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OPTIMUM DESIGN AND MISS DISTRIBUTIONS OF HOMING MISSILES

by

RICHARD C. BOOTON, JR.
DYNAMIC ANALYSIS AND CONTROL LABORATORY

31 MARCH 1950

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FOREWORD

Among the objectives of an analysis of an automatic control system are first, the reduction of the problem to its important elements, second, the determination of the optimum characteristics of those elements within the control of the designer, and finally, the determination of the performance of the system with optimum or near optimum parameter adjustments. Such an analysis is very profitable, even though the physical system may include many unconsidered secondary effects, since such a study can provide a comprehension of the operation of the over-all system and a yardstick for measuring the effects of complicating parameters. Because of the infancy of the techniques involved in such a study, many important systems under development have never been so analysed, and one of these is the homing type of guided missile. The objective of this report is to carry out such a dynamic analysis, and it is felt that the study described makes a significant contribution.

Many problems in the homing system are not considered here, as indeed they cannot be in any basic analysis. Among the more important of these are secondary smoothing effects, the effect of nonlinearities, particularly saturation in the missile acceleration, and three-dimensional intercoupling problems. While all of these problems are important, it is not expected that when these matters are considered, improvements in the results presented here will be found. On the other hand, there is some hope that most of these additional problems, if properly handled, will not have a substantially deleterious effect on the system performance. Missile-acceleration saturation is probably a notable exception in this respect.

It is hoped that this report will be of use to those working in and directing the efforts of the guided missile field, not only from the standpoint of the techniques developed and the results obtained, but also as an example of the value of a system dynamic analysis.

Albert C. Hall

ABSTRACT

The homing missile kinematic equations are linearized to permit the application of general optimization techniques. The minimum rms values of miss are calculated for two sets of conditions: (1) white radar noise and an intelligent target maneuver and (2) white radar noise and a random target maneuver.

The linearized equation for the trajectory followed by a missile guided by a proportional-navigation-with-simple-time-lag control system is solved for certain values of the navigation constant. Optimum control-system parameters are calculated for the two cases considered in the general study, and the minimum rms miss for this system is compared with the minimum rms miss for any linear control system. Values of miss caused by an error in initial heading and by an initial lateral acceleration are calculated.

1. INTRODUCTION

1.1. Statement of Homing Problem.

The homing missile is characterized by its use of line-of-sight measurements to obtain information concerning the position of its target. The quantities utilized by the homing missile are its own heading relative to some fixed reference, the relative bearing of the target, and the missile-to-target range. The range measurement, often omitted, is less important than the two angular measurements.

One missile may use in succession several means of navigation. For example, a missile may initially receive its guidance from a radar beam controlled by the parent aircraft and then utilize homing for the terminal guidance. This study is concerned with the missile whose terminal guidance may be classified as homing and considers only the terminal phase of the flight. The initial positions, velocities, etc., considered here are the conditions at the moment homing flight commences, whether these be the initial launching conditions or the conditions existing as the navigation is changed to homing from some other system.

1.2. Assumptions.

This study is restricted to the case where the target and missile move with constant speed in a plane. It is assumed that the available lateral acceleration of the missile is so high that effects of acceleration limiting are negligible. Very little analytical work has been done on this general problem. Nevertheless, important results have been obtained using the general kinematic equations for the case in which the missile time lag can be neglected.[1,2] The effect of time lag has been the subject of several studies on the M.I.T. Flight Simulator.[3,4]

To make possible further analytical study of the homing problem, it is assumed that during the homing phase of the problem the changes in the angles considered are so small that the trigonometric functions may be replaced by linear functions. The justification for this approximation is the extent to which solutions of the linearized equation agree with the solutions of the general equations. It has been shown by comparison with numerical solutions that for the range of parameters and variables important in the homing problem, the approximation is valid. The target maneuvers considered are a single, constant-lateral-acceleration turn and a statistical target maneuver. The radar noise considered is the error in apparent target center caused by glint effects, because this type of radar noise probably produces greater miss distance than other noise in the system. It is assumed that the noise has statistical properties which enable it to be considered a stationary random variable (considered as an error in distance). This is thought to be a reasonable assumption for the noise, particularly when target information comes from monopulse radar.

1.3. Objectives.

This study has two primary objectives. The first is the establishment of certain general results for the linear equation. The second is the investigation of the homing missile in which the control system is characterized by the proportional-navigation-with-simple-time-lag equation, with the aim of specifying values of control-system parameters for optimum performance. The two factors considered in the optimization procedure are target maneuvers and radar noise. The optimization neglects the miss caused by initial errors, since a study of this miss shows that it may be made relatively small by making the initial range large enough. The results of the study of initial error effects (Sec. 3.4) can be used for the cases in which the initial range is such that initial errors are important.

[1] Clymer, A. B., *Proportional Navigation Trajectories*, Meteor Report No. 24, Cambridge, Mass.: Mass. Inst. of Tech., June 30, 1949 (Secret).

[2] [illegible], C. W., Jr., *Trajectory Analyses and Stability Criteria*, Meteor Report No. 27, Cambridge, Mass.: Mass. Inst. of Tech., December 31, 1948 (Secret).

[3] Jones, Thomas F., Jr., Richard C. Booton, Jr., and George H. Hart, *The Effect of Time Lag on Proportional Navigation Trajectories*, Meteor Report No. 38, Cambridge, Mass.: Mass. Inst. of Tech., July, 1949 (Confidential).

[4] Booton, Richard C., Jr., *Effect of Circular Target Maneuvers on Homing Missile Trajectories*, Meteor Report No. 47, Cambridge, Mass.: Mass. Inst. of Tech., November, 1949 (Secret).

2. GENERAL LINEAR RELATIONS

2.1. Derivation of the Linear Equation.

If the available lateral acceleration of the missile is sufficient to insure that no acceleration limiting occurs, the motion of the homing missile is described approximately by a linear equation, provided certain small-angle approximations may be made. These small-angle approximations are valid if the initial heading of the missile is approximately on a collision course and if the deviations of all angles from their initial values are small. The notation used is indicated in Fig. 1.

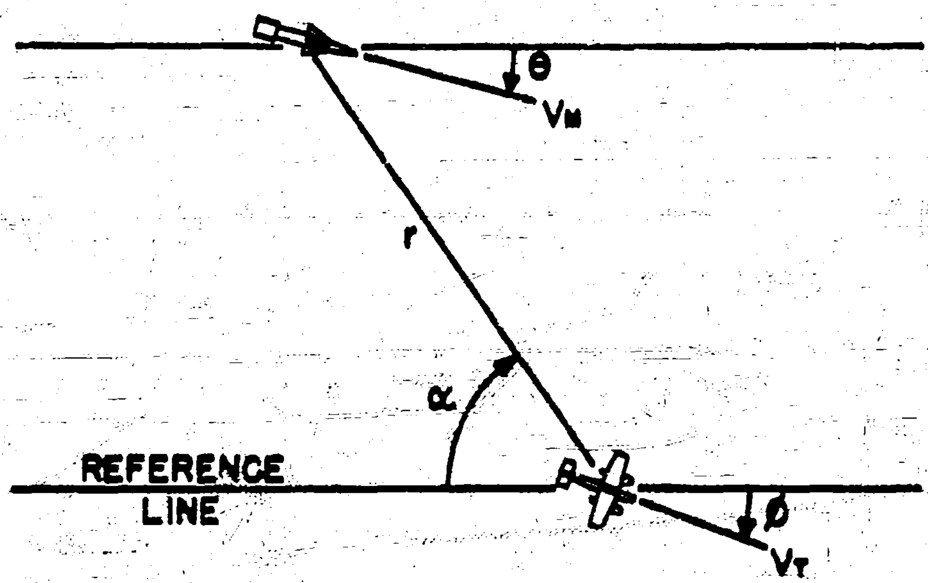

Fig. 1. Geometry.

The kinematic equations for the geometry of Fig. 1 may be written in polar form as

$$\dot{r} = V_T \cos(\alpha - \phi) - V_M \cos(\alpha - \theta) \quad (1a)$$

$$r\dot{\alpha} = -V_T \sin(\alpha - \phi) + V_M \sin(\alpha - \theta). \quad (1b)$$

If the above-mentioned small-angle approximations may be made, $\dot{r}$ is approximately constant, and

$$\dot{r} = V_T \cos(\alpha_0 - \phi_0) - V_M \cos(\alpha_0 - \theta_0). \quad (2)$$

With $\dot{r} = -V_R$, then

$$r = r_0 - V_R t. \quad (3)$$

If each angle is written as the sum of the initial value and a small transient term, Eq. (1b) may be written as

$$\begin{aligned} r\dot{\alpha}_t &= -V_T[\sin(\alpha_0 - \phi_0) + (\alpha_t - \phi_t)\cos(\alpha_0 - \phi_0)] + V_M[\sin(\alpha_0 - \theta_0) + (\alpha_t - \theta_t)\cos(\alpha_0 - \theta_0)] \\ &= -V_T \sin(\alpha_0 - \phi_0) + V_M \sin(\alpha_0 - \theta_0) + [-V_T\cos(\alpha_0 - \phi_0) + V_M \cos(\alpha_0 - \theta_0)]\,\alpha_t \\ &\quad + \phi_t V_T \cos(\alpha_0 - \phi_0) - \theta_t V_M \cos(\alpha_0 - \theta_0). \end{aligned} \quad (4)$$

If the reference direction is so chosen that $\phi_0 = 0$, then using Eq. (3) and $\beta_0 = \alpha_0 - \theta_0$, Eq. (4) may be written as

$$r\dot{\alpha}_t + \dot{r}\alpha_t = r_0 \dot{\alpha}_0 + \phi_t V_T \cos\alpha_0 - \theta_t V_M \cos\beta_0 \quad (5)$$

Letting

$$x_T = V_T \cos \alpha_0 \int_0^t \phi_t(\tau)\, d\tau \tag{6a}$$

and

$$x_M = V_M \cos \beta_0 \int_0^t \theta_t(\tau)\, d\tau - r_0 \dot{\alpha}_0 t \tag{6b}$$

where τ is a time variable of integration,

$$r\alpha_t = x_T - x_M. \tag{7}$$

Equations (6a) and (6b) are equivalent to referring the missile and target positions to a rectangular coordinate system moving (without rotation) with a velocity equal to the initial velocity of the target. The Y axis of this coordinate system is always parallel to the initial line of sight. The difference in missile and target y coordinates is approximately equal to the range, except near the end of the run. The minimum value of the range (miss distance) occurs at approximately the time the y coordinates are equal and is thus approximately $x_T - x_M$ when $t = r_0/V_R$.

The homing missile control system utilizes the apparent relative position of missile and target (measured by the apparent range r_A and the apparent line-of-sight angle α_A) to determine the missile velocity vector angle θ. The effect of errors in the apparent range may be neglected ($r_A = r$), since the primary guidance information comes from measurements of α. The error in α_A is considered as an error in the apparent value of x_T. The error may arise from several sources, among which the shift of the radar cg of the target (assuming radar is used to obtain the homing information) appears to be the most serious. Regardless of the source of this error it is hereinafter referred to as "noise" and denoted by x_N. The equation corresponding to Eq. (7) is then

$$r\alpha_{At} = x_T + x_N - x_M. \tag{8}$$

The expression relating θ, r, and α_A (which may include the effects of missile dynamics) is known as the missile-control equation. If the missile-control equation is a differential equation which is linear in θ and α_A (not necessarily in r), it follows from Eqs. (3), (6), and (8) that x_M and $x_T + x_N$ are related by a linear differential equation. Let G (r, t, p) be a linear integrodifferential operator. That is, G is a rational function of p = d/dt, and the coefficients are functions of r and t. Then the general form of the missile-control equation which is linear in θ and α_A is

$$G(r, t, p)\, \theta = \alpha_A + f_1(r, t) \tag{9}$$

where f_1 is an arbitrary function of r and t.

Substitution of Eqs. (3), (6), and (8) into Eq. (9) results in the linear equation relating x_M and $x_T + x_N$,

$$L(t, p)\, x_M = x_T + x_N + f_2(t), \tag{10}$$

where the linear operator L is given by

$$L(t, p) = \frac{r_0 - V_R t}{V_M \cos \beta_0} G(r_0 - V_R t, t, p)\, p + 1 \tag{11}$$

and

$$f_2(t) = (r_0 - V_R t)\left[\alpha_0 + f_1(r_0 - V_R t, t) - G(r_0 - V_R t, t, p)\left(\theta_0 + \frac{r_0 \dot{\beta}_0}{V_M \cos \beta_0}\right)\right]. \tag{12}$$

For the general proportional-navigation system,

$$D(p)\, \dot{\theta} = \dot{\alpha}, \tag{13}$$

the corresponding equation is

$$\left[\frac{r_0 - V_R t}{V_M \cos \beta_0} D(p)\, p + 1\right] x_M = x_T + x_N + C(r_0 - V_R t) \tag{14}$$

where C is a constant depending upon the initial conditions (C = 0 if the missile is initially on a straight-line collision course, that is, if $\dot{\alpha} = 0$, $t < 0$).

2.2. Superposition Integral.

Since x_M, x_T, and x_N are related by a linear equation, the response of the missile to any target motion and noise may be expressed in terms of a generalized step-response function by means of the superposition integral. Let $A\ (t_1, t)$ be the response from rest to a step applied at time t_1. That is, let

$$\left.\begin{array}{l} A\ (t_1, t) = 0 \text{ if } t < t_1 \\ \text{and} \quad L\ (t, p)\ A\ (t_1, t) = 1 \text{ if } t > t_1. \end{array}\right\} \tag{15}$$

Then if[1]

$$\left.\begin{array}{l} x_M\ (t) = 0 \text{ if } t < 0 \\ \text{and} \quad L\ (t, p)\ x_M = f\ (t) \text{ if } t > 0, \\ x_M\ (t) = A\ (t, t)\ f\ (t) - \int_0^t \left[\frac{\partial}{\partial t_1} A\ (t_1, t)\right] f\ (t_1)\ dt_1. \end{array}\right\} \tag{16}$$

For the equation considered here, Eq. (10), $f\ (t) = x_T\ (t) + x_N\ (t) + f_2\ (t)$.

Thus to determine the entire trajectory, a function of two variables must generally be known. However, to determine the terminal value of x_M (and hence the miss distance), a function of one variable only need be known. That is,

$$x_M\left(\frac{r_0}{V_R}\right) = A\left(\frac{r_0}{V_R}, \frac{r_0}{V_R}\right) f\left(\frac{r_0}{V_R}\right) - \int_0^{\frac{r_0}{V_R}} \left[\frac{\partial}{\partial t_1} A\left(t_1, \frac{r_0}{V_R}\right)\right] f\ (t_1)\ dt_1. \tag{17}$$

Let

$$a\left(\frac{r_0}{V_R} - t_1\right) = A\left(t_1, \frac{r_0}{V_R}\right). \tag{18}$$

Then $a'\left(\frac{r_0}{V_R} - t_1\right) = -\frac{\partial}{\partial t_1} A\left(t_1, \frac{r_0}{V_R}\right)$, and

$$x_M\left(\frac{r_0}{V_R}\right) = a(0)\ f\left(\frac{r_0}{V_R}\right) + \int_0^{\frac{r_0}{V_R}} a'\left(\frac{r_0}{V_R} - t_1\right) f\ (t_1)\ dt_1. \tag{19}$$

The step-response function will herein be known as $a(\tau)$. For physical systems, $a\ (0) = 0$. In this case, Eq. (19) reduces to

$$\begin{aligned} x_M\left(\frac{r_0}{V_R}\right) &= \int_0^{\frac{r_0}{V_R}} a'\left(\frac{r_0}{V_R} - t_1\right) f\ (t_1)\ dt_1 \\ &= \int_0^{\frac{r_0}{V_R}} a'\ (\tau)\ f\left(\frac{r_0}{V_R} - \tau\right) d\tau. \end{aligned} \tag{20}$$

The miss distance is

$$x_T\left(\frac{r_0}{V_R}\right) - x_M\left(\frac{r_0}{V_R}\right) = x_T\left(\frac{r_0}{V_R}\right) - \int_0^{\frac{r_0}{V_R}} a'\ (\tau)\ f\left(\frac{r_0}{V_R} - \tau\right) d\tau. \tag{21}$$

Substitution of $f\ (t) = x_T\ (t) + x_N\ (t) + f_2\ (t)$ results in

$$x_T\left(\frac{r_0}{V_R}\right) - x_M\left(\frac{r_0}{V_R}\right) = M_T + M_N - \int_0^{\frac{r_0}{V_R}} a'\ (\tau)\ f_2\left(\frac{r_0}{V_R} - \tau\right) d\tau \tag{22}$$

where

$$M_T = x_T\left(\frac{r_0}{V_R}\right) - \int_0^{\frac{r_0}{V_R}} a'\ (\tau)\ x_T\left(\frac{r_0}{V_R} - \tau\right) d\tau \tag{23}$$

[1]The response of a system not initially at rest can be derived by a change of variable. See Sec. 3.1 for an example of this.

and

$$M_N = -\int_0^{\frac{r_0}{V_R}} a'(\tau)\, x_N\left(\frac{r_0}{V_R} - \tau\right) d\tau \tag{24}$$

denote, respectively, the miss caused by target maneuver and the miss caused by noise. The third term of Eq. (22), $\int_0^{\frac{r_0}{V_R}} a'(\tau)\, f_2\left(\frac{r_0}{V_R} - \tau\right) d\tau$, is a function only of the control system and the initial conditions and is generally zero if the missile is initially on a straight-line collision course. This term is omitted until the launching errors and control-system drift are discussed.

