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NON-LINEAR VIBRATION PROBLEMS TREATED BY THE AVERAGING METHOD OF W. RITZ

Part II

Single Degree of Freedom Systems
Single Term Approximations

BY

KARL KLOTTER

TECHNICAL REPORT NO. 17, PART II

PREPARED UNDER CONTRACT N6onr-251, TASK ORDER 2
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FOR
OFFICE OF AIR RESEARCH
AND
OFFICE OF NAVAL RESEARCH

DIVISION OF ENGINEERING MECHANICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA

JUNE 1951

Air Force Flight Control Lab.
Wright Field

ERRATA

Technical Report No. 17
(Part I)

p. 9 eq. (9) first term should be

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}}\right) -$$

p. 10 in eq. (11), $y(x)$ should read $\tilde{y}(x)$.

p. 12 eq. (14) should read

$$\frac{\partial I}{\partial a_k} = \left[\psi_k \frac{\partial F}{\partial \tilde{y}''}\right]_{x_0}^{x_1} - \left[\psi_k' \frac{\partial F}{\partial \tilde{y}''} - \psi_k \frac{d}{dx}\left(\frac{\partial F}{\partial \tilde{y}''}\right)\right]_{x_0}^{x_1} + - \cdots\cdots$$

$$+ \int_{x_0}^{x_1} \psi_k \left\{\frac{\partial F}{\partial \tilde{y}} - \frac{d}{dx}\left(\frac{\partial F}{\partial \tilde{y}'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial \tilde{y}''}\right) - + \cdots (-1)^n \frac{d^n}{dx^n}\left(\frac{\partial F}{\partial \tilde{y}^{(n)}}\right)\right\} dx .$$

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Stanford, California

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STANFORD UNIVERSITY, DIVISION OF ENGINEERING MECHANICS, CALIF. (TECHNICAL REPORT NO. 17, PART II)

NON-LINEAR VIBRATION PROBLEMS TREATED BY THE AVERAGING METHOD OF W. RITZ - PART II - SINGLE DEGREE OF FREEDOM SYSTEMS SINGLE TERM APPROXIMATIONS

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A. INTRODUCTION

1. The Averaging Method; Single Term Approximation.

In Part I of the present report the Averaging Method of W. Ritz was described and discussed. It was emphasized there that this method has been rather extensively applied to statical problems, especially for elastic bodies (i.e., rods, plates, shells, etc.). In this Part II of the report we shall apply the method to dynamical problems, with special reference to certain types of vibration problems. [1]

The advantages of the method were also discussed in some detail in Part I, and it was pointed out that the results obtained from the application of the method appear in the closed form of an equation (or a system of equations). This latter feature is of importance in that it readily allows a discussion of the influence of the parameters involved.

For convenience we shall repeat here the technique of the method in the form of what almost may be called a " recipe". In terms of a dynamical problem the " recipe" consists of the two statements:

Given

$$E[q(t)] = 0 \tag{1.1}$$

---

[1] Dynamical problems have hardly ever been attacked by this method before; only one effort in this direction is found in the literature. See, " On One-Term Approximations of Forced Nonharmonic Vibrations" by G. Schwesinger; Journal of Applied Mechanics; vol. 17, June 1950; pp. 202-208.

as the differential equation of the problem, we assume the solution to be of the form

$$\tilde{q}(t) = a_1 \psi_1(t) + a_2 \psi_2(t) + \cdots + a_n \psi_n(t) \qquad (1.2)$$

where the $\psi_k(t)$ are a given set of functions, properly chosen, and the coefficients a_k are to be determined. The "best" solution will be that one in which the coefficients a_k are determined from the set of equations

$$\int_{t_0}^{t_1} E[\tilde{q}(t)]\, \psi_k(t)\, dt = 0 \qquad (1.3)$$

where t_0 and t_1 are fixed values.

In this Part II we shall be dealing chiefly with approximations which consist of a single term only; i.e.

$$\tilde{q}(t) = a\,\psi(t) \qquad (1.2a)$$

The single coefficient a, then, is to be determined from the condition

$$\int_{t_0}^{t_1} E[a\,\psi(t)]\, \psi(t)\, dt = 0 \qquad (1.3a)$$

Inasmuch as this Part II of the report will deal with periodic oscillations, it will be appropriate to choose the coordinate function ψ to be either $\cos\tau$ or $\sin\tau$ where τ denotes the product of some frequency with the time t. In some cases, however, the harmonic function will have to be chosen as $-\cos(\tau-\varepsilon)$ which in turn is equivalent to

$\cos\tau\cdot\cos\varepsilon + \sin\tau\cdot\sin\varepsilon$. Therefore, the most general form of assumption will be

$$\tilde{q} = A\cos\tau + B\sin\tau \quad . \tag{1.2b}$$

The equations from which the coefficients A and B are to be determined are then:

$$\begin{cases} \int_0^{2\pi} E[A\cos\tau + B\sin\tau]\cos\tau\, d\tau = 0 \\ \int_0^{2\pi} E[A\cos\tau + B\sin\tau]\sin\tau\, d\tau = 0 \quad . \end{cases} \tag{1.3b}$$

Because of the equivalence of (1.2b) to the expression $C\cos(\tau-\varepsilon)$ we shall continue to speak of a "single term approximation" in those cases where the two coordinate functions $\cos\tau$ and $\sin\tau$ appear (but both of the same frequency).

2. Systems Treated in Part II.

First, in Chapter B, the Averaging Method will be applied to cases where exact solutions are available, or can be obtained. This will allow a comparison between the exact values and those furnished by the approximate method.

There are two classes of systems, and motions, for which exact solutions can be found: First, the undamped vibrations of systems whose restoring forces depend upon a power, or a polynomial, of the displacement [see eqs. (4.9) and (4.21)]; these

are treated in Div. a. Second, the free or forced vibrations in the large variety of systems for which the plot of the restoring force versus the displacement (the so-called " characteristic line") is composed of straight line segments (see Figs. 7/1 and 8/1); these are treated in Div. b.

For both classes, we will show explicitly how the exact solutions are obtained. Then we will use the Averaging Method to produce the approximate solution, using a single term, and then we will compare the results. We shall see that the approximate results are in very good agreement with the exact ones, in most cases, and that the deviations which do occur can be readily explained. It will be obvious that the approximate results are obtained much easier than are the exact ones.

In Chapter C we consider a case for which no exact solution is known; in particular, we deal with the forced vibrations of a system having an arbitrary restoring force and an arbitrary damping force [see, e.g., eq. (12.1)]. The solution by the Averaging Method for this very important and general system is given in a closed form which makes it most valuable. It is hoped that the discussion of the results and especially of the response curves (amplitude-frequency-curves) will constitute a definite contribution to the knowledge of the behavior of such systems.

The discussion of the response curves of this system will center about the " resonance amplitude" and about certain curves which separate special regions in the plane of the diagram. These

special regions will later be proved to contain the representative points for either stable or unstable motion. A very simple stability criterion will then be established, but its proof and discussion of the details of the stability problems will be deferred to Part III of the report.

Finally, Chapter D deals with a few cases which are intended to indicate examples of further applications of the method.

3. Acknowledgments

During the course of the work relating to the problems treated in this part of the report, the author enjoyed the assistance of a number of collaborators. Most valuable were the contributions of two of them: Mr. H. Norman Abramson contributed especially to discussion of the response curves and to the preparation of the text, and Mr. James A. Plough furnished the bulk of the numerical results contained herein, either by direct evaluation of the respective formulae or by supervising the computing staff, and by preparing the figures.

B. UNDAMPED VIBRATIONS IN SYSTEMS WHERE EXACT SOLUTIONS ARE AVAILABLE FOR COMPARISON

a. The Restoring Force Depends Upon Pure Powers or Upon Polynomials of the Displacement

4. The Exact Solutions

α) General Expressions

It is well known that the solution of the differential equation of motion for the free undamped vibrations [1]

$$\ddot{q} + \kappa^2 f(q) = 0 \tag{4.1}$$

can be expressed in the form of quadratures. The results are given in Eqs. (4.4a') and (4.7). Nevertheless, we repeat here the steps which lead to those results.

First, by making use of the identity

$$\ddot{q} = \frac{1}{2}\frac{d(v^2)}{dq}$$

where v is equivalent to $\dot{q}$, the differential equation (4.1) is transformed into

$$\frac{d(v^2)}{dq} + 2\kappa^2 f(q) = 0 \tag{4.2}$$

---

1) For convenience q is considered to be a dimensionless quantity.

This differential equation is one of first order with v^2 as the dependent variable and q as the independent variable. Its integral is

$$v^2 = 2k^2 \int_q^Q f(u)\,du$$

where Q denotes the displacement q when $v = 0$. From this we obtain

$$|v| = k\sqrt{2}\sqrt{\int_q^Q f(u)\,du} \qquad (4.3)$$

Now, upon replacing v by $\frac{dq}{dt}$ this equation is again a first order differential equation (q is now the dependent variable and t is the independent variable). The variables are separable, thus leading to the integral

$$t - t_0 = \frac{1}{k\sqrt{2}} \int_0^q \frac{ds}{\sqrt{\int_s^Q f(u)\,du}} \qquad (4.4a)$$

where $t = t_0$ for $q = 0$, or to the integral

$$t - t_0 = \frac{1}{k\sqrt{2}} \int_q^Q \frac{ds}{\sqrt{\int_s^Q f(u)\,du}} \qquad (4.4b)$$

where $t = t_0$ for $q = Q$.

