 0,\ \lambda = \lambda_n} = 1 \tag{3.21}$$

When Eq. 3.20 holds, it is seen that the fundamental problem is that of determining the V_n and the λ_n. For the most part, one has

$$\lambda_n = \sigma_n - i\tau_n,$$

$$\sigma_n \sim k,$$

$$\tau_n > 0$$

so that for sufficiently large values of kr,

$$H_0^{(2)}(\lambda_n r) \sim \left(\frac{2}{\pi}\right)^{\frac{1}{2}} e^{-i(kr - \frac{\pi}{4})} \frac{e^{-\tau_n r}}{\sqrt{kr}}$$

On the other hand, the functions $V_n(z)$ ultimately grow exponentially for positive τ_n, the ratio of growth increasing with increasing τ_n. It thus appears that the series Eq. 3.20 is useful for computation only for small z and large r, and then under such velocity structure that n(z) is a non-decreasing function. These conditions are those which apply to the shadow region near the free surface, which is caused by downward refraction of the rays in consequence of the monotone character of n(z). When the convergence of Eq. 3.20 is not satisfactory, recourse must be had to the methods of geometrical optics, which will be considered in Section 6.

For the determination of λ_n, we have the condition that $V_n(0) = 0$. In general, there would seem to be no satisfactory method for the numerical evaluation of λ_n, the particular scheme to be employed depending on the character of n(z) and upon the value of λ_n itself. When n(z) can by continuation be considered as an analytic function of the complex variable z, one has recourse to the W. K. B. phase integral method (6), (9), (13), giving

$$\int_0^{z^{(n)}} (k^2 n^2 - \lambda_n^2)^{\frac{1}{2}} dz = (n - \frac{1}{4})\pi \tag{3.22}$$

where

$$kn(z^{(n)}) = \lambda_n \tag{3.23}$$

If n(z) has not this character, the method fails, but can be extended by the following scheme: Let n(z) be piecewise analytic in the real variable z, i.e.,

$$n(z) = n_i(z) \qquad z_i < z < z_{i+1}$$

each $n_i(z)$ being analytic in z. Let condition 3.23 give in this case

$$z_j < |z^{(n)}| z_{j+1}$$

Then, Eq. 3.22 is to be replaced by the equation

$$\int_0^{z_j} (k^2n^2 - \lambda_n^2)^{\frac{1}{2}} dz + \int_{z_j}^{z^{(n)}} (k^2n_j^2 - \lambda_n^2)^{\frac{1}{2}} dz = (n - \frac{1}{4})\pi \qquad 3.24$$

It has been empirically demonstrated that Eq. 3.24 gives λ_n accurately if τ_n is sufficiently large, and τ_n accurately otherwise. σ_n may be very much in error when τ_n is small, and under these circumstances, other methods must be used if necessary. In many cases a knowledge of σ_n is not necessary; in others, it can be easily calculated by a direct numerical integration of the differential equation, once given τ_n.

The methods of perturbation and variation of parameters may prove useful at times, and are considered in some detail in (17), Vol. I. In subsequent sections we shall consider the utility of integral equations and contour integrals.

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The methods of perturbation and variation of parameters may prove useful at times, and are considered in some detail in (17), Vol. I. In subsequent sections we shall consider the utility of integral equations and contour integrals.

SECTION 4

THE EQUIVALENT INTEGRAL EQUATION. USE FOR DIRECT COMPUTATION. LIOUVILLE-NEUMANN SERIES.

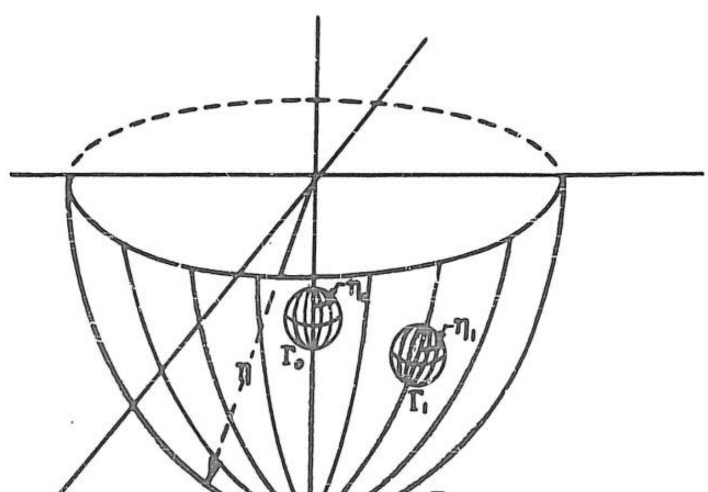
An alternative formulation of the boundary value problem may be obtained by the use of an appropriately chosen Green's function. One obtains an integral equation for the function $\beta(\lambda, z)$ which may be used for perturbing the eigen functions corresponding to the Green's function in question (17, Vol. I),(21). We prefer, however, to use a Green's function having a continuum of eigenvalues, namely, that corresponding to a homogeneous medium. The integral equation may then be solved by iteration in general yielding a solution complementary to that of the normal mode solution. Let us write

$$n^2 = \delta^2 + \varepsilon(z) \;, \qquad \delta = \text{const.}$$

so that φ is a function of ε. Now when $\varepsilon = 0$, the solution to Eq. 3.1 is

$$\psi(r, \theta, z, z_0) = \frac{e^{-ik'R}}{R} - \frac{e^{-ik'R'}}{R'} \;, \quad k' = k\delta$$

where R' is the distance between $(0, 0, -z_0)$ and (r, θ, z). Let now T be the volume enclosed between a hemispherical surface Γ, and small spherical surfaces Γ_0, Γ_1 centered respectively at $(0, 0, z_0)$, (r_1, θ_1, z_1), as shown in the following diagram.



We then have

$$\int_T \psi(\underline{R}_1, \underline{R})\,\Delta\varphi(\underline{R}, \underline{R}_0)\,d\tau - \int_T \varphi(\underline{R}, \underline{R}_0)\,\Delta\psi(\underline{R}_1, \underline{R})\,d\tau = \int_\Sigma \left\{ \psi(\underline{R}_1, \underline{R})\nabla\varphi(\underline{R}, \underline{R}_0) - \varphi(\underline{R}, \underline{R}_0)\nabla\psi(\underline{R}_1, \underline{R}) \right\} \cdot d\sigma \qquad 4.1$$

where the position vectors of the points are used for convenience to indicate the appropriate arguments. The surface integral is

$$\int_{\Sigma} = -\int_{\Gamma_0} - \int_{\Gamma_1} + \int_{\Gamma} \qquad 4.2$$

We shall assume (since k has a negative imaginary part) that for large η, φ is $O(\eta)$ or Γ . Then, as $\eta \to \infty$, the integral over Γ vanishes. For small n_0, $|\nabla\varphi|$ is $O(n_0^2)$, so that the contribution of the integral over Γ_0 is just $-4\pi\psi(R_1, R_0)$. Similarly, the integral over Γ_1 yields $-4\pi\varphi(R_1, R_0)$. Then Eq. 4.2 is, in the limit,

$$\int_{\Sigma} = 4\pi[\psi(R_1,R_0) - \varphi(R_1,R_0)]$$

Using the differential equations which ψ, φ satisfy, the left-hand side of Eq. 4.1 is

$$\int_T - k^2n^2\psi(R_1,R)\varphi(R,R_0)\,d\tau + \int_T k'^2\varphi(R,R_0)\psi(R_1,R)\,d\tau$$

so that we obtain, using

$$n^2 = \delta^2 + \varepsilon$$

$$\varphi(R_1,R_0) = \psi(R_1,R_0) + \frac{k^2}{4\pi}\int_T \varepsilon\psi(R_1,R)\varphi(R,R_0)\,d\tau \qquad 4.3$$

If ψ be written out in full,

$$\psi(R_1,R) = \frac{e^{-ik'\sqrt{\omega^2 + (z - z_1)^2}}}{\sqrt{\omega^2 + (z - z_1)^2}} - \frac{e^{-ik'\sqrt{\omega^2 + (z + z_1)^2}}}{\sqrt{\omega^2 + (z + z_1)^2}}$$

$$\omega^2 = r^2 + r_1^2 - 2rr_1\cos(\theta_1 - \theta)$$

Hence

$$\psi(R,R_1) = \int_0^\infty \frac{J_0(\mu\omega)\,e^{-\sqrt{\mu^2 - k'^2}|z - z_1|}}{\sqrt{\mu^2 - k'^2}}\mu d\mu - \int_0^\infty \frac{J_0(\mu\omega)\,e^{-\sqrt{\mu^2 - k'^2}|z + z_1|}}{\sqrt{\mu^2 - k'^2}}\mu d\mu$$

where the path of integration has an upward indentation at the point $\mu = k'$ (27, p. 416)

We may also obtain an integral equation for $\beta(\lambda, z, z_0)$. The result of substituting and carrying out the integration with respect to θ is

$$\int_0^\infty J_0(\lambda r_1)\beta(\lambda,z,z_0)\,d\lambda = \int_0^\infty \frac{J_0(\mu r_1)\,e^{-\sqrt{\mu^2 - k'^2}\,|z_0 - z_1|}}{\sqrt{\mu^2 - k'^2}}\,\mu d\mu - \int_0^\infty \frac{J_0(\mu r_1)\,e^{-\sqrt{\mu^2 - k'^2}\,|z_0 + z_1|}}{\sqrt{\mu^2 - k'^2}}\,\mu d\mu$$

$$+ \frac{k^2}{2}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty J_0(\mu r)J_0(\mu r_1)\,\varepsilon(z_1)\beta(\lambda,z_1,z_0)$$

$$\left\{\frac{e^{-\sqrt{\mu^2 - k^2}\,|z - z_1|} - e^{-\sqrt{\mu^2 - k^2}\,|z + z_1|}}{\sqrt{\mu^2 - k'^2}}\right\}\mu r\, d\mu d\lambda dz_1 dr$$

which reduces by virtue of the Fourier-Bessel theorem to

$$\int_0^\infty J_0(\lambda r)\left[\beta(\lambda,z,z_0) - \frac{\lambda e^{-\alpha|z - z_0|}}{\alpha} + \frac{\lambda e^{-\alpha|z + z_0|}}{\alpha} - \frac{k^2}{2}\int_0^\infty\left\{\frac{e^{-\alpha|z - z_1|}}{\alpha} - \frac{e^{-\alpha|z + z_1|}}{\alpha}\right\}\varepsilon(z_1)\beta(\lambda,z_0,z_1)\,dz_1\right]d\lambda = 0$$

where $\alpha = \sqrt{\lambda^2 - k'^2}$. This formula, which holds for general values of r, shows that the function within the square brackets must be a null function. Hence, we obtain an integral equation for β :

$$\beta(\lambda,z,z_0) = \frac{\lambda}{\alpha}\left[e^{-\alpha|z - z_0|} - e^{-\alpha|z + z_0|}\right]$$
$$+ \frac{k^2}{2\alpha}\int_0^\infty \varepsilon(z_1)\left[e^{-\alpha|z - z_1|} - e^{-\alpha|z + z_1|}\right]\beta(\lambda,z_0,z_1)\,dz_1 \qquad 4.4$$

We may now explain the boundary conditions contained in Eqs. 3.8 through 3.11, the first and last of which are evident upon inspection. Since β (λ, z, z_o) is symmetric in z, z_o, it is clearly continuous at z_o. Further, each term in Eq. 4.4, with the exception of the first, has a continuous derivative with respect to z. Therefore,

$$\left[\frac{\partial\beta}{\partial z}\right]_{z = z_o} = \lambda - (-\lambda) = 2\lambda$$

which is the required property for a source of unit strength.

If this be written in the abbreviated form

$$\beta(\lambda,z,z_o) = K(\lambda,z,z_o) + \frac{k^2}{2\lambda}\int_0^\infty \varepsilon(z_1)K(\lambda,z,z_1)\beta(\lambda,z_1,z_0)\,dz_1$$

then a (unique) solution (28)

$$\beta(\lambda, z, z_0) = \sum_{n=0}^{\infty} \left(\frac{k'^2}{2\lambda}\right)^n \gamma_n(\lambda, z, z_0) \qquad 4.5$$

exists under some circumstances. We shall suppose that $\varepsilon(z)$ is different from zero only in the interval $0 \le z \le \xi$. This restriction will not limit the practical use to which we shall pose our formulae and will, at the same time, yield a simpler condition under which Eq. 4.5 converges. The coefficients are to be calculated from the formula

$$\gamma_0(\lambda, z, z_0) = K(\lambda, z, z_0)$$

$$\gamma_n(\lambda, z, z_0) = \int_0^{\xi} \varepsilon(z_1) K(\lambda, z, z_0) \gamma_{n-1}(\lambda, z_0, z_1)\, dz_1 \qquad 4.6$$

where, it will be recalled,

$$K(\lambda, z, z_1) = \frac{\lambda}{\alpha}\left\{e^{-\alpha|z - z_1|} - e^{-\alpha|z + z_1|}\right\}$$

A sufficient condition for the convergence of Eq. 4.5 is

$$\left|\frac{k^2 \varepsilon_0 \xi N}{2}\right| < 1 \qquad 4.7$$

ε_0 and N are such that the inequalities

$$\left|\varepsilon(z)\right| \le \varepsilon_0$$

$$\left|\frac{1}{\lambda} K(\lambda, z, z_1)\right| \le N$$

The number N will have a value, in general independent of λ, which depends on the form of the contour to be used in calculating $\varphi(r, z, z_0)$ by transformation of $\beta(\lambda, z, z_0)$. For example, β can have no singularities in the first quadrant of λ provided

$$\left|k^2 \varepsilon_0 \xi^2\right| < 1 \text{ or } \left|k^2 \varepsilon_0 z \xi\right| \quad 1 \qquad 4.8$$

according as ξ is or is not greater than z. In this case the series obtained for φ by term-wise integration converges with sufficient speed to permit calculation. Alternatively, we may choose the contour of integration such that

$$|\alpha| \ge k'$$

and in this case Eq. 4.5 converges if

$$\left|k \varepsilon_0 \xi\right| < 1 \text{ or } \left|k \varepsilon_0 z\right| < 1$$

again depending on the relative magnitude of ξ, z. However, the series obtained for φ in this case converges very slowly, unless the Condition 4.8 is also satisfied.