A target maneuver of special interest is the constant-lateral-acceleration turn starting at time $t_1 = (r_0 - r_1)/V_R$.

$$\begin{aligned} x_T(t) &= 0 && \text{for } t < t_1 \\ &= \frac{1}{2} a_T (t - t_1)^2 \cos \alpha_0 && \text{for } t > t_1. \end{aligned} \tag{25}$$

For this maneuver, M_T can be expressed somewhat more simply.

$$M_T = \frac{1}{2} a_T \left(\frac{r_1}{V_R}\right)^2 \cos \alpha_0 - \int_0^{\frac{r_1}{V_R}} a'(\tau)\, \frac{1}{2} a_T \left(\frac{r_1}{V_R} - \tau\right)^2 \cos \alpha_0 \, d\tau; \tag{26}$$

$$\begin{aligned} \frac{M_T}{a_T \cos \alpha_0} &= \frac{1}{2}\left(\frac{r_1}{V_R}\right)^2 - \int_0^{\frac{r_1}{V_R}} a'(\tau)\, \frac{1}{2}\left(\frac{r_1}{V_R} - \tau\right)^2 d\tau \\ &= \frac{1}{2}\left(\frac{r_1}{V_R}\right)^2 - \int_0^{\frac{r_1}{V_R}} \int_0^{\tau_2} a(\tau_1)\, d\tau_1\, d\tau_2. \end{aligned} \tag{27}$$

The noise term must be considered statistically. It is sufficient for the present purpose to consider the case in which x_N is a Gaussian random process. The distribution of M_N is then Gaussian, and all information concerning M_N is contained in its standard deviation σ_N. It can be shown that

$$\sigma_N^2 = \int_{-\infty}^{\infty} \left| \int_0^{\frac{r_0}{V_R}} a'(\tau)\, e^{-i\omega\tau} d\tau \right|^2 \Phi_N(\omega)\, d\omega \tag{28}$$

where $\Phi_N(\omega)$ is the spectral density of $x_N(t)$. If $x_N(t)$ is such that $\Phi_N(\omega)$ can be considered constant (white noise), Eq. (28) reduces to

$$\sigma_N^2 = \Phi_N \int_{-\infty}^{\infty} \left| \int_0^{\frac{r_0}{V_R}} a'(\tau)\, e^{-i\omega\tau} d\tau \right|^2 d\omega. \tag{29}$$

This may also be expressed as

$$\sigma_N^2 = 2\pi\, \Phi_N \int_0^{\frac{r_0}{V_R}} [a'(\tau)]^2 d\tau. \tag{30}$$

2.3. Minimum RMS Miss for White Noise and a Fixed Target Maneuver.

Among the several criteria that may be used to determine the effectiveness of a control system, one of the more convenient mathematically is the rms value of miss. Target motion may be considered statistically, or $x_T(t)$ may be considered to be a fixed function. In this study, $x_T(t)$ is generally considered as a simple fixed function (in particular, a constant-lateral-acceleration turn is employed). Until reliable experimental data are obtained concerning the form of the noise spectral density $\Phi_N(\omega)$, attention may profitably be concentrated on the case in which the noise may be considered white. To determine the efficiency of a control system for a particular target maneuver and noise spectrum, it is useful to know the minimum rms miss when no restriction is placed upon the form of the control system. In this section, the minimum rms miss is calculated for the case in which the noise may be considered white and the target motion is a fixed function. In the following section, the minimum rms miss is calculated for white noise and a statistical target maneuver.

The miss caused by a target maneuver $x_T(t)$ is given by

$$M_T = x_T\left(\frac{r_0}{V_R}\right) - \int_0^{\frac{r_0}{V_R}} a'(\tau)\, x_T\left(\frac{r_0}{V_R} - \tau\right) d\tau. \tag{23}$$

The deviation σ_N of the miss caused by white noise is given by

$$\sigma_N^2 = 2\pi\,\Phi_N \int_0^{\frac{r_0}{V_R}} [a'(\tau)]^2\, d\tau. \tag{30}$$

The rms value M of the total miss is given by

$$M^2 = M_T^2 + \sigma_N^2. \tag{31}$$

The step-response function $a'(\tau)$ minimizes M if, and only if, for each function $m(\tau)$, $M^2(a' + m) \geqq M^2(a')$. Now

$$M^2(a' + m) = \left\{ x_T\left(\frac{r_0}{V_R}\right) - \int_0^{\frac{r_0}{V_R}} [a'(\tau) + m(\tau)]\, x_T\left(\frac{r_0}{V_R} - \tau\right) d\tau \right\}^2 + 2\pi\,\Phi_N \int_0^{\frac{r_0}{V_R}} [a'(\tau) + m(\tau)]^2\, d\tau. \tag{32}$$

But this is

$$\begin{aligned} M^2(a' + m) = M^2(a') &+ \left[\int_0^{\frac{r_0}{V_R}} m(\tau)\, x_T\left(\frac{r_0}{V_R} - \tau\right) d\tau\right]^2 \\ &+ 2\left[\int_0^{\frac{r_0}{V_R}} a'(\rho)\, x_T\left(\frac{r_0}{V_R} - \rho\right) d\rho - x_T\left(\frac{r_0}{V_R}\right)\right]\left[\int_0^{\frac{r_0}{V_R}} m(\tau)\, x_T\left(\frac{r_0}{V_R} - \tau\right) d\tau\right] \\ &+ 2\pi\,\Phi_N \int_0^{\frac{r_0}{V_R}} 2\, a'(\tau)\, m(\tau)\, d\tau \\ &+ 2\pi\,\Phi_N \int_0^{\frac{r_0}{V_R}} m^2(\tau)\, d\tau. \end{aligned} \tag{33}$$

For any $m(\tau)$ not identically zero $M^2(a' + m) > M^2(a')$ if

$$2\left[\int_0^{\frac{r_0}{V_R}} a'(\rho)\, x_T\left(\frac{r_0}{V_R} - \rho\right) d\rho - x_T\left(\frac{r_0}{V_R}\right)\right]\left[\int_0^{\frac{r_0}{V_R}} m(\tau)\, x_T\left(\frac{r_0}{V_R} - \tau\right) d\tau\right] + 2\pi\,\Phi_N \int_0^{\frac{r_0}{V_R}} 2a'(\tau)\, m(\tau)\, d\tau = 0. \tag{34}$$

This may be rewritten as

$$\int_0^{\frac{r_0}{V_R}} m(\tau) \left\{ x_T\left(\frac{r_0}{V_R} - \tau\right)\left[\int_0^{\frac{r_0}{V_R}} a'(\rho)\, x_T\left(\frac{r_0}{V_R} - \rho\right) d\rho - x_T\left(\frac{r_0}{V_R}\right)\right] + 2\pi\,\Phi_N\, a'(\tau) \right\} d\tau = 0. \tag{35}$$

This is true for each function m if for each τ such that $0 \leqq \tau \leqq r_0/V_R$:

$$x_T\left(\frac{r_0}{V_R} - \tau\right)\left[\int_0^{\frac{r_0}{V_R}} a'(\rho)\, x_T\left(\frac{r_0}{V_R} - \rho\right) d\rho - x_T\left(\frac{r_0}{V_R}\right)\right] + 2\pi\,\Phi_N\, a'(\tau) = 0, \tag{36}$$

that is,

$$a'(\tau) = K\, x_T\left(\frac{r_0}{V_R} - \tau\right). \tag{37}$$

The constant K is determined by substituting into Eq. (36)

$$x_T\left(\frac{r_0}{V_R}-\tau\right)\left[K\int_0^{\frac{r_0}{V_R}} x_T^2\left(\frac{r_0}{V_R}-\rho\right)d\rho - x_T\left(\frac{r_0}{V_R}\right)\right] + 2\pi\,\Phi_N\,K\,x_T\left(\frac{r_0}{V_R}-\tau\right) = 0. \tag{38}$$

Solving for K,

$$K = \frac{x_T\left(\frac{r_0}{V_R}\right)}{2\pi\,\Phi_N + \int_0^{\frac{r_0}{V_R}} x_T^2\left(\frac{r_0}{V_R}-\rho\right)d\rho} = \frac{x_T\left(\frac{r_0}{V_R}\right)}{2\pi\,\Phi_N + \int_0^{\frac{r_0}{V_R}} x_T^2\,(\rho)\,d\rho}. \tag{39}$$

The minimum value of M^2 is then obtained by substituting Eq. (37), where K has the value given by Eq. (39), into Eq. (31).

$$M_{MIN}^2 = \left[x_T\left(\frac{r_0}{V_R}\right) - K\int_0^{\frac{r_0}{V_R}} x_T^2\,(\rho)\,d\rho\right] + 2\pi\,\Phi_N\,K^2\int_0^{\frac{r_0}{V_R}} x_T^2\,(\rho)\,d\rho. \tag{40}$$

From Eq. (39)

$$x_T\left(\frac{r_0}{V_R}\right) - K\int_0^{\frac{r_0}{V_R}} x_T^2\,(\rho)\,d\rho = 2\pi\,\Phi_N\,K. \tag{41}$$

Then

$$M_{MIN}^2 = 2\pi\,\Phi_N\,K^2\left[2\pi\,\Phi_N + \int_0^{\frac{r_0}{V_R}} x_T^2\,(\tau)\,d\tau\right] = \frac{x_T^2\left(\frac{r_0}{V_R}\right)}{1 + \frac{1}{2\pi\,\Phi_N}\int_0^{\frac{r_0}{V_R}} x_T^2\,(\tau)\,d\tau}. \tag{42}$$

For the constant target turn starting at time $t_1 = (r_0 - r_1)/V_R$,

$$\left.\begin{aligned} x_T\,(t) &= 0 && \text{if } t < t_1 \\ &= \frac{1}{2}\,a_T\,(t-t_1)^2\cos\alpha_0 && \text{if } t > t_1, \end{aligned}\right\} \tag{43}$$

$$M_{MIN}^2 = \frac{\frac{1}{4}\,a_T^2\left(\frac{r_1}{V_R}\right)^4\cos^2\alpha_0}{1 + \left(\frac{1}{2\pi\,\Phi_N}\right)\frac{1}{20}\,a_T^2\left(\frac{r_1}{V_R}\right)^5\cos^2\alpha_0}. \tag{44}$$

The value M_{MIN} reaches a maximum (with respect to r_1) where $\partial M_{MIN}^2/\partial r_1 = 0$, that is, where

$$\left[1 + \left(\frac{1}{2\pi\,\Phi_N}\right)\frac{1}{20}\,a_T^2\left(\frac{r_1}{V_R}\right)^5\cos^2\alpha_0\right] a_T^2\left(\frac{r_1}{V_R}\right)^3\cos^2\alpha_0 = \frac{1}{4}\,a_T^2\left(\frac{r_1}{V_R}\right)^4\cos^2\alpha_0\left[\frac{1}{2\pi\,\Phi_N}\,\frac{1}{4}\,a_T^2\left(\frac{r_1}{V_R}\right)^4\cos^2\alpha_0\right]. \tag{45}$$

This is true for

$$\frac{r_1}{V_R} = \left[\frac{80}{a_T^2}\,\frac{(2\pi\,\Phi_N)}{\cos^2\alpha_0}\right]^{\frac{1}{5}}. \tag{46}$$

Substitution of this value of r_1/V_R into Eq. (42) gives the maximum value of M_{MIN} as

$$(M_{MIN})_M = 1.2904\ (2\pi\ \Phi_N)^{\frac{1}{2}}\ (a_T\ \cos^2 \alpha_0)^{\frac{1}{2}}. \tag{47}$$

The form of the minimizing $a(\tau)$ is probably of no importance, but the maximum value of M_{MIN} is a lower bound for the maximum value of rms miss for all control systems. The value of $(M_{MIN})_M$ given by Eq. (47) is used in Sec. 3.6 to obtain a figure of merit for a particular control system.

2.4. Minimum RMS Miss for White Noise and a Statistical Target Maneuver.

In the present section, the minimum rms miss is calculated for the case where white noise and a statistical target maneuver cause the miss. It is assumed that x_T and x_N are independent, stationary random variables and that the initial range can be considered infinite.

If the spectral density of the target motion $x_T(t)$ is denoted by $\Phi_T(\omega)$, the rms miss σ_T caused by the target maneuver is given by

$$\sigma_T^2 = \int_{-\infty}^{\infty} |1 - F\ (i\omega)|^2\ \Phi_T\ (\omega)\ d\omega \tag{48}$$

where

$$F\ (i\omega) = \int_0^{\infty} e^{-i\omega\tau}\ a'\ (\tau)\ d\tau. \tag{49}$$

The rms value of the total miss $M_T + M_N$ is given by

$$\sigma^2 = \int_{-\infty}^{\infty} |1 - F\ (i\omega)|^2\ \Phi_T\ (\omega)\ d\omega + \int_{-\infty}^{\infty} |F\ (i\omega)|^2\ \Phi_N\ (\omega)\ d\omega. \tag{50}$$

The problem of finding the $a(\tau)$ that minimizes σ has been solved by Wiener.[1] It is important to note that the present discussion considers the problem from a different viewpoint. Since the value of x_M at only one time (r_0/V_R) is of interest, the rms value of $x_T - x_M$ at this time is taken with respect to the ensemble of target motion and noise functions. The optimum operator is determined from the class of all linear operators, including time-variant operators. The linear operator is not determined uniquely by the optimization process, because more than one A (t_1, t) can yield the same $a\ (t_f - t_1) = A\ (t_1, t_f)$. As the interest here is in the minimum value of σ and not in possible forms of the optimum operator, the problem of synthesizing the operators that yield the desired $a(\tau)$ is not discussed.

The $a(\tau)$ that minimizes σ is determined by

$$\left.\begin{aligned} F\ (i\omega) &= \frac{1}{2\pi\ \Gamma\ (\omega)} \int_0^{\infty} e^{-i\omega t} \int_{-\infty}^{\infty} \frac{\Phi_T\ (\gamma)}{\overline{\Gamma}\ (\gamma)}\ e^{i\gamma t}\ d\gamma\ dt \\ \text{where} \quad & \\ |\Gamma\ (\omega)|^2 &= \Phi_T\ (\omega) + \Phi_N\ (\omega) \end{aligned}\right\} \tag{51}$$

and Γ has no poles or zeros in the upper half plane.

If the target maneuver consists of arcs of constant lateral acceleration $\pm a_T$ with random switching at an average rate of $k/2$, the spectral density of x_T is given by

$$\Phi_T\ (\omega) = \frac{k\ (a_T\ \cos\ \alpha_0)^2}{\pi\ \omega^4\ (\omega^2 + k^2)}. \tag{52}$$

It is assumed that $\Phi_N(\omega)$ can be considered constant. Then

$$\sigma^2 = \frac{k\ (a_T\ \cos\ \alpha_0)^2}{\pi} \int_{-\infty}^{\infty} |1 - F\ (i\omega)|^2\ \frac{1}{\omega^4\ (\omega^2 + k^2)}\ d\omega + \Phi_N \int_{-\infty}^{\infty} |F\ (i\omega)|^2\ d\omega. \tag{53}$$

[1] Wiener, Norbert, *Extrapolation, Interpolation, and Smoothing of Stationary Time Series with Engineering Applications*. New York: John Wiley & Sons, Inc., 1949. Chap. III.

Statement of the problem in nondimensional form reduces the number of independent parameters to one. With $\omega = hv$,

$$\sigma^2 = \frac{k\,(a_T \cos \alpha_0)^2}{\pi\, h^5} \int_{-\infty}^{\infty} |1 - F\,(ihv)|^2 \frac{1}{v^4\,(v^2 + m^2)}\, dv + \Phi_N\, h \int_{-\infty}^{\infty} |F\,(ihv)|^2\, dv \tag{54}$$

where

$$h = \left[\frac{k\,(a_T \cos \alpha_0)^2}{\pi\, \Phi_N}\right]^{\frac{1}{6}} \tag{55}$$

and

$$m = \frac{k}{h}. \tag{56}$$

Since $k\,(a_T \cos \alpha_0)^2/\pi\, h^5 = \Phi_N\, h$,

$$E^2 = \int_{-\infty}^{\infty} |1 - F\,(ihv)|^2 \frac{1}{v^4\,(v^2 + m^2)}\, dv + \int_{-\infty}^{\infty} |F\,(ihv)|^2\, dv \tag{57}$$

where

$$E^2 = \frac{\sigma^2}{\Phi_N\, h}. \tag{58}$$

If

$$|\Psi\,(v)|^2 = \frac{1}{v^4\,(v^2 + m^2)} + 1 \tag{59}$$

and Ψ has no zeros or poles in the lower half plane, the optimum F is given by

$$F\,(ihv) = \frac{1}{2\pi\, \Psi\,(v)} \int_0^{\infty} e^{-iv\tau} \int_{-\infty}^{\infty} \frac{e^{i\tau y}}{y^4\,(y^2 + m^2)\, \overline{\Psi}\,(y)}\, dy\, d\tau. \tag{60}$$

With a change of variable $u = i\,y$, Eq. (60) becomes

$$F\,(ihv) = \frac{1}{2\pi\, i\, \Psi\,(v)} \int_0^{\infty} e^{-iv\tau} \int_{-i\infty}^{i\infty} \frac{e^{\tau u}}{u^4\,(-u^2 + m^2)\, \overline{\Psi}\,(-iu)}\, du\, d\tau. \tag{61}$$

Actually, because of the $1/\omega^4$ factor in the $\Phi_T(\omega)$ considered here, Ψ must have a pole of second order at the origin. Certain formal difficulties can be removed if the present case is considered as the limiting case of $\Phi_T\,(\omega) = k\,(a_T \cos \alpha_0)^2/\pi\,(\omega^2 + \epsilon^2)^2\,(\omega^2 + k^2)$ as $\epsilon \to 0$. Then all poles or zeros are definitely in one or the other half plane. Expressed in terms of $u = iv$,

$$\left|\Psi\left(\frac{u}{i}\right)\right|^2 = \frac{u^6 - u^4\, m^2 - 1}{u^4\,(u^2 - m^2)}. \tag{62}$$

With u_1, u_2, u_3 the left half-plane zeros of $u^6 - u^4\, m^2 - 1$, Ψ may be taken as

$$\Psi\left(\frac{u}{i}\right) = \frac{p\,(u)}{u^2\,(u + m)} \tag{63}$$

where $p\,(u) = (u - u_1)\,(u - u_2)\,(u - u_3)$ and the pole caused by $1/u^2$ shall be considered in the left half plane, in accordance with the remark above. Then,

$$\frac{1}{u^4\,(-u^2 + m^2)\, \overline{\Psi}\left(\frac{u}{i}\right)} = \frac{-1}{u^2\,(u + m)\, p\,(-u)} = \frac{p\,(u) - u^2\,(u + m)}{u^2\,(u + m)} + \frac{q\,(u)}{p\,(-u)} \tag{64}$$

where the first term has right half-plane poles and the second term has left half-plane poles.