Putting $t_0 = 0$ and denoting the undetermined integral over $f(q)$ by $I(q)$, the two equations (4.4a) and (4.4b)

become

$$t = \frac{1}{K\sqrt{2}} \int_0^{q} \frac{ds}{\sqrt{I(Q) - I(s)}} \tag{4.4a'}$$

and

$$t = \frac{1}{K\sqrt{2}} \int_q^{Q} \frac{ds}{\sqrt{I(Q) - I(s)}} \tag{4.4b'}$$

Assuming that the motion is oscillatory in character and denoting the time from zero displacement $(q = 0)$ to maximum displacement $(q = Q)$ by t_1, we obtain

$$t_1 = \frac{1}{K\sqrt{2}} \int_0^{Q} \frac{ds}{\sqrt{I(Q) - I(s)}} .$$

For subsequent cases we will assume that $f(q)$ is an odd function of q, i.e.

$$f(-q) = -f(q) \tag{4.5}$$

or, in a slightly different notation,

$$f(q) = (\operatorname{sgn} q) \cdot f(|q|) . \tag{4.6}$$

If the restoring force is an odd function of the displacement, the time t_1 is equivalent to the quarter period of the motion, $\frac{T}{4}$. The result may, therefore, be written in the form

$$(4.7) \qquad \frac{T}{4} = \frac{1}{\kappa\sqrt{2}} \int_0^Q \frac{ds}{\sqrt{I(Q) - I(s)}}$$

We will now look for those cases where the integral in eq. (4.7) can be expressed explicitly in terms of Q. Thus we will be dealing with cases where exact solutions are obtainable.

(a) Pure Powers of the Displacement

One class of systems for which exact solutions may be obtained, as defined above, are those having restoring forces dependent upon a pure power of the displacement. This means that, with respect to eq. (4.6), we have

$$(4.8) \qquad f(q) = (\operatorname{sgn} q) \cdot |q|^n$$

For positive values of q (which is all that we need because of the supposed symmetry) this equation will be equivalent to

$$(4.9) \qquad f(q) = q^n$$

The integral $I(q)$ then becomes

$$(4.10) \qquad I(q) = \frac{q^{n+1}}{n+1}$$

and, therefore, the expression (4.7) for the quarter period is

(4.11a) $$\frac{KT}{4} = \frac{\sqrt{n+1}}{\sqrt{2}} \int_0^Q \frac{ds}{\sqrt{Q^{n+1} - s^{n+1}}}$$

or, by setting $u = \frac{s}{Q}$,

(4.11b) $$\frac{KT}{4} = \frac{1}{\sqrt{Q^{n-1}}} \left(\sqrt{\frac{n+1}{2}} \int_0^1 \frac{du}{\sqrt{1-u^{n+1}}} \right) .$$

We now denote the expression within the parentheses, which depends upon the parameter n alone, by $\psi(n)$; thus

(4.12) $$\psi(n) = \sqrt{\frac{n+1}{2}} \int_0^1 \frac{du}{\sqrt{1-u^{n+1}}} .$$

Therefore, we have

(4.13) $$\frac{KT}{4} = \frac{1}{\sqrt{Q^{n-1}}} \psi(n) ,$$

Now we will introduce the quantity $\frac{2\pi}{\omega}$ for the period T, thereby obtaining

(4.14) $$\frac{\omega^2}{K^2} = \left[\frac{\pi^2/4}{\psi^2(n)} \right] Q^{n-1} .$$

The equations (4.13) and (4.14) show explicitly how the period T and the frequency ω depend upon the maximum displacement Q of the oscillation. The factor in brackets in

eq. (4.14) will be designated by $\chi(n)$:

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These equations contain the quantity $\psi(n)$, as given by eq. (4.12), whose numerical value may be determined by numerical integration for particular values of n. In special cases the expression reduces to certain values of known functions.

If $n = 0$ the expression $\psi(0)$ can be easily integrated giving

$$\psi(0) = \sqrt{\tfrac{1}{2}} \int_0^1 \frac{du}{\sqrt{1-u}} = \sqrt{2} \quad ; \tag{4.15a}$$

for $n = 1$, which corresponds to the well known linear case, we have

$$\psi(1) = \int_0^1 \frac{du}{\sqrt{1-u^2}} = \arcsin 1 = \frac{\pi}{2} \quad . \tag{4.15b}$$

The cases $n = 2$ and $n = 3$ lead to certain elliptic integrals [1]:

$$\psi(2) = \sqrt{\frac{3}{2}} \int_0^1 \frac{du}{\sqrt{1-u^3}} = \frac{\sqrt{3/2}}{\sqrt[4]{3}} F\left(75^\circ, \arccos \frac{\sqrt{3}-1}{\sqrt{3}+1}\right) \tag{4.15c}$$

$$= 1.7157$$

---

1) See e.g., Jahnke-Emde, Tables of Functions with Formulae and Curves, Dover Publications, 1943, p. 59.

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$$\frac{KT}{4} = \frac{1}{\sqrt{Q^{n-1}}} \left(\sqrt{\frac{n+1}{2}} \int_0^1 \frac{du}{\sqrt{1-u^{n+1}}} \right) . \tag{4.11b}$$

We now denote the expression within the parentheses, which depends upon the parameter n alone, by $\gamma(n)$; thus

$$\gamma(n) = \sqrt{\frac{n+1}{2}} \int_0^1 \frac{du}{\sqrt{1-u^{n+1}}} . \tag{4.12}$$

Therefore, we have

$$\frac{KT}{4} = \frac{1}{\sqrt{Q^{n-1}}} \gamma(n) . \tag{4.13}$$

Now we will introduce the quantity $\frac{2\pi}{\omega}$ for the period T, thereby obtaining

$$\frac{\omega^2}{K^2} = \left[\frac{\pi^2/4}{\gamma^2(n)} \right] Q^{n-1} . \tag{4.14}$$

The equations (4.13) and (4.14) show explicitly how the period T and the frequency ω depend upon the maximum displacement Q of the oscillation. The factor in brackets in

eq. (4.14) will be designated by $\chi(n)$:

$$\chi(n) = \frac{n^2/4}{\psi^2(n)} \quad . \tag{4.14a}$$

These equations contain the quantity $\psi(n)$, as given by eq. (4.12), whose numerical value may be determined by numerical integration for particular values of n. In special cases the expression reduces to certain values of known functions.

If $n = 0$ the expression $\psi(0)$ can be easily integrated giving

$$\psi(0) = \sqrt{\frac{1}{2}} \int_0^1 \frac{du}{\sqrt{1-u}} = \sqrt{2} \quad ; \tag{4.15a}$$

for $n = 1$, which corresponds to the well known linear case, we have

$$\psi(1) = \int_0^1 \frac{du}{\sqrt{1-u^2}} = \arcsin 1 = \frac{\pi}{2} \quad . \tag{4.15b}$$

The cases $n = 2$ and $n = 3$ lead to certain elliptic integrals [1]:

$$\psi(2) = \sqrt{\frac{3}{2}} \int_0^1 \frac{du}{\sqrt{1-u^3}} = \frac{\sqrt{3/2}}{\sqrt[4]{3}} F\left(75^\circ, \arccos \frac{\sqrt{3}-1}{\sqrt{3}+1}\right) \tag{4.15c}$$

$$= 1.7157$$

---

1) See e.g., Jahnke-Emde, Tables of Functions with Formulae and Curves, Dover Publications, 1943, p. 59.

$$(4.15d)\qquad \psi(3) = \sqrt{2}\int_0^1 \frac{du}{\sqrt{1-u^4}} = K\left(\frac{1}{\sqrt{2}}\right) = 1.8541.$$

If n is an integer greater than 3, or if n is not an integer, the integral in (4.12) must be evaluated numerically. However, one must note that the integral is an improper one; therefore, the usual methods of numerical integration may require modification here. As an example, we will now show how Simpson's rule can be employed for evaluating this improper integral.

Simpson's rule applied to the integral

$$(4.16a)\qquad S = \int_a^b \varphi(x)\,dx$$

gives

$$(4.16b)\qquad S = \frac{h}{3}\left[\varphi(a) + 4\varphi(a+h) + 2\varphi(a+2h) + 4\varphi(a+3h) + \varphi(b)\right]$$

if we divide the interval $b-a$ into four parts of length $h = \frac{1}{4}(b-a)$.