We shall suppose, therefore, that Eq. 4.5 converges by virtue of Condition 4.8. If the calculation of the γ_n be carried out, there results

$$\beta(\lambda, z, z_0) = K(\lambda, z, z_0) \left\{ 1 + \frac{k^2}{2\alpha} [A(z_0) + f(z)] + \frac{k^4}{4\alpha^2} [A^2(z_0) + A(z_0) f(z) + \int_{z_0}^{\infty} U(z_1) f(z_1) dz_1 \right.$$

$$\left. + \int_0^{\infty} V(z, z_1) f(z + z_1) dz_1] + \ . \ . \ . \right\}$$

$$+ e^{-\alpha(z + z_0)} \left\{ \frac{k^2\lambda}{2\alpha^2} \beta(z_0) + \frac{k^4\lambda}{4\alpha^3} [A(z_0)\beta(z_0) + f(z_0)\beta(z_0) + f(z)\beta(z_0) \right.$$

$$\left. + \int_0^{z_0} W(z_1) A(z_1) dz_1 + \int_0^{z_0} U(z_1) \beta(z_1) dz_1] + \ . \ . \ . \right\} \qquad 4.9$$

Eq. 4.9 holds for $z > z_0$, and is symmetric in z, z_0. The following symbols have been used:

$$U(z_1) = \varepsilon(z_1)[1 - e^{-2\alpha z_1}]$$

$$V(z, z_1) = \varepsilon(z + z_1)[e^{-2\alpha z_1} - 1]$$

$$W(z_1) = \varepsilon(z_1)[e^{2\alpha z_1} + e^{-2\alpha z_1} - 2]$$

$$A(z_0) = \int_{z_0}^{\infty} U(z_1) dz_1$$

$$f(z) = \int^{\infty} V(z, z_1) dz_1$$

$$\beta(z_0) = \int^{z_0} W(z_1) dz_1$$

It is convenient at this point to calculate the functions A, f, B, etc. in ascending series in α . This is permissible, since $\varepsilon(z)$ vanishes for $z > \xi$. We then get

$$\beta(\lambda, z, z_0) = K(\lambda, z, z_0) \left\{ 1 + k^2C_{01} + k^4C_{02} + \ldots + \alpha[k^2C_{11} + k^4C_{12} + \ldots] \right.$$

$$\left. + \alpha^2[k^2C_{21} + k^4C_{22} + \ldots] + \ldots \right\}$$

$$+ e^{-\alpha(z + z_0)}\lambda \left\{ k^2C'_{01} + k^4C'_{02} + \ldots + \alpha[k^2C'_{11} + k^4C'_{12} + \ldots] + \ldots \right\}$$

Where the coefficients C_{ij}, C'_{ij} are functions of z, z_0 but not of α . In the calculation of the velocity potential we then have

$$\varphi(r,z) = \sum_{m=0}^{\infty} \int_0^{\infty} J_0(\lambda r) K(\lambda, z, z_0) \alpha^m \sum_{n=0}^{\infty} k^{2n} C_{mn} d\lambda + \sum_{m=0}^{\infty} \int_0^{\infty} J_0(\lambda r) e^{-\alpha(z+z_0)} \alpha^m \sum_{n=0}^{\infty} k^{2n} C'_{mn} d\lambda \qquad 4.10$$

We are thus faced with the problem of evaluating the remaining integrals, which will be considered as special cases of slightly more general functions to be found useful in Section 5. Other special forms of these functions have been used by Van der Pol (25).

We shall digress, therefore, to consider the functions

$$F_{MN}(r,k,z;\alpha_1,\alpha_2,\cdots,\alpha_M;\beta_1,\beta_2,\cdots,\beta_N) = \int_0^{\infty} \frac{J_0(\lambda r)\lambda e^{-\alpha z} \prod_1^M (\alpha - \alpha_m)}{\prod_1^N (\alpha - \beta_n)} d\lambda$$

$$\alpha = \sqrt{\lambda^2 - k^2}, \quad \arg \alpha \to \arg \lambda, \qquad 4.11$$

$$\lambda \to \infty$$

The path of integration is taken to have an upward indentation at any of the points k, β_n which are real and positive. It is first assumed that the β_n are distinct and non-zero (except in the trivial case $N = 1$, $\beta_1 = 0$). Therefore, if

$$f_N(\alpha) = \prod_1^N (\alpha - \beta_n)$$

we can write Eq. 4.11 in the form

$$F_{MN}(r,k,z;\alpha_1,\alpha_2,\cdots,\alpha_M;\beta_1,\beta_2,\cdots,\beta_N) = \sum_1^N f'(\beta_n) \int_0^{\infty} \frac{J_0(\lambda r)\lambda e^{-\alpha z} \prod_1^M (\alpha - \alpha_m) d\lambda}{\alpha - \beta_n}$$

$$= \sum_1^N f'(\beta_n) F_{M1}(r,k,z;\alpha_1,\alpha_2,\cdots,\alpha_N;\beta_n)$$

$$= (-)^M \sum_1^N f'(\beta_n) \prod_1^M e^{-\alpha_n z} \frac{d e^{\alpha_m z}}{dz} F_{01}(r,k,z;\beta_n) \qquad 4.12$$

Now

$$\int_0^{\infty} \frac{J_0(\lambda r)\lambda e^{-\alpha z}}{\alpha} d\lambda = \frac{e^{-ik\sqrt{r^2+z^2}}}{\sqrt{r^2+z^2}} = F_{01}(r,k,z;0) \qquad 4.13$$

Hence we have

$$F_{01}(r,k,z;\beta_n) = e^{-\beta_n z} \int_{\infty \exp i\theta}^{z} e^{\beta_n t} \frac{\partial}{\partial t} \frac{e^{-ik\sqrt{r^2+t^2}}}{\sqrt{r^2+t^2}} dt \qquad 4.14$$

provided

$$\pi + \arg k - \arg \beta_n \leq \theta \leq 2\pi - \arg \beta_n$$

We may now consider that roots are repeated. Obviously, if α_n is a repeated root, the above analysis needs no modification, since each repetition simply yields one term in the sum, Eq. 4.12. If β_n is a non-zero repeated root, then repeated integrations of the type in Eq. 4.14 are required. Since this procedure is excessively laborious and roots of this type are not found in most physical problems, we shall not consider them here. Accordingly, we write

$$f_N(\alpha) = \alpha^P \sum_1^{N-p} (\alpha - \beta_n) = \alpha^p g(\alpha)$$

so that

$$\frac{1}{f_N(\alpha)} = \frac{1}{\alpha^p} \sum_1^{N-p} \frac{g'(\beta_n)}{\alpha - \beta_n}$$

It follows then that

$$F_{MN}(r,k,z;\alpha_1,\alpha_2,\cdot\cdot,\alpha_M;\beta_1,\beta_2,\cdot\cdot,\beta_N) = (-)^M \sum_{n=1}^{N-p} \frac{g'(\beta_n)}{\beta_n} \prod_{m=1}^{M} e^{\alpha_m z} \frac{d}{dz} e^{-\alpha_m z} F_{o1}(r,k,z;\beta_n)$$

$$- (-)^M \sum_{n=1}^{N=p} \frac{g'(\beta_n)}{\beta_n} \sum_{q=1}^{p} \beta_{q-1} \prod_{m=1}^{M} e^{-\alpha_m z} \frac{d}{dz} \Big[e^{\alpha_m z}$$

$$F_{o2}(r,k,z;0,0,\cdot\cdot,0) \Big]$$

Now

$$\frac{d}{dz} \int_o^\infty \frac{J_o(\lambda r)\lambda e^{-\alpha z}}{\alpha^p} d\lambda = -\int_o^\infty \frac{J_o(\lambda r)\lambda e^{-\alpha z}}{\alpha^{p-1}} d\lambda$$

and

$$\frac{d}{dk} \int_o^\infty \frac{J_o(\lambda r)\lambda e^{-\alpha z}}{\alpha^p} d\lambda = k \left[z\int_o^\infty \frac{J_o(\lambda r)\lambda e^{-\alpha z}}{\alpha^{p+1}} d\lambda + p\int_o^\infty \frac{J_o(\lambda r)\lambda e^{-\alpha z}}{\alpha^{p+2}} d\lambda \right]$$

It is evident that we can calculate these functions stepwise by repeated differentiation if we are given the functions $F_{o1}(r,k,z;\beta_n)$ and $F_{o2}(r,k,z;0,0)$. In lieu of using Eq. 4.14 for the calculation of these functions, it is convenient to consider a transformation of the defining Eq. 4.9. We shall consider both functions simultaneously in the form

$$F_{o2}(r,k,z;0,\beta_n) \equiv F_{o2} = \int_o^\infty \frac{J_o(\lambda r)\lambda e^{-\alpha z}}{\alpha(\alpha - \beta_n)} d\lambda$$

If we use the identity

$$2J_0(\lambda r) = H_0^{(1)}(\lambda r) + H_0^{(2)}(\lambda r)$$

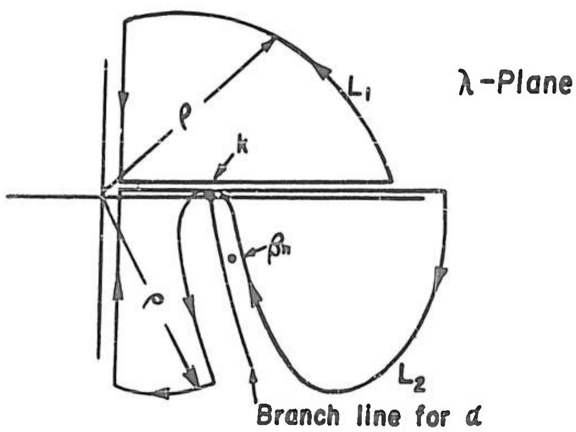
then

$$F_{02} = \tfrac{1}{2}I_1 + \tfrac{1}{2}I_2$$

$$I_1 = \int_0^\infty \frac{H_0^{(1)}(\lambda r)\lambda e^{-\alpha z}}{\alpha(\alpha - \beta_n)}\, d\lambda \qquad 4.15$$

$$I_2 = \int_0^\infty \frac{H_0^{(2)}(\lambda r)\lambda e^{-\alpha z}}{\alpha(\alpha - \beta_n)}\, d\lambda \qquad 4.16$$

Now the integrand in Eq. 4.15 is an analytic function of λ in $0 \leq \arg \lambda \leq \frac{\pi}{2}$, and that in Eq. 4.16 is analytic in the cut space $-\frac{\pi}{2} \leq \lambda \leq 0$, the cut extending from $\lambda = k$ to $\lambda = \infty \exp i\gamma$, $-\frac{\pi}{2} < \gamma < 0$, except for a simple pole at $\alpha = \beta_n$. (See diagram below.)



To the right of the cut arg α tends to arg λ for large $|\lambda|$, while to the left of the cut arg α tends to arg $\lambda + \pi$ for large $|\lambda|$. Accordingly,

$$\int_{L_1} \frac{H_0^{(1)}(\lambda r)\lambda e^{-\alpha z}}{\alpha(\alpha - \beta_n)}\, d\lambda = 0 = \int_{L_2} \frac{H_0^{(2)}(\lambda r)\lambda e^{-\alpha z}}{\alpha(\alpha - \beta_n)}\, d\lambda$$

Now as ρ tends to infinity, the integrals along the semi-circles tend to zero since $H_0^{(1)}(\lambda r)$ vanishes exponentially in the first quadrant and $H_0^{(2)}(\lambda r)$ in the fourth. The integrals along the imaginary axis are equal and opposite, since α is analytic along the imaginary axis, and $H_0^{(1)}(ix) = H_0^{(2)}(-ix)$, if x be real. We shall suppose that the integral around the branch point $\lambda = k$ starts at $a - i\infty$, encircles k in the counterclockwise sense, and returns to $b - i\infty$, a and b being positive. This contour will be denoted by L_3. Then if L_3 separates the points $k, \lambda_0 = \sqrt{k^2 + \beta_n}$, $-\frac{\pi}{2} < \arg \lambda_0 < 0$, we have

$$2F_{02} = 2\pi i\, H_0^{(2)}(\lambda r)\, e^{-\beta_n z} - \int_{L_3} \frac{H_0^{(2)}(\lambda r)\, e^{-\alpha z}}{\alpha(\alpha - \beta_n)} \lambda\, d\lambda \qquad 4.17$$

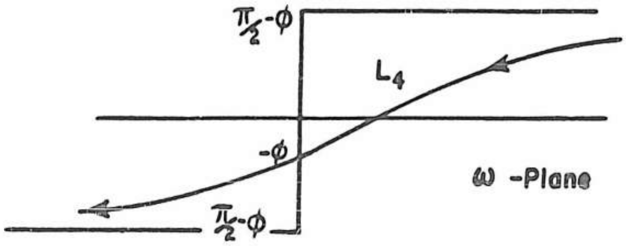
by Cauchy's theorem. The branch line integral in Eq. 4.17 may be evaluated by using the asymptotic formula for $H_0^{(2)}(\lambda r)$ (27, p. 198)

$$H_0^{(2)}(\lambda r) \sim \left(\frac{2}{\pi\lambda r}\right)^{1/2} e^{-i(\lambda r - \pi/4)} \sum_0^\infty \frac{\Gamma(\frac{1}{2} + n)}{\Gamma(\frac{1}{2} - n)} \frac{(2i\lambda r)^{-n}}{n!} \qquad 4.18$$

We have therefore to consider the integrals of the type

$$J_m = \int_{L_3} \frac{e^{-i\lambda r - \alpha z}}{\alpha(\alpha - \beta_n)} \lambda^{m/2}\, d\lambda \qquad 4.19$$

In this integral, write $\lambda = \cosh w$, $w = u - iv$ so that as λ moves along L_3, w traces out L_4, as may be seen in the following diagram.



We may write

$$J_m = \frac{2k^{\frac{m-2}{2}}}{m+2} \int_{-\infty - i\frac{\pi}{2} - \varphi}^{\infty + i\frac{\pi}{2} - \varphi} \frac{e^{-ikr\cosh w - kz \sinh w}}{\sinh w - \beta_n/k}\, d(\cosh w)^{\frac{m+2}{2}} \qquad 4.20$$

Now put $z = R\sin\varphi$, $r = R\cos\varphi$, $x = kR$, and write $g(w)$ for the exponent in Eq. 4.20, so that

$$g(w) = -ix\cosh(w - i\varphi) = -x(\xi + i\eta) \qquad 4.21$$

We observe that the point $\lambda = k$ maps into the point $w = 0$, so that the contour L_4 may be deformed in a suitable manner, provided that it passes below the origin.

According to the principle of stationary phase, (27, Ch. 8) the major contribution to the integral in Eq. 4.20 comes from the vicinity of the point at which $g(w) = 0$, i.e., at $w = i\varphi$. The whole of the contour is then to be taken so that on it the imaginary part of $g(= -\eta x)$ is constant. Accordingly, L_4 is specifically taken to be such that

$$\eta = \cosh u \cos(v + \varphi) = 1$$

$$\xi = \sinh u \sin(v + \varphi) \qquad 4.22$$

We then have

$$J_m = \frac{2}{m+2} e^{-ikR_k} \overset{\frac{m-2}{2}}{} \int_0^\infty e^{-x\xi} (y_1 - y_2)\, d\xi \qquad 4.23$$

where

$$y_{1,2} = \frac{1}{\sinh w - \beta_n/k} \frac{d}{d\xi} (\cosh w)^{\frac{m+2}{2}}$$

$$\cosh w = \cos \varphi (1 - i\xi) \mp \sin \varphi (\xi^2 + 2i\xi)^{\frac{1}{2}}$$

$$\sinh w = i \sin\varphi (1 - i\xi) \pm \cos \varphi (\xi^2 + 2i\xi)^{\frac{1}{2}}$$

The upper signs refer to y_1 and the lower to y_2. That square root is to be taken which tends to ξ for large ξ.