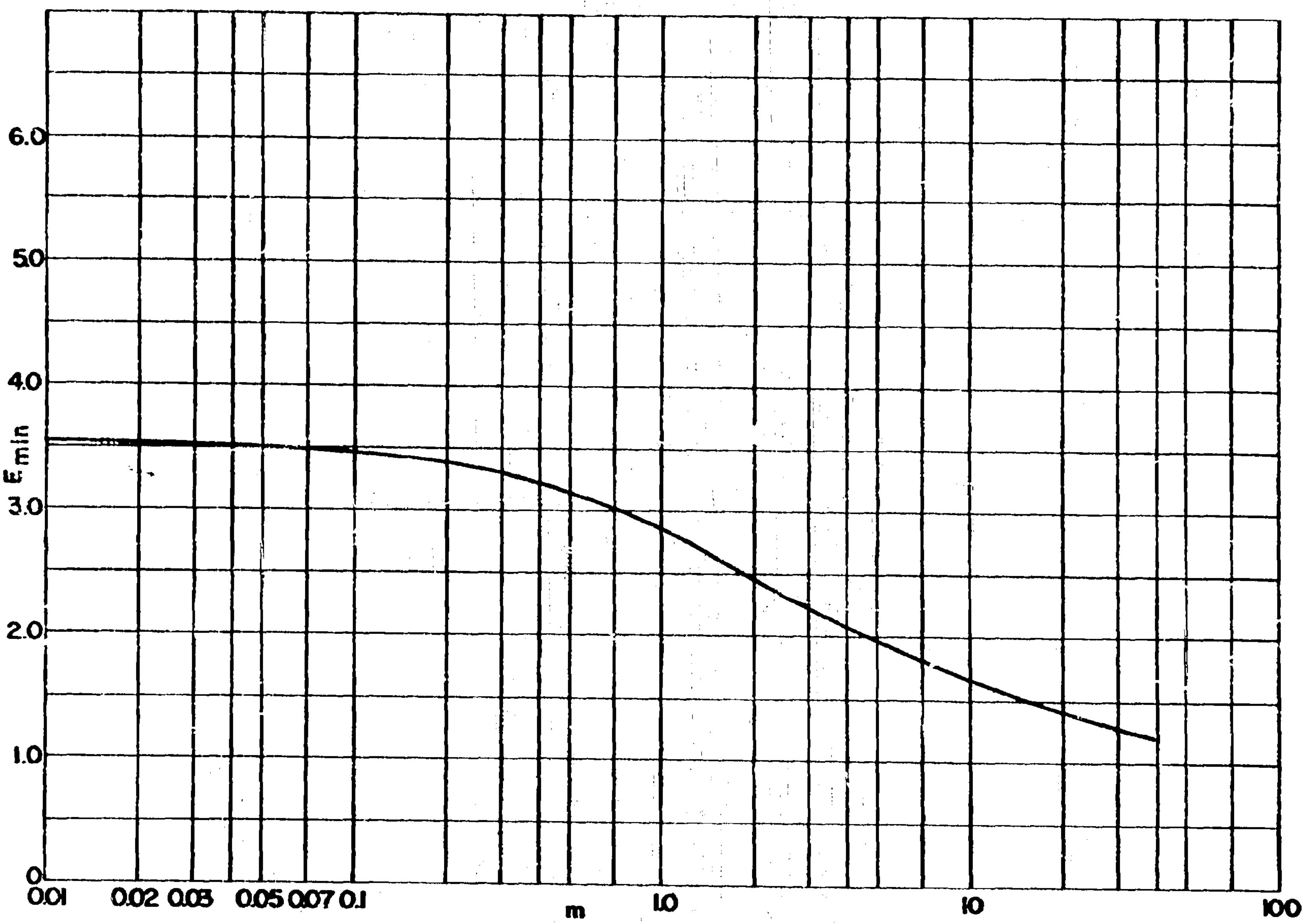

Fig. 2. Minimum rms miss (nondimensionalized) for white noise and a statistical target maneuver.

Because of the positions of the poles of the terms in Eq. (64), the two integrations of Eq. (61) yield $2\pi i$ times the first term of Eq. (64). That is,

$$F(hu) = \frac{u^2 (u+m)}{2\pi i\, p(u)} \cdot 2\pi i \frac{p(u) - u^2 (u+m)}{u^2 (u+m)}$$
$$= 1 - \frac{u^2 (u+m)}{p(u)}. \qquad (65)$$

This expression for the optimum F is substituted into Eq. (61) to give the minimum rms miss E.

Expressed in terms of $b_1 = u_1 u_2 + u_1 u_3 + u_2 u_3$ and $b_2 = u_1 + u_2 + u_3$, these integrals reduce to

$$E^2 = \pi \frac{b_2}{b_1 b_2 - 1} + \pi \frac{b_1 (b_2 - m)^2 + b_1^2 - b_2 + 2m}{b_1 b_2 - 1}. \qquad (66)$$

The minimum E is plotted as a function of m in Fig. 2.

3. PROPORTIONAL NAVIGATION WITH SIMPLE TIME LAG

3.1. Solution of the Linearized Equation.

The proportional-navigation-with-simple-time-lag control system is characterized by the equation

$$A\ddot{\theta} + \dot{\theta} = (b+1)\,\dot{\alpha}. \qquad (67)$$

In the notation of Eq. (18), $D(p) = (Ap + 1) / (b + 1)$, and the resulting linearized equation is

$$\left[\frac{r_0 - V_R t}{V_M \cos\beta_0} \frac{Ap+1}{b+1} p + 1\right] x_M = x_T + x_N + C (r_0 - V_R t)$$

where

$$C = \frac{A\dot{\theta}_0}{b+1} - \frac{r_0 \dot{\alpha}_0}{(b+1) V_M \cos\beta_0}$$
$$= \frac{1}{(b+1) V_M \cos\beta_0}\left[A\ddot{x}_M(0) + \dot{x}_M(0)\right]. \qquad (68)$$

If $n = [(b+1) V_M \cos\beta_0] / V_R$, Eq. (68) reduces to

$$\frac{A}{n}\left(\frac{r_0}{V_R} - t\right)\ddot{x}_M + \frac{1}{n}\left(\frac{r_0}{V_R} - t\right)\dot{x}_M + x_M = x_T + x_N + C (r_0 - V_R t). \qquad (69)$$

For integral values of n, this equation may be solved analytically.[1] It is sufficient to consider the case in which $x_M(t) = 0$ for $t < t_1$ and $x_T(t) + x_N(t) + C(r_0 - V_R t) = 1$ for $t > t_1$, since the general solution can be expressed in terms of the response from rest to a unit step applied at time t_1. Then, after multiplication by n, differentiation of each side of Eq. (69) results in

$$A\left(\frac{r_0}{V_R} - t\right)\dddot{x}_M + \left(\frac{r_0}{V_R} - A - t\right)\ddot{x}_M + (n-1)\,\dot{x}_M = 0. \qquad (70)$$

The coefficient of the lowest derivative of x_M is decreased by 1. Repeated differentiation results in

$$A\left(\frac{r_0}{V_R} - t\right) x_M^{(i+2)} + \left(\frac{r_0}{V_R} - iA - t\right) x_M^{(i+1)} + (n-i)\, x_M^{(i)} = 0. \qquad (71)$$

[1]Only positive values of n are of interest here. For negative integral values of n, Eq. (69) may be solved by repeated integration in a manner similar to that followed here for positive n. For nonintegral values of n, a solution can be found in the form of an infinite series. D. L. Bobroff of D.A.C.L. has shown the Laplace transform $F(s) = \int_0^\infty e^{-s\tau} a'(\tau)\, d\tau$ is given by $1 - (As)^n/(As+1)^n$ for all positive values of n, although the series defining the step response $a(\tau)$ does not terminate.

After n differentiations, the equation is of the form

$$x_M^{(2+n)} + \left(\frac{1}{A} - \frac{n}{\frac{r_0}{V_R} - t}\right) x_M^{(1+n)} = 0. \tag{72}$$

This may be regarded as a first-order equation in $x_M^{(1+n)}$. If each side of Eq. (72) is multiplied by the integrating factor[1] $\left(\frac{r_0}{V_R} - t\right)^n e^{\frac{t}{A}}$, Eq. (72) becomes

$$\frac{d}{dt}\left[x_M^{(1+n)}\left(\frac{r_0}{V_R} - t\right)^n e^{\frac{t}{A}}\right] = 0. \tag{73}$$

Integration results in[2]

$$x_M^{(1+n)}\ (t)\ \left(\frac{r_0}{V_R} - t\right)^n e^{\frac{t}{A}} = C_1 \tag{74}$$

where the constant C_1 is evaluated by setting $t = t_1$ as

$$x_M^{(1+n)}\ (t_1)\ \left(\frac{r_0}{V_n} - t_1\right)^n e^{\frac{t_1}{A}} = C_1. \tag{75}$$

Solution of Eq. (74) for $x_M^{(1+n)}$ gives

$$x_M^{(1+n)}\ (t) = C_1 \left(\frac{r_0}{V_n} - t\right)^{-n} e^{-\frac{t}{A}}. \tag{76}$$

The quantity x_M is then obtained by repeated integration as

$$x_M\ (t) = C_1 \int_{t_1}^{t}\int_{t_1}^{\tau_{n+1}} \ldots \int_{t_1}^{\tau_2} \left(\frac{r_0}{V_R} - \tau_1\right)^{-n} e^{-\frac{\tau_1}{A}}\, d\tau_1 \ldots d\tau_{n+1} + \sum_{j=2}^{n} \frac{x_M^{(j)}\ (t_1)}{j!}\, t^j. \tag{77}$$

Substitution of the value of C_1 obtained from Eq. (75) and expression of the repeated integral in Eq. (77) as a single integral[3] result in

$$x_M\ (t) = x_M^{(1+n)}\ (t_1)\left(\frac{r_0}{V_n} - t_1\right)^n e^{\frac{t_1}{A}} \frac{1}{n!}\int_{t_1}^{t} (t-\tau)^n \left(\frac{r_0}{V_n} - \tau\right)^{-n} e^{-\frac{\tau}{A}}\, d\tau + \sum_{j=2}^{n} \frac{x_M^{(j)}\ (t_1)}{j!}\, t^j. \tag{78}$$

Thus, if the response from rest to a unit step applied at time t_1 is denoted by $A\ (t_1, t)$, $A\ (t_1, t)$ is the right side of Eq. (78).

If $\dot{x}_M\ (0) = 0$, the general solution of Eq. (69) is given by the superposition integral. See Eq. (14). If $\dot{x}_M\ (0) \neq 0$, a preliminary change of variable is necessary. Let

$$s\ (t) = x_M\ (t) - \dot{x}_M\ (0)\ t. \tag{79}$$

---

[1] If each side of the first-order differential equation

$$\frac{dy}{dx} + P\ (x)\ y = Q\ (x)$$

is multiplied by the integrating factor $e^{\int P(x)dx}$, the left side becomes the derivative of $y\ e^{\int P(x)dx}$; and the equation may then be solved by integration. That is,

$$e^{\int P(x)dx}\frac{dy}{dx} + e^{\int P(x)dx}\ P\ (x)\ y = Q\ (x)\ e^{\int P(x)dx}$$

$$y e^{\int P(x)dx} = \int Q\ (x)\ e^{\int P(x)dx}\, dx.$$

[2] In this and the following equations, $t \geq t_1$, since $x_M\ (t) = 0$ for $t < t_1$.

[3] It may be shown by integration by parts that

$$\int_{t_1}^{t}\int_{t_1}^{\tau_{n+1}} \ldots \int_{t_1}^{\tau_2} f\ (\tau_1)\ d\tau_1 \ldots d\tau_{n+1} = \frac{1}{n!}\int_{t_1}^{t} (t-\tau)^n\ f\ (\tau)\ d\tau.$$

If this substitution is made in Eq. (69), the following differential equation for z (t) results:

$$A\left(\frac{r_0}{V_R}-t\right)\ddot{z}+\left(\frac{r_0}{V_R}-t\right)\dot{z}+n\,z=n\,x_T+n\,x_N+n\,C\,(r_0-V_Rt)-\left(\frac{r_0}{V_R}-t\right)\dot{x}_M\,(0)-n\,\dot{x}_M\,(0)\,t. \tag{80}$$

Substitution of the value of C from Eq. (68) results in

$$A\left(\frac{r_0}{V_R}-t\right)\ddot{z}+\left(\frac{r_0}{V_R}-t\right)\dot{z}+n\,z=n\,x_T+n\,x_N+\frac{r_0A}{V_R}\ddot{x}_M\,(0)-[A\ddot{x}_M\,(0)+n\dot{x}_M\,(0)]\,t. \tag{81}$$

Since $\dot{z}$ (0) = 0, z can be expressed by the superposition integral as

$$z\,(t)=-\int_0^t\left[\frac{\partial}{\partial t_1}A\,(t_1,t)\right]f\,(t_1)\,dt_1$$

where

$$f\,(t_1)=x_T\,(t_1)+x_N\,(t_1)+\frac{r_0\,A}{(b+1)\,V_M\cos\beta_0}\ddot{x}_M\,(0)-\left[\frac{AV_R\,\ddot{x}_M\,(0)}{(b+1)\,V_M\cos\beta_0}+\dot{x}_M\,(0)\right]t_1. \tag{82}$$

Substitution of Eq. (79) results in

$$x_M\,(t)=\dot{x}_M\,(0)\,t-\int_0^t\left[\frac{\partial}{\partial t_1}A\,(t_1,t)\right]f\,(t_1)\,dt_1. \tag{83}$$

Application of Eq. (83) to determine the complete missile trajectory requires explicit expressions for A (t_1, t), which can be calculated from Eq. (78). The resulting expressions have been omitted here because complexity restricts their practical use.

As is shown in Sec. 2.2, determination of the miss distance requires knowledge of a function of one variable only, which is herein known as the step-response function a (τ). As in the derivation of Eq. (16), let $a\,(r_0/V_R-t_1)=A\,(t_1,r_0/V_R)$. Then $a'\,(r_0/V_R-t_1)=-\frac{\partial}{\partial t_1}A\,(t_1,r_0/V_R)$.

Substitution of this expression into Eq. (82) results in

$$\begin{aligned}x_M\left(\frac{r_0}{V_R}\right)&=\dot{x}_M\,(0)\,\frac{r_0}{V_R}+\int_0^{\frac{r_0}{V_R}}a'\left(\frac{r_0}{V_R}-t_1\right)f\,(t_1)\,dt_1\\&=\dot{x}_M\,(0)\,\frac{r_0}{V_R}+\int_0^{\frac{r_0}{V_R}}a'\,(\tau)\,f\left(\frac{r_0}{V_R}-\tau\right)d\tau.\end{aligned} \tag{84}$$

The miss distance M is given by $M=x_T\,(r_0/V_R)-x_M\,(r_0/V_R)$, and use of Eqs. (81) and (84) results in

$$\begin{aligned}M=x_T\left(\frac{r_0}{V_R}\right)&-\int_0^{\frac{r_0}{V_R}}a'\,(\tau)\,x_T\left(\frac{r_0}{V_R}-\tau\right)d\tau-\int_0^{\frac{r_0}{V_R}}a'\,(\tau)\,x_N\left(\frac{r_0}{V_R}-\tau\right)d\tau\\&-\frac{r_0\,A}{(b+1)\,V_M\cos\beta_0}\ddot{x}_M\,(0)\int_0^{\frac{r_0}{V_R}}a'\,(\tau)\,d\tau\\&+\left[\frac{AV_R}{(b+1)\,V_M\cos\beta_0}\ddot{x}_M\,(0)+\dot{x}_M\,(0)\right]\int_0^{\frac{r_0}{V_R}}a'\,(\tau)\left(\frac{r_0}{V_R}-\tau\right)d\tau-\dot{x}_M\,(0)\,\frac{r_0}{V_R}.\end{aligned} \tag{85}$$

The total miss M is thus the sum of four terms: M_T, M_N, M_E, and M_s. The miss M_T caused by target motion is given by

$$M_T=x_T\left(\frac{r_0}{V_R}\right)-\int_0^{\frac{r_0}{V_R}}a'\,(\tau)\,x_T\left(\frac{r_0}{V_R}-\tau\right)d\tau. \tag{86}$$

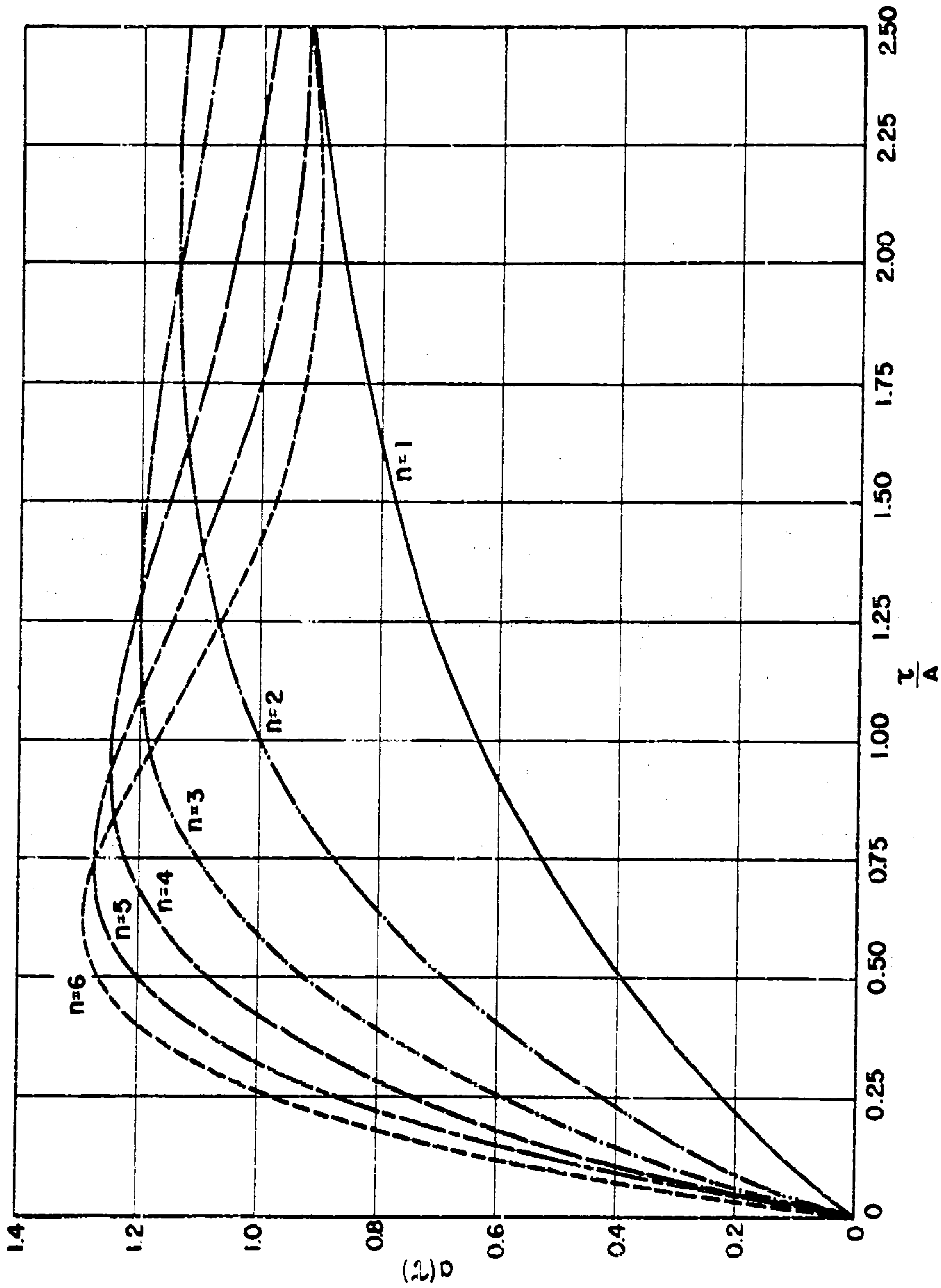

Fig. 3. Step-response function for proportional-navigation-with-simple-time-lag control system.