However, in attempting to use this rule for evaluating the expression

$$(4.17)\qquad \psi(n) = \sqrt{\frac{n+1}{2}}\int_0^1 \frac{du}{\sqrt{1-u^{n+1}}}$$

we find that at the upper boundary the value $\varphi(b)$ of the integrand is unlimited and therefore cannot be introduced in (4.16b). In order to find an integral whose integrand remains bounded we apply a transformation of variables:

$$z^2 = 1-u \qquad u = 1-z^2$$
$$2z\,dz = -du \qquad u^{n+1} = (1-z^2)^{n+1} \tag{4.18}$$

With this transformation the integral on the right hand side of (4.17) may be written as

$$\int_0^1 \frac{du}{\sqrt{1-u^{n+1}}} = 2\int_0^1 \frac{z\,dz}{\sqrt{1-(1-z^2)^{n+1}}}, \tag{4.19}$$

In this form the integrand is limited at $z=0$, but it is undefined; however, this value can be determined, and will be found to be

$$\varphi(0) = \frac{1}{\sqrt{n+1}}.$$

Therefore, Simpson's rule (4.16b), applied to the right hand side of (4.19), allows us to write, because of (4.17)

$$\psi(n) = 2\sqrt{\frac{n+1}{2}}\int_0^1 \frac{z\,dz}{\sqrt{1-(1-z^2)^{n+1}}}$$

$$\approx \frac{1}{6}\sqrt{\frac{n+1}{2}}\left[\varphi(0) + 4\varphi\left(\tfrac{1}{4}\right) + 2\varphi\left(\tfrac{1}{2}\right) + 4\varphi\left(\tfrac{3}{4}\right) + \varphi(1)\right]$$

and finally,

$$(4.20)\qquad \psi(n) = \frac{1}{\delta}\sqrt{\frac{n+1}{2}}\left[\frac{1}{\sqrt{n+1}} + \frac{1}{\sqrt{1-\left(\frac{15}{16}\right)^{n+1}}} + \frac{1}{\sqrt{1-\left(\frac{3}{4}\right)^{n+1}}} + \frac{3}{\sqrt{1-\left(\frac{7}{16}\right)^{n+1}}} + 1\right].$$

Table I gives values of $\psi(n)$, as defined by (4.12) and employed in (4.13), and values of $\chi(n)$, as defined by (4.14a) and employed in (4.14), for various values of n. For n less than or equal to 3 the results of equations (4.15) are tabulated and for n greater than 3 the $\psi(n)$ were obtained from (4.20).

γ) Polynomials of the Displacement

We may apply the procedures given in the preceding section not only to the case of pure powers but also those cases where we deal with polynomials in $f(q)$. Therefore, we have a second broad class of systems for which exact solutions may be found. In particular, we will treat a binomial [see (4.21)]; the generalization will then be obvious. We assume

$$(4.21)\qquad f(q) = q^n + \mu q^m \qquad , \quad m > n \neq 0$$

and again consider only odd functions [see (4.5)]. The expression for $I(q)$ then becomes

$$(4.22)\qquad I(q) = \frac{1}{n+1} q^{n+1} + \frac{\mu}{m+1} q^{m+1} .$$

Instead of our previous equation (4.11a) for the quarter period we now have

$$(4.23a)\qquad \frac{KT}{4} = \frac{1}{\sqrt{2}} \int_0^Q \frac{ds}{\sqrt{\frac{1}{n+1}Q^{n+1} + \frac{\mu}{m+1}Q^{m+1} - \frac{1}{n+1}s^{n+1} - \frac{\mu}{m+1}s^{m+1}}}$$

or, corresponding to (4.11b)

$$(4.23b)\qquad \frac{KT}{4} = \frac{1}{\sqrt{Q^{n-1}}} \left\{ \sqrt{\frac{n+1}{2}} \int_0^1 \frac{du}{\sqrt{\left[1+\bar{\mu}(Q)\right] - \left[u^{n+1} + \bar{\mu}(Q)\,u^{m+1}\right]}} \right\}$$

where $\bar{\mu}(Q)$ is defined by

$$(4.24)\qquad \bar{\mu}(Q) = \mu \frac{n+1}{m+1} Q^{m-n} .$$

Again, we may use ψ to represent the expression within the braces of (4.23b). In this case ψ will, of course, depend not only upon n, but also upon m and $\bar{\mu}$; the latter quantity, according to (4.24), contains the maximum displacement Q.

The equation (4.23b) may be evaluated by means of elliptic integrals if neither m nor n exceeds 3. As an example, let us evaluate the expression for the case of $m = 3$ and $n = 1$.

From (4.23b) we have

$$\psi = \int_0^1 \frac{du}{\sqrt{(1+\frac{1}{2}\mu Q^2) - (u^2 + \frac{1}{2}\mu Q^2 u^4)}}$$

and with

(4.25) $$\theta = \mu Q^2$$

we have

(4.26) $$\frac{\kappa T}{4} = \frac{1}{\sqrt{1+\theta}} K\left(\frac{1}{\sqrt{2(1+\frac{1}{\theta})}}\right)$$

or

(4.27) $$\eta^2 = \frac{1+\theta}{\frac{4}{\pi^2} K^2\left(\frac{1}{\sqrt{2(1+\frac{1}{\theta})}}\right)} ,$$

where

(4.28) $$\eta = \frac{\omega}{\kappa} .$$

In case the exponent m is greater than 3 the integral in (4.23) may be evaluated, as before, by applying Simpson's rule. As the procedure is not difficult it may be sufficient here to note only the result [corresponding to (4.20)]:

$$(4.29)\qquad \gamma = \frac{1}{6}\sqrt{\frac{n+1}{2}}\,\frac{1}{\sqrt{1+\bar{\mu}}}\left[\frac{1}{\sqrt{1+\frac{n+\bar{\mu}m}{1+\bar{\mu}}}} + \frac{1}{\sqrt{1-\frac{(15/16)^{n+1}+\bar{\mu}(15/16)^{m+1}}{1+\bar{\mu}}}}\right.$$

$$\left. + \frac{1}{\sqrt{1-\frac{(3/4)^{n+1}+\bar{\mu}(3/4)^{m+1}}{1+\bar{\mu}}}} + \frac{3}{\sqrt{1-\frac{(7/16)^{n+1}+\bar{\mu}(7/16)^{m+1}}{1+\bar{\mu}}}} + 1\right].$$

5. The Approximate Solutions

α) Pure Powers of the Displacement

The differential equation of motion of the system which we shall treat is

$$(5.1)\qquad \ddot{q} + \kappa^2 f(q) = 0$$

with

$$(5.2)\qquad f(q) = (\operatorname{sgn} q)\,|q|^n$$

which for positive values of q reduces to

$$(5.2a)\qquad f(q) = q^n .$$

In accordance with the discussion of section 1 the assumed form of solution will be taken to be [see (1.2a)]

$$\tilde{q} = Q \cos \tau \quad , \quad \tau = \omega t \quad . \tag{5.3}$$

Introducing this assumed solution into the differential equation (5.1), with $f(q)$ given by (5.2a), we obtain from (1.3a) and considering that $f(q)$ is odd

$$\int_0^{\frac{\pi}{2}} \left(-\omega^2 Q \cos^2 \tau + \kappa^2 Q^n \cos^{n+1} \tau \right) d\tau = 0 \quad . \tag{5.4}$$

From this equation we derive the relation

$$\frac{\omega^2}{\kappa^2} = Q^{n-1} \varphi(n) \tag{5.5}$$

where $\varphi(n)$ denotes

$$\varphi(n) = \frac{4}{\pi} \int_0^{\frac{\pi}{2}} \cos^{n+1} \tau \, d\tau \quad . \tag{5.6}$$

If n is an odd integer we may express $\varphi(n)$ as

$$n \geq 3 \; ; \quad \varphi(n) = \frac{n}{n+1} \cdot \frac{n-2}{n-1} \cdot \cdots \cdot \frac{3}{4} \quad , \tag{5.6a}$$

and if n is an even integer

$$n \geq 2 \; ; \quad \varphi(n) = \frac{n}{n+1} \cdot \frac{n-2}{n-1} \cdot \cdots \cdot \frac{2}{3} \cdot \frac{4}{\pi} \quad . \tag{5.6b}$$

Table 1 shows the values of $\varphi(n)$ for the integer arguments from $n=0$ to $n=7$.

β) Polynomials of the Displacement.

Again we consider the equation of motion to be given by (5.1). In this section we designate the restoring force as

$$f(q) = (\operatorname{sgn} q)\left(|q|^n + \mu |q|^m\right) \tag{5.7}$$

or, for positive values of q,

$$f(q) = q^n + \mu q^m . \tag{5.7a}$$

Corresponding to (5.4) for pure powers, we now have

$$\int_0^{\frac{\pi}{2}} \left[-\omega^2 Q \cos^2 \tau + \kappa^2 Q^n \cos^{n+1} \tau + \kappa^2 \mu Q^m \cos^{m+1} \tau \right] d\tau = 0 . \tag{5.8}$$

Therefore, as the final expression we obtain

$$\frac{\omega^2}{\kappa^2} = Q^{n-1}\left[\varphi(n) + \mu Q^{m-n} \varphi(m)\right] \tag{5.9}$$

which is, of course, a generalization of (5.5).