We have now to expand y_1 and y_2 in ascending half-integral powers of ξ. The procedure clearly must depend on the relative magnitude of the quantities $\sin \varphi$ and β_n/k. Now the point $\lambda = \lambda_o$ maps into the point w_o so that $\sinh w_o = \beta_n/k$. It will be recalled that the residue at $\lambda = \lambda_o$ was explicitly included in Eq. 4.17 under the supposition that L_3 separated the points k, λ_o. This supposition then implies that L_4 separates the origin and w_o. It is necessary to determine whether this is true, and also whether $\sin \varphi < |\beta_n/k|$, in order to expand the quantity $\frac{1}{\sinh w - \beta_n/k}$ about $\xi = 0$. We shall not give a complete account of this situation, but shall merely impose the sufficient condition

$$|kz| < |\beta_n r| \qquad 4.24$$

in order that the explicit form of the residuein Eq. 4.17 may be retained.

This may be seen from the fact that L_4 may be written

$$\cosh u \cos(v + \varphi) = 1$$

Thus, near the origin,

$$\frac{dv}{du} = 1$$

while for positive u,

$$\frac{d^2 y}{du^2} = -[\tanh u \csc(v + \varphi) + \operatorname{csch} u \cot(v + \varphi)] < 0$$

so that L_4 lies below the line $u = v + \varphi$ for $u > 0$ and above the line $u = v$ for $u < 0$. Hence, if $\beta_n/k = a - ib, a, b > 0$ the condition required is satisfied if

$$\sin^{-1}\frac{z}{r} < a + b$$

or if

$$|kz| < |\beta_n r|$$

It may be well to note, however, that if w_0 lies close to L_4, the asymptotic series is not at all satisfactory, and the variation of the quantity $\alpha - \beta_n$ must be explicitly taken into account.

If Eq. 4.24 holds, we have for y_1

$$\sinh \alpha - \beta_n/k = i\sin\varphi - \beta_n/k + \cos\varphi\sqrt{2i}\,\xi^{\frac{1}{2}} + \sin\varphi\,\xi + \cos\varphi\frac{\sqrt{2i}}{4i}\xi^{\frac{3}{2}} + \ldots$$

so that

$$\frac{1}{\sinh\alpha - \beta_n/k} = \frac{1}{i\sin\varphi - \beta_n/k} - \frac{\cos\varphi\sqrt{2i}\,\xi^{\frac{1}{2}}}{(i\sin\varphi - \beta_n/k)^2} + \left[\frac{4\,i\cos^2\varphi}{(i\sin\varphi - \beta_n/k)^3} - \frac{\sin^2\varphi}{(i\sin\varphi - \beta_n/k)^2}\right]\xi + \ldots$$

$$\cosh w = \cos\varphi - \sqrt{2i}\,\xi^{\frac{1}{2}} - i\cos\varphi\,\xi - \sin\varphi\frac{\sqrt{2i}}{4i}\xi^{\frac{3}{2}} + \ldots$$

$$\frac{d}{d\xi}(\cosh w)^{\frac{m+2}{2}} = (\cos\varphi)^{\frac{m+2}{2}}\left\{-\frac{m+2}{4\cos\varphi}\sqrt{2i}\,\xi^{\frac{1}{2}} + \left(-\frac{i(m+2)}{2\cos\varphi} + \frac{(m^2-4)}{2\cos^2\varphi}i\right) + \ldots\right\}$$

In the calculation for y_2, we have to change the sign of all odd half-integral powers of ξ in the above. If this is done, we have finally

$$y_1 - y_2 = A_{0,m}\xi^{-\frac{1}{2}} + A_{1,m}\xi^{\frac{1}{2}} + \ldots$$

where

$$A_{0,m} = -\cos\varphi^{\frac{m+2}{2}}\frac{(m+2)\sqrt{2i}}{2\cos\varphi(i\sin\varphi - \beta_n/k)}$$

$$A_{1,m} = (\cos\varphi)^{\frac{m+2}{2}}\left\{-\frac{(m+2)\sqrt{2i}}{i\cos\varphi}\left[\frac{4\,i\cos^2\varphi}{(i\sin\varphi - \beta_n/k)^3} - \frac{\sin^2\varphi}{(i\sin\varphi - \beta_n/k)^2}\right]\right.$$

$$\left. - \frac{\cos\varphi\ 2i}{(i\sin\varphi - \beta_n/k)^2}\left[-\frac{i(m+2)}{\cos\varphi} + \frac{i(m^2-4)}{\cos^2\varphi}\right]\right\}$$

Returning now to Eqs. 4.17 and 4.18, we have

$$2F_{o2} \sim 2\pi i\, H_o^{(2)}(\lambda_o r) e^{-\beta_n z} - \left(\frac{2}{\pi kr}\right)^{\frac{1}{2}} e^{-i(kR - \frac{\pi}{4})} \sum_0^\infty \beta_n \int_0^\infty e^{-x\xi}\, \xi^{\frac{2n-1}{2}} d\xi$$

$$= 2\pi i\, H_o^{(2)}(\lambda_o r) e^{-\beta_n z} - \left(\frac{2}{\pi kr}\right)^{\frac{1}{2}} e^{-i(kR - \frac{\pi}{4})} x^{-\frac{1}{2}} \sum_0^\infty \frac{\beta_n \Gamma\left(\frac{2n+1}{2}\right)}{x^n}$$

according to a lemma of Watson (27, p. 236). The coefficients in this expansion are

$$B_0 = \tfrac{2}{3} A_{0,1}$$

$$B_1 = \tfrac{2}{3} A_{1,1} - \frac{A_{0,1}}{4kr}$$

If like terms be collected, there finally results for the dominant terms in the calculation

$$2F_{o2}(r,k,z;0,\beta_n) \sim 2\pi i\, H_o^{(2)}(\lambda_o r) e^{-\beta_n z} - \frac{2}{R^2} e^{-ikR} \cdot \frac{1}{\sin\varphi + i\beta_n/k}$$

$$+ \frac{1}{R^3} e^{-ikR} \left\{ \frac{i \cos^2\varphi}{(i\sin\varphi - \beta_n/k)^3} - \frac{i\sin^2\varphi - \cos\varphi - 1}{(i\sin\varphi - \beta_n/k)^2} \right\} + \cdots \qquad 4.25$$

This series converges rapidly, provided the quantity $|i\sin\varphi - \beta_n/k|$ is not too small. We shall now develop a series which converges rapidly when $|i\sin\varphi - \beta_n/k|$ is small.

If we return to Eq. 4.14, and integrate once by parts, we obtain

$$F_{o2}(r,k,z;0,\beta_n) = -e^{-\beta_n z} \int_{\infty \exp i\vartheta}^{z} \frac{e^{\beta_n t - ik\sqrt{r^2 + t^2}}}{\sqrt{r^2 + t^2}} dt \qquad 4.26$$

If we integrate repeatedly by parts, a series similar to Eq. 4.25 may be obtained, but the residue term is not explicity contained. For small z, β_n we write

$$t = r\cosh u,\ \sinh \tau = z/r,\ kr = y\cosh v,\ \beta_n r = iy\sinh v$$

$$F_{c2} = e^{-\beta_n z} \int_\tau^{\infty + i\vartheta} e^{-iy\cosh(u - v)} du$$

$$= e^{-\beta_n z} \left[\int_0^{\infty + i\vartheta} - \int_0^{\tau - v} \right] e^{-iy\cosh u} du$$

$$= \frac{\pi i}{2} e^{-\beta_n z} H_o^{2}(y) - e^{-\beta_n z} \int_0^{\tau - v} e^{-ix\cosh u} du \qquad 4.27$$

The last step may be verified by reference to Watson (27, p. 180) or can be shown by evaluating

$$\int_0^\infty \frac{J_0(\lambda r)\lambda d\lambda}{\alpha^2}$$

by use of the contour L_3. Now the quantity $|\tau - \nu|$ is presumed small, so that if we put $\xi = \tau - \nu$, then

$$\int_0^\xi e^{-iy\cosh u} du = e^{-iy\cosh\xi}\left[\xi - \frac{iy\xi^2}{2}\cosh\xi - \frac{(iy\cosh\xi + 4y^2\cosh^2\xi)\xi^4}{24}\right.$$

$$\left. - \frac{(iy\cosh\xi + 16y^2\cosh^2\xi - 12iy^3\cosh^3\xi)\xi^6}{720} + \cdots\right] \qquad 4.28$$

This series is convergent (absolutely) provided

$$|\xi| < 1$$

but for purposes of computation, the quantity $|y\xi^2|$ must be small. It is thus seen that Eqs. 4.25 and 4.28 are complementary.

A series which converges more rapidly than Eq. 4.28 can be used if tables of the function.

$$F(\xi) = \int_0^\xi e^{-iu^2} du$$

are available. $F(\xi)$ is easily obtained from the complex error function:

$$F(\xi) = 1 - \frac{\sqrt{\pi}}{2}\operatorname{Erfc}(\xi\sqrt{i}) \qquad 4.29$$

If we expand the quantity $e^{-iy[\cosh u - 1 - \frac{u^2}{2}]}$ in powers of U, then

$$\int_0^\xi e^{-iy\cosh u} du = e^{-iy}\int_0^\xi e^{-iy\frac{u^2}{2}}\left[1 - \frac{iyu^4}{24} - \frac{iyu^6}{720} - \cdots\right]du \qquad 4.30$$

The successive terms in this series may be obtained by differentiation:

$$\int_0^\xi e^{-iy\frac{u^2}{2}} u^4 du = -4\frac{\partial^2}{\partial y^2}\int_0^\xi e^{-iy\frac{u^2}{2}} du = -\frac{3}{y^2}\int_0^\xi e^{-iy\frac{u^2}{2}} du + \left(\frac{i\xi^3}{y} + \frac{3\xi}{y^2}\right)e^{-i\frac{\xi y^2}{2}} \qquad 4.31$$

If we substitute Eqs. 4.29 and 4.31 into Eq. 4.30, then

$$\int_0^{\xi} e^{-iy\cosh u}\,du = e^{-iy}\sqrt{\frac{2}{iy}}\left\{\operatorname{Erfc}\left(\xi\sqrt{\frac{i4}{2}}\right)\left[1 + \frac{1}{8y} + \frac{1}{48y^2} + \cdots\right]\right. \tag{4.32}$$

$$\left. + e^{-i\xi\frac{u^2}{2}}\left[-\frac{\xi}{8y}\left(1 + \frac{1}{6y} + \cdots\right) + \frac{\xi^3}{24}\left(1 - \frac{1}{8y} + \cdots\right)\right]\right\}$$

Eq. 4.32 is useful when $|y\xi^4|$ is small. It is to be observed that $\operatorname{Erfc}\sqrt{(i\xi)}$ is the Fresnel integral when ξ is real, and is tabulated (8) for $\xi < 50$. A table is given by Norton (18, 19) from which $\operatorname{Erfc}(-i\sqrt{\xi})$ can be deduced if $0 \leq |\xi| \leq 50$, $0 \leq \arg \xi \leq \pi$. A very interesting discussion by Rosser (23) gives general methods for the calculation of the error integral and allied functions.

Having considered methods for the evaluation of the integrals defined by Eq. 4.11, we may now return to the series Eq. 4.10. By virtue of the Condition 4.8 each of the series in ascending powers in k^2 converges, and by using Eq. 4.12 we have

$$\varphi(r,z) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-)^m k^{2n} C_{mn}\frac{\partial^m}{\partial z^m}\left[\frac{e^{-ikR}}{R} - \frac{e^{-ikR'}}{R'}\right] + \sum_{m=1}^{\infty}\sum_{n=0}^{\infty}(-)^m k^{2n} C'_{mn}\frac{\partial^m}{\partial z^m}\frac{e^{-ikR}}{R} \tag{4.33}$$

where

$$R^2 = r^2 + (z - z_0)^2\ ;\ R'^2 = r^2 + (z + z_0)^2$$

If we consider that k is small enough so that the series in Eq. 4.33 converges with sufficient speed to permit numerical evaluation, then it is evident that the multipole terms present will contribute little to the field. Thus at such low frequencies, the inhomogeneous character of the medium has very little influence on propagation. At higher frequencies, some solution to Eq. 4.4 other than the Liouville-Neumann series must be obtained. The Schmidt series (26) is equivalent to the eigen function expansion obtained in Section 3, and so holds no advantage for present purposes.

An attempt has been made to obtain modified Liouville-Neumann series, and although this effort has not been completed, the method will be outlined as being of possible value. Eq. 4.4 may be written in operational form

$$\beta = K + \varepsilon K\beta$$

and solved algebraically, yielding

$$\beta = \frac{K}{1 - \varepsilon K}$$

We seek to decompose the operator εK into a multiplicative operator A and an integral operator f, so that

$$\beta = \frac{K}{1 - A - f} = \frac{K}{(1 - A)\left(1 - \frac{f}{1 - A}\right)}$$

$$= \frac{1}{1 - A}\sum_{0}^{\infty}\left(\frac{f}{1 - A}\right)^n K \tag{4.34}$$

The series in Eq. 4.34 may be expected to converge satisfactorily when A is not in the neighborhood of unity. The series of this type representing the direct radiation (i.e., associated with the factor $e^{-\alpha|z - z_0|}$) can in fact be exhibited. If we write

$$\beta(\lambda, z, z_0) = F(\lambda, z, z_0)\, e^{-\alpha|z - z_0|} + G(\lambda, z, z_0)\, e^{-\alpha|z + z_0|}$$

then

$$F(\lambda, z, z_0) = \frac{K}{1 - A} \sum_0^\infty \left(\frac{\psi}{1 - A}\right)^n + \frac{K\Omega}{(1 - A)^2} \sum_0^\infty n \left(\frac{\psi}{1 - A}\right)^{n-1}$$

where

$$\psi^n = \int_0^\infty V(z + z_1) \int_0^\infty V(z + z_1 + z_2) \int_0^\infty \cdots \int_0^\infty f(z_1 + z_2 + \cdots + z_n)\, dz_n \cdots dz_1$$

$$\Omega = \int_{z_0}^\infty U(z_1)[\quad]\, dz_1 .$$

in the notation of Eq. 4.9.

The construction of a similar series for the reflected radiation G is a matter of great complexity, and it is not known at present whether such a series exists.

Finally, the integral equation may be approximated by a system of linear equations, and the solution obtained algebraically. In this connection, the method of Lanczos (12) would seem suitable, although there are difficulties arising from the dependence of the kernel $K(\lambda, z, z_0)$ upon the eigen parameter λ.

SECTION 5

SPECIAL CASES. APPLICATION OF THE STOKES EQUATION. APPLICATION OF THE BESSEL POLYNOMIALS.