The miss M_N caused by noise is given by

$$M_N = -\int_0^{\frac{r_0}{V_R}} a'(\tau)\, x_N\left(\frac{r_0}{V_R} - \tau\right) d\tau. \tag{87}$$

The miss M_E caused by an initial error in missile heading is given by

$$\begin{aligned} M_E &= \dot{x}_M(0) \int_0^{\frac{r_0}{V_R}} a'(\tau)\left(\frac{r_0}{V_R} - \tau\right) d\tau \\ &= \dot{x}_M(0)\left[\int_0^{\frac{r_0}{V_R}} a(\tau)\, d\tau - \frac{r_0}{V_R}\right]. \end{aligned} \tag{88}$$

The miss M_a caused by an initial lateral acceleration of the missile is given by

$$M_a = \dot{\theta}_0\left[\frac{AV_R}{b+1}\int_0^{\frac{r_0}{V_R}} a(\tau)\, d\tau - \frac{r_0 A}{b+1}\, a\left(\frac{r_0}{V_R}\right)\right]. \tag{89}$$

Application of these four formulas requires explicit expressions for a (τ), which in the general case can be calculated from A (t_1, t) by setting $t = r_0/V_R$ and $t_1 = r_0/V_R - \tau$. The step response a (τ) can be more conveniently calculated directly from Eq. (78) by making the above substitutions. Solutions for $n = 1, \ldots, 6$ are given below.

$$\left.\begin{aligned} a(\tau) &= 1 - e^{-l} && \text{for } n = 1 \\ &= 1 + e^{-l}(l - 1) && \text{for } n = 2 \\ &= 1 - \frac{e^{-l}}{2!}(l^2 - 4l + 2) && \text{for } n = 3 \\ &= 1 + \frac{e^{-l}}{3!}(l^3 - 9l^2 + 18l - 6) && \text{for } n = 4 \\ &= 1 - \frac{e^{-l}}{4!}(l^4 - 16l^3 + 72l^2 - 96l + 24) && \text{for } n = 5 \\ &= 1 + \frac{e^{-l}}{5!}(l^5 - 25l^4 + 200l^3 - 600l^2 + 600l - 120) && \text{for } n = 6 \end{aligned}\right\} \tag{90}$$

where $l = \tau/A$. These step-response functions are plotted in Fig. 8.

3.2. Miss Caused by Constant Target Turn.

In Sec. 2.2 it is shown that the miss caused by a constant-lateral-acceleration target turn starting at range r_1 is given by

$$\frac{M_T}{a_T \cos \alpha_0} = \frac{1}{2}\left(\frac{r_1}{V_R}\right)^2 - \int_0^{\frac{r_1}{V_R}}\int_0^{\tau_2} a(\tau_1)\, d\tau_1\, d\tau_2. \tag{27}$$

Letting $l = r_1/AV_R$, this is

$$\frac{M_T}{A^2\, a_T \cos \alpha_0} = \frac{1}{2}\, l^2 - \int_0^{l}\int_0^{\gamma_2} a(A\gamma_1)\, d\gamma_1\, d\gamma_2. \tag{91}$$

For the proportional-navigation-with-simple-time-lag system, a (τ) has been calculated for certain integral values of $[(b+1)\, V_M \cos \beta_0]/V_R$. See Eq. (90). Substitution of these response functions into Eq. (91) results in the following expressions for $M_T/A^2\, a_T \cos \alpha_0$:

$$\left.\begin{aligned} \frac{M_T}{A^2\, a_T \cos \alpha_0} &= e^{-l} + l - 1 && \text{for } n = 1\\ &= 1 - e^{-l}(l+1) && \text{for } n = 2\\ &= \frac{e^{-l}}{2!}\, l^2 && \text{for } n = 3\\ &= -\frac{e^{-l}}{3!}\,(l^3 - 3l^2) && \text{for } n = 4\\ &= \frac{e^{-l}}{4!}\,(l^4 - 8l^3 + 12l^2) && \text{for } n = 5\\ &= -\frac{e^{-l}}{5!}\,(l^5 - 15l^4 + 60l^3 - 60l^2) && \text{for } n = 6 \end{aligned}\right\} \tag{92}$$

where $l = r_1/AV_R$. These curves of the miss are plotted in Fig. 4.

As the maximum value of $M_T/A^2\, a_T \cos \alpha_0$ is of special importance, a formula will be developed to approximate it. In Fig. 5, $M_T/A^2\, a_T \cos \alpha_0$ is plotted as a function of n. The values for n = 3, 4, 5, 6 lie very close to a straight line (on log-log paper) with a slope of −2.5. Hence it is reasonable to approximate $M_T/A^2\, a_T \cos \alpha_0$ by $Cn^{-2.5}$ for values of n from 3 to 6. The approximation becomes worse as n decreases below 3, but this region is of less practical interest. With the use of Fig. 4 to evaluate the constant C, the formula becomes

$$\left.\begin{aligned} M_T &= \sqrt{K_T}\, A^2\, (b+1)^{-2.5}\\ \text{where}\quad \sqrt{K_T} &= 4.30\, a_T \cos \alpha_0 \left(\frac{V_R}{V_M \cos \beta_0}\right)^{\cdots} \end{aligned}\right\} \tag{93}$$

3.3. Deviation of Miss Caused by White Noise.

It is shown in Sec. 2.3 that the deviation σ_N of the miss caused by white noise is given by

$$\sigma_N^2 = 2\pi\, \Phi_N \int_0^{\frac{r_0}{V_R}} [a'(\tau)]^2\, d\tau. \tag{80}$$

If the initial range is large enough to be considered infinite,[1] $\sigma_N^2 = 2\pi\, \Phi_N \int_0^{\infty} [a'(\tau)]^2\, d\tau.$

In the present section a formula is developed to approximate σ_N for the proportional-navigation-with-simple-time-lag system. With the step-response functions developed in Sec. 3.1, $\int_0^{\infty} [a'(\tau)]^2\, d\tau$ can be calculated for integral values of n. For n = 1, . . . , 6 these values of n are given below.

$$\left.\begin{aligned} \int_0^{\infty} [a'(\tau)]^2\, d\tau &= 0.5\left(\frac{1}{A}\right) && \text{for } n = 1\\ &= 1.25\left(\frac{1}{A}\right) && \text{for } n = 2\\ &= 2.0625\left(\frac{1}{A}\right) && \text{for } n = 3\\ &= 2.90625\left(\frac{1}{A}\right) && \text{for } n = 4\\ &= 3.76953125\left(\frac{1}{A}\right) && \text{for } n = 5\\ &= 4.6496093375\left(\frac{1}{A}\right) && \text{for } n = 6 \end{aligned}\right\} \tag{94}$$

[1]That is, r_0/V_R is such that the difference between

$$\int_0^{\infty} [a'(\tau)]^2\, d\tau \text{ and } \int_0^{\frac{r_0}{V_R}} [a'(\tau)]^2\, d\tau \text{ is small.}$$

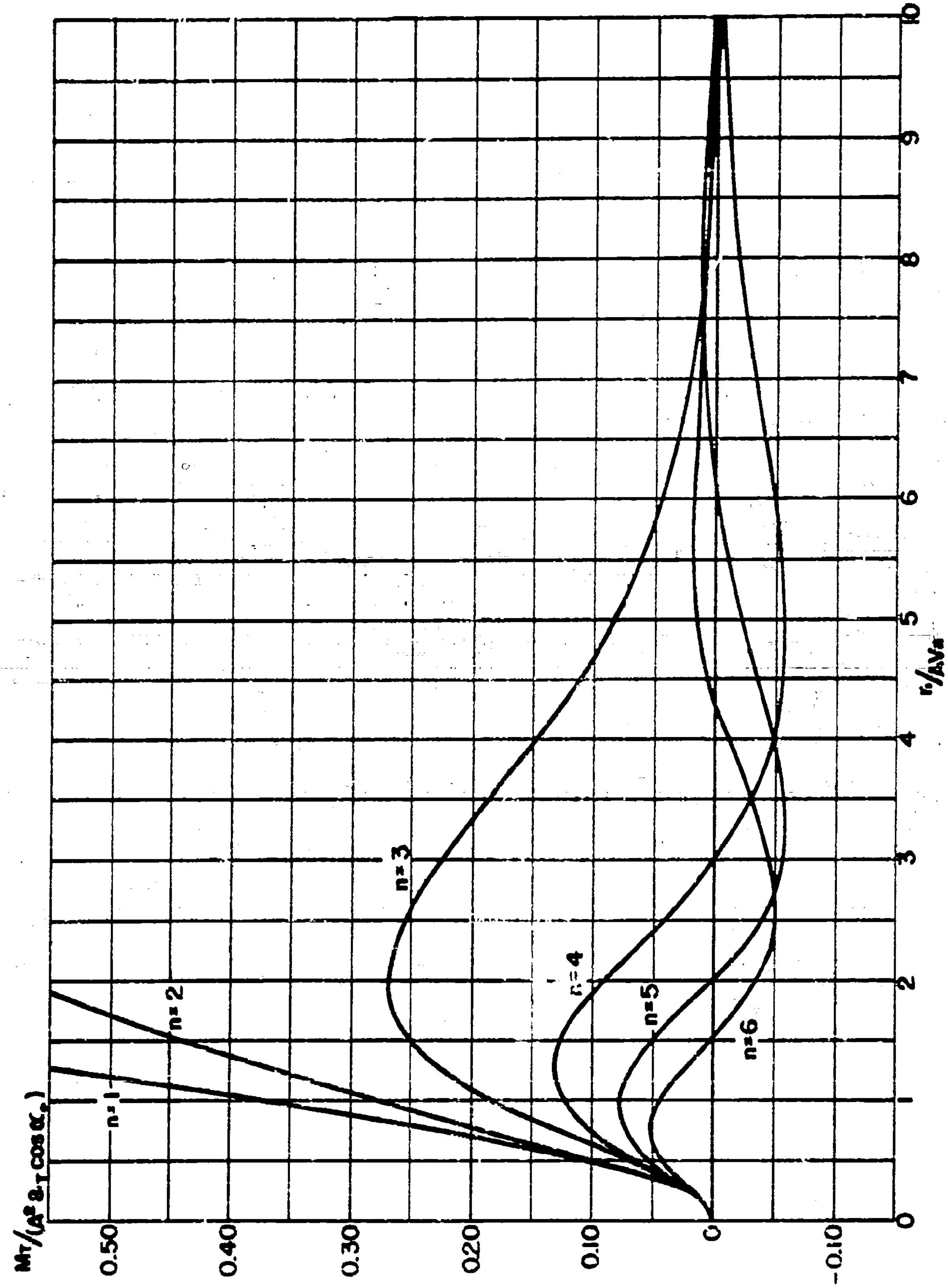

Fig. 4. Miss caused by constant-lateral-acceleration target turn.

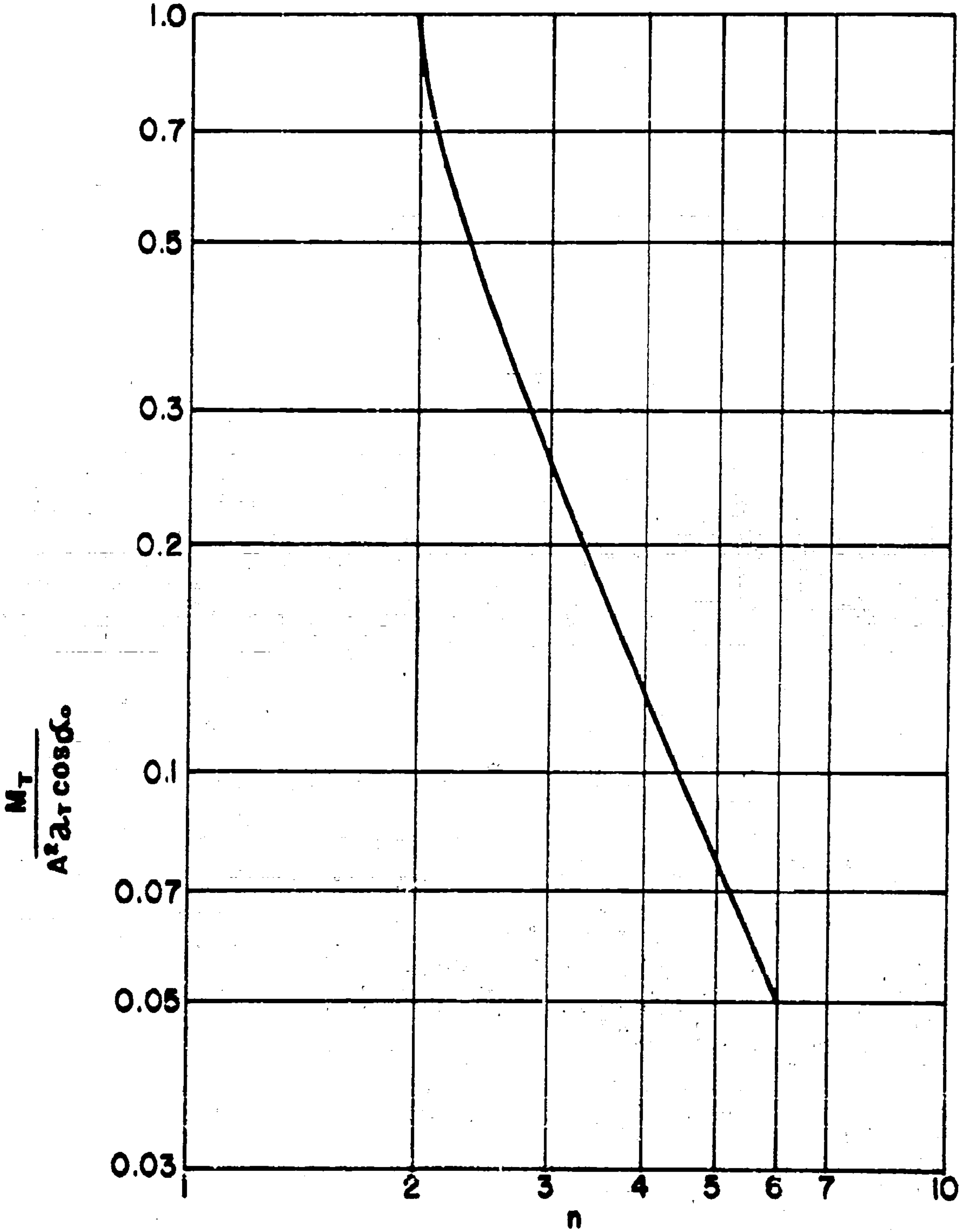

Fig. 6. Maximum value of the miss caused by constant-lateral-acceleration target turn.

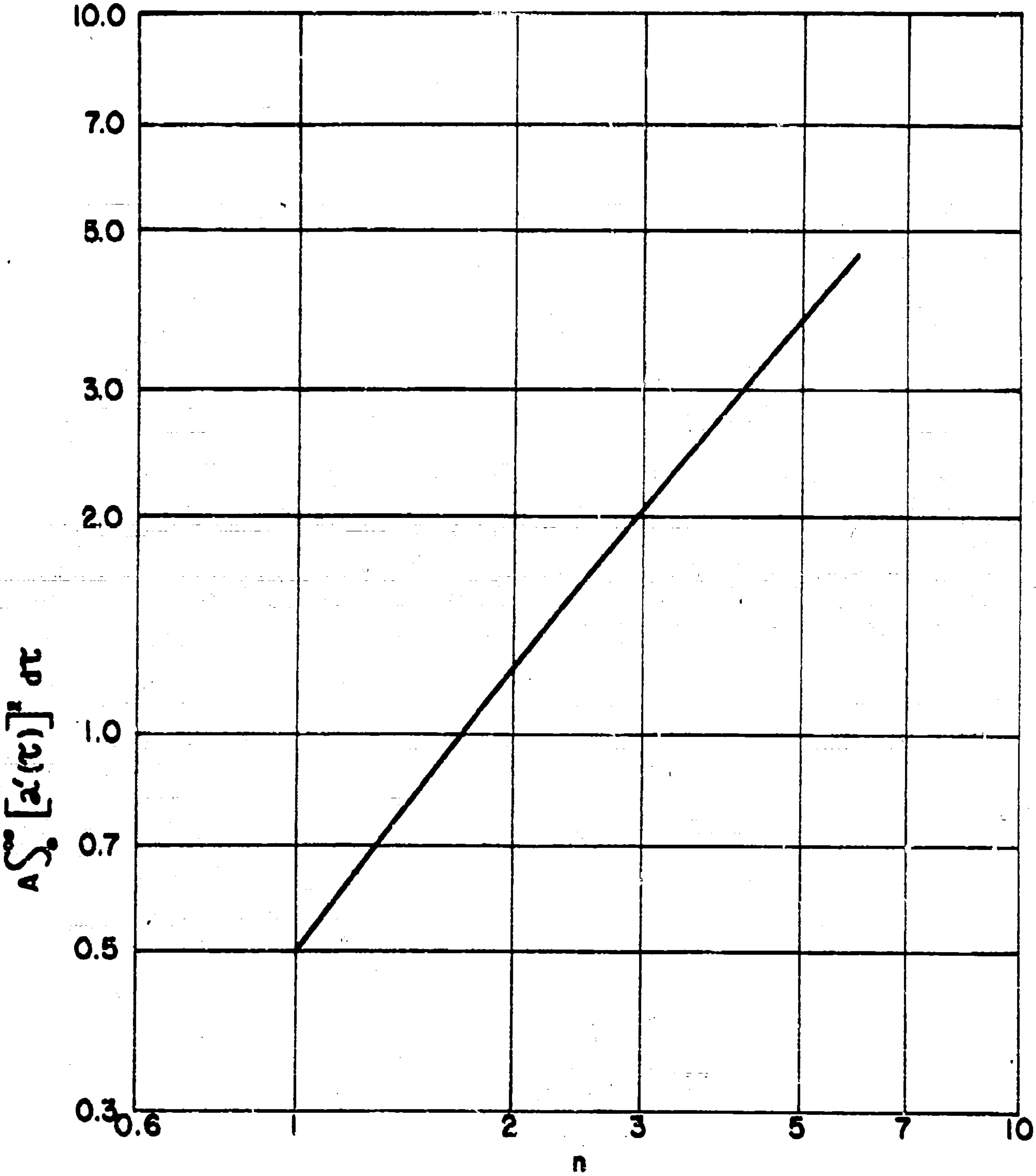

Fig. 6. Miss caused by noise (nondimensionalized).