If $n = 1$, this equation reduces to

$$\frac{\omega^2}{\kappa^2} = 1 + \mu Q^{m-1} \varphi(m) \; ; \tag{5.9a}$$

and if we take, as examples, $m = 3$ and $m = 5$ we obtain

$$\frac{\omega^2}{\kappa^2} = 1 + \frac{3}{4} \mu Q^2 \tag{5.9b}$$

and

$$\frac{\omega^2}{\kappa^2} = 1 + \frac{5}{8} \mu Q^4 \; . \tag{5.9c}$$

We might draw attention here to the very concise form of the results which the Averaging Method furnishes, as compared with exact, or "nearly exact", results of section 4.

It is not difficult to extend these results for a two term polynomial to expressions involving more than two terms. Let us assume that, instead of eq. (5.7a) we have

$$f(q) = \sum_{\lambda} \mu_\lambda q^\lambda \; , \quad \text{with} \quad \mu_1 = 1 \; . \tag{5.10}$$

Following through the same procedure as before, we may obtain

$$\frac{\omega^2}{\kappa^2} = \sum_{\lambda} \mu_\lambda \varphi(\lambda) Q^{\lambda - 1} \; . \tag{5.11}$$

As an example, let us expand the function $\sin\varphi$ in a Taylor's series; taking only the first four terms we may write

$$f(\varphi) = \varphi - \frac{1}{3!}\varphi^3 + \frac{1}{5!}\varphi^5 - \frac{1}{7!}\varphi^7 \qquad (5.12)$$

$$= \sum_{n=1}^{4} (-1)^{n-1} \varphi^{2n-1} \frac{1}{(2n-1)!} .$$

Also

$$\mu_\lambda = \mu_{2n-1} = (-1)^{n-1} \frac{1}{(2n-1)!} . \qquad (5.13)$$

We therefore obtain, from (5.11),

$$\frac{\omega^2}{\kappa^2} = 1 - \frac{1}{8}Q^2 + \frac{1}{192}Q^4 - \frac{1}{9.2^{10}}Q^6 . \qquad (5.14)$$

6. Comparisons.

The results of the exact, or "nearly exact", analyses have been given in section 4, and in section 5 we have given the results obtained from the application of the Averaging Method. When comparing these results, however, we must keep one point in mind: the factor Q, in the exact formulae, denotes the maximum displacement (for $\dot{\varphi} = 0$), whereas in the approximate formulae Q denotes the amplitude of the assumed harmonic solution.

If the restoring force is expressed as a pure power of the displacement, the frequency-amplitude relationship is given

by (4.14),

$$\frac{\omega^2}{\kappa^2} = \chi(n)\, Q^{n-1} \tag{6.1}$$

in the exact case, and by (5.5),

$$\frac{\omega^2}{\kappa^2} = \varphi(n)\, Q^{n-1} \tag{6.2}$$

in the approximate case. The respective formulae for the two cases are strikingly similar: Q appears in each, raised to the same power, and is multiplied in each case by a numerical constant. The numerical coefficients, $\chi(n)$ for the exact case and $\varphi(n)$ for the approximate case, are both shown in Table 1. Also shown there is the ratio $\frac{\varphi(n)}{\chi(n)}$ and the "percentage error" $\left[\frac{\varphi(n)}{\chi(n)} - 1\right] 100$. The same results are shown graphically by the curves of fig. 6/1.

It is worth noting here that for $f(q) = (\sin q)\, q^2$ the error does not exceed 1.5%; for $f(q) = q^3$ it remains within 4.5% and even for such an extreme nonlinearity as $f(q) = q^7$ the error is not greater than 21%. The curves in fig. 6/2 show the general shape of the relations (6.1) and (6.2); the numerical values correspond to eq. (6.1). The curves corresponding to (6.2) would show the same general behavior but with slightly different numerical values.

The agreement between exact (or "nearly exact") and approximate values is even better for polynomials in $f(q)$.

As a numerical example take the case of $n=1$, $m=3$.

The approximate solution [as expressed by (5.9b)] gives

$$\frac{\omega^2}{K^2} = 1 + \frac{3}{4}\mu Q^2 \quad ;$$

for the exact solutions we must employ eq. (4.27). If we expand the expression $\frac{2}{\pi}K(\gamma)$, which appears in (4.27). in powers of χ and retain only the quadratic terms, we find [1]

$$\frac{2}{\pi}K = 1 + \frac{1}{4}\chi^2 + \ldots$$

and, therefore,

$$\frac{4}{\pi^2}K^2 = 1 + \frac{1}{2}\chi^2 \quad .$$

Introducing into this the notation of (4.27) we have

$$\frac{4}{\pi^2}K^2 = 1 + \frac{1}{4}\frac{\theta}{1+\theta}$$

so that finally, eq. (4.27) gives

$$\frac{\omega^2}{K^2} = 1 + \frac{3}{4}\theta \quad .$$

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[1] See, Jahnke Emde, Tables of Functions with Formulae and Curves, Dover Publications, 1943, p. 73.

When we note, from eq. (4.25), that

$$\theta = \mu Q^2 ,$$

we see that this result agrees exactly with the approximate one (5.9b).

b. The Characteristic Line is Composed of Straight Line Segments.

7. Exact Solutions for Forced Vibrations: General Expressions.

As was indicated earlier, another broad class of systems for which exact solutions may be obtained are those whose characteristic line (i.e., the plot of restoring force versus displacement) is composed of segments of straight lines. We will restrict the analysis to those characteristic lines which have not more than two different slopes. Again we will assume that the restoring force $R(q)$ is an odd function of the displacement q; i.e.,

$$R(-q) = -R(q) \quad . \tag{7.1}$$

We can obtain exact solutions for the forced vibrations as well as for the free vibrations because the differential equations are linear in their respective range. We will treat the forced vibrations first.

The differential equations for the forced vibrations of a system having the characteristic line shown in fig. 7/1 are

(for positive values of q_j)

$$(7.2)\quad \begin{cases} (*) \quad a\ddot{q}_j + c_1 q_j = \pm P \sin \Omega t & \text{for } q_j \leq q_{j_1} \\ (**) \quad a\ddot{q}_j + (c_1 - c_2) q_{j_1} + c_2 q_j = \pm P \sin \Omega t & \text{for } q_j \geq q_{j_1} . \end{cases}$$

The upper sign on the right hand side corresponds to a force which is " in phase" with the displacement while the lower one to a force " opposite" in phase [see fig. 7/2]. Thus we are allowed to deal with positive values of q_j only.

The following notation will be employed:

$$(7.3)\quad \Omega t = \tau \ , \quad \frac{c_1}{a} = \omega_1^2 \ , \quad \frac{c_2}{a} = \omega_2^2 \ , \quad \frac{c_2}{c_1} = \gamma \ .$$

Unless Ω coincides with either ω_1 or ω_2, the solutions q_j^* and q_j^{**}, which belong to the corresponding equations above, have the general form

$$(7.4)\quad \begin{cases} q_j^* = Q_1 \sin \tau + A_1 \cos \omega_1 t + B_1 \sin \omega_1 t \\ q_j^{**} = Q_2 \sin \tau + A_2 \cos \omega_2 t + B_2 \sin \omega_2 t + C_2 \end{cases}$$

where A_1, A_2, B_1, and B_2 are arbitrary constants to be determined by the initial conditions (for each of the regions); and Q_1, Q_2, and C_2 are constants to be determined from the differential equations themselves. These

last three constants can be shown to be

$$(7.5)\quad \begin{cases} Q_1 = \dfrac{\pm \frac{P}{c_1}}{1 - \frac{\Omega^2}{\omega_1^2}}\,, \qquad Q_2 = \dfrac{\pm \frac{P}{c_2}}{1 - \frac{\Omega^2}{\omega_2^2}}\,, \\ C_2 = \left(1 - \dfrac{c_1}{c_2}\right) q_{f_1}\,. \end{cases}$$

As to the initial conditions, we assume that the zeros of the motion and of the driving force coincide. The instant at which q_f passes from the region $q_f < q_{f_1}$ to the region $q_f > q_{f_1}$ will be indicated by $\tau_1 = \frac{\tau_1}{\Omega}$. Therefore, $\frac{(\pi - \tau_1)}{\Omega}$ indicates the instant at which q_f again passes the q_{f_1} line.

In view of the preceding statements we may write the following conditions which the motion must satisfy:

$$(7.6)\quad \begin{cases} a) \quad q_f^*(0) = 0 \\ b) \quad q_f^*(\tau_1) = q_{f_1} \\ c) \quad q_f^{**}(\tau_1) = q_{f_1} \\ d) \quad \dot{q}_f^*(\tau_1) = \dot{q}_f^{**}(\tau_1) \\ e) \quad \dot{q}_f^{**}\left(\frac{\pi}{\Omega}\right) = 0 \end{cases}$$

Therefore, we have five conditions which will enable us to determine the five unknowns, A_1, A_2, B_1, B_2, and τ_1.