It was seen in Section 3 that under certain circumstances, the wave potential is expressible as a series of orthogonal functions arising from a Sturm-Liouville problem. In this section we shall consider some special cases of these functions as a guide to the understanding of their uses and limitations. One of the most widely studied and exhaustively tabulated problems of the type applicable to our investigation is based on Stokes' equation

$$\frac{d^2W}{d\zeta^2} + \zeta W = 0 \qquad 5.1$$

We shall denote two linearly independent solutions of this by the symbols $U(\zeta)$, $V(\zeta)$; U denotes the solution appropriate for upgoing waves and V for downgoing.

Let the phase velocity, over certain depths, be

$$c = c_0 + \gamma z$$

and, if the fractional change in c be small, we have nearly

$$\left(\frac{c_0}{c}\right)^2 = 1 - \frac{2\gamma}{c_0} z$$

Thus, if we set $\alpha^2 = \lambda^2 - k^2$, Eq. 3.6 becomes, for this special case,

$$\frac{d^2\beta}{dz^2} + (gz - \alpha^2)\beta = 0 \qquad 5.2$$

where

$$g = -\frac{2k^2\gamma}{c_0} \qquad 5.3$$

and Stokes' equation may be obtained by substituting

$$\zeta = g^{-2/3}[gz - \alpha^2]$$

Stokes' equation thus is useful when the phase velocity changes in a nearly linear manner with depth. If the velocity changes in this manner over all depths under contemplation, a simple solution to the problem may be obtained. Before proceeding to a more detailed study of the bilinear gradient, in which two ranges of depth are associated with different velocity gradients, we shall discuss briefly the two cases $g > 0$ and $g < 0$.

If $g > 0$, the velocity decreases with depth and rays are bent downward from the surface, so that ultimately a shadow zone is formed.

The solutions U, V to Eq. 5.1 are in this case analytic functions of λ, so that according to the principles set forth in Section 3, we have for the eigenvalues

$$V(-\alpha_n^2 g^{-2/3}) = 0 \qquad 5.4$$

and for the functions

$$U_n = \frac{V(g^{1/3}z - \alpha_n^2 g^{-2/3})}{V'(-\alpha_n^2 g^{2/3})} \qquad 5.5$$

while the wave potential is

$$\varphi = \sum_n H^{(2)}(\lambda_n r)\frac{V(g^{1/3}z - \alpha_n^2 g^{-2/3})\,V(g^{1/3}z_0 - \alpha_n^2 g^{-2/3})}{[V'(-\alpha_n^2 g^{-2/3})]^2} \qquad 5.6$$

The W. K. B. solution to Eq. 5.4 (6) is simply

$$\alpha_n^2 = e^{\frac{5\pi i}{3}}\left\{\tfrac{3}{2}g(n - \tfrac{1}{4})\pi\right\}^{2/3}$$

so that

$$\lambda_n = (k^2 + \alpha_n^2)^{1/2}$$

$$\sim k + \frac{\alpha_n^2}{2k}$$

$$\sim k - \frac{\sqrt{3}}{1k}\left\{\frac{3}{2}g(n - \frac{1}{4})\pi^{\,2/3}\right\}i$$

$$\sigma_n - i\tau_n \sim k - \frac{\sqrt{3}}{4}\left\{\frac{3}{2}(n - \frac{1}{4})\pi\right\}^{2/3} k^{1/3}\gamma^{2/3} i$$

and it appears that the horizontal attenuation, which is determined by the imaginary part of λ_n, increases as the one-third power of the frequency.

If we use the appropriate asymptotic forms of $H_o^{(2)}$, V, viz.:

$$H_o^{(2)}(\lambda_n r) \sim \frac{e^{-i\lambda_n r}}{\sqrt{i\lambda_n r}}$$

$$V(g^{1/3}z - \alpha_n^2 g^{-2/3}) \sim \frac{e^{-i\frac{2}{3}(g^{1/3}z - \alpha_n^2 g^{-2/3})^{3/2}}}{[g^{1/3}z - \alpha_n^2 g^{-2/3}]^{1/4}}$$

a typical term of Eq. 5.6, apart from a factor varying slowly with r, z, is

$$e^{-\tau_n\left(r - \frac{z^{1/2} + z_0^{1/2}}{2\gamma^{1/2}}\right)}$$

for sufficiently large values of z. If Eq. 5.6 is to converge rapidly enough to be useful for numerical application, we must have

$$r > \frac{z^{1/2} + z_0^{1/2}}{2\alpha^{1/2}}.$$

It will be seen in Section 6 that this inequality is satisfied by points beyond the boundary of the shadow zone. At sufficiently great ranges, then, the field is characterized by a large horizontal attenuation, which in practical cases is generally much greater than attenuation due to other causes, such as scattering and absorption.

It is to be noted further that each mode of Eq. 5.6 increases exponentially with depth. While this has no significance at points short of the shadow boundary, at sufficiently great ranges it serves to characterize the vertical structure of the field. Thus, at points sufficiently remote from the shadow boundary, there is a great increase in the field intensity with increasing source depth.

If $g < 0$, the velocity increases with depth, so that the rays are bent upward from the source to the surface, whereupon repeated reflection and refraction may be expected to produce a channel in the vicinity of the surface. For this situation, the W. K. B. method gives

$$\alpha_n^2 = e^{\pi i} \left\{ \frac{3}{2} g \left(n - \frac{1}{4}\right) \pi \right\}^{2/3}$$

whence

$$\lambda_n = (k^2 + \alpha_n^2)^{1/2}$$

is real provided

$$\frac{3}{2} g \left(n - \frac{1}{4}\right) \pi < k^3$$

or

$$n < \frac{2k}{3\pi\gamma} + \frac{1}{4} \qquad 5.7$$

In practice, this inequality is satisfied for values of n into the thousands, so that there is no horizontal damping to be associated with this type of propagation. The modes ultimately decrease exponentially with depth, although it is not clear that this implies a similar character for the entire field. It is manifestly out of the question to make a numerical study of this case using the above method.

We shall now turn our attention to the bilinear gradient. We shall suppose that ultimately the velocity decreases with depth, but shall permit either type of surface gradient. Specifically, let

$$\left.\begin{aligned} k^2 n^2 &= k^2 + g_0 z & 0 \leq z \leq z_1 \\ &= k^2 + (g_0 - g_1) z_1 + g_1 z & z_1 \leq z \\ g_1 &> 0 & \end{aligned}\right\} \qquad 5.8$$

Let

$$
\left.\begin{aligned}
\zeta_0 &= g_0^{1/3} z - \alpha^2 g_0^{-2/3} \\
\zeta_1 &= g_1^{1/3} z - \alpha_1^2 g_1^{-2/3} \\
\alpha^2 &= \lambda^2 - k^2 \\
\alpha_1^2 &= \alpha^2 - (g_0 - g_1) z_1
\end{aligned}\right\} \qquad 5.9
$$

so that we have

$$\frac{d^2\beta}{d\zeta_i^2} + \zeta_1 \beta = 0 \qquad i = 0, 1$$

and the following boundary conditions

$$
\begin{aligned}
\beta &= 0 & \zeta_0 &= 0 \\
[\beta] &= 0 & \zeta_0 &= \zeta_1 \\
\left[\frac{d\beta}{dz}\right] &= 2\lambda & \zeta_0 &= \zeta_1 \\
\beta &= CV(\zeta_1) & z &> z_1
\end{aligned}
$$

Thus, if

$$
\begin{aligned}
\beta &= AU(\zeta_0) + BV(\zeta_0) & 0 \le z \le z_1 \\
&= CV(\zeta_1) & z_1 \le z
\end{aligned}
$$

we have for the determinant of the system

$$\Delta = U_{00}\left\{\left(\frac{g_0}{g_1}\right)^{1/3} V_{01}' \; V_{11} - V_{01} \; V_{11}'\right\} - V_{00}\left\{\left(\frac{g_0}{g_1}\right)^{1/3} U_{01}' \; V_{11} - U_{01} \; V_{11}'\right\}$$

where

$$U_{00} = U(\zeta_0 \text{ at } z = 0)$$

$$U_{01} = U(\zeta_1 \text{ at } z = z_1), \text{ etc.}$$

and the primes refer to differentiation with respect to the argument. If we set

$$x = \frac{-\alpha^2}{|g_0|\, z_1} \qquad M = |g_0|^{1/3} z_1 \qquad a = \frac{g_1 - g_0}{g_1}$$

then, apart from a non-vanishing factor, the determinant is asymptotically equivalent to the formula

$$x^{3/2} - a(x - 1)^{3/2} = \frac{3}{2}\pi M^{-3/2}\left(n - \frac{1}{4}\right) \qquad 5.10$$

if $g_0 < 0$, and

$$x^{3/2} - a(x + 1)^{3/2} = -\frac{3}{2}\pi M^{-3/2}\left(n - \frac{1}{4}\right) \qquad 5.11$$

if $g_0 > 0$, n being any positive integer.

Formulae 5.10 and 5.11 are considered to be valid provided that $|x| M >> 1$, $\arg x > \frac{\pi}{3}$. These formulae may be obtained by use of the appropriate asymptotic developments of the solutions to Stokes equation or by a formal application of the W. K. B. phase integral to the velocity structure under consideration (6, 13). There has been a considerable amount of discussion in the literature (6, 17) which is relevant to Eqs. 5.10 and 5.11 and to related equations using different (but asymptotically equivalent) forms of the solutions to Stokes equation. A certain degree of multiplicity is connected even with the proper roots of these equations. Care must be taken to select those roots which tend to the proper limiting values in extreme cases.

It has been found that Eqs. 5.10 and 5.11 give accurately the imaginary part of x for all modes, except when $\arg x \sim 0$. This is gratifying, since it is this part which must be known in order to determine the horizontal damping. The real part of x as determined from these equations is not accurate for $\arg x < \frac{\pi}{3}$.

When $\arg x \sim 0$ the formulae (6)

$$\mathrm{Re}(x) = \frac{1}{M}\left[\frac{3\pi}{2}\left(n - \frac{1}{4}\right)\right]^{2/3}$$

$$\mathrm{Im}(x) = \frac{1}{kz_1}\left[\frac{-2}{3\left(n - \frac{1}{4}\right)\pi g_0}\right]^{1/3} e^{-\frac{4}{3}\left(1 - \frac{g_0}{g_1}\right)\left\{M - \left[\frac{3\left(n - \frac{1}{4}\right)\pi}{2}\right]^{2/3}\right\}^{3/2}} \qquad 5.12$$

may be used provided

$$M^{3/2} > \frac{3\left(n - \frac{1}{4}\right)\pi}{2} \qquad 5.13$$

It will be seen that Im(x) as determined from Eq. 5.12 is quite small, so that Eq. 5.13 may be regarded as yielding the number of modes characterized by small damping (trapped modes).

The transition region $0 < \arg \lambda < \frac{\pi}{3}$ is not covered by any of the above formulae. Since the arguments of the functions U, V are small in this region, asymptotic expansions are not satisfactory, and an appeal must be made to accurate tables of these functions. In general, there will be, at most, one mode not covered by Eqs. 5.10 through 5.13. For a discussion of this transitional region, reference (17) may be consulted.

Eqs. 5.10 and 5.11 are convenient in form for calculation, but the results of such calculation are more conveniently tabulated in terms of the quantities

$$\left.\begin{aligned} Y &= \left|a - 1\right| \; x \\ L &= \frac{3\pi}{2}\left(n - \frac{1}{4}\right) \left|a - 1\right| M^{-3/2} \end{aligned}\right\} \qquad 5.14$$

In terms of these quantities, the horizontal attenuation coefficient is

$$\tau_n = \frac{1}{2k}\left|\frac{g_o}{1 - a}\right|^{2/3} M \, \mathrm{Im}(Y) \qquad 5.15$$

Curves of Y qua function of L with parameter a are shown in Fig. 5.1. These curves were computed from Eq. 5.10 or Eq. 5.11, as appropriate, by the following process. If $x = re^{i\theta}$, then we can get

$$a = \frac{r^{3/2} \sin \frac{3\theta}{2}}{\sin \psi (r^2 + 1 \mp 2r \cos \theta)^{3/4}}$$

$$L = \pm \frac{r^{3/2} \sin (\psi - \frac{3\theta}{2})}{\sin \psi} \left|a - 1\right|$$

with

$$\psi = \frac{3}{2} \tan^{-1} \frac{r \sin \theta}{r \cos \theta \mp 1}$$

The upper sign holds for $a > 1$, and the lower, for $a < 1$. Values of a, L were computed from these formulae for various values of r and θ, and the curves of Fig. 5.1 were prepared graphically by cross-fairing between these accurately known points.

Applications of the bilinear case to practical problems will be considered in Section 7.

Another fundamental differential equation is that generating the spherical Bessel functions, or modified functions of order integer + $\frac{1}{2}$. This equation is

$$\frac{d^2\varphi}{dz^2} + c^2\varphi = \frac{p(p + 1)}{z^2}\varphi \qquad 5.16$$

p being integral. We have the result that linearly independent solutions of Eq. 5.16 are

$$\left.\begin{aligned} U &= (-2icz)^{-p}\, e^{icz}\, Q_p(-2icz) \\ V &= \bar{U} \qquad \text{(complex conjugate of } U, cz \text{ real)} \end{aligned}\right\} \qquad 5.17$$

where

$$Q_p(t) = \sum_{r=0}^{p} \frac{(p+r)!}{r!\,(p-r)!}\, t^{p-r}$$

is known as the Bessel polynomial of order p. Fig. 5.2, which gives the zeros of the first four of the Q_p, may be of passing interest. It may be noted that these zeros have a negative real part, so that there are no residue terms in the determination of the potential. If c has a positive real part, then evidently V(z) is that solution to be used in representing downgoing waves (positive z). Pekeris (20) has considered a similar equation in which p is not an integer. The similarity of the form of the equation is entirely superficial, however, as the solutions are diametral in analytic character.