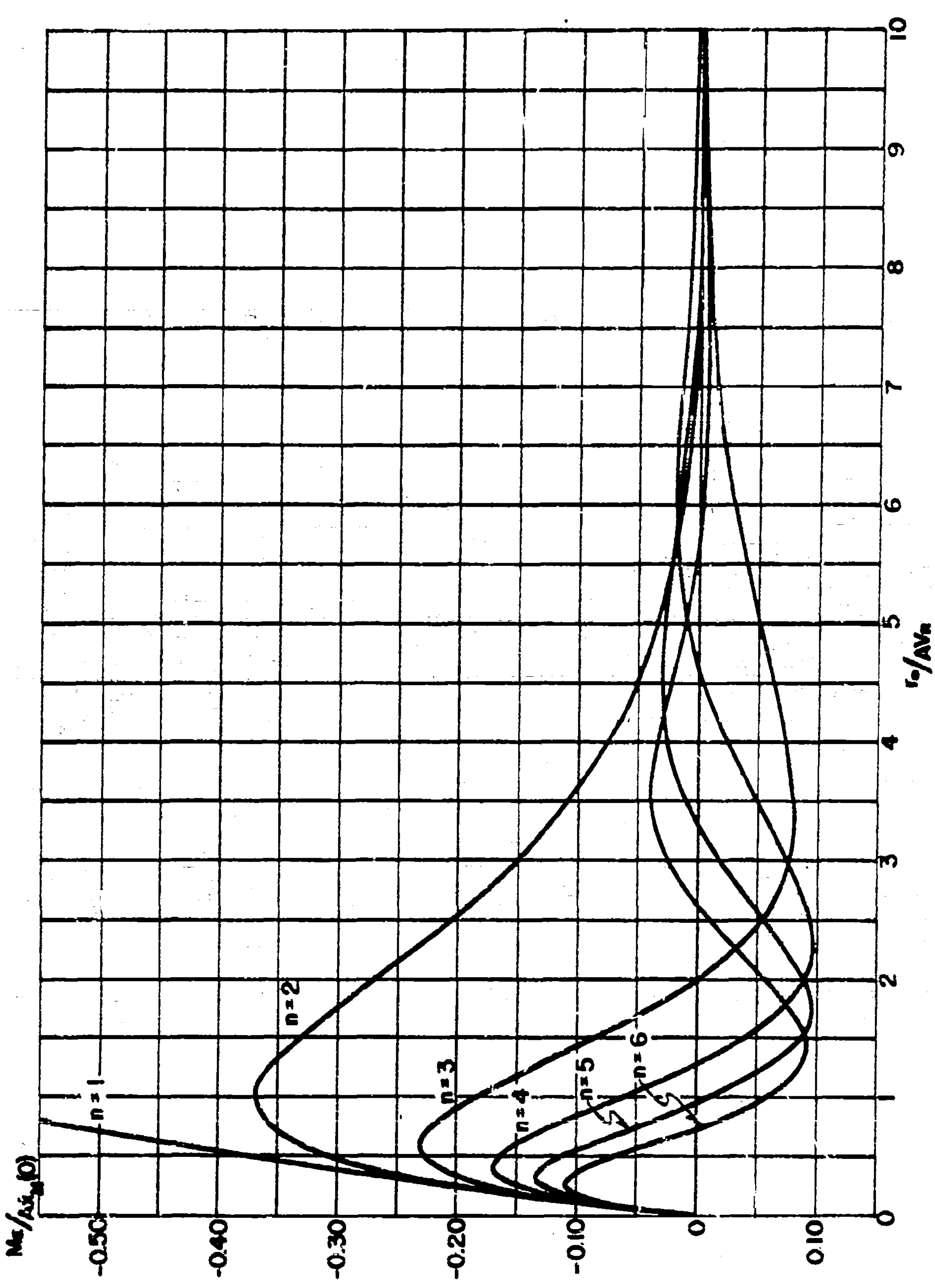

Fig. 7. Miss caused by initial error in heading.

In Fig. 6, $A\int_0^\infty [a'(\tau)]^2\, d\tau$ is plotted as a function of n for n = 1, 2, . . . , 6. These values lie approximately on a straight line (on log-log paper) with slope 1.25. Hence, over this region of values of n, it is reasonable to approximate the integral by

$$\int_0^\infty [a'(\tau)]^2\, d\tau = C n^{1.25} A^{-1}$$
$$= 0.5\, n^{1.25} A^{-1}. \tag{95}$$

Since

$$\sigma_N^2 = 2\pi\, \Phi_N \int_0^\infty [a'(\tau)]^2\, d\tau$$

and

$$n = \frac{(b+1)\, V_M \cos\beta_0}{V_R},$$

$$\left.\begin{aligned} \sigma_N^2 &= 0.5\,(2\pi\,\Phi_N)\left(\frac{V_R}{V_M \cos\beta_0}\right)^{-1.25}(b+1)^{1.25} A^{-1} \\ &= K_N A^{-1}(b+1)^{1.25} \\ \text{where}\quad K_N &= \pi\,\Phi_N \left(\frac{V_R}{V_M\cos\beta_0}\right)^{-1.25}. \end{aligned}\right\} \tag{96}$$

3.4. Miss Caused by Initial Conditions.

It is shown in Sec. 3.1 that the miss caused by an initial error in missile heading is given by

$$M_E = \dot{x}_M(0)\left[\int_0^{\frac{r_0}{V_R}} a(\tau)\, d\tau - \frac{r_0}{V_R}\right] \tag{88}$$

or

$$\frac{M_E}{A\dot{x}_M(0)} = \int_0^{\frac{r_0}{AV_R}} a(A\gamma)\, d\gamma - \frac{r_0}{AV_R}.$$

For each n, $M_E / A\dot{x}_M(0)$ is a function of r_0/AV_R alone and can be calculated by substituting into the above expression the functions a (τ) given by Eq. (90). Solutions for n = 1, . . ., 6 are given below.

$$\left.\begin{aligned} \frac{M_E}{A\dot{x}_M(0)} &= e^{-l} - 1 && \text{for } n = 1 \\ &= -\,l\, e^{-l} && \text{for } n = 2 \\ &= \frac{e^{-l}}{2!}(l^2 - 2l) && \text{for } n = 3 \\ &= -\frac{e^{-l}}{3!}(l^3 - 6l^2 + 6l) && \text{for } n = 4 \\ &= \frac{e^{-l}}{4!}(l^4 - 12l^3 + 36l^2 - 24l) && \text{for } n = 5 \\ &= -\frac{e^{-l}}{5!}(l^5 - 20l^4 + 120l^3 - 240l^2 + 120l) && \text{for } n = 6 \end{aligned}\right\} \tag{97}$$

where $l = r_0/AV_R$. These functions are plotted in Fig. 7.

It is shown in Sec. 3.1 that the miss caused by an initial lateral acceleration of the missile is given by

$$M_a = \dot{\theta}_0\left[\frac{AV_R}{b+1}\int_0^{\frac{r_0}{V_R}} a(\tau)\, d\tau - \frac{r_0 A}{b+1}\, a\left(\frac{r_0}{V_R}\right)\right] \tag{89}$$

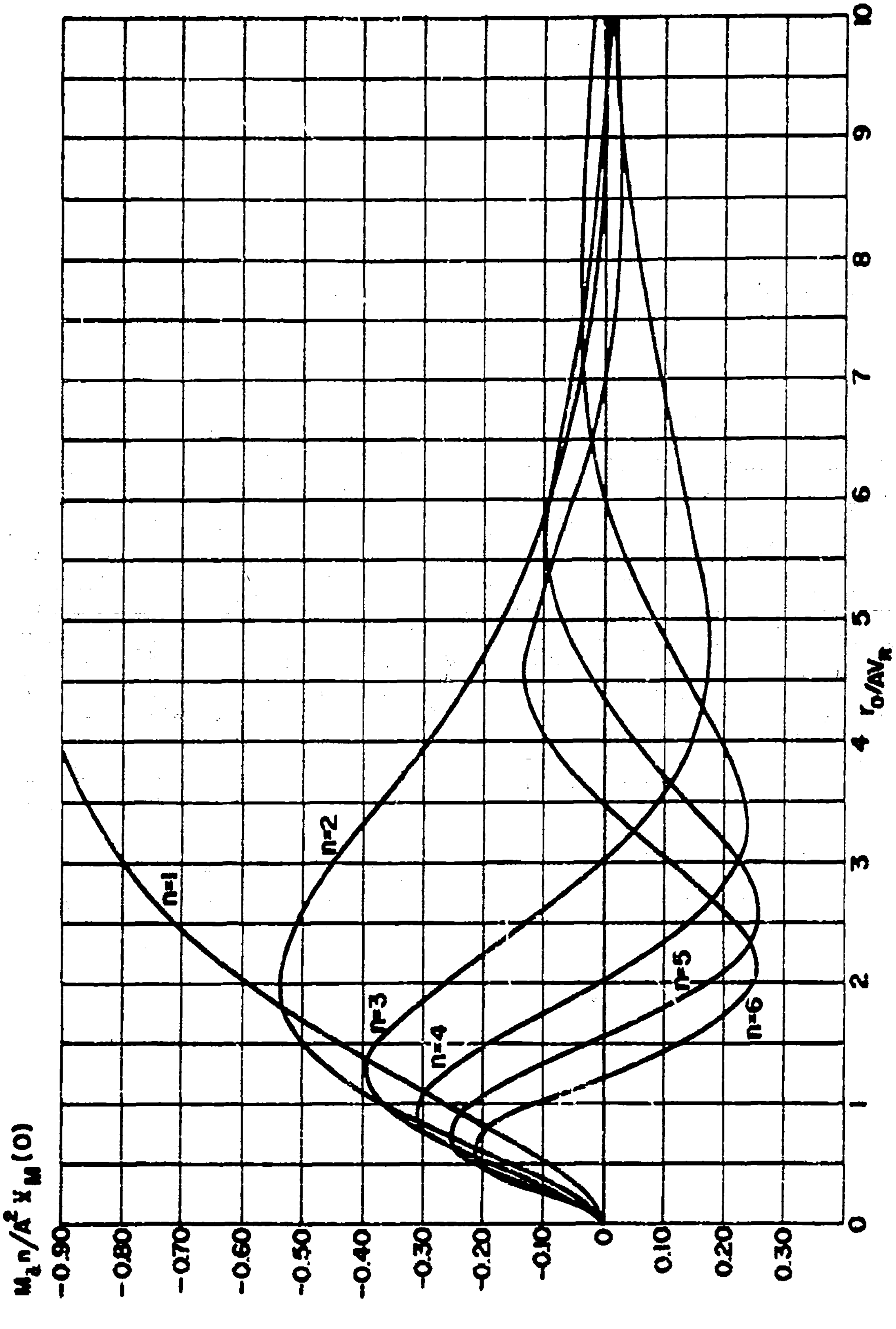

Fig. 8. Miss caused by initial lateral acceleration

or

$$\frac{M_a}{A^2 \ddot{x}_M(0)} = \frac{1}{n}\left[\int_0^{\frac{r_0}{AV_R}} a(A\gamma)\, d\gamma - \frac{r_0}{AV_R}\, a\left(\frac{r_0}{V_R}\right)\right].$$

For each n, $M_a/A^2 \ddot{x}_M(0)$ is a function of r_0/AV_R alone and can be calculated by substituting the functions $a(\tau)$ given by Eq. (90). Solutions for n = 1, . . ., 6 are given below.

$$
\left.
\begin{aligned}
\frac{M_a}{A^2 \ddot{x}_M(0)} &= e^{-l}(l+1) - 1 && \text{for } n = 1\\
&= -\frac{e^{-l}}{2!}(l^2) && \text{for } n = 2\\
&= \frac{e^{-l}}{3!}(l^3 - 3\,l^2) && \text{for } n = 3\\
&= -\frac{e^{-l}}{4!}(l^4 - 8\,l^3 + 12\,l^2) && \text{for } n = 4\\
&= \frac{e^{-l}}{5!}(l^5 - 15\,l^4 + 60\,l^3 - 60\,l^2) && \text{for } n = 5\\
&= -\frac{e^{-l}}{6!}(l^6 - 24\,l^5 + 180\,l^4 - 480\,l^3 + 360\,l^2) && \text{for } n = 6
\end{aligned}
\right\} \quad (98)
$$

where $l = r_0/AV_R$. There functions multiplied by n are plotted in Fig. 8.

3.5. Optimization for White Noise and Constant-Lateral-Acceleration Target Turn.

The proportional-navigation-with-simple-time-lag system is characterized by two constants, A and b + 1. With m (A, b + 1) as a measure of effectiveness of the system, the best system is determined by the equations $\partial m/\partial A = 0$ and $\partial m/\partial(b+1) = 0$. If these equations are not independent, the optimization may furnish only an optimum value for some function of A and b + 1. This occurs in the case to be considered in this section.

The control-system parameters may be optimized for the case in which target motion is either a fixed function or a statistical function. In this section the minimum rms miss is calculated for the proportional-navigation-with-simple-time-lag control system for the case in which target motion is a constant-lateral-acceleration turn. In Sec. 3.7 the same is done for the statistical-target-maneuver case. For these two cases the minimum rms miss for any linear control system is calculated in Secs. 2.3 and 2.4, respectively.

For each system, let M_T denote the maximum value of miss caused by a single constant-lateral-acceleration turn. Let σ_N denote the deviation of the miss caused by white noise in the case in which the initial range is considered infinite. The three assumptions — single target turn, white noise, and infinite initial range — are not essential to this method, but they are convenient simplifications.

A reasonable criterion of effectiveness of the control system is the rms miss M, and the optimum system may be defined as the system that minimizes M. The rms miss is chosen primarily for reasons of mathematical convenience. A more logical criterion is the probability of the miss being less than some fixed lethal radius. It is shown later that for the problem of this section, the system with the parameters that minimize rms miss yields approximately the maximum probability.

Since M_T and σ_N are also functions of the target acceleration a_T, the noise level Φ_N, the target-to-missile speed ratio λ, and the angle of approach α_0, the optimum system is expressed in terms of these variables.

It is shown in Sec. 3.2 that M_T is approximated by

$$M_T = \sqrt{K_T}\, A^2 (b+1)^{-2.5}$$

(for $b + 1 > 2\, V_R/V_M \cos\beta_0$) where

$$\sqrt{K_T} = 4.30\, a_T \cos\alpha_0 \left(\frac{V_R}{V_M \cos\beta_0}\right)^{2.5}. \quad (93)$$

It is shown in Sec. 3.3 that

$$\left.\begin{aligned} \sigma_N^2 &= K_N\, A^{-1}\, (b+1)^{1.25} \\ \text{where} \quad K_N &= \pi\, \Phi_N \left(\frac{V_R}{V_M \cos \beta_0}\right)^{-1.25}. \end{aligned}\right\} \qquad (96)$$

It can be shown that the rms miss is given by $M^2 = M_T^2 + \sigma_N^2$. Therefore,

$$M^2 = K_T\, A^4\, (b+1)^{-5} + K_N\, A^{-1}\, (b+1)^{1.25}. \qquad (99)$$

Now as M is a function of $A\,(b+1)^{-1.25}$, minimisation of M furnishes an optimum value of $A\,(b+1)^{-1.25}$. Letting $Z = A\,(b+1)^{-1.25}$, $M^2 = K_T\, Z^4 + K_N\, Z^{-1}$. Then $\partial M^2/\partial Z = 4\, K_T\, Z^3 - K_N\, Z^{-2}$.

M is minimum for $\partial M^2/\partial Z = 0$. With $\left.\frac{\partial M^2}{\partial Z}\right|_{Z=Z_0} = 0$,

$$\left.\begin{aligned} 4\, K_T\, Z_0^3 &= K_N\, Z_0^{-2}; \\ Z_0 &= \left(\frac{K_N}{4\, K_T}\right)^{1/5} \\ &= \frac{0.368\, (2\pi\, \Phi_N)^{1/5} \left(\frac{V_R}{V_M \cos \beta_0}\right)^{-1.25}}{(a_T \cos \alpha_0)^{1/5}}. \end{aligned}\right\} \qquad (100)$$

$$\begin{aligned} M_{MIN}^2 &= K_T \left(\frac{K_N}{4\, K_T}\right)^{4/5} + K_N \left(\frac{K_N}{4\, K_T}\right)^{-1/5} \\ &= \left(K_T \frac{K_N}{4\, K_T} + K_N\right) \left(\frac{K_N}{4\, K_T}\right)^{-1/5} \\ &= \frac{5}{(4)^{4/5}}\, K_N^{4/5}\, K_T^{1/5} \\ &= 1.650\, K_N^{4/5}\, K_T^{1/5}. \\ M_{MIN} &= 1.285\, K_N^{2/5}\, K_T^{1/10} \\ &= 1.303\, (2\pi\, \Phi_N)^{2/5}\, (a_T \cos \alpha_0)^{1/10}. \end{aligned} \qquad (101)$$

It is shown in Sec. 2.3, Eq. (47), that M_{MIN} cannot be less than $1.2904\, (2\pi\, \Phi_N)^{2/5}\, (a_T \cos \alpha_0)^{1/10}$ for any control system. Hence, on the basis of the criterion used here, the proportional-navigation-with-simple-time-lag control system is as good as the best possible linear control system (including time-variant systems) within the accuracy of the approximation formulas, Eqs. (93) and (96).

It is interesting to note that for this control system with typical values of $a_T \cos \alpha_0$, $2\pi\, \Phi_N$, and the lethal radius, the system with the parameter values that minimize the rms miss yields approximately the maximum probability of the miss distance which is less than a fixed distance (the lethal radius).

If, as before, the total miss is the sum of the maximum miss M_T caused by a constant-acceleration target turn and a normally distributed (with deviation σ_N) miss caused by noise, the probability of a miss between $-p$ and p is

$$P(p) = \frac{1}{\sigma_N \sqrt{2\pi}} \int_{-p}^{p} e^{-\frac{(x-M_T)^2}{2\sigma_N^2}}\, dx \qquad (102)$$

where M_T and σ_N are given as functions of $Z = A\,(b+1)^{-1.25}$ by Eq. (93) and Eq. (96), respectively.