From (7.6a) it follows that $A_1 = 0$; eq. (7.6b) gives B_1 in terms of τ_1; eqs. (7.6c) and (7.6e) give A_2 and B_2

in terms of τ_1 ; therefore, if we now substitute these values into eq. (7.6d) we will obtain the following (transcendental) equation for τ_1 :

$$(7.7)\qquad \left[1 \mp \frac{\bar{A}_1}{1-\eta_1^2}\sin\tau_1\right]\cot\frac{\tau_1}{\eta_1} - \sqrt{\gamma}\left[\frac{1}{\gamma} \mp \frac{\bar{A}_1}{\gamma-\eta_1^2}\sin\tau_1\right]\tan\left(\sqrt{\gamma}\,\frac{\frac{\pi}{2}-\tau_1}{\eta_1}\right) = \mp\frac{(1-\gamma)\,\bar{A}_1}{(\gamma-\eta_1^2)(1-\eta_1^2)}\,\eta_1\cos\tau_1$$

where the following notation is used:

$$(7.8)\qquad \frac{\Omega}{\omega_1} = \eta_1 \qquad \frac{\Omega}{\omega_2} = \eta_2 \qquad \bar{A}_1 = \frac{P}{c_1 g_1} .$$

The upper sign in eq. (7.7) corresponds to the driving force $+P\sin\Omega t$, and the lower sign corresponds to $-P\sin\Omega t$. Furthermore, the frequency ratios are related by the equation

$$(7.8a)\qquad \eta_2^2 = \frac{1}{\gamma}\,\eta_1^2 .$$

After having solved the transcendental equation (7.7) for τ_1 we may write the results in the following form:

for $\varphi_b \equiv \varphi_{b_1}$

$$(7.9)\qquad \frac{\varphi_b^*}{g_1} = \frac{\mp\bar{A}_1}{1-\eta_1^2}\left[\sin\tau - \sin\tau_1\,\frac{\sin\frac{\tau}{\eta_1}}{\sin\frac{\tau_1}{\eta_1}}\right] + \frac{\sin\frac{\tau}{\eta_1}}{\sin\frac{\tau_1}{\eta_1}}$$

for $q_\beta \geq q_{\beta_1}$

$$(7.9)\quad \frac{q_\beta^{**}}{q_{\beta_1}} = \frac{\pm \bar{A}_1}{\frac{1}{\gamma}-\gamma_1^2}\sin\tau + \left(1-\frac{1}{\gamma}\right) + \left[\frac{1}{\gamma} \mp \frac{\bar{A}_1}{\frac{1}{\gamma}-\gamma_1^2}\sin\tau_1\right]\frac{\cos\left[\sqrt{\gamma}\,\frac{(\tau-\frac{\pi}{2})}{\gamma_1}\right]}{\cos\left[\sqrt{\gamma}\,\frac{(\tau_1-\frac{\pi}{2})}{\gamma_1}\right]}\,.$$

Now we must deal with the two special cases which were excluded in the analysis above; i.e., $\Omega = \omega_1$, and $\Omega = \omega_2$.

If $\Omega = \omega_1$, or, in other terms, $\gamma_1 = 1$, the first of equations (7.4) has to be replaced by

$$(7.4a)\quad q_\beta^* = Q_1 \sin\Omega t + A_1\cos\Omega t - \frac{tB_1}{2\Omega}\cos\Omega t\,.$$

The equation for τ_1 (7.7) now becomes

$$(7.7a)\quad \pm\frac{\bar{A}_1}{2}\left[\tau_1\sin\tau_1 - \cos\tau_1\right] + \sqrt{\gamma}\left[\frac{1}{\gamma} \mp \frac{\bar{A}_1}{\gamma-1}\sin\tau_1\right]\tan\left[\sqrt{\gamma}\,(\tau_1-\tfrac{\pi}{2})\right]$$
$$= \pm\frac{\bar{A}_1}{\gamma-1}\cos\tau_1 - \cot\tau_1\left[1 \pm \frac{\bar{A}_1\tau_1}{2}\cos\tau_1\right]$$

and the final equations, which replace (7.9), are

$$(7.9a)\quad \begin{cases} \dfrac{q_\beta^*}{q_{\beta_1}} = \pm\dfrac{\bar{A}_1}{2}\left[\dfrac{\sin\tau}{\sin\tau_1}\,\tau_1\cos\tau_1 - \tau\cos\tau\right] + \dfrac{\sin\tau}{\sin\tau_1} \\[2ex] \dfrac{q_\beta^{**}}{q_{\beta_1}} = \pm\dfrac{\bar{A}_1}{\gamma-1}\left\{\sin\tau - \sin\tau_1\dfrac{\cos\left[\sqrt{\gamma}\,(\tau-\frac{\pi}{2})\right]}{\cos\left[\sqrt{\gamma}\,(\tau_1-\frac{\pi}{2})\right]}\right\} \\ \qquad + \left(1-\dfrac{1}{\gamma}\right) + \dfrac{1}{\gamma}\dfrac{\cos\left[\sqrt{\gamma}\,(\tau-\frac{\pi}{2})\right]}{\cos\left[\sqrt{\gamma}\,(\tau_1-\frac{\pi}{2})\right]} \end{cases}$$

Proceeding in a similar manner for the case $\Omega = \omega_2$, or $\eta_2 = 1$, we see that the second of eqs. (7.4) must be replaced by

$$(7.4b) \qquad q_f^{**} = Q_2 \sin \Omega t + A_2 \cos \Omega t - \frac{t B_2}{2\Omega} \cos \Omega t + C_2 \ .$$

The transcendental equation (7.7) now becomes

$$(7.7b) \qquad \frac{1}{\eta_1} \cot \frac{\pi_1}{\eta_1} \left[1 \mp \frac{\bar{A}_1}{1-\eta_1^2} \sin \pi_1 \right] \pm \frac{\bar{A}_1}{2\gamma} \left[\cos \pi_1 + \left(\frac{\pi}{2} - \pi_1 \right) \sin \pi_1 \right]$$
$$= \cot \pi_1 \left[\pm \frac{\bar{A}_1}{2\gamma} \left(\pi_1 - \frac{\pi}{2} \right) \cos \pi_1 + \frac{1}{\gamma} \right] \mp \frac{\bar{A}_1}{1-\eta_1^2} \cos \pi_1 \ .$$

The final results are

$$(7.9b) \qquad \begin{cases} \dfrac{q_b^*}{q_{b_1}} = \pm \dfrac{\bar{A}_1}{1-\eta_1^2} \left[\sin \tau - \sin \pi_1 \dfrac{\sin \frac{\tau}{\eta_1}}{\sin \frac{\pi_1}{\eta_1}} \right] + \dfrac{\sin \frac{\tau}{\eta_1}}{\sin \frac{\pi_1}{\eta_1}} \\ \dfrac{q_b^{**}}{q_{b_1}} = \pm \dfrac{\bar{A}_1}{2\gamma} \left[\left(\frac{\pi}{2} - \tau \right) \cos \tau - \left(\frac{\pi}{2} - \pi_1 \right) \cos \pi_1 \dfrac{\sin \tau}{\sin \pi_1} \right] \\ \qquad + \dfrac{1}{\gamma} \dfrac{\sin \tau}{\sin \pi_1} + \left(1 - \dfrac{1}{\gamma} \right) \tau \end{cases}$$

8. Exact Solutions for Forced Vibrations: Special Cases.

We now proceed to a specialization of the general results of the preceding section. Only those cases where essentially new features appear will be studied.

α) Characteristic Line as Shown in Fig. 8/1a; ($c_1 = 0$)

For this particular system the first of equations (7.2) becomes

$$\alpha \ddot{q} = \pm P \sin \Omega t \quad . \tag{8.1}$$

The first of equations (7.4) then is

$$q^* = Q_1 \sin \Omega t + A_1 t + B_1 \quad . \tag{8.2}$$

Following the same procedure as in section 7, we obtain the equation for τ_1 ,

$$\frac{\eta_2}{\tau_1}\left[1 \pm \frac{\bar{\Delta}_2}{\eta_2^2} \sin \tau_1\right] \mp \left[\frac{\frac{\bar{\Delta}_2}{1-\eta_2^2} \sin \tau_1}{\cot \frac{\pi/2}{\eta_2} \cos \frac{\tau_1}{\eta_2} + \sin \frac{\tau_1}{\eta_2}}\right]\left[\cot \frac{\pi/2}{\eta_2} \sin \frac{\tau_1}{\eta_2} - \cos \frac{\tau_1}{\eta_2}\right] \tag{8.3}$$

$$= \pm \frac{\bar{\Delta}_2}{\eta_2(1-\eta_2^2)} \cos \tau_1$$

and, therefore, the solutions are found to be

$$\begin{cases} \dfrac{q_b^*}{q_{b1}} = \dfrac{\tau}{\tau_1}\left[1 \pm \dfrac{\bar{\Delta}_2}{\eta_2^2} \sin \tau_1\right] \mp \dfrac{\bar{\Delta}_2}{\eta_2^2} \sin \tau \\ \dfrac{q_b^{**}}{q_{b1}} = \pm \dfrac{\bar{\Delta}_2}{1-\eta_2^2} \sin \tau \mp \dfrac{\bar{\Delta}_2}{1-\eta_2^2} \sin \tau_1 \left[\dfrac{\cot \frac{\pi/2}{\eta_2} \cos \frac{\tau}{\eta_2} + \sin \frac{\tau}{\eta_2}}{\cot \frac{\pi/2}{\eta_2} \cos \frac{\tau_1}{\eta_2} + \sin \frac{\tau_1}{\eta_2}}\right] + 1 \quad . \end{cases} \tag{8.4}$$

The resonance case $\eta_2 = 1$ is of no importance here because it admits only infinite values for $q_f(\frac{\pi}{2})$.