Let z measure depth from the free surface as usual, and put

$$x = \omega z + \xi$$

$$\lambda^2 = \omega^2(a^2 - \mu^2)$$

so that Eq. 3.6 may be written

$$\frac{d^2\beta}{dx^2} + \mu^2\beta = (a^2 - \frac{1}{c^2})\beta$$

If the velocity is given by the relation

$$c^2 = \frac{(\omega z + \xi)^2}{a^2(\omega z + \xi)^2 - p(p+1)} \qquad 5.18$$

we have Eq. 5.16. The surface velocity is

$$c_o = \frac{\xi}{[a^2\xi^2 - p(p+1)]^{1/2}}$$

and c decreases monotonically from this value to the limiting value $\frac{1}{a}$, provided

$$a\xi > [p(p+1)]^{1/2} > 0$$

If the surface gradient (with respect to z) be γ, we evidently have

$$\xi = c_o\left[\frac{\omega p(p+1)}{\gamma}\right]^{1/3}$$

$$a^2 = \frac{1 + \left[\frac{\gamma^2}{\omega^2}\, p(p+1)\right]^{1/3}}{c_o^2}$$

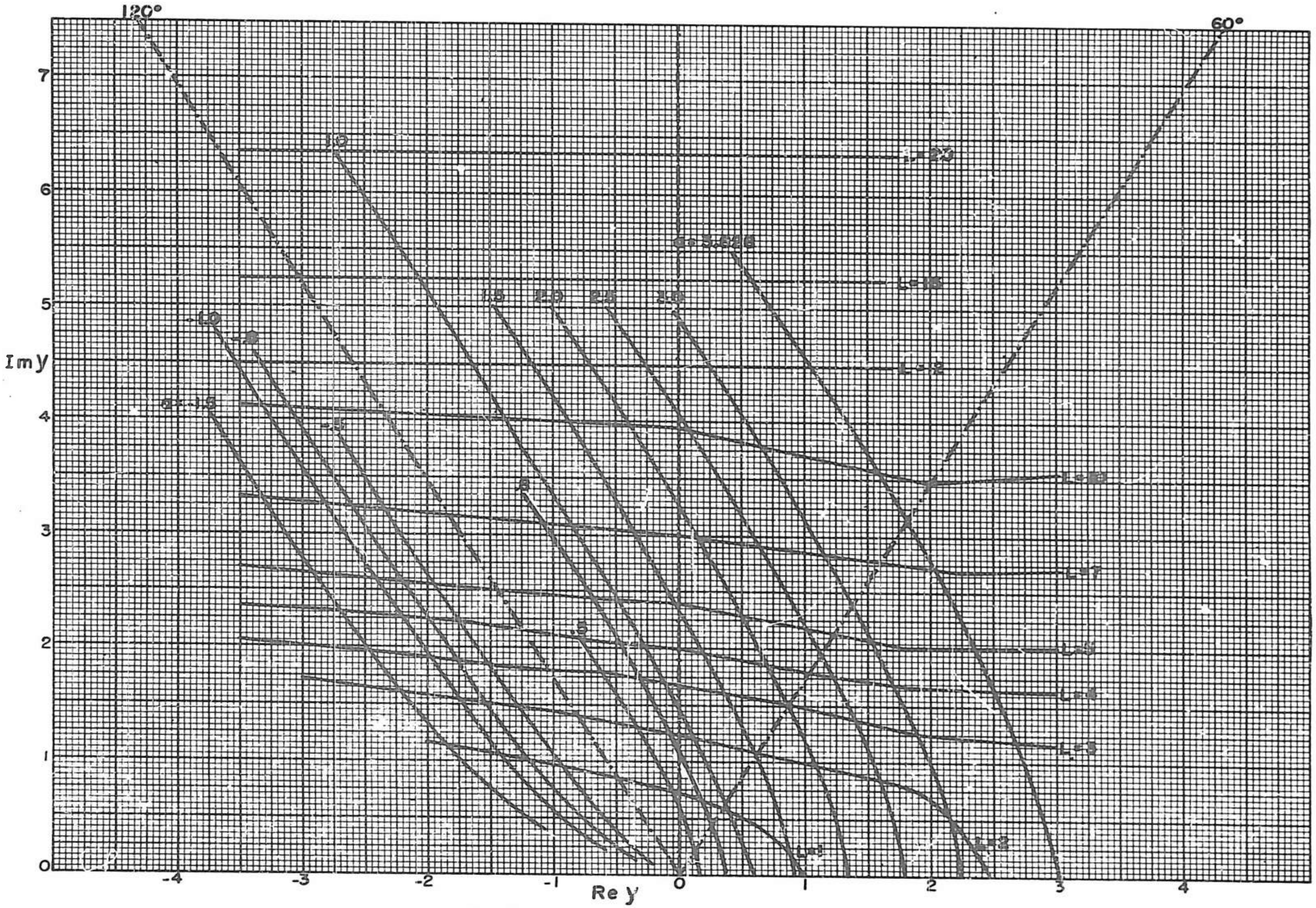


Fig. 5.1 - Eigenvalues for the Bilinear Gradient

Let the source depth be z_o, and put $x_o = \omega z_o + \xi$. The conditions 3.8 through 3.11 then become for this case

$$AU(2i\mu\xi) + BV(2i\mu\xi) = 0$$

$$AU(2i\mu x_o) + BV(2i\mu x_o) = CV(2i\mu x_o)$$

$$AU'(2i\mu x_o) + BV'(2i\mu x_o) = CV'(2i\mu x_o) + \frac{\lambda}{i\mu\omega}$$

The Wronskian $UV' - VU'$ of these solutions is easily seen to be -1. Thus, upon solving for A, B, and C, we obtain

$$\beta(\lambda, z) = \frac{\lambda V(2i\mu x_o)}{i\mu V(2i\mu\xi)}[U(2i\mu x)V(2i\mu\xi) - V(2i\mu x)U(2i\mu\xi)] \qquad \xi \leq x \leq x_o$$

$$= \frac{\lambda V(2i\mu x)}{i\mu V(2i\mu\xi)}[U(2i\mu x_o)V(2i\mu\xi) - V(2i\mu x_o)U(2i\mu\xi)] \qquad x \geq x_o \qquad 5.19$$

Therefore, β has poles at the zeros of $Q_p(2i\mu\xi)$, and a branch point at $\mu = 0$. $\beta(\lambda, z)$ can evidently be written

$$\beta(\lambda, z) = \lambda e^{-i\mu(x + \xi - x_o)} \Sigma\frac{a_r}{\mu^r} \Sigma\frac{b_s}{\mu - m_s}$$

if $x > x_o$, a similar expression holding if $x < x_o$.

A typical term of this series is

$$a_r b_s \frac{e^{-i\mu(x + \xi - x_o)}}{\mu^r(\mu - m_s)}$$

so that in constructing the wave potential, we must consider integrals of the type

$$\int_0^\infty \frac{J_o(\lambda r) e^{-i\mu(x + \xi - x_o)} \lambda d\lambda}{\mu^r(\mu - m_s)} \qquad 5.20$$

which have been considered in Section 4. If the degree of Q_p is not too great, these integrals are very easily evaluated, and one has the advantage of obtaining the field at all points with equal facility. It may be seen that p is approximately given by $M^{9/2}$. This method is therefore most attractive for low frequencies and moderate gradients.

It is also evident that the velocity structure

$$c = \frac{\omega z - \xi_1}{[a_1{}^2(\omega z - \xi_1) - p(p+1)]^{1/2}} \qquad 0 \le z \le \xi < \xi_1$$

$$= \frac{\omega z + \xi_2}{[a_2{}^2(\omega z + \xi_2)^2 - p(p+1)]^{1/2}} \qquad \xi \le z, \; \xi_1, \; \xi_2 > 0$$

can be handled with equal ease. This amounts to having a layer of positive velocity gradient from the surface (taken to be at 0) to a depth ξ , over the negative gradient previously discussed. Thus, the conditions of a surface channel also can be investigated by the use of these polynomials. A simple numerical example will be given in Section 7.

For analytic details concerning these polynomials, the attention of the reader is invited to (10).

We have shown in this section by means of special cases that the use of normal modes (and associated branch line integrals) is limited to sound fields exhibiting a shadow zone. If the sound velocity decreases monotonically from the surface, then the field within the shadow zone may be determined from the first few modes alone. In the case of a surface channel, the sound field is characterized by a small number of modes only under such conditions that diffraction of energy out of the channel (leakage) is strong.

p	Q_p	
1	$t + 2$	-2
2	$t^2 + 6t + 12$	$-3 \pm 1.732051\ i$
3	$t^3 + 12t^2 + 60t + 120$	-4.64434; -3.67783 $\pm$ 3.50876 i
4	$t^4 + 20t^3 + 180t^2 + 840t + 1680$	$-4.207580 \pm 5.314834\ i$ $-5.792420 \pm 1.734474\ i$

Fig. 5.2 - The Zeros of $Q_p(t)$

SECTION 6

RAY ACOUSTICS

It is of interest to deduce the equations of ray acoustics from the wave equation. The line of reasoning employed here is similar to that used in the optical case (16) The method of characteristics is perhaps more suitable for a unified treatment of the subject, but it is preferred here to use a simple approach to the basic formulae for the field intensity in the limit of high frequencies. Bremmer (5) gives an illuminating treatment of the ray, or geometrical, approximation by the use of the method of steepest descents.

If the wave potential φ be written in the form

$$\varphi = \psi e^{ik\theta}$$

then the Eq. 3.1 becomes

$$\Delta\psi + 2ik\nabla\psi\cdot\nabla\theta + ik\psi\Delta\theta + k^2\psi(\nabla\theta\cdot\nabla\theta - n^2) \qquad 6.1$$

We restrict θ by supposing that

$$\nabla\theta\cdot\nabla\theta = n^2 \qquad 6.2$$

Consider a skew curve $\Gamma: x, y, z = x, y, z(\tau)$ satisfying the conditions

$$\left.\begin{aligned} &\frac{d}{d\tau}(n\dot{x}) = \frac{\partial n}{\partial x}, \text{ etc.} \\ &\dot{x} = \frac{dx}{d\tau} \end{aligned}\right\} \qquad 6.3$$

Then,

$$n\dot{x} = \frac{\partial\theta}{\partial x}, \text{ etc.}$$

Along such a curve we have

$$\dot{\psi} = \dot{x}\frac{\partial\psi}{\partial x} + \dot{y}\frac{\partial\psi}{\partial y} + \dot{z}\frac{\partial\psi}{\partial z}$$

$$= \frac{1}{n}\nabla\psi\cdot\nabla\theta$$

Hence, Eq. 6.1 becomes

$$\Delta\psi + ik(2n\dot{\psi} + \psi\Delta\theta) = 0$$

and in the limit of large k,

$$2n\psi + \psi\Delta\theta = 0$$

from which it follows that

$$\psi = e^{-\frac{1}{2}\int^{\tau}\frac{\Delta\theta}{n}d\tau} \qquad 6.4$$

and

$$\theta = \int_{0}^{\tau} n d\tau \qquad 6.5$$

Consider an elementary volume v with ends of area s_A, s_B and generators along the curves Γ. Then,

$$\frac{\psi_B}{\psi_A} = e^{-\frac{1}{2}\int_A^B \frac{\Delta\theta}{n}d\tau}$$

Now, Green's theorem gives

$$\int_v \Delta\theta dv = [\int_{s_A} + \int_{s_B} + \int_{s_C}]\nabla\theta\ ds$$

$$= \int_{s_B} n_B ds_B - \int_{s_A} n_A ds_A$$

if s_C be the area corresponding to the generating surface.

Let $$ds_B = K_B ds_o,\ ds_A = K_A ds_o$$

Then, $$\int_v \Delta\theta dv = \int_{s_o}\int_A^B \frac{\partial nK}{\partial\tau} d\tau ds_o$$

On the other hand, $dv = KdS_o d\tau$, so that

$$\int_v \Delta\theta dv = \int_v \frac{1}{K}\frac{\partial nK}{\partial\tau} dv$$

and

$$\frac{\Delta\theta}{n} = \frac{1}{nK}\frac{\partial nK}{\partial\tau} = \frac{\partial}{\partial\tau}\log nK ,$$

$$\frac{\psi_B}{\psi_A} = \left(\frac{n_B K_B}{n_A K_A}\right)^{\frac{1}{2}} \qquad 6.6$$

Formula 6. 6 shows that the intensity, which is proportional to ψ^2, decreases along Γ in proportion to the increase of the area cut out by the generators lying initially within a specified solid angle.

It is of interest to make a specific calculation of ψ for one case of especial importance, namely, that of constant velocity gradient, in which we take

$$n = \frac{z_o}{z} \qquad 6.7$$

z_o thus represents the distance from the source to the depth at which the velocity c becomes zero. This situation is nearly equivalent to the velocity structure chosen for the case of the Stokes' equation in Section 5, provided $(z - z_o) << 1$. Take the origin of coordinates at the level of zero velocity. Then,

$$\dot{x} = c_1 z_1 \qquad \dot{z} = (1 - c_1^2 z^2)^{\frac{1}{2}}$$

c_1 being a constant. Since

$$\frac{dx}{dz} = \frac{c_1 z}{(1 - c_1^2 z^2)^{\frac{1}{2}}}$$

we may specify c_1 by requiring that at $z = z_o$, $\frac{dx}{dz} = \cot\sigma$, σ being the initial latitude of the ray in question. If we write

$$uz_o = x, \ vz_o = z, \ tz_o = \tau$$

then

$$\left.\begin{aligned} u &= \frac{\sin(t\cos\sigma + \sigma) - \sin\sigma}{\cos\sigma} \\ v &= \frac{\cos(t\cos\sigma + \sigma)}{\cos\sigma} \\ u^2 &+ v^2 + 2u\tan\sigma = 1 \end{aligned}\right\} \qquad 6.8$$

Evidently,

$$\theta = z_o \int_0^t ndt = z_o \cosh^{-1}\frac{u^2 + v^2 + 1}{2v}$$

If $r = z_0(u^2 + v^2)^{\frac{1}{2}}$, then

$$\int_0^t \frac{1}{n}\Delta\theta dt = \log n + 2 \log r - 2 \log z_0$$

so that

$$\psi = \frac{n^{\frac{1}{2}} z_0}{r} \qquad 6.9$$

Eq. 6.9 is subject to multiplication by an arbitrary function of σ fixing the strength of the source. Evidently, for small r one wants $\psi \sim \frac{\cos \sigma}{r}$, so that we take

$$\psi = \frac{\cos \sigma}{r} \qquad 6.10$$

We thus have the conclusion that in a medium of constant velocity gradient, the field is characterized by essentially inverse square spreading.

Calculation of ray intensities is readily extended to the velocity structure consisting of segments of constant gradient, as mentioned in Section 3. Thus, let

$$n(z) = n_i(z) \qquad z_i < z < z_{i+1}$$

$$n_i(z) = \delta_i + \frac{g_i}{z}$$

with the continuity condition

$$\delta_i + \frac{g_i}{z_i} = \delta_{i-1} + \frac{g_{i-1}}{z_i}$$

For convenience, let the source lie between z_0 and z_1, and let σ be the initial angle. Along this ray, let σ_i be the angle at $z = z_i$. Let Δx_i be the increment in x corresponding to the increment $\zeta_i = z_{i+1} - z_i$ in z. Then ψ in this case is given by

$$\psi^2 = \frac{\cos^2\sigma}{x \sin \sigma \quad \sin \sigma_i \; \Sigma \dfrac{\Delta x_i}{\sin \sigma_{i-1} \sin \sigma_i}} \qquad 6.11$$

and this formula is readily adapted to computation.

The boundary condition at the surface is evidently to be taken into account by including rays reflected with a phase change of 180°. Since rays do not penetrate the surface, however, there may be regions penetrated by neither direct nor reflected rays. Thus, for the n given by Eq. 6.7, it is evident that no rays reach points for which

$$\left(\frac{x}{2z_0}\right) > (z_0 - z_1)^{1/2} + (z_0 - z)^{1/2} \qquad 6.12$$

the surface being at z_0 and the source at z_1. Such points lie outside the limiting ray namely, that for which

$$\cos \sigma = \frac{z_1}{z_0}$$

Condition 6.12 is seen to correspond to the condition for the convergence of the normal mode solution (Section 5). That region for which Eq. 6.12 is a near equality can not be treated with accuracy by either method. Accordingly, one must interpolate between ray intensities calculated well within the region of "normal" propagation and the diffraction field beyond the limiting ray.