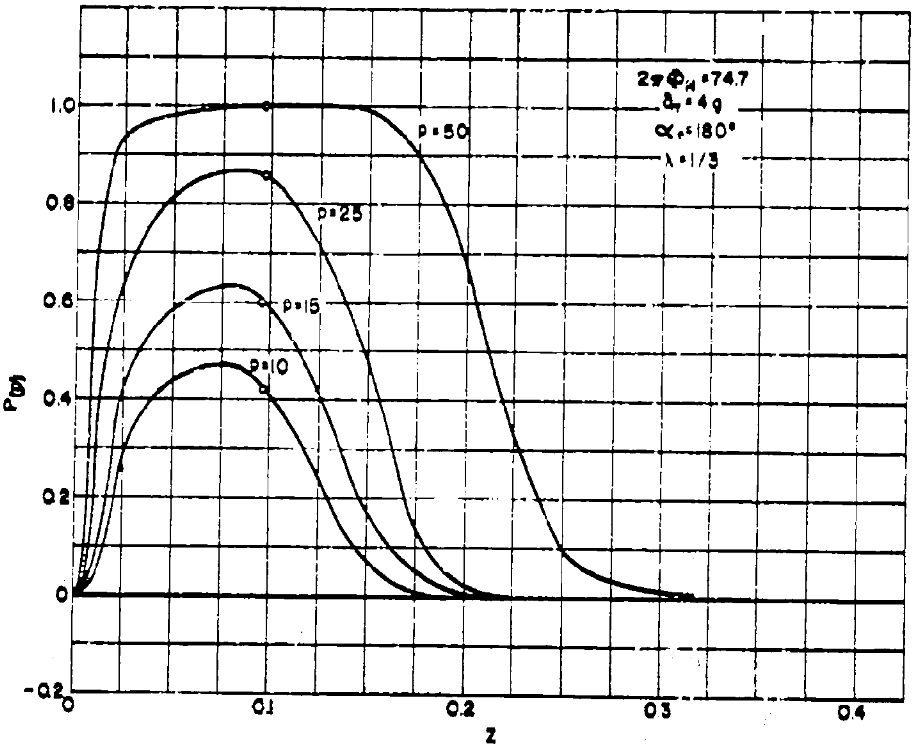

Fig. 9. Probability of hit (miss less than lethal radius).

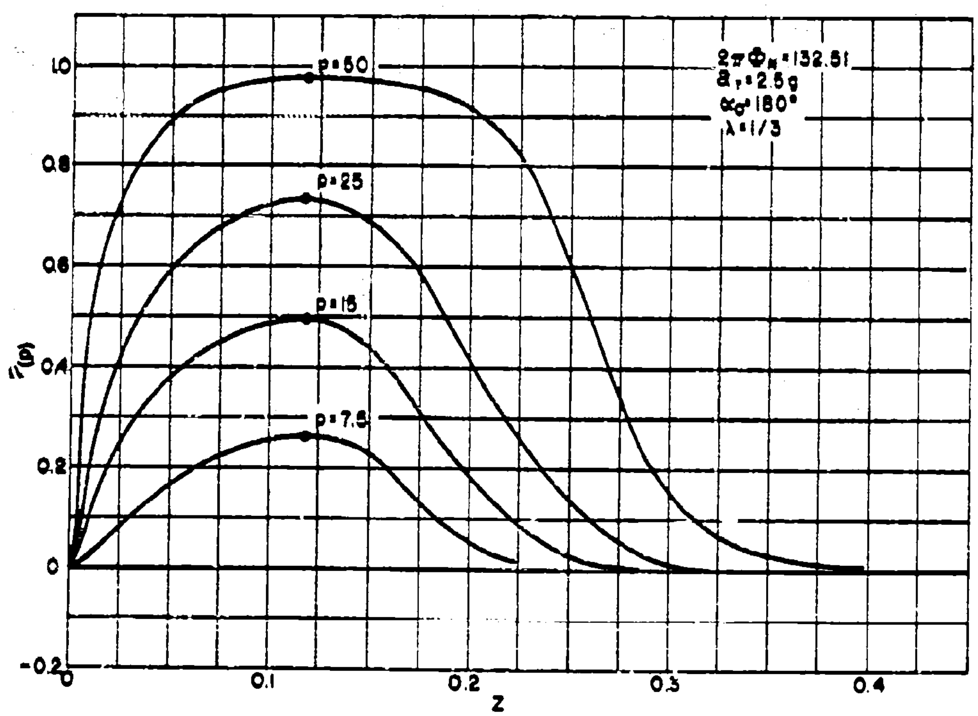

Fig. 10. Probability of hit (miss less than lethal radius).

Figures 9 and 10 are plots of the probability P as a function of Z for several values of ρ. Figure 9 is based upon the following values of various parameters:

$$a_T = 4\text{ g}$$
$$\alpha_0 = 180°$$
$$\lambda = 1/3$$
$$2\pi\ \Phi_N = 74.7$$

Figure 10 is based upon the following values:

$$a_T = 2.5\text{ g}$$
$$\alpha_0 = 180°$$
$$\lambda = 1/3$$
$$2\pi\ \Phi_N = 132.51$$

The points corresponding to the system that minimizes the rms value of miss are indicated by circles in Figs. 9 and 10. It is seen that the minimum rms system yields approximately the maximum probability. Thus, it can be concluded that the rms miss is a fairly reasonable criterion of control-system effectiveness.

3.6. Applications.

Any application of the above optimization formulas depends upon knowledge of the noise density Φ_N. Unfortunately, reliable experimental data concerning Φ_N are almost nonexistent. For this reason, the applications presented here should be viewed more as illustrations of the possible application of the optimization formulas than as reliable calculations. However, several qualitative conclusions which would seem to be fairly independent of accurate knowledge of Φ_N may be reached.

Although little is definitely known about the variation of Φ_N with the angle of approach α_0, Φ_N is probably a complicated function of α_0. A rough picture of its probable behavior is obtained by assuming an elliptical variation of $\sqrt{\Phi_N}$ with α_0. That is,

$$\Phi_N = \frac{C_1}{1 + C_2 \cos^2 \alpha_0} \tag{103}$$

where C_1 and C_2 are to be determined from experimental data for head-on and beam approaches.

It has been observed experimentally with an antenna system passing approximately the first half cycle of noise that for a head-on approach the rms noise is approximately 1/20 the wing span S and for a beam approach the rms noise is approximately 1/6 the aircraft length L. If Φ_N is assumed constant with respect to frequency,

$$\begin{aligned} \Phi_N &= \frac{(1/20\ S)^2}{4\pi\ (0.5)} \text{ for } \alpha_0 = 180° \\ &= \frac{(1/6\ L)^2}{4\pi\ (0.5)} \text{ for } \alpha_0 = 90° \ . \end{aligned} \tag{104}$$

The ratio of wing span to aircraft length is approximately 1.4 for present bombers. If $S/L = 1.4$ and Φ_N assumes the values given by Eq. (104), Eq. (103) becomes

$$\Phi_N = \frac{0.00226\ S^2}{1 + 4.669 \cos^2 \alpha_0} \ . \tag{105}$$

With Eq. (105) as an expression for Φ_N, the results of the preceding section are applied to determine rms miss for two typical bombers, the B36 and B29 (Figs. 11 and 12, respectively). An estimated lateral acceleration a_T of 2.5g is used for each target, and a speed ratio $\lambda = 1/3$ is assumed. For the B36 bomber, the wing span $S = 230$ feet; and for the B29 bomber, $S = 141$ feet.

Four curves are shown for each case. The M_{MIN} curve is the minimum value of rms miss for each value of α_0 (developed in Sec. 3.5). Curves of rms miss are shown for systems optimized at a particular value of α_0. Values used are $\alpha_0 = 0°$, $\alpha_0 = 180°$, and $\alpha_0 = 120°$. The 120° system is selected as the system that gives the best all-round performance.

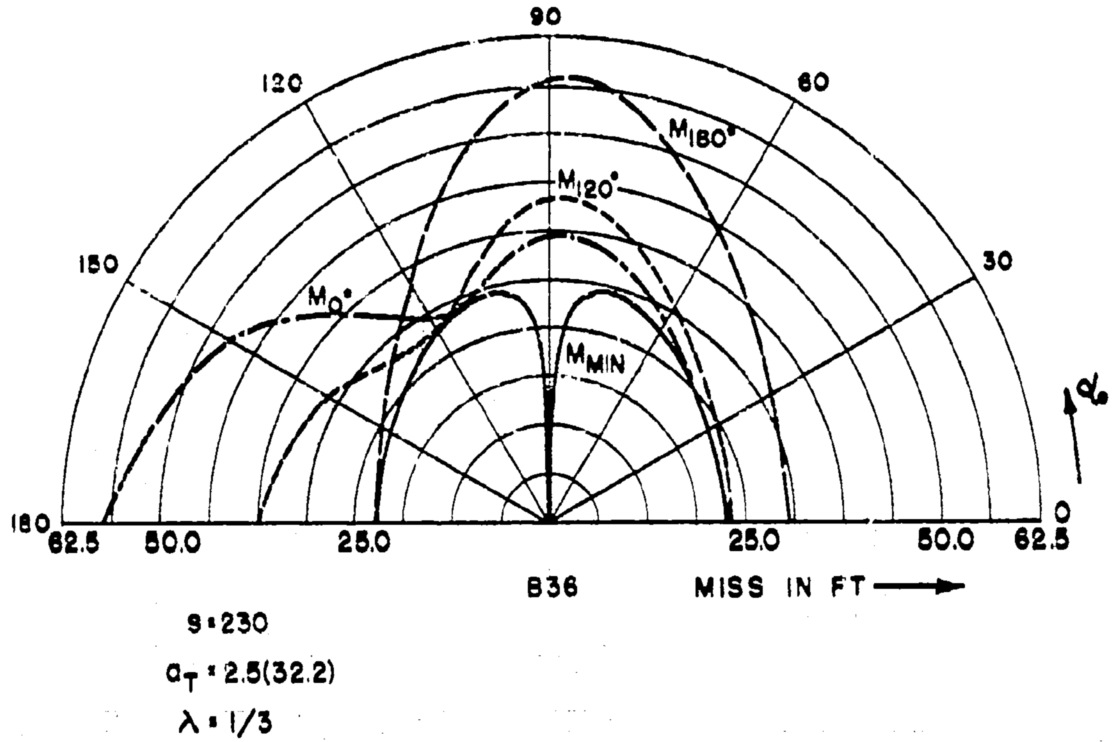

Fig. 11. RMS miss for B36 target.

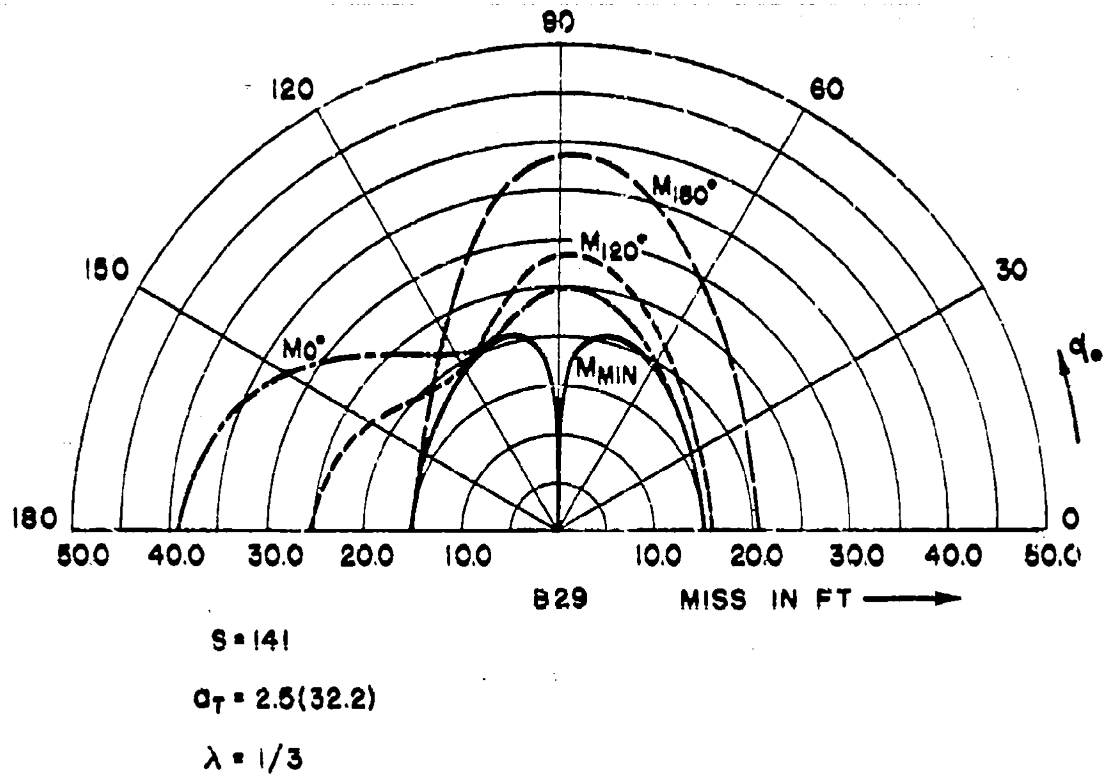

Fig. 12. RMS miss for B29 target.

3.7. Optimization for White Noise and Statistical Target Maneuver.

In Sec. 3.5 the optimum parameters are calculated for the proportional-navigation-with-simple-time-lag control system on the basis of white noise and a single target turn. The minimum rms miss for this system under these conditions is compared with the minimum rms miss for any linear system under the same conditions [calculated in Sec. 2.3, Eq. (47)].

In the present section, the same is done for the case of white noise and statistical target maneuvers. The statistical maneuver considered is the sequence of arcs of constant lateral acceleration with random switching, for which the minimum rms miss for any linear system is calculated in Sec. 2.4 (Fig. 2). As in Sec. 2.4, the rms miss σ is given by Eq. (53) as

$$\sigma^2 = \frac{k\ (a_T \cos a_0)^2}{\pi} \int_{-\infty}^{\infty} |\ 1 - F\ (i\omega)\ |^2\ \frac{1}{\omega^4\ (\omega^2 + k^2)}\ d\omega + \Phi_N \int_{-\infty}^{\infty} |\ F\ (i\omega)\ |^2\ d\omega \tag{53}$$

where F (iω) is the frequency characteristic defined by

$$F\ (i\omega) = \int_0^{\infty} e^{-i\omega\tau}\ \dot{a}'\ (\tau)\ d\tau. \tag{49}$$

It is again convenient to consider the problem in nondimensional form. Following the nondimensionalising procedure of Sec. 2.4, a nondimensional miss E is given by

$$E^2 = \int_{-\infty}^{\infty} |\ 1 - F\ (ihv)\ |^2\ \frac{1}{v^4\ (v^2 + m^2)}\ dv + \int_{-\infty}^{\infty} |\ F\ (ihv)\ |^2\ dv \tag{57}$$

where

$$E^2 = \frac{\sigma^2}{\Phi_N\ h}, \tag{58}$$

$$h = \left[\frac{k\ (a_T \cos a_0)^2}{\pi\ \Phi_N}\right]^{\frac{1}{5}}, \tag{55}$$

and

$$m = \frac{k}{h}. \tag{56}$$

For the proportional-navigation-with-simple-time-lag control system, expressions for a (τ) are given by Eq. (90). The frequency characteristic F (s) associated with a (τ) is

$$F\ (s) = 1 - \frac{s^n}{\left(s + \frac{1}{A}\right)^n}\ . \tag{106}$$

The nondimensional miss E is then evaluated by substitution of Eq. (106) into Eq. (57). The detailed calculations are rather lengthy and are omitted here. For n = 1, E is infinite. For n = 2, 3, 4, 5, the miss E is given by

$$\left.\begin{aligned}
E^2 &= \frac{2\pi}{4m^5}\left[\frac{l^4\ (l + 2)}{(l + 1)^2} + \frac{5m^6}{l}\right] && \text{for } n = 2 \\
&= \frac{2\pi}{16m^5}\left[\frac{l^6\ (l + 3)}{(l + 1)^3} + \frac{33m^6}{l}\right] && \text{for } n = 3 \\
&= \frac{2\pi}{32m^5}\left[\frac{l^5\ (l^2 + 4l + 1)}{(l + 1)^4} + \frac{93m^6}{l}\right] && \text{for } n = 4 \\
&= \frac{2\pi}{256m^5}\left[\frac{l^5\ (5l^3 + 25l^2 + 15l + 3)}{(l + 1)^5} + \frac{965m^6}{l}\right] && \text{for } n = 5
\end{aligned}\right\} \tag{107}$$

where $l = h\ A\ m = k\ A$.

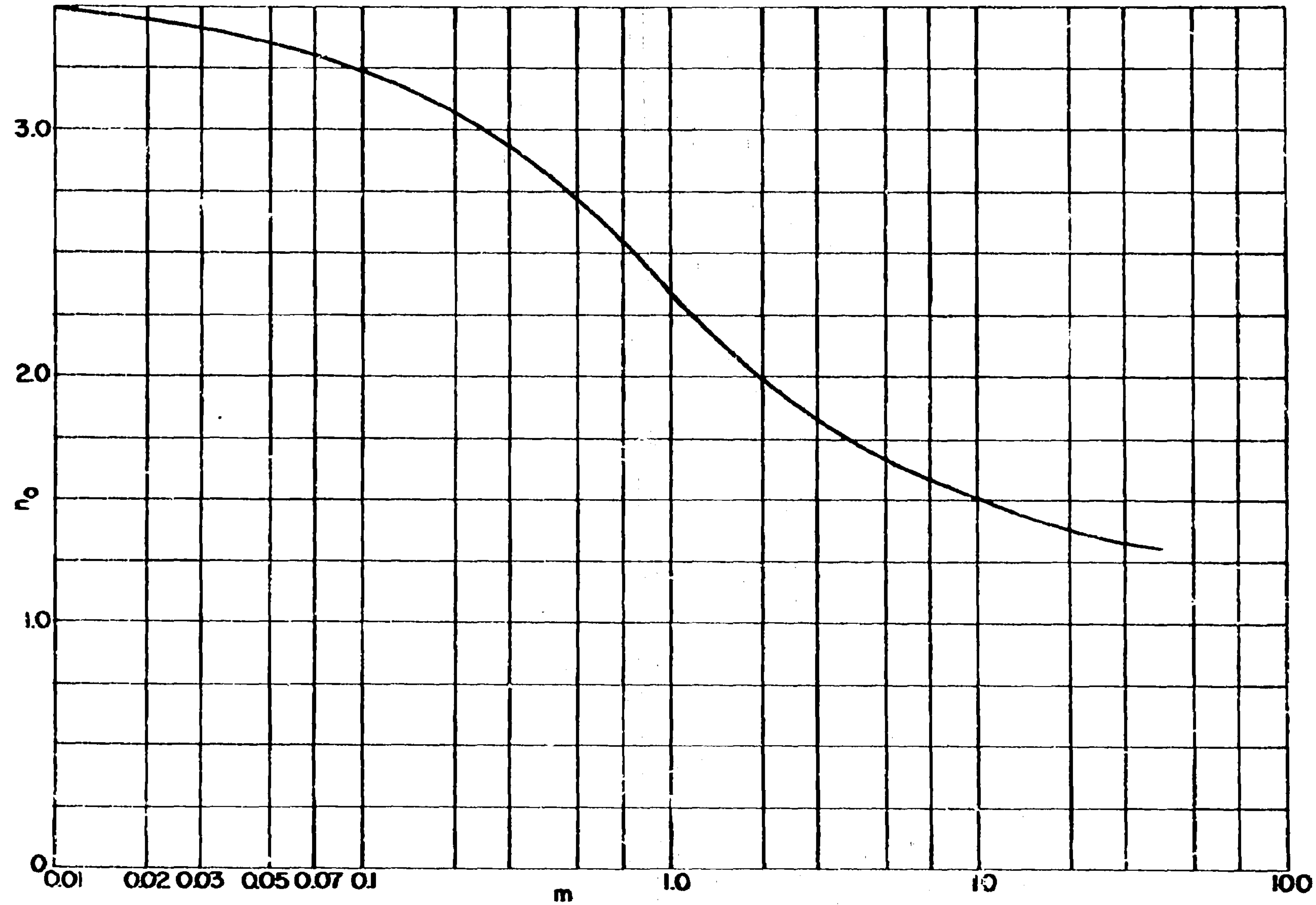

Fig. 13. Optimum value of navigation constant times speed ratio, $n = [(b + 1) V_M \cos \beta_0]/V_R$.