β) Characteristic Line as Shown in Fig. 8/1b: ($c_2 = 0$)

The set of differential equations corresponding to (7.2) is, for this case (again for $q_f > 0$) —

$$(8.5)\qquad \begin{cases} a\ddot{q}_f + c_1 q_f = \pm P \sin \Omega t & q_f \le q_{f1} \\ a\ddot{q}_f + c_1 q_{f1} = \pm P \sin \Omega t & q_f \ge q_{f1} \end{cases}$$

and the form of solutions is

$$(8.6)\qquad \begin{cases} q_f^{*} = Q_1 \sin \Omega t + A_1 \cos \omega_1 t + B_1 \sin \omega_1 t \\ q_f^{**} = Q_2 \sin \Omega t + A_2 + B_2 t + C_2 t^2 \end{cases}$$

Therefore, we obtain as the transcendental equation for τ_1 —

$$(8.7)\qquad \eta_1 \cot \frac{\tau_1}{\eta_1} = \frac{\frac{\pi}{2} - \tau_1 \mp \frac{\Sigma_1}{1-\eta_1^2} \cos \tau_1}{1 \mp \frac{\Sigma_1}{1-\eta_1^2} \sin \tau_1}$$

and as the solutions

$$(8.8)\quad \begin{cases} \dfrac{q^{*}}{q_{0_1}} = \pm \dfrac{\bar{A}_1}{1-\eta_1^2}\sin\tau + \dfrac{\sin\frac{\tau}{\eta_1}}{\sin\frac{\pi}{\eta_1}}\left[1 \mp \dfrac{\bar{A}_1}{1-\eta_1^2}\sin\tau_1\right] \\ \dfrac{q^{**}}{q_{0_1}} = \pm \dfrac{\bar{A}_1}{\eta_1^2}\left[\sin\tau_1 - \sin\hat{\tau}\right] + \dfrac{1}{2}\dfrac{\tau}{\eta_1^2}(\pi-\tau) + \dfrac{1}{2}\dfrac{\tau_1}{\eta_1^2}(\tau_1-\pi) + 1 \end{cases}$$

For $\eta_1 = 1$, the resonance case, the first of equations (8.6) is replaced by

$$(8.6a)\quad q^{*} = Q\sin\Omega t + A_1\cos\Omega t - \frac{B_1 t}{2\Omega}\cos\Omega t .$$

Equation (8.7) then becomes

$$(8.7a)\quad \pm\frac{\bar{A}}{2}\left[\hat{\tau}_1\sin\hat{\tau}_1 - \cos\tau_1 + \hat{\tau}_1\cos\tau_1\cot\tau_1\right] \pm \bar{A}_1\cos\hat{\tau}_1$$

$$+\left(\hat{\tau}_1 - \frac{\pi}{2} + \cot\tau_1\right) = 0 .$$

The solutions will be

$$(8.8a)\quad \begin{cases} \dfrac{q^{*}}{q_{0_1}} = \dfrac{\sin\tau}{\sin\tau_1}\left[\pm\dfrac{\bar{A}_1\hat{\tau}_1}{2}\cos\tau_1 + 1\right] \mp \dfrac{\bar{A}_1\hat{\tau}}{2}\cos\hat{\tau} \\ \dfrac{q^{**}}{q_{0_1}} = \pm\bar{A}_1\left(\sin\tau_1 - \sin\hat{\tau}\right) + \left(1 - \dfrac{\pi}{2}\hat{\tau}_1 + \dfrac{\hat{\tau}_1^2}{2}\right) + \left(\dfrac{\pi}{2}\tau - \dfrac{\tau^2}{2}\right) \end{cases}$$

γ) Characteristic Line as shown in Fig. 8/1c: ($c_1 = \infty$)

Here the differential equation is

$$(8.9) \qquad a\ddot{q} + c_2 q + R_0 = \pm P \sin \Omega t$$

The general solution of this equation is

$$(8.10) \qquad q = Q \sin \Omega t + A \cos \omega_2 t + B \sin \omega_2 t + C$$

where Q and C are (as obtained from the differential equation)

$$(8.10a) \qquad Q = \frac{\lambda_2}{1-\eta_2^2} \quad , \quad C = -\frac{R_0}{c_2}$$

and

$$(8.11) \qquad \lambda_2 = \frac{P}{c_2 q_0} = \frac{P}{R_0} \; ; \; \eta_2^2 = \frac{\Omega^2}{c_2/a} \; ; \; \omega_2^2 = \frac{c_2}{a}$$

The conditions for determining A and B are

$$(8.12) \qquad \begin{cases} \alpha) & q(0) = 0 \\ \beta) & \dot{q}\left(\frac{\pi}{2}\right) = 0 \end{cases}$$

The solution then becomes

$$(8.13) \qquad \frac{q}{q_0} = \pm \frac{\lambda_2}{1-\eta_2^2} \left[\sin \tau + \tan \frac{\pi \eta_2}{2} \sin \frac{\tau}{\eta_2} + \cos \frac{\tau}{\eta_2} - 1 \right]$$

Here, again, the resonance condition is of no special significance.

∂) Characteristic Line as Shown in Fig. 8/1d: ($c_2 = \infty$)

In this case, the differential equation is simply the first of eqs. (8.5) with the first of eqs. (8.6) as its solution. As before, Q_1 is determined by the equation of motion and A_1, B_1, and τ_1 follow from the conditions

$$(8.14)\quad \begin{cases} (a) & q(0) = 0 \\ (b) & q(\tau_1) = q_1 \\ (c) & \dot{q}(\pi) = 0 \end{cases}$$

From these conditions the solution becomes

$$\frac{q}{q_1} = \pm \frac{\bar{s}_1}{1-\eta_1^2}\left[\sin t - \sin t_1 \frac{\sin \frac{t}{\eta_1}}{\sin \frac{\tau_1}{\eta_1}}\right] + \frac{\sin \frac{t}{\eta_1}}{\sin \frac{\tau_1}{\eta_1}}$$

Here, again, the resonance condition is without special significance.

9. Exact Solution for Free Vibrations.

The free vibrations are, of course, simply given by the equations (7.2) when the right hand side is set equal to zero. One could perhaps think of merely taking the results which were obtained in section 8 and set A or $\bar{A}$ equal to zero; however, it would be seen that difficulties appear which necessitate considerable attention. We will not discuss these in detail. In view of this fact, we prefer to deal with the free vibrations

separately, and, furthermore, we shall restrict our efforts to finding the period or the frequency of the free vibrations without attempting to obtain the displacement-time relationship in detail.

In the present section, we shall use the notation of fig. 7/1 and the quantities ω_1 and ω_2 as defined previously.

The quarter period $\frac{T}{4}$ is the time required by the system to traverse the distance q_2 to 0 when the motion initially has zero velocity and is acted upon by the restoring force of fig. 7/1. Under these circumstances the quarter period can be found to be

$$(9.1) \qquad \frac{T}{4} = \frac{1}{\omega_1} \arctan \frac{c_2 q_1}{\frac{\omega_2}{\omega_1}\sqrt{R_2^2 - R_1^2}} + \frac{1}{\omega_2} \arccos \frac{R_1}{R_2}$$

where c_1 and c_2 are neither zero nor infinity.

The special cases in figs. 8/1a and 8/1c are easily dealt with and follow directly from eq. (9.1). The first special case has

$$(9.2) \qquad \frac{T}{4} = \frac{1}{\omega_2}\left[\frac{\pi}{2} + \frac{q_1}{q_2 - q_1}\right] ,$$

the second has

$$(9.3) \qquad \frac{T}{4} = \sqrt{\frac{a q_2}{R_2 - R_0}} \arccos \frac{R_0}{R_2} .$$

The case given by fig. 8/1b requires that we use eq. (8.5), but with the right-hand side set equal to zero. The result is found to be

$$(9.4)\qquad \frac{T}{4} = \frac{1}{\omega_1}\left[\sqrt{2\left(\frac{Q}{g_1}-1\right)} + \arctan\frac{1}{\sqrt{2\left(\frac{Q}{g_1}-1\right)}}\right].$$

For the case given by fig. 8/1d the period is seen to depend upon some parameter; for instance, upon the velocity v_0 at $g = 0$. We find immediately that

$$(9.5)\qquad \frac{T}{4} = \frac{1}{\omega_1}\arccos\frac{v_0}{\omega_1}.$$

We will require no formulas other than those given in this section.