U. S. Navy Underwater Sound Laboratory Report No. 111

Theory of the Anamolous Propagation of Acoustic Waves in the Ocean, by H. W. Marsh, Jr.

12 May 1950 64 pp. illus. UNCLASSIFIED

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SECTION 7

NUMERICAL DISCUSSION OF SPECIAL CASES

The use of normal modes for the numerical evaluation of field strength is limited to those circumstances in which only a very few modes contribute significantly. For moderate depths and sufficient ranges, the first (least damped) mode will dominate. For purposes of illustration, a special numerical example is portrayed in Figs. 7.1 through 7.7. The values $M=2$, $a = 2$, $kz_1^2 = 8 \times 10^3$ were selected. For acoustic waves in water, these values are equivalent to a frequency of 637 cycles/sec., a layer depth of 100 feet and a gradient (equal and opposite gradients in surface and deep layer) of 1.25×10^{-5} per foot. Of course, any other values of the physical quantities can be used which are consistent with the quoted values of M, a and kz_1^2.

The normalized depth functions $|U_n(z)|^2 |U_n(z_o)|^2$ are shown in Fig. 7.1 as functions of the dimensionless depth parameter $t = \frac{z}{z_1}$. The depth gain of the higher modes is strongly contrasted with the loss of the first mode. In Figs. 7.2 through 7.5 are shown contours of constant values of the various modes, with the horizontal gain factor included. These contours are referred to the value of the mode at the point $z = \frac{1}{2}z_1, r = 1$. From these figures, the relative contribution of each mode can be seen at once. A source depth corresponding to $t = \frac{1}{2}$ was used.

In Figs. 7.6 and 7.7 are shown the effects upon the horizontal attenuation of the first mode of varying singly the frequency, layer depth, and surface gradient. The variation of these parameters is shown by the dimensionless ratios referred to the standard values of frequency, layer depth, and surface gradient. These standard values are such that the values $M = 2$, $a = 2$, $kz_1^2 = 8 \times 10^3$ are satisfied.

Figs. 7.7 and 7.8 are of interest because they indicate under what combination of circumstances the character of propagation is sensitive to changes in the physical parameters of the problem.

In the notati n of Section 5, the functions $U_n(t)$ for this case of $\rho < 1$ are

$$\left.\begin{aligned} U_n(t) &= \frac{1}{C_n} V(B_1) \frac{U(A_o)V(A_t) - V(A_o)U(A_t)}{U(A_o)V(A_1) - V(A_o)U(A_1)}, \quad t < 1 \\ &= \frac{1}{C_n} V(B_t) \qquad\qquad\qquad\qquad\qquad\quad , \quad t > 1 \end{aligned}\right\} \quad 7.1$$

$$\left.\begin{aligned} B_t &= M\left[\frac{t-1}{|\rho|} + \rho^2(x-1)\right] \\ A_t &= Me^{-\pi i/3}(t - x_n) \\ C_n^2 &= \frac{z_1(\rho^3 - 1)}{M^3}\left\{M^3(1-x)U_n^2(1) + \left(\frac{dU_n}{dt}\right)^2_{t=1} + \frac{1}{\rho^3 - 1}\left(\frac{dU_n}{dt}\right)^2_{t=0}\right\} \end{aligned}\right\} \quad 7.2$$

The functions U, V are identical with those designated as h_1, h_2 respectively, in (3), and the reader's attention is directed to that source for a detailed discussion of their properties.

We wish to note only that if y be real, then V(y) has the following qualitative character:

$$V(y) \sim e^{-y} \qquad , \; y < 0$$

$$\sim \frac{1}{y^{1/4}} \qquad , \; y > 0$$

so that for strictly trapped modes, the field diminishes exponentially with depth up to such a point that

$$z > z_1 + |\rho|^2 (1 - x_n) z_1 \tag{7.3}$$

This is characteristic of the exponential decrease in the strength of the diffracted field upon increasing penetration into the shadow zone. Thus, that mode with the smallest real part (and in this case of trapping, the smallest imaginary part) will suffer the least loss in the shadow zone. At sufficiently great ranges, this mode will dominate the field up to depths so great that horizontal diffraction into the shadow zone again becomes important.

When several modes are trapped, the calculation of the sound field becomes impractical. We have shown in Section 5 that a particular velocity structure is characterized by a finite number of modes, in addition to functions representing the fields of multipole sources. This velocity structure is illustrated by a numerical example portrayed in Figs. 7.8 and 7.9. The values $p = 1$, $z_0 = 50$, $z^1 = \frac{\xi}{\omega} = 350$ were selected. If these numbers are in units of feet, then the associated frequency is 16 cycles/sec. Fig. 7.8 shows the velocity profile for this case.

Eq. 5.19 becomes for this case

$$\beta(\lambda, z) = \frac{\lambda}{(z_0 + z^1)(z + z^1)} \Bigg\{ e^{-\alpha|z_0 - z|} \left[\frac{(z_0 + z^1)(z + z^1)}{\alpha} + \frac{|z_0 - z|}{\alpha^2} - \frac{1}{\alpha^3} \right]$$

$$- e^{-\alpha(z_0 + z)} \left[\frac{(z_0 + z^1)(z + z^1)}{\alpha} + \frac{2 z z_0}{\alpha(\alpha z^1 + 1)} + \frac{z_0 + z}{\alpha^2} - \frac{1}{\alpha^3} \right] \Bigg\} \tag{7.4}$$

The terms representing the direct sound field are evident, in addition to the mode corresponding to the residue at $\lambda = \frac{1}{z^1}\sqrt{1 + (kz^1)^2}$ and the multipole terms. Since the eigenvalue of the mode is essentially real (the imaginary part being due to the attenuation associated with k), this mode is completely trapped and is of importance at great ranges. The mode decays exponentially with depth as e^{-z/z^1} and may be interpreted as representing a surface wave (24), (5).

Detailed calculations of the various functions in Eq. 7.4 were made using Eq. 4.30 for the basic formula. The eight terms of this series which are given in Eq. 4.30 were sufficient to yield an accuracy of 1 part in 10^{-6}, which was considered necessary in view of the strong interference between the direct and reflected terms in Eq. 7.4.

In Fig. 7.9 are shown contours of constant intensity. The decibel levels corresponding to these contours are referred to the field at one yard from the source.

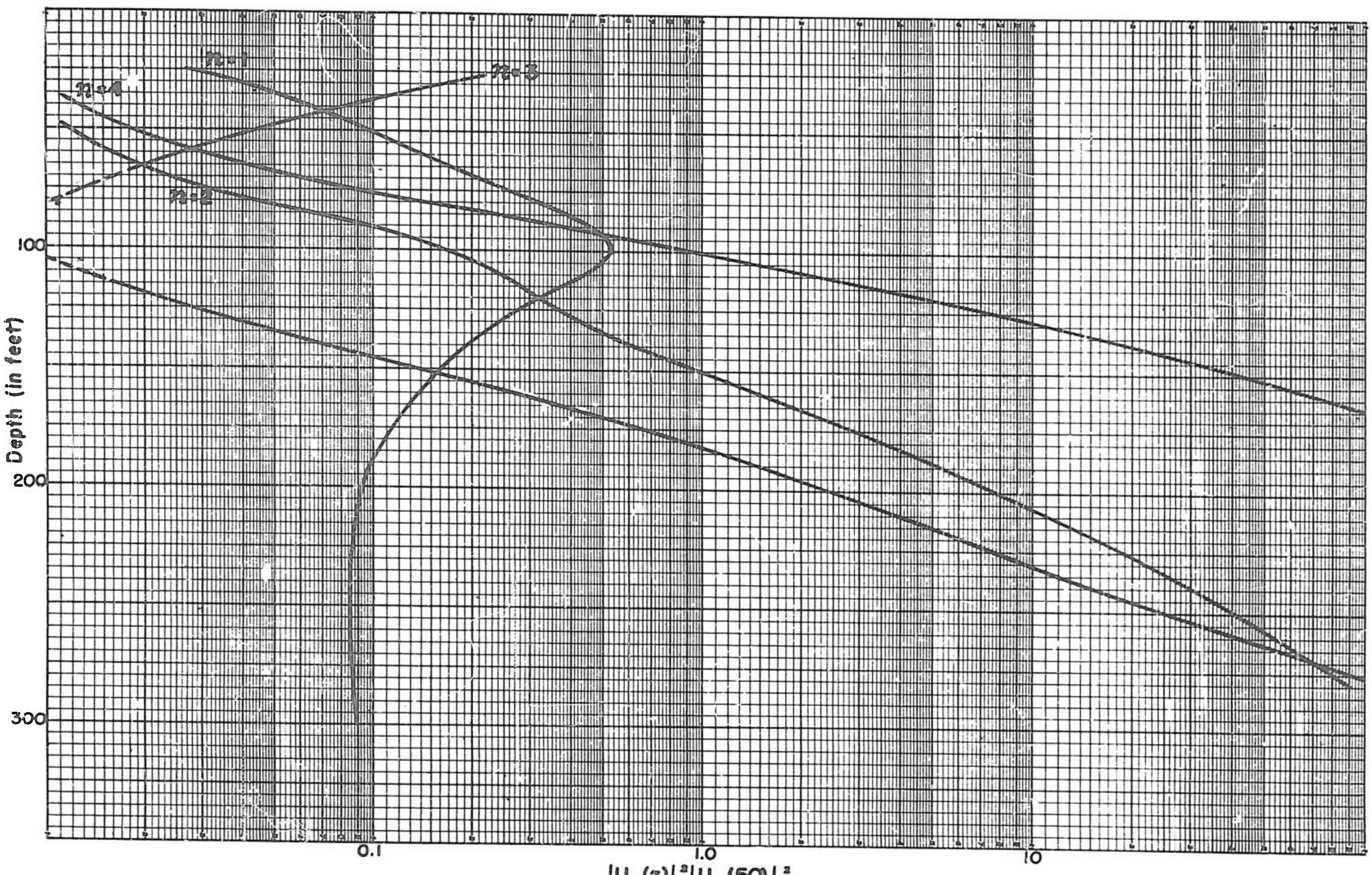

Fig. 7.1 - Normalized Depth Functions (First Four Modes)

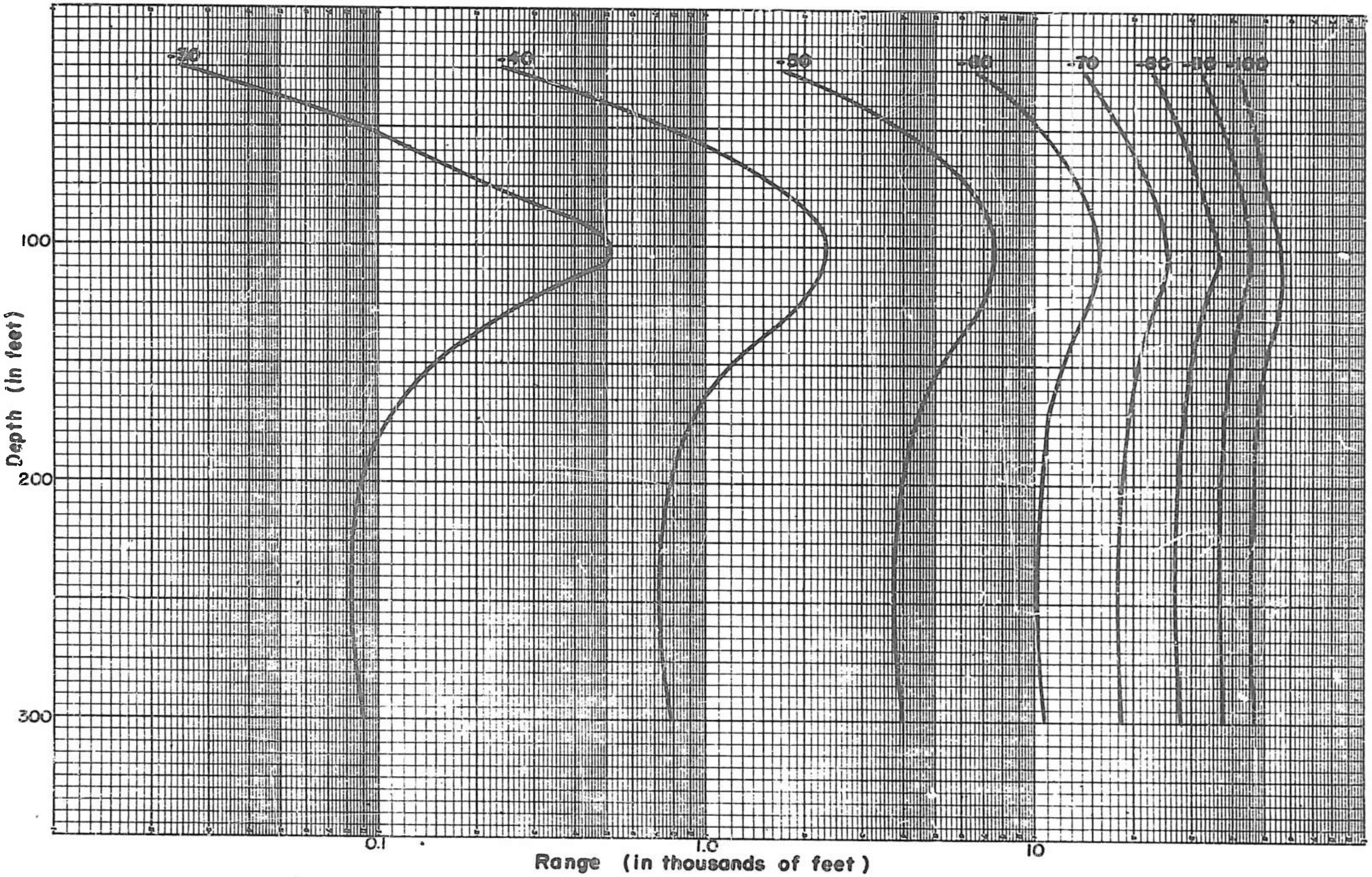

Fig. 7.2 - Iso-Intensity Contours (First Mode)