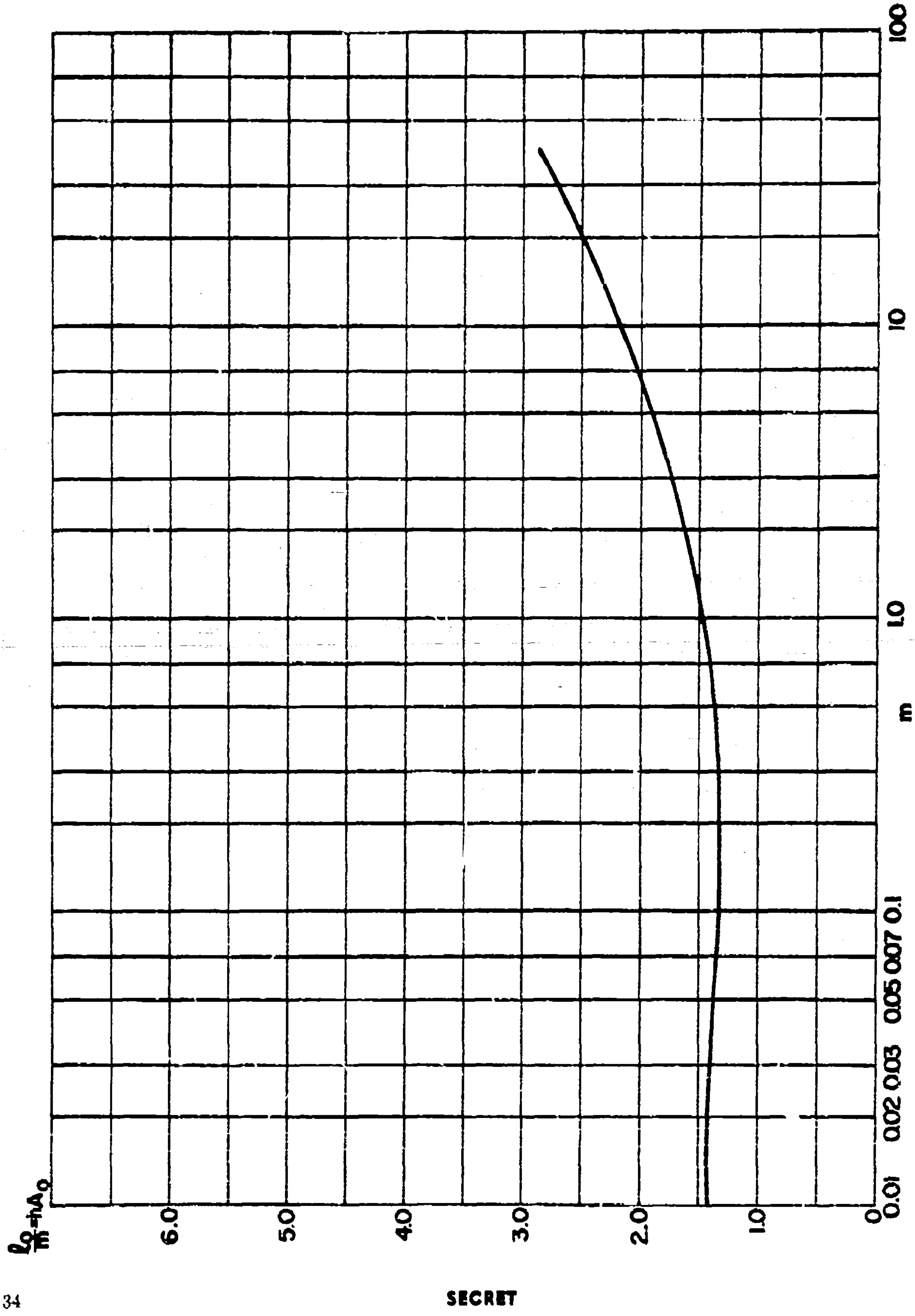

Fig. 14. Optimum value of nondimensionalized time lag hA.

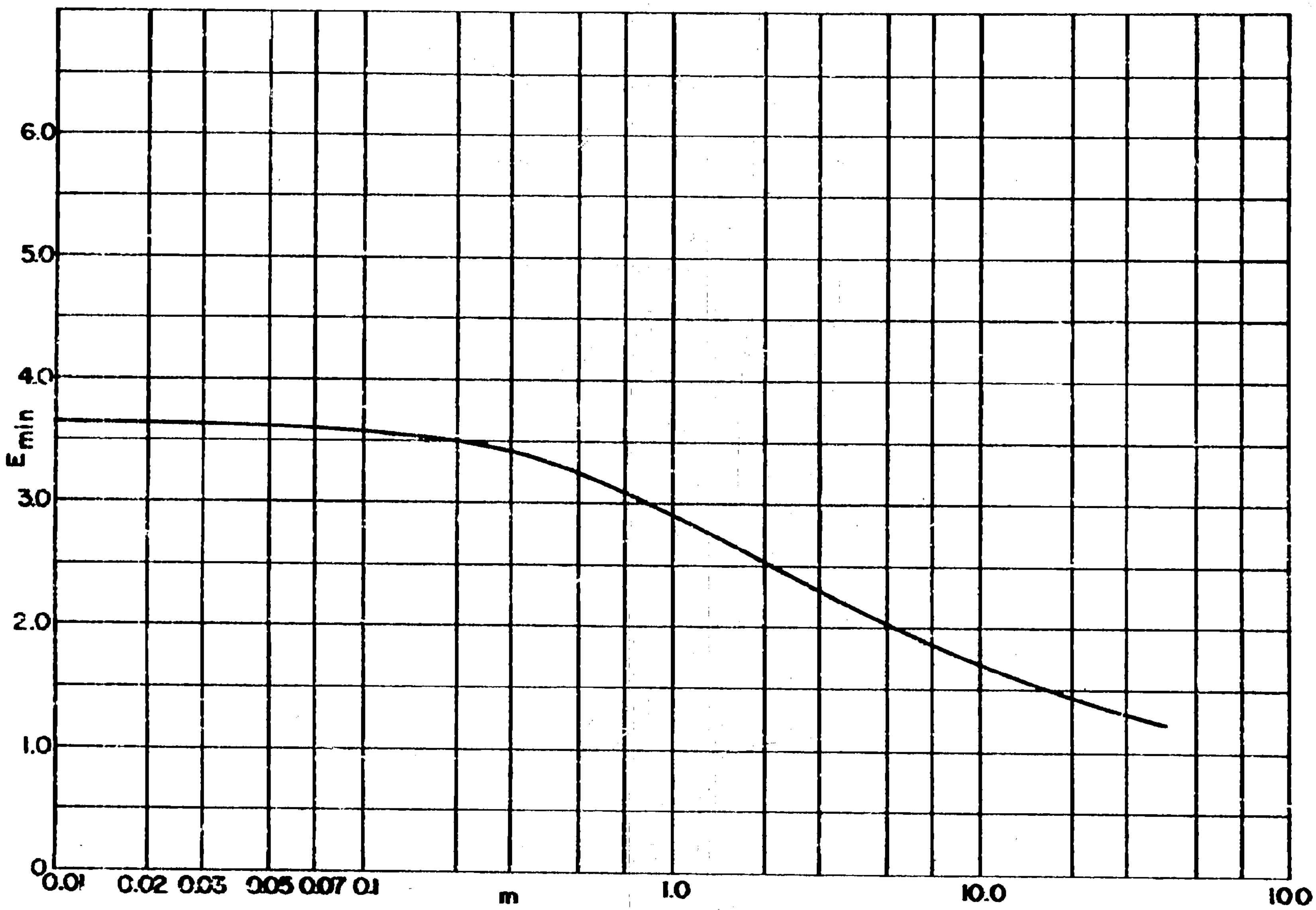

Fig. 15. Minimum rms miss (nondimensionalized) for white noise and a statistical target maneuver, with the missile guided by proportional-navigation-with-simple-time-lag control system.

The parameters of the optimum proportional-navigation-with-simple-time-lag system are then determined by so choosing l and n as to minimize E^2. Since expressions for E^2 are available only for integral values of n, the optimum value of n is obtained by interpolation.

Figures 13 and 14 are plots of the optimum values of n and $l/m = hA$ as functions of m. The minimum value of miss E_{MIN} is shown as a function of m in Fig. 15. It can be seen by comparison with Fig. 2 that the minimum rms miss for the proportional-navigation-with-simple-time-lag control system is only about 3 per cent higher than the minimum rms miss for any linear control system.

4. CONCLUSIONS

To every control equation which is linear between α and θ, there corresponds a linear equation relating missile position to target position and to disturbing factors, such as noise. The linearisation of the kinematic equations rests upon the assumption that angular changes are small. This restricts the study to a time interval so short that the changes in target heading are not large. Fortunately for the applicability of linear techniques to the homing missile problem, enough of the terminal portion of the flight lies in this interval with the result that miss distances calculated from the linearised equations are very close to the actual misses obtained.

The linearity of the equation which determines missile position permits use of several powerful mathematical techniques of optimization. The control system is optimized for an intelligent target maneuver, that is, a maneuver so made as to maximise miss distance. For a statistical maneuver, the optimization technique of Wiener is extended to time-variant systems. The efficient use of optimization techniques requires further study of the relative probability of target maneuvers under various tactical situations.

A study of the rms miss for the missile guided by the proportional-navigation-with-simple-time-lag control system without limiting indicates that for a given set of conditions (that is, a given angle of approach to target, target maneuver, and noise) the miss, by proper choice of navigation constant and time constant, can be reduced to a value only 2 or 3 per cent higher than the minimum for any linear system operating under the same conditions. The control-system parameters so obtained are functions of the given tactical situation. There is need for control-system optimisation based on a class of tactical situations rather than on a single situation. This was not attempted here, because of lack of time and because of the present uncertainties concerning probability of target maneuvers and the behavior of radar noise as a function of target size and angle of approach.

Radar dead space, one of the problems not considered in this study, can seriously limit the performance of homing missiles. Usually, there is a minimum range at which reliable radar data can be obtained. Below this range, the control system must stabilise the missile on a path determined only by past data. For example, when radar contact is lost, the missile may maintain its previous angle or angular velocity. The mathematical problem is essentially one of prediction of the final position of the target from signals that stop before the end of the flight. The optimisation techniques introduced here can be extended to cover the prediction problem.

Throughout this report the emphasis is on the value of miss distance. Other factors of great importance in the design of a missile control system are the power required to drive control surfaces and the power required to overcome drag caused by lateral acceleration (or, for the gliding missile, the decrease in speed caused by such drag). While the ultimate purpose is hitting the target, power consumption must be considered for practical reasons. Although it is perhaps obvious, the fact should be pointed out that there can be no meaningful power minimization that disregards miss distance, because the minimum power consumption (zero) is obtained with locked control surfaces.

The power-consumption problem can be studied from several approaches. As was noted in Sec. 2.4 and is apparent from the step-response approach to the time-variant system, any optimization with respect to miss distance does not uniquely determine the control equation. A useful approach to the power-consumption problem would be to select from all control equations yielding a given miss characteristic the control equation that minimizes some measure of power consumption, for example, $\int(\dot{\theta})^2\,dt$. An alternate approach is to select from all control equations yielding a given measure of power consumption the control equation that minimizes the rms miss.

The most serious limitation of the present study is its restriction to linear systems. The conclusions concerning miss distance and control-system parameters herein presented are valid only for missiles in which nonlinearities (lateral-acceleration limiting probably being the most significant) can be neglected. There are really two questions involved here:

a. How closely can the performance of a missile with prescribed nonlinearities be made to approach the performance of a linear system?

b. To what extent is the optimum linear system the optimum system (linear or nonlinear)?

The first question has not yet been satisfactorily answered. However, the unpublished work of W. W. Seifert, using the M.I.T. Flight Simulator, indicates that the acceleration limiting probably imposes serious limitations on missile performance, regardless of the form of the control equation.

A more definite answer can be given to the second question. If the target position x_T and the radar noise x_N have normal (Gaussian) distributions, the optimum linear system is the optimum system. It seems reasonable that the noise will be near-normal. On the other hand, the distribution of x_T depends upon the type of maneuver expected of the target. If a random maneuver which leads to approximately Gaussian distribution in x_T is used to evaluate the system, the optimum linear system approximates the optimum system. If the target maneuver is of a different kind, such as a single intelligent turn, there is no reason to expect the optimum system to be linear. It is only logical that the best system for a single target turn be designed to recognize the existence of a target turn and then to guide the missile to the collision point. Such a system certainly would not be linear.

The probability of various maneuvers involves tactical considerations independent of the mathematical problem. It seems reasonable to design missile-control systems for a random target maneuver of the type considered in this study, unless the tactical considerations indicate a higher probability of maneuvers of a different nature. For the random maneuver investigated here (sequence of arcs), it is approximately correct to regard the optimum linear system as the ideal system which can be approached by removing nonlinearities from the actual systems.

APPENDIX A. — EFFECT OF CONTROL-SYSTEM DRIFT ON THE PROPORTIONAL-NAVIGATION-WITH-SIMPLE-TIME-LAG CONTROL SYSTEM

If, as is usually the case, the missile control system uses gyro information to establish a space reference, gyro drift causes a miss not previously considered. Gyro drift enters the control equation primarily because the apparent line-of-sight angle α_A is determined by adding to the apparent relative target bearing β_A the missile heading as determined by a gyro, but it may also enter because the missile autopilot stabilization derives from a gyro (often the same gyro). The manner in which stabilization is obtained determines the relative magnitude of the drift term d in the control equation.

Factors other than control-system drifts and unbalances may contribute to the drift term. For example, if the missile weight is balanced by a fixed control-surface deflection, an increase or decrease of speed produces an unbalanced force. This effect may often be approximated by the addition of a drift term to the control equation.

Drift effects cause the proportional-navigation-with-simple-time-lag control equation to be of the form

$$A\ddot{\theta} + \dot{\theta} = B\ (\dot{\alpha} + d). \tag{A-1}$$

The linearized equation corresponding to Eq. (69) is then

$$\frac{A}{n}\left(\frac{r_0}{V_R} - t\right)\ddot{x}_M + \frac{1}{n}\left(\frac{r_0}{V_R} - t\right)\dot{x}_M + x_M = x_T + x_N + C\ (r_0 - V_R t) + d\ (r_0 - V_R t)\ t. \tag{A-2}$$

With application of the superposition theorem, the miss M_d caused by the drift is given by

$$M_d = \int_0^{\frac{r_0}{V_R}} a'\ (\tau)\ d\ (r_0 - V_R \tau)\ \tau\ d\tau. \tag{A-3}$$

Letting $\gamma = \tau/A$ and $l = r_0/AV_R$, Eq. (A-3) becomes

$$M_d = d\ V_R\ A^2 \int_0^l \left[\frac{d}{d\gamma}\ a\ (A\gamma)\right]\ (l - \gamma)\ d\gamma \tag{A-4}$$

or

$$\frac{M_d}{d\ V_R\ A^2} = \int_0^l \left[\frac{d}{d\gamma}\ a\ (A\gamma)\right]\ (l - \gamma)\ d\gamma. \tag{A-5}$$

With use of the expressions for a (τ) presented in Eq. (90), the integral in Eq. (A-4) may be evaluated to yield

$$\left.\begin{aligned}
\frac{M_d}{d\ V_R\ A^2} &= (l + 2)\ e^{-l} + l - 2 && \text{for } n = 1\\
&= -\ (l^2 + 2l + 2)\ e^{-l} + 2 && \text{for } n = 2\\
&= l^3\ \frac{e^{-l}}{2\,!} && \text{for } n = 3\\
&= -\ (l^4 - 4l^3)\ \frac{e^{-l}}{3\,!} && \text{for } n = 4\\
&= (l^5 - 10l^4 + 20l^3)\ \frac{e^{-l}}{4\,!} && \text{for } n = 5\\
&= -\ (l^6 - 18l^5 + 90l^4 - 120l^3)\ \frac{e^{-l}}{5\,!} && \text{for } n = 6
\end{aligned}\right\} \tag{A-6}$$

The curves of this nondimensionalized miss are plotted in Fig. A-1.