10. The Approximate Solutions Given by the Averaging Method

We now turn to the Averaging Method for an approximate solution of the differential equations of the system having characteristic lines which are composed of segments of straight lines (see fig. 7/1).

The motion of the system is described by the set of two differential equations (7.2). We repeat them here in a slightly different form:

$$(10.1)\qquad \begin{cases} E_1 \equiv a\ddot{q} + c_1 q \mp P \sin \Omega t = 0 \\ E_2 \equiv a\ddot{q} + c_2 q - (c_2 - c_1) q_1 \mp P \sin \Omega t = 0 \end{cases} .$$

The assumed form of solution will be

$$(10.2)\qquad \tilde{q} = Q \sin \tau \quad , \qquad \tau = \Omega t$$

The Ritz condition is, then,

$$(10.3)\qquad \int_0^{2\pi} E(\tilde{q}) \sin \tau \, d\tau = 0$$

Because the characteristic line is symmetrical with respect to the origin, the upper limit in (10.3) may be changed to $\frac{\pi}{2}$. The quantity E will represent the expression E_1 of (10.1) between the limits 0 and τ_1, and the expression E_2 of (10.1) between the limits τ_1 and $\frac{\pi}{2}$, with $q_1 = Q \sin \tau_1$. Thus, (10.3) may be written in the more complete form

$$(10.3a)\qquad \int_0^{\tau_1} E_1(\tilde{q}) \sin \tau \, d\tau + \int_{\tau_1}^{\frac{\pi}{2}} E_2(\tilde{q}) \sin \tau \, d\tau = 0$$

or, in a still more explicit form,

$$(10.3b)\qquad \int_0^{\tau_1}\left[(c_1-a\Omega^2)Q \mp P\right]\sin^2\tau\,d\tau + \int_{\tau_1}^{\frac{\pi}{2}}\left[(c_2-a\Omega^2)Q \mp P\right]\sin^2\tau\,d\tau$$

$$-(c_2-c_1)\,q_{b1}\int_{\tau_1}^{\frac{\pi}{2}}\sin\tau\,d\tau = 0 \quad .$$

Upon evaluating these integrals and introducing the notations

$$\frac{c_2}{c_1} = \gamma \quad , \quad \frac{c_1}{a} = \omega_1^2 \quad , \quad \frac{c_2}{a} = \omega_2^2 \quad , \quad \frac{\Omega}{\omega_1} = \eta_1$$

we find

$$(10.4)\qquad \eta_1^2 = \gamma - \frac{2}{\pi}(\gamma-1)\left[\frac{q_{b1}}{Q}\sqrt{1-\left(\frac{q_{b1}}{Q}\right)^2} + \arcsin\left(\frac{q_{b1}}{Q}\right)\right] \mp \frac{A_1}{Q} \quad .$$

Introducing the dimensionless variables

$$\bar{Q} = \frac{Q}{q_{b1}} \qquad \text{and} \qquad \bar{A}_1 = \frac{A_1}{q_{b1}} = \frac{P}{c_1 q_{b1}}$$

eq. (10.4) becomes

$$(10.4a)\qquad \eta_1^2 = \gamma - \frac{2}{\pi}(\gamma-1)\left[\frac{1}{\bar{Q}}\sqrt{1-\left(\frac{1}{\bar{Q}}\right)^2} + \arcsin\left(\frac{1}{\bar{Q}}\right)\right] \mp \frac{\bar{A}_1}{\bar{Q}} \quad .$$

The corresponding expression for the free vibrations follows from either (10.4) or (10.4a) by setting Δ_1, or $\bar{\Delta}_1$, equal to zero. An equation, equivalent to (10.4a), can be found which uses, instead of η_1 and Δ_1, the parameters

$$(10.5) \qquad \eta_2^2 = \frac{1}{\mu}\eta_1^2 \quad , \quad \bar{\Delta}_2 = \frac{P}{c_2 \delta_1} = \frac{1}{\mu}\bar{\Delta}_1 \quad .$$

By introducing (10.5) into eq. (10.4), and dividing the entire equation by μ, we find, as an equivalent form of (10.4),

$$(10.6) \qquad \eta_2^2 = 1 - \frac{2}{\pi}\left(1 - \frac{1}{\mu}\right)\left[\frac{\eta_1}{\varrho}\sqrt{1-\left(\frac{\eta_1}{\varrho}\right)^2} + \arcsin\left(\frac{\eta_1}{\varrho}\right)\right] + \frac{\bar{\Delta}_2}{\varrho} \quad .$$

We will now proceed to establish the equations which correspond to (10.4) or (10.6) for the special cases listed in fig. 8/1.

In the case of fig. 8/1a the spring constant c_1 assumes the value zero, therefore, $\frac{1}{\mu} = \frac{c_1}{c_2} = 0$. From (10.6) we easily obtain

$$(10.7) \qquad \eta_2^2 = 1 - \frac{2}{\pi}\left[\frac{\eta_1}{\varrho}\sqrt{1-\left(\frac{\eta_1}{\varrho}\right)^2} + \arcsin\left(\frac{\eta_1}{\varrho}\right)\right] + \frac{\bar{\Delta}_2}{\varrho} \quad .$$

In the case of fig. 8/1b, the spring constant c_2, and therefore μ, assumes the value zero. From (10.4) we have

immediately,

$$(10.8)\qquad \eta_1^2 = \frac{2}{\pi}\left[\frac{q_1}{Q}\sqrt{1-\left(\frac{q_1}{Q}\right)^2} + \arcsin\left(\frac{q_1}{Q}\right)\right] \mp \frac{A_1}{Q} \quad .$$

A little more work is involved for finding the limiting formula for the case shown in fig. 8/1c. Here, after putting $\frac{a}{v_1} = \frac{R_0}{c_1}$ eq. (10.6) becomes

$$\eta_2^2 = 1 - \frac{2}{\pi}\left(1-\frac{c_1}{c_2}\right)\left[\frac{R_0}{c_1 Q}\sqrt{1-\left(\frac{R_0}{c_1 Q}\right)^2} + \arcsin\left(\frac{R_0}{c_1 Q}\right)\right] \mp \frac{A_2}{Q} \quad .$$

Now, because c_1 increases indefinitely, we first expand the small terms $\frac{R_0}{c_1 Q}$ inside the bracket. The value of the bracket tends toward $\frac{2R_0}{c_1 Q}$; therefore, we obtain

$$(10.9)\qquad \eta_2^2 = 1 + \frac{4}{\pi}\frac{R_0}{c_2 Q} \mp \frac{A_2}{Q} \quad .$$

Considerations similar to those above are necessary in the last case, which is represented by fig. 8/1d. We begin with eq. (10.4a). If c_2 tends to increase indefinitely, γ will be large also, and the dimensionless amplitude $\bar{Q}$ will barely exceed unity. By setting

$$\bar{Q} = 1 + \varepsilon \quad , \quad \varepsilon \ll 1$$

we find [by expanding the terms inside of the bracket of eq. (10.4a) in terms of powers of ε] the value of the bracket to be $\frac{\pi}{2} - \varepsilon\sqrt{2\varepsilon}$. From (10.4a), thus, we have

$$\eta_2^2 = 1 + \frac{\gamma - 1}{\pi} (2\varepsilon)^{3/2} \mp \gamma A_2 (1 - \varepsilon)$$

which, for $\gamma >> 1$, reduces to

$$\eta_2^2 = 1 + \frac{\gamma}{\pi} (2\varepsilon)^{3/2} \mp \gamma A_2 (1 - \varepsilon) \quad . \tag{10.10}$$

For the free vibrations this equation then simplifies to

$$\eta_2^2 = 1 + \frac{\gamma}{\pi} (2\varepsilon)^{3/2} \quad . \tag{10.11}$$

11. Discussion of the Results

α) Comparison Between the Exact and the Approximate Solutions

First, let us consider the free vibrations ($\varkappa = 0$). Table 2 compares the values of $\psi_{approx} = |\bar{\Phi}|$ (amplitudes) for the approximate solutions, as given by (10.4a), with the values of $\psi_{exact} = \left(\frac{\beta}{\beta_1}\right)_{max}$ (maximum displacements) for the exact solutions, as given by (7.9) (with $\tau = \frac{\pi}{2}$). Three sample systems $\left[\gamma = 2\,,\ \gamma = \frac{1}{2}\,,\ \gamma = 0\ (c_2 = 0)\right]$ are compared by listing the frequency ratios η which belong to coinciding values of ψ.

For the first of these systems ($\gamma = 2$) both sets of numerical values are plotted (see fig. 11/1). The agreement is so good that the curves drawn through the two sets of points cannot be distinguished. The same is true of the other two cases; therefore, the plots are omitted.