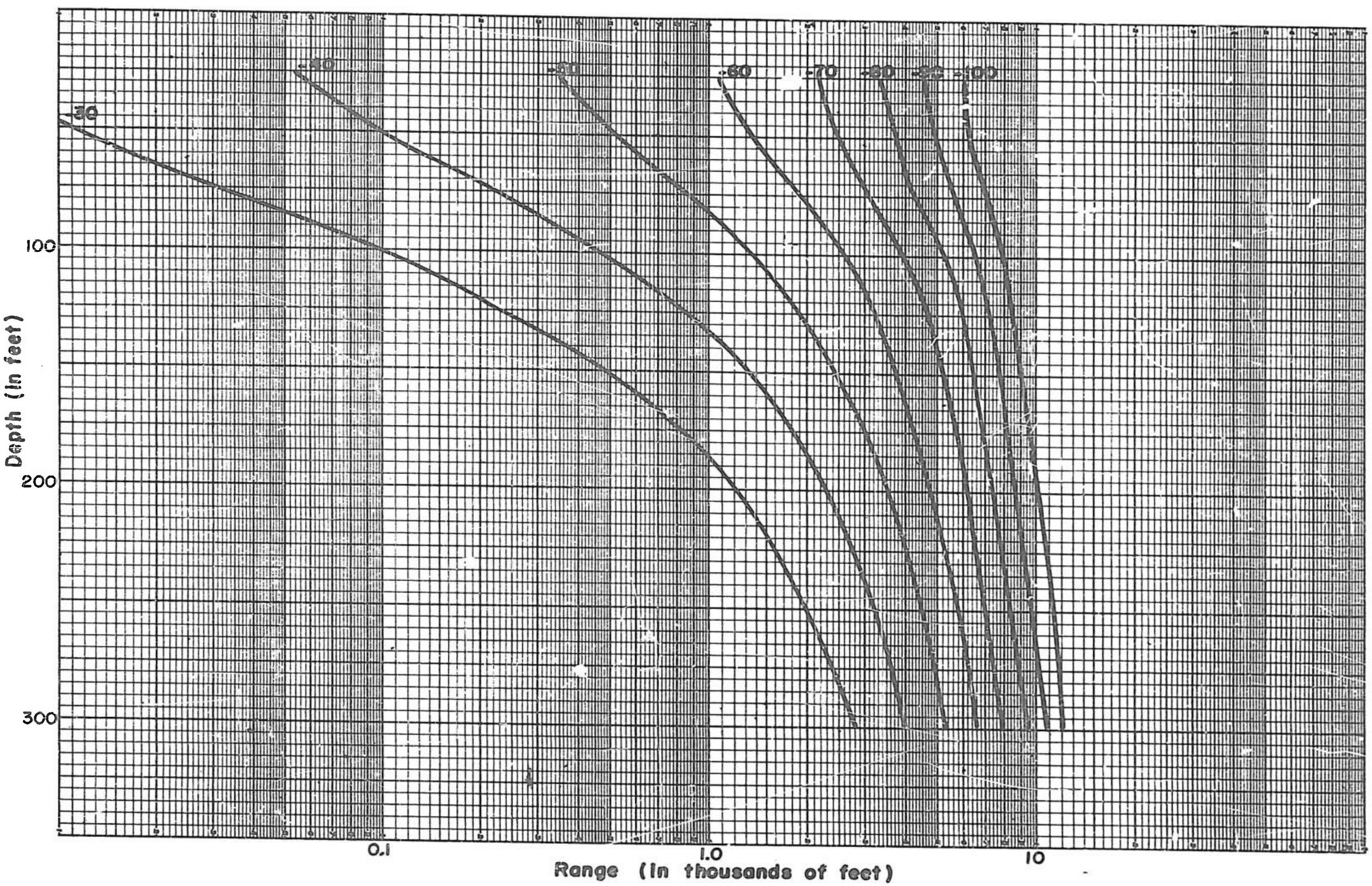

Fig. 7.3 - Iso-Intensity Contours (Second Mode)

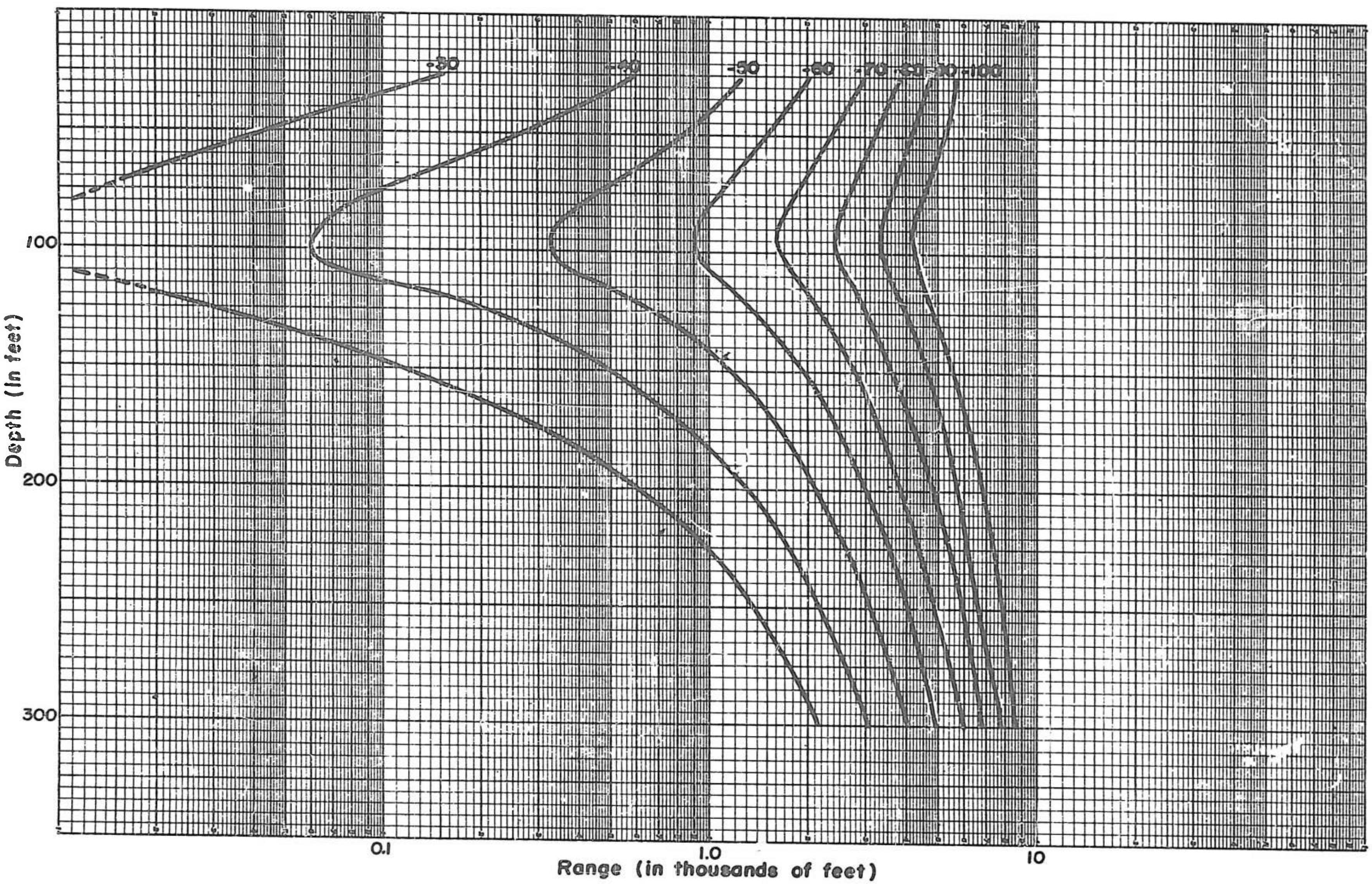

Fig. 7.4 - Iso-Intensity Contours (Third Mode)

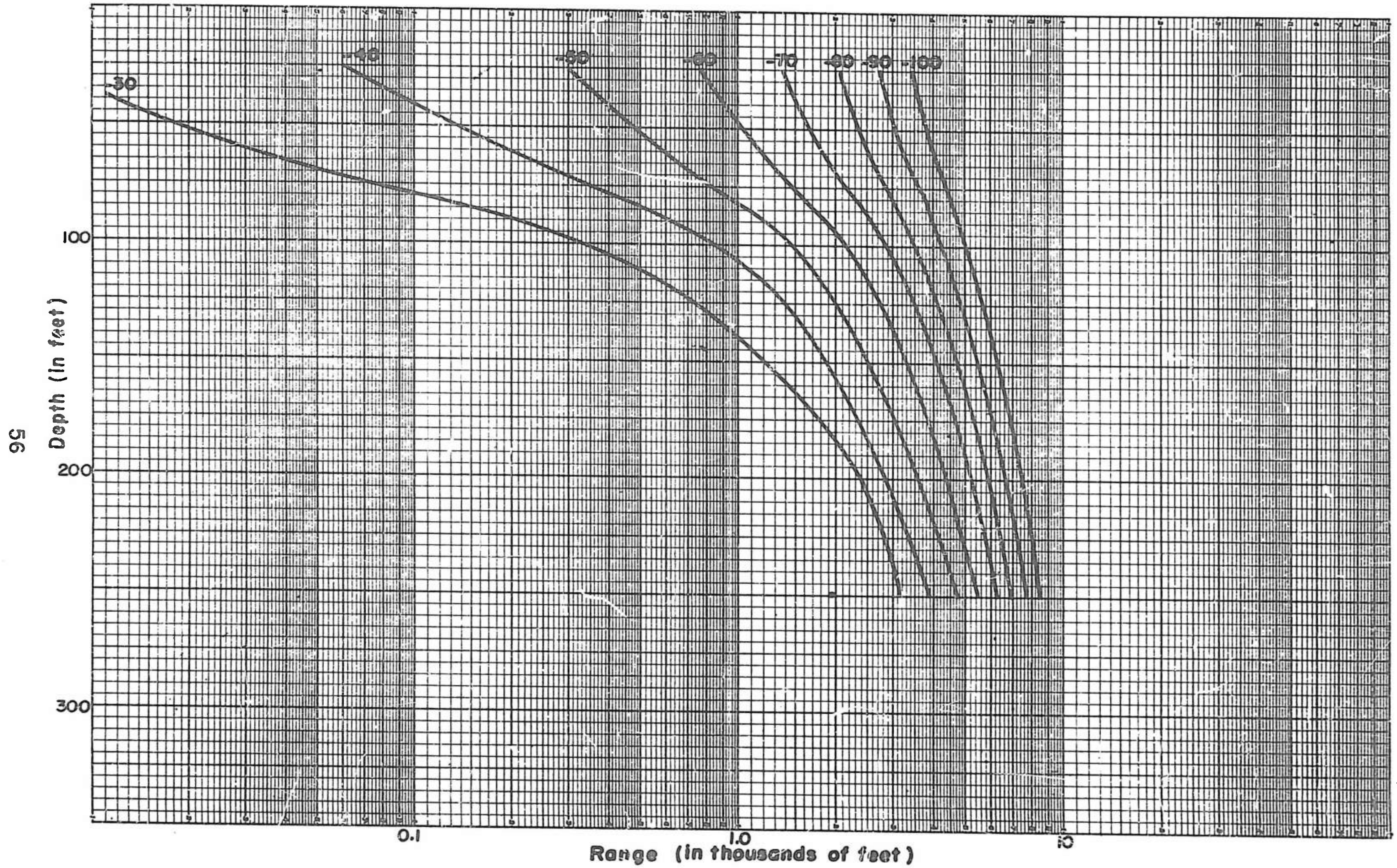

Fig. 7.5 - Iso-Intensity Contours (Fourth Mode)