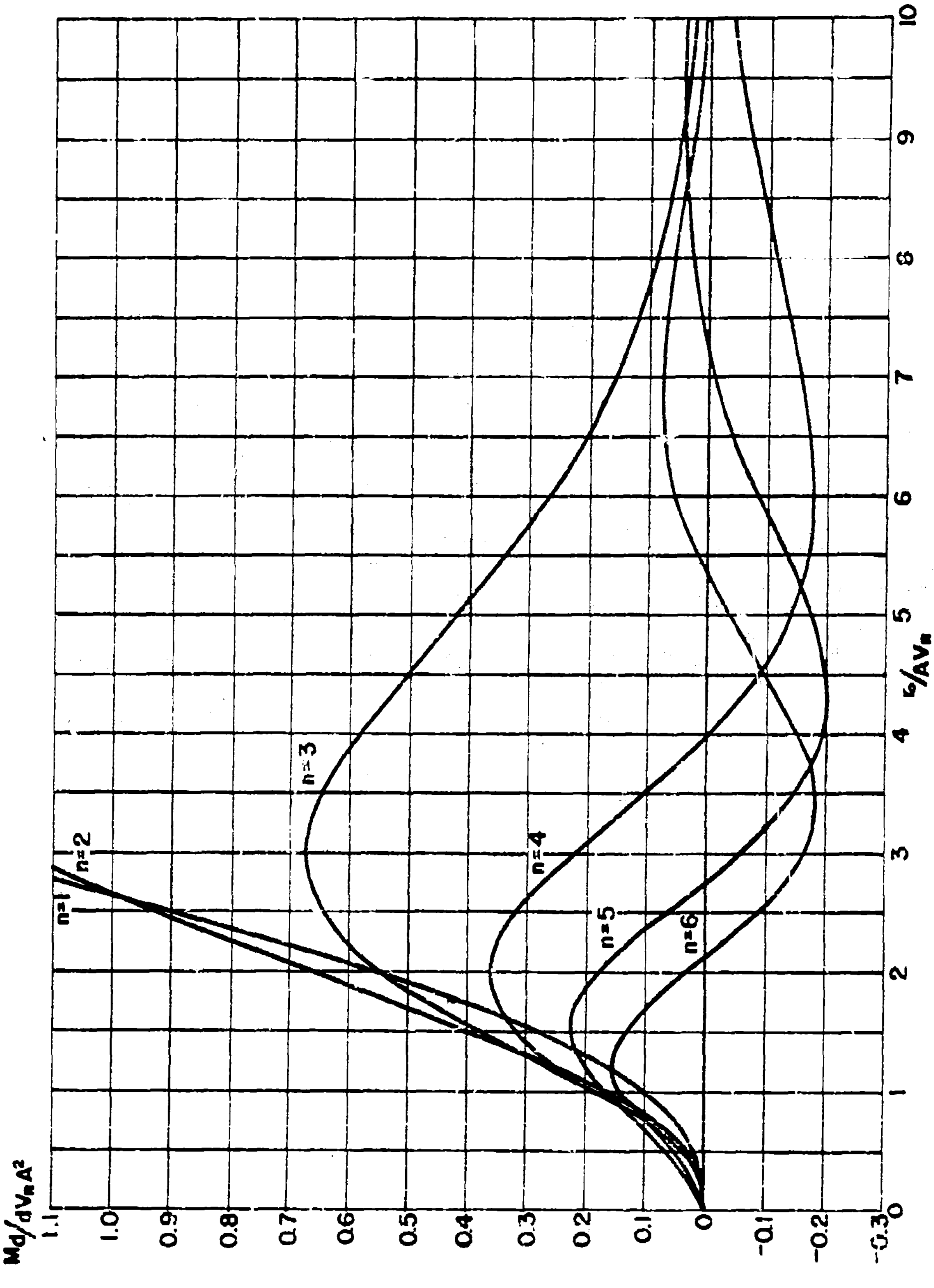

Fig. A-1. Nondimensionalized miss caused by control-system drift.

APPENDIX B. — EFFECT OF TIME LAG OF THE TARGET

The effect of time lag of the target on target maneuvers should at least be briefly considered, since so much of this study is concerned with target maneuvers characterized by sudden changes in the lateral acceleration of the target.

For the single lateral-acceleration turn considered in Sec. 2.3, the minimum value of rms miss caused by the worst (starting at the range that maximizes the miss caused by target turn) maneuver must be at least

$$(M_{MIN})_M = 1.2904 \; (2\pi \Phi_N)^{1/5} \; (a_T \cos \alpha_0)^{2/5}. \tag{47}$$

With the conditions, $2\pi \Phi_N = 100$ and $a_T \cos \alpha_0 = 80.5$, the value of miss is equal to 19.6 feet.

It is of interest to determine the amount this value of miss is reduced by the assumption that a time interval is required for the target to reach full lateral acceleration. For any target maneuver x_T, the minimum rms miss is given by Eq. (42) as

$$M_{MIN}^2 = \frac{x_T^2 \left(\frac{r_0}{V_B}\right)}{1 + \frac{1}{2\pi \Phi_N} \int_0^{\frac{r_0}{V_B}} x_T^2 (\tau)\, d\tau}. \tag{42}$$

If, instead of reaching full lateral acceleration instantaneously, the target increases its lateral acceleration linearly until full acceleration a_T is reached, then

$$\left.\begin{aligned} x_T (t) &= \frac{1}{6} \left(\frac{a_T \cos \alpha_0}{t_2}\right) (t - t_1)^3 & t_1 < t < t_2 \\ &= (a_T \cos \alpha_0)\, t_2^2 \left[\frac{1}{6} - \frac{1}{2}\left(\frac{t - t_1}{t_2}\right) + \frac{1}{2}\left(\frac{t - t_1}{t_2}\right)^2\right] & t_2 < t \end{aligned}\right\} \tag{B-1}$$

where t_1 is the time at which the maneuver starts and t_2 is the time at which full lateral acceleration a_T is reached.

Substitution of Eq. (B-1) into Eq. (42) gives the minimum value of rms miss as

$$M_{MIN}^2 = (a_T \cos \alpha_0)^2 \, t_2^4 \, \frac{N}{D}$$

where

$$\left.\begin{aligned} N &= \frac{1}{36} + \frac{1}{6}(p-1) + \frac{5}{12}(p-1)^2 + \frac{1}{2}(p-1)^3 + \frac{1}{4}(p-1)^4, \\ D &= 1 + K\left[\frac{1}{252} + \frac{1}{36}(p-1) + \frac{1}{12}(p-1)^2 + \frac{5}{36}(p-1)^3 + \frac{1}{8}(p-1)^4 + \frac{1}{20}(p-1)^5\right], \\ p &= \frac{t_1}{t_2}, \\ K &= \frac{(a_T \cos \alpha_0)^2 \, t_1^4}{2\pi \Phi_N}. \end{aligned}\right\} \tag{B-2}$$

For $t_2 - t_1 = 1$ second and for the same values of $a_T \cos \alpha_0$ and Φ_N that are used for the case in which the target acceleration is an instantaneous step, the maximum value of M_{MIN} (with respect to t_1) given by Eq. (B-2) is found to be 18.6 feet or 0.95 times the M_{MIN} value for the instantaneous-step case. As would be expected intuitively, the worst (causing maximum miss) time to begin the linear increase in target acceleration is approximately $(t_2 - t_1)/2$ seconds before the start of worst maneuver of the instantaneous step in acceleration. For the values of $a_T \cos \alpha_0$ and Φ_N considered here, the worst step maneuver starts with 1 second of flight time remaining. With $t_2 - t_1 = 1$ second, the worst maneuver starts with 1.5 seconds of flight time remaining.

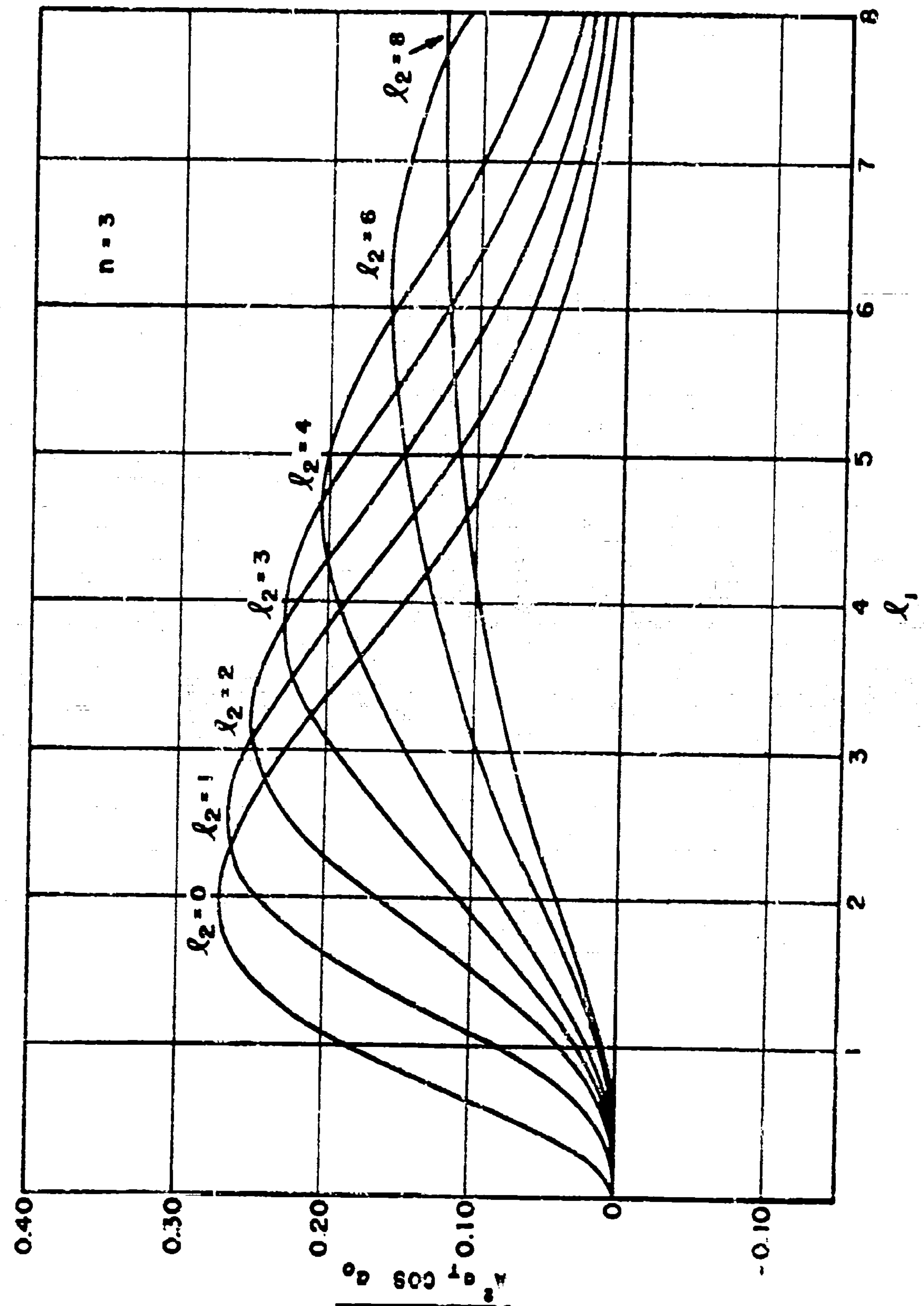

Fig. B-1. Miss caused by the maneuver of a target with lag.

Because these calculations are for an optimum system in the presence of noise, the question arises whether the decrease in miss caused by the more realistic target maneuver is masked by the miss caused by noise. It is instructive to consider the miss caused by target maneuver alone. Calculations are made for the proportional-navigation-with-simple-time-lag control system. (Similar calculations, not included here, can be made for other control systems.)

In Sec. 2.2 the miss caused by a target motion $x_T(t)$ is given by

$$M_T = x_T\left(\frac{r_0}{V_R}\right) - \int_0^{\frac{r_0}{V_R}} a'(\tau)\, x_T\left(\frac{r_0}{V_R} - \tau\right) d\tau. \tag{23}$$

For the target maneuver considered above, $x_T(t)$ is given by Eq. (B-1). For the proportional-navigation-with-simple-time-lag control equation with $n = [(b+1)\, V_M \cos \beta_0]/V_R = 3$, $a(\tau)$ is given as

$$a(\tau) = 1 - \frac{e^{-l}}{2}\,(l^2 - 4l + 2) \tag{90}$$

where $l = \tau/A$. Substitution of these expressions for $a(\tau)$ and $x_T(t)$ into Eq. (23) results in

$$\left.\begin{aligned} \frac{M_T}{A^2\, a_T \cos \alpha_0} &= \frac{1}{l_2} - \frac{1}{2l_2}\, e^{-l_1}\,(l_1^2 + 2l_1 + 2) && \text{for } l_1 < l_2 \\ &= \frac{1}{2l_2}\, e^{-(l_1 - l_2)}\,[(l_1 - l_2)^2 + 2\,(l_1 - l_2) + 2] \\ &\quad - \frac{1}{2l_2}\, e^{-l_1}\,(l_1^2 + 2l_1 + 2) && \text{for } l_2 < l_1 \end{aligned}\right\} \tag{B-3}$$

where $l_1 = (r_0/V_R - t_1)/A$ and $l_2 = (t_2 - t_1)/A$.

In Fig. B-1 this nondimensional miss $M_T/A^2\, a_T \cos \alpha_0$ is plotted as a function of l_1 for several values of l_2. Even though the target requires a time equal to four times the missile time constant to reach full lateral acceleration, the maximum value of the miss is reduced only 23 per cent from the miss caused by an instantaneous step in target acceleration.

The results indicate that assumption of an instantaneous step in target acceleration leads to values of miss representative of a wide class of target maneuvers both physically realisable and probable.

APPENDIX C. — STABILITY

As the concept of stability for time-variant systems is not widely utilized, a brief discussion of its meaning appears useful. Stability, in its usual mathematical sense, implies continuity with respect to initial conditions and driving functions.

At a particular time, a given system (or differential equation) may be stable or unstable, depending on the variable considered. For example, a system may be stable with respect to one variable and not with respect to its derivative. As time varies, a system may pass from stability to instability.

The concept of stability is most conveniently defined in terms of the frequency characteristic—the Laplace transform of the impulse response (derivative of step response). If the frequency characteristic has poles in the right half plane, the system is said to be unstable. The concept of relative stability for stable systems can be introduced in terms of the position of the poles in the left half plane.

A homing system is stable with respect to its miss distance if the frequency characteristic defined by

$$F(s) = \int_0^{\infty} e^{-s\tau} a'(\tau)\, d\tau$$

has no poles in the right half plane. For a finite time of flight r_0/V_R, the associated finite transform

$$F_f(s) = \int_0^{\frac{r_0}{V_R}} e^{-s\tau} a'(\tau)\, d\tau \tag{C-1}$$

has no poles anywhere; but if it assumes very large values near any points in the right half plane, for practical purposes the system may be said to be unstable.

If the coefficients of the differential equation change slowly enough, the frequency characteristic of the system bears approximately the same relation to the polynomial associated with the instantaneous values of the coefficients of the differential equation as the frequency characteristic of a constant-coefficient system bears to its associated polynomial. Thus with slowly changing coefficients, the location of the zeros of the associated polynomial determines the instability or relative stability of the system.

However, it must be kept in mind that the preceding remarks apply only when the coefficients are changing slowly. For proportional-navigation systems the changes in coefficients caused by the closing range are quite large toward the end of the flight. Consequently, any interpretation of the positions of the zeros of the associated polynomial must be viewed with question, if not disregarded.

APPENDIX D. — GLOSSARY OF SYMBOLS

A	—	Missile time constant (sec).
a	—	Missile step response.
a_T	—	Target lateral acceleration (ft/sec^2).
$b + 1$	—	Navigation constant in missile-control equation.
b_1, b_2	—	Coefficients in optimum transfer function.
C, C_1, C_2	—	Constants
D	—	Operator characterizing proportional-navigation system.
d	—	Drift term in control equation (rad/sec).
E	—	Nondimensional miss distance.
E_{MIN}	—	Minimum value of nondimensional miss E.
F	—	Transfer function, Laplace transform of a′(t).
f, f_1, f_2	—	Functions of time.
G	—	Operator characterizing general homing equation.
h	—	Nondimensionalizing variable.
j	—	Summation index.
K	—	Constant evaluated in determination of rms miss.
K_N	—	Constant in expression for miss caused by noise.
K_T	—	Constant in expression for miss caused by target motion.
k	—	Twice the average turning rate of the target.
L	—	Aircraft length.
L	—	Linear operator.
l, l_1	—	Nondimensional time.
l_2	—	Nondimensional target time lag, $(t_2 - t_1)/A$.
M	—	Miss distance.
M_E	—	Miss caused by error in initial heading of missile.
M_N	—	Miss caused by radar noise.
M_T	—	Miss caused by target maneuver.
M_a	—	Miss caused by initial lateral acceleration of missile.
M_d	—	Miss caused by drift.
M_{MIN}	—	Minimum value of miss.
$(M_{MIN})_M$	—	Maximum value of M_{MIN}.
m	—	Twice the nondimensional turning rate of target.
n	—	Navigation constant times speed ratio, $[(b + 1)V_M \cos \beta_0]/V_R$.
P	—	Probability of miss less than fixed distance p.
p	—	Fixed miss distance (lethal radius).
p	—	Operator symbol d/dt.
p, q	—	Polynomials.
r	—	Missile-to-target range.
r_A	—	Apparent value of r.
r_0	—	Initial value of r.
r_1	-	Range at which target turn begins.
[illegible]	—	Target wing span.
s	—	Complex frequency variable, Laplace transform variable.
[illegible]	—	Time variable.

t_1 — Time at which target turn begins.

t_2 — Time at which target reaches full lateral acceleration.

u — Nondimensionalized complex frequency variable.

V_M — Missile velocity (fps).

V_R — Relative velocity (fps).

V_T — Target velocity (fps).

v — Nondimensionalized frequency variable (usually a variable of integration).

x_M — Lateral movement of missile position from reference line.

x_N — Lateral value of noise from reference line.

x_T — Lateral movement of target position from reference line.

y — Variable of integration.

Z — Function of missile time lag and gain, $A(b + 1)^{-1.25}$.

z — Missile position referred to zero initial velocity reference.

α — Missile-to-target line-of-sight angle.

α_A — Apparent value of α.

α_{At} — Transient value of α_A.

α_t — Transient value of α.

α_0 — Initial value of α.

β_A — Apparent value of the angle between missile heading and line of sight.

β_0 — Initial value of the angle between missile heading and line of sight.

Γ — Spectral density factor.

$\gamma, \gamma_1, \gamma_2$ — Nondimensional time variables (usually variables of integration).

ϵ — Variable which approaches zero in limiting case of spectral density of target motion.

θ — Missile-velocity vector angle.

θ_0 — Initial value of θ.

λ — Target-to-missile speed ratio.

ρ — Variable of integration.

σ — RMS value of miss.

σ_N — RMS value of miss caused by noise.

σ_T — RMS value of miss caused by target motion.

τ — Time variable (usually as variable of integration).

τ_1, τ_2, etc. — Time variables of integration.

Φ_N — Spectral density of radar noise x_N (defined for $-\infty < \omega < \infty$).

Φ_T — Spectral density of target motion x_T (defined for $-\infty < \omega < \infty$)

ϕ — Target-velocity vector angle.

ϕ_0 — Initial value of ϕ.

ϕ_t — Transient value of ϕ.

Ψ — Spectral density factor.

ω — Frequency variable (rad/sec).

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ABSTRACT:

A dynamic analysis is made of the guided missile homing problem. The homing missile kinematic equations are linearized to permit the application of general optimization techniques. The minimum rms values of miss are calculated for two sets of conditions: white radar noise and an intelligent target maneuver, and white radar noise and a random target maneuver. The linearized equation for the flight path of a missile guided by a proportional-navigation-with-simple-time-lag control system is solved for certain values of the navigation constant. Optimum control system parameters are calculated for the two cases considered, and the minimum rms miss for this system is compared with the minimum rms miss for any linear control system. Values of miss caused by an error in initial heading and by an initial lateral acceleration are calculated.

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