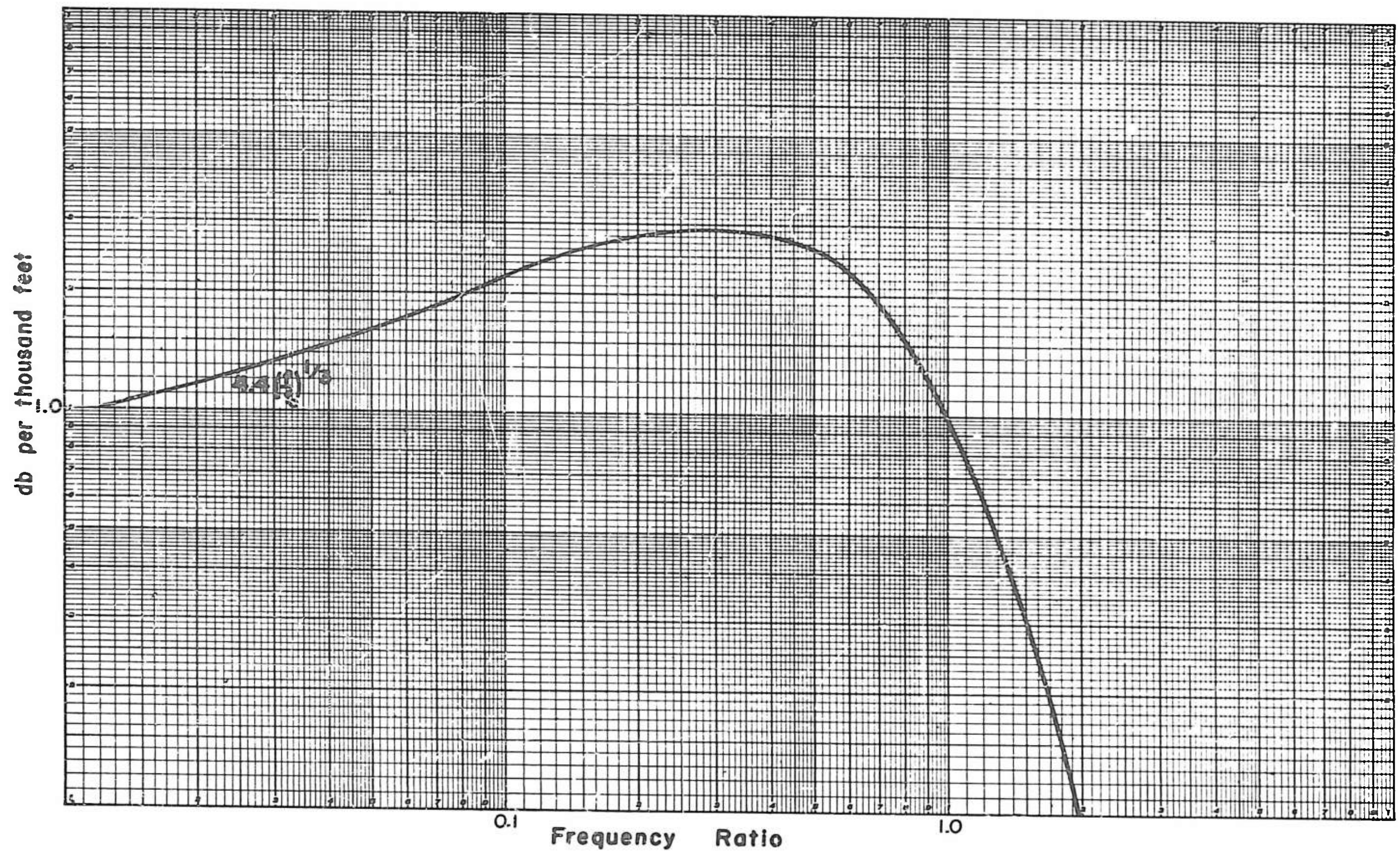

Fig. 7.6 - Attenuation Coefficient vs. Frequency

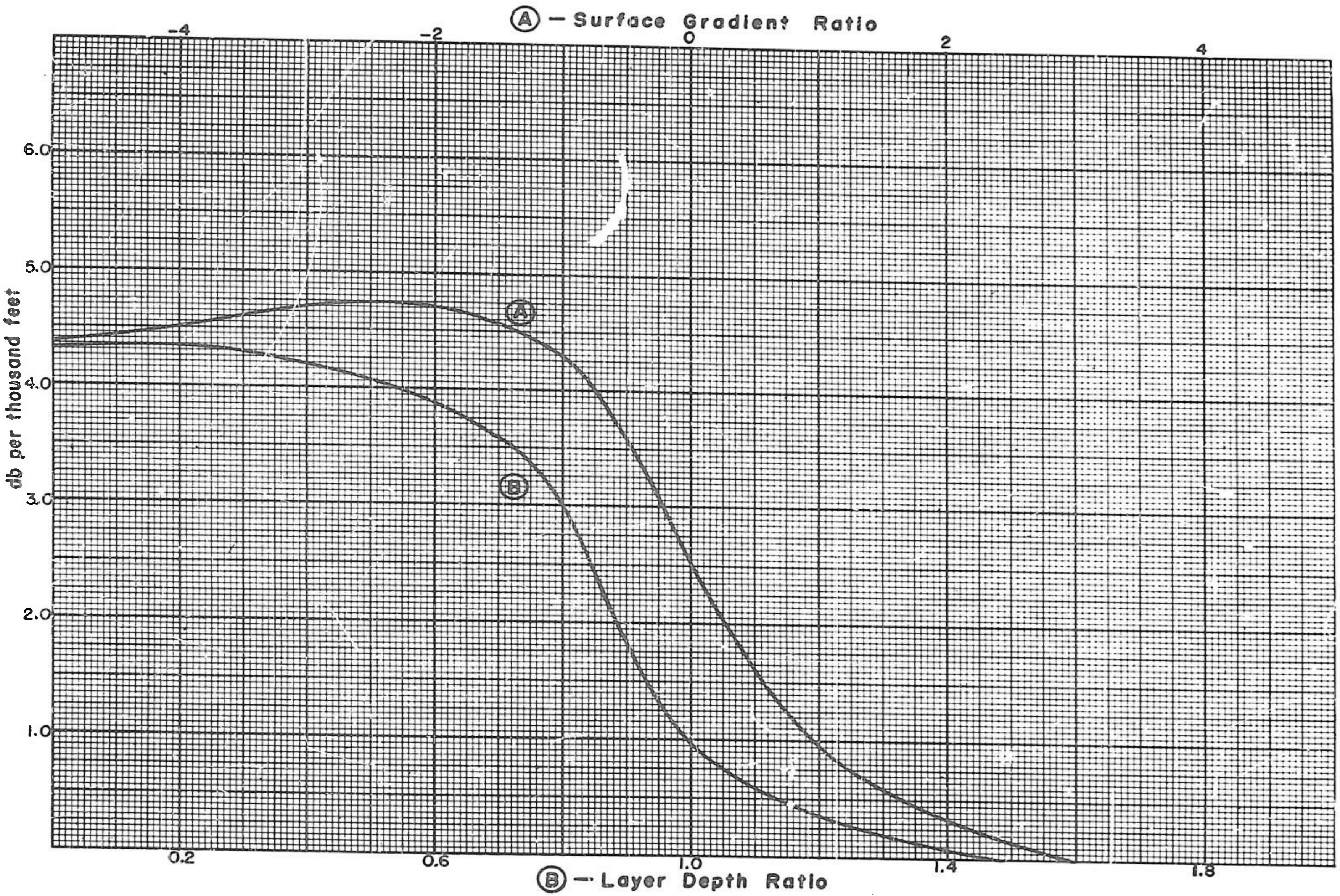

Fig. 7.7 - Attenuation Coefficient vs. Surface Gradient and Layer Depth

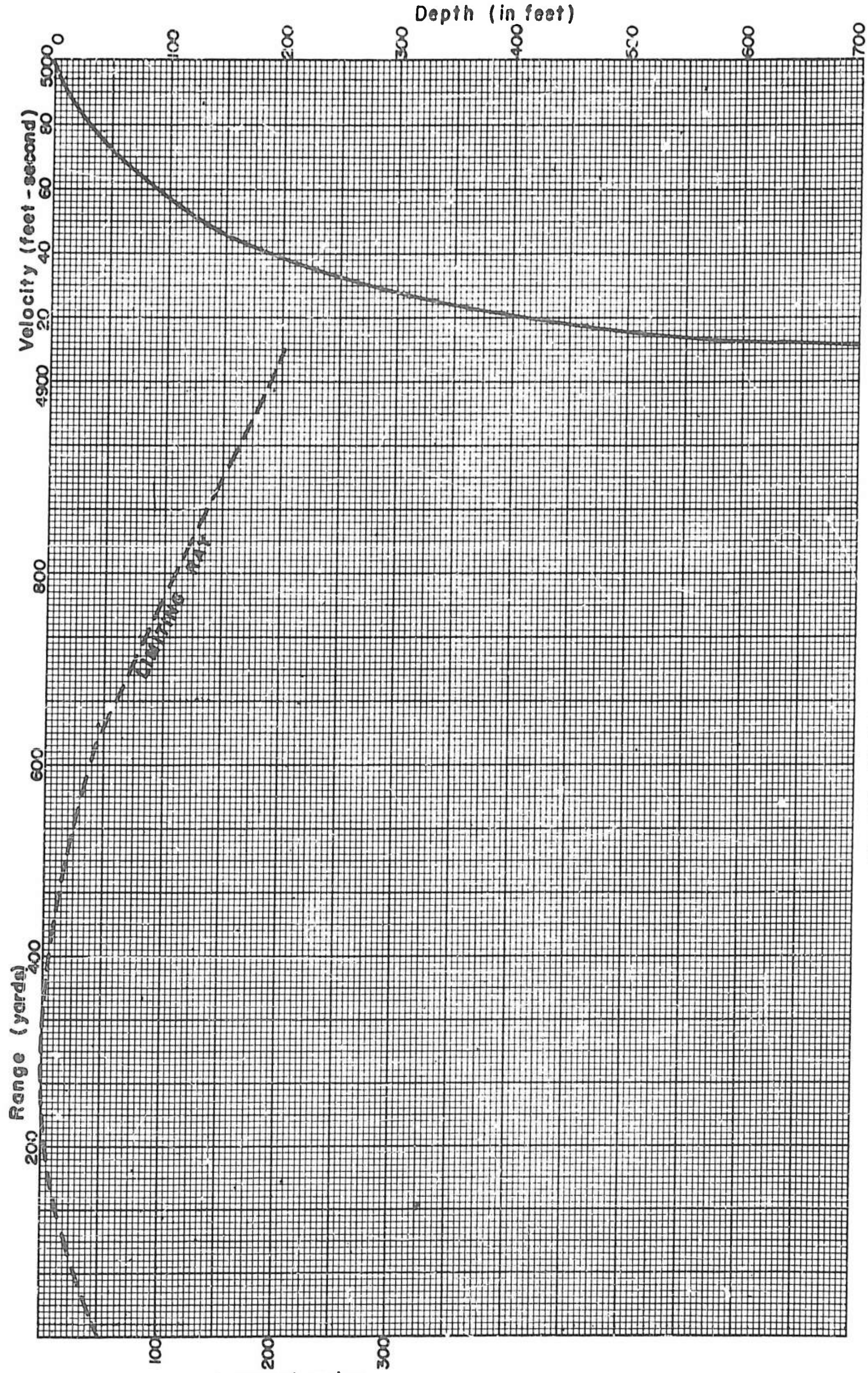

Fig. 7.8 - Limiting Ray and Velocity Profile

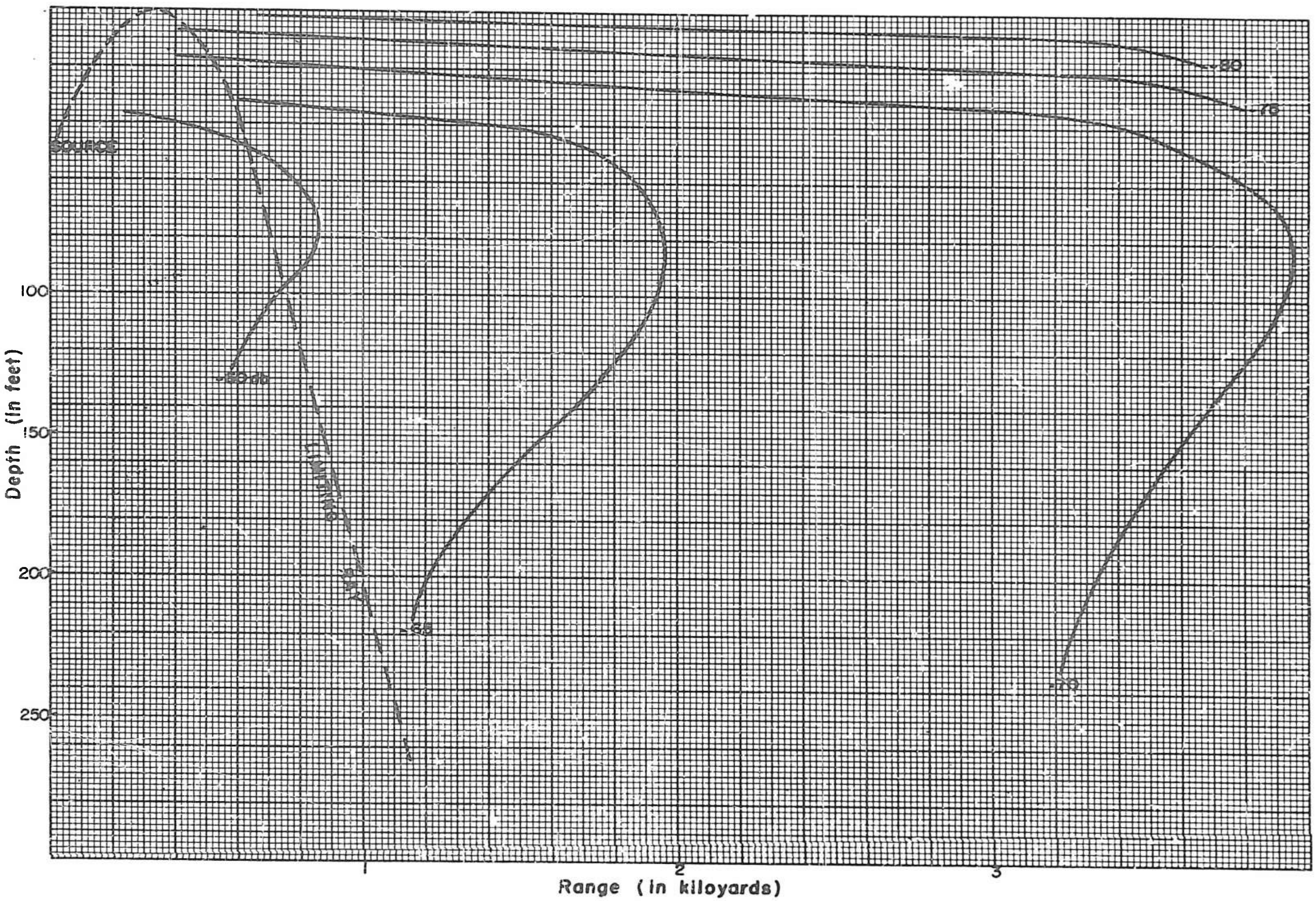

Fig. 7.9 - Iso-Intensity Contours

SECTION 8

SUMMARY

It was shown in Section 2 that the wave equation

$$\Delta\varphi + \frac{\omega^2}{C^2}\varphi = 0 \qquad 8.1$$

is suitable for the study of acoustic waves characterized by the potential φ. For the boundary conditions defining the field due to a simple source in the vicinity of a reflecting surface, the calculation of φ is made to depend on the determination of its Fourier-Bessel transform, $\beta(\lambda, z)$, so that

$$\varphi(r,z) = \int_0^\infty J_0(\lambda r)\beta(\lambda,z)\,d\lambda \qquad 8.2$$

$\beta(\lambda, z)$ was shown to satisfy the equation

$$\frac{d^2}{dz^2}\beta(\lambda,z) + (k^2n^2 - \lambda^2)\beta(\lambda,z) = 0 \qquad 8.3$$

or the equivalent integral equation

$$\beta(\lambda,z,z_0) = \frac{e^{-\alpha|z-z_0|} - e^{-\alpha|z+z_0|}}{\alpha}\lambda + \frac{k^2}{2\alpha}\int_0^\infty \varepsilon(z_1)\left[e^{-\alpha|z-z_1|} - e^{-\alpha|z+z_1|}\right]\beta(\lambda,z_0,z_1)\,dz_1 \qquad 8.4$$

which includes the boundary condition on β.

The evaluation of Eq. 8.2 was effected by expansion in a series of orthogonal functions satisfying Eq. 8.3 and requiring an additional integral term in certain cases ($\frac{dn}{dz} = 0$). Special cases of this expansion were considered in Sections 5 and 7.

It was found that the Liouville-Neumann series obtained for the solution of Eq. 8.4 was of little use, except for frequencies so low that the inhomogeneities present in the medium had little effect on propagation. Possible methods of extending the use of Eq. 8.4 by an operational solution were suggested.

No discourse of this sort would be complete without some recommendation for future research. One obvious possibility would be that of broadening the scope of the problem, by allowing three dimensional or temporal variations. The desirability of such an increase in the complexity of the problem must be measured by the expected narrowing of the gap between prediction and experiment. In view of the character of experimental conditions to be encountered, it is likely that little is to be gained along these lines. It would appear that the existing theory gives very satisfactory qualitative answers, and could be materially improved by a decrease in the complexity of numerical work required to evaluate it. The gap between the ray theory and that of diffraction is large because of the numerical difficulty in interpreting them. It was hoped initially that the use of integral equations would narrow this gap, but this expectation proved to be naive and unfounded. However, the integral equation does lead to a formulation which can be evaluated numerically in certain portions of the ray field.

One velocity structure which permits a direct comparison of rays and waves is considered in Section 5. Although this structure was set up artificially to permit this comparison, it does possess some of the important realistic features. It is possible that much more can be done along the lines of constructing velocity structures which yield wave fields capable of simple evaluation, and the use of the Bessel polynomials in Section 5 can be regarded as a first step in this direction. Langer (14) has considered an alternative method of constructing "solvable" equations.

The question of boundary conditions must be considered with great care. In most acoustic problems the effect of earth's curvature is negligible, as is the departure of the reflection coefficient from unity at the sea-air interface. Of far greater effect is the presence of the boundary at the bottom of the ocean. This fact precludes the possibility of extending the present theory to acoustic propagation at ranges of more than a few miles, except as a lower limit to the expected field intensity.

It was originally intended that a comparison between experiment and the theory presented herein would be included, but this was not possible. It may be stated, however, that quantitative agreement is generally very poor, at frequencies above a few kilocycles. The trends indicated by the theory in the formation of shadow zones, and the dependence of the field upon depth are in qualitative accord with experimental evidence, but it would appear that the theory as presented in these pages is oversimplified. This is unfortunate, since even this theory is very tedious to interpret numerically. It is probable that the theory will be found in closer agreement with experiment at lower frequencies, where the simplified model of the field may be expected to be more realistic.

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