Next, let us look at a set of thirteen figures (11/2a through 11/6b) which give the displacement - time curves for the forced vibrations. These diagrams refer to five different systems whose characteristic lines are sketched on the respective figures. The diagrams numbered a, b, c, in each set (i.e. for each system) are distinguished by the values of the frequency ratio γ_1 or γ_2; the values have been chosen as 1/2, 1, and 2 wherever feasible. In each diagram there appears a family of curves with A_1 (or A_2) as a parameter; this parameter assumes the values 1, 2, 5 wherever feasible. In any case, the solid lines represent the exact solutions and the broken lines represent the approximate solutions.

All of the above diagrams, with the exception of three, show very good agreement between the results. The three which show a considerable difference between the results are 11/4a, 11/5a, and 11/6a; in these cases the frequency ratio is $\gamma = \frac{1}{2}$, i.e. $\Omega = \frac{1}{2}\omega_1$. Here, quite obviously the exact solution contains at least a third order harmonic term which has a considerable amplitude. This third order term (of negative amplitude in 11/4a and 11/5a and positive amplitude in 11/6a) has not, of course, been included in the approximate solution. The fact that the exact solution contains a third order term of appreciable amplitude is readily explicable by remarking that the driving frequency $\Omega = \frac{1}{2}\omega_1$ is close to the resonance frequency $\frac{\omega_1}{3}$

of this third order term.

In all other cases the higher order terms are quite negligible, even for $\gamma = \frac{1}{2}$; in a number of cases the curves are indistinguishable. The slight differences which do appear in the other cases can again be explained in the same way as we did for the large differences in figs. 11/4a, 11/5a and 11/6a. The differences are essentially due to the third order terms; and the sign of this term, in each case, is exactly that which would be expected, as may be checked very readily.

The comparison shows, therefore, that we may expect the approximate solutions obtained by the Averaging Method to represent the exact solution very closely in all cases where the driving frequency is not close to the resonant frequency of a higher order harmonic.

β) Response Curves

Now that we have compared a number of results and found very good agreement between approximate and exact solutions, we may proceed to a study of other results which have been obtained from application of the Averaging Method.

Figs. 11/7 through 11/11 show (for the same five systems considered previously) the families of response curves (they are plotted with γ^2 as the abscissa and $|\bar{Q}|$ as the ordinate). These curves were obtained from eq. (10.4a), or the corresponding eqs. (10.7) and (10.9). The individual curves in each family belong to values of the parameter $\bar{A}$ which runs the sequence

of values indicated in the respective diagrams. The curves for $\mathcal{A} = 0$ represent the amplitude-frequency relation for the free vibrations (resonance curves, or "backbone" curves - see section 13)

The response curves for all these systems show all of the expected features (asymptotic tangents, etc.) and, therefore, do not require special discussion, except, perhaps for figs. 11/10 and 11/11. In these cases it might be mentioned that the separating value of the parameter $\mathcal{A}_2$ for curves approaching the line $\eta_2^2 = 1$ from the left, or right, side is $\bar{\mathcal{A}}_2 = \frac{4}{3\pi}$ in both cases; this may be easily derived from the pertinent equations.

In addition, fig. 11/12 shows the family of response curves for a system in which the characteristic line has three different slopes. The abscissas for the breaks in the slopes are $\eta_1 = 1$ and $\eta_2 = 2$, and the ratios of the spring constants are $\gamma_2 \equiv \frac{c_2}{c_1} = 2$ and $\gamma_3 \equiv \frac{c_3}{c_1} = 4$. Since the pertinent equations are derived according to patterns which we have already discussed several times, they are omitted here.

In all cases we find that the features of the response curves are as expected. The Averaging Method provides a simple tool for obtaining the equations which describe them.

Finally, figs. 11/13 and 11/14 provide some special information regarding the system shown in fig. 8/1d. In fig. 11/13 a few displacement-time curves for forced vibrations are drawn, as obtained from the exact solution (8.15). Fig. 11/14 deals

with the free vibrations of the general system and indicates how the "backbone" curve tends toward the horizontal when δ increases - i.e. the system approaches that of the type shown in fig. 8/1d. Eq. (10.11) describes these curves.

C. FORCED VIBRATIONS WITH ARBITRARY RESTORING AND DAMPING FORCES

12. General Case; Amplitude and Phase Angles

In the preceding chapter we treated systems having differential equations for which we could obtain exact (or "nearly exact") solutions; however, we are even more interested in applying the Averaging Method to those systems whose differential equations cannot be treated readily by other means. The prototype of such a system is the single degree of freedom system acted upon by arbitrary restoring and damping forces. The differential equation of motion of such a system may be written as

$$a\ddot{q} + b\,g(\dot{q}) + c\,f(q) = P\cos\Omega t \,. \tag{12.1}$$

Here again q denotes the dependent coordinate; the coefficients a, b, and c are chosen so as to make each term have the dimension of a force, where

$$\dim q = \dim \dot{q}$$

$$\dim f = \dim q$$

P denotes the amplitude, Ω the frequency of the driving force. [It may be well to keep in mind that the importance of the differential equation (12.1) is not the same in the non-linear case as in the linear one. In the latter case the

driving force may be supposed to be a harmonic component of any periodic force, thus leading to the corresponding term in the resulting motion.]

The functions $g(\dot{q})$ in the damping force term and $f(q)$ in the restoring force term are arbitrary functions, and shall be subjected only to the restriction of being odd functions of their respective arguments:

$$(12.2) \qquad \begin{cases} g(-\dot{q}) = -g(\dot{q}) \\ f(-q) = -f(q) \end{cases}$$

This restriction does not necessarily mean that terms with even powers of the arguments are now allowed; if we insist only upon a change of sign when the system passes through zero argument, then such terms may be admitted. For instance,

$$(12.2a) \qquad \begin{cases} g(\dot{q}) = (\operatorname{sgn}\dot{q})\,\nu\dot{q}^2 \\ f(q) = (\operatorname{sgn} q)\,\mu q^2 \end{cases}$$

are odd functions and may therefore be admitted.

Before proceeding to a treatment of the differential equation (12.1) let us first write it in a slightly different form by dividing through by the factor a :

$$(12.3) \qquad E \equiv \ddot{q} + 2Dk\,g(\dot{q}) + k^2 f(q) - p\cos\tau = 0$$

where

$$(12.4)\qquad \begin{cases} 2D\kappa = \dfrac{b}{a} & p = \dfrac{P}{a} \\ \kappa^2 = \dfrac{c}{a} & \tau = \Omega t \end{cases}$$

It is form (12.3) of the equation of motion with which we shall be dealing.

We assume the solution to be approximated by

$$(12.5)\qquad \tilde{q} = Q\cos(\tau - \varepsilon)$$

which is equivalent to

$$(12.5a)\qquad \tilde{q} = A\cos\tau + B\sin\tau$$

where

$$(12.5b)\qquad A = Q\cos\varepsilon \quad , \quad B = Q\sin\varepsilon \quad .$$

Therefore, we are to determine Q and ε or, equivalently, A and B.

The Averaging Method, then, furnishes the following two conditions:

$$(12.6a)\qquad \int_0^{2\pi} E\cos\tau\, d\tau = 0$$

and

$$(12.6b) \qquad \int_0^{2\pi} E \sin\tau \, d\tau = 0$$

Now, introducing (12.5) into the differential equation (12.3) [with the condition (12.2)], and then applying equations (12.6) we obtain the following two equations:

$$(12.7a) \qquad -\Omega^2 Q \int_0^{2\pi} \cos(\tau-\varepsilon)\cos\tau \, d\tau - 2Dk \int_0^{2\pi} g\left(\Omega Q \sin(\tau-\varepsilon)\right)\cos\tau \, d\tau$$

$$+ k^2 \int_0^{2\pi} f\left(Q\cos(\tau-\varepsilon)\right)\cos\tau \, d\tau - p\int_0^{2\pi} \cos^2\tau \, d\tau = 0$$

$$(12.7b) \qquad -\Omega^2 Q \int_0^{2\pi} \cos(\tau-\varepsilon)\sin\tau \, d\tau - 2Dk \int_0^{2\pi} g\left(\Omega Q \sin(\tau-\varepsilon)\right)\sin\tau \, d\tau$$

$$+ k^2 \int_0^{2\pi} f\left(Q\cos(\tau-\varepsilon)\right)\sin\tau \, d\tau = 0$$

Here we introduce the abbreviations

$$(12.8a) \qquad F(Q) = \frac{1}{Q\pi}\int_0^{2\pi} f(Q\cos\sigma)\cos\sigma \, d\sigma$$

$$= \frac{4}{Q\pi}\int_0^{\frac{\pi}{2}} f(Q\cos\sigma)\cos\sigma \, d\sigma$$

(12.8b) $$G(\Omega, Q) = \frac{1}{\kappa Q \pi} \int_0^{2\pi} g(\Omega Q \sin\sigma) \sin\sigma \, d\sigma$$

$$= \frac{4}{\kappa Q \pi} \int_0^{\frac{\pi}{2}} g(\Omega Q \sin\sigma) \sin\sigma \, d\sigma$$

and we note that, since

(12.8c) $$\sigma = \tau - \varepsilon \ ,$$

we may write

(12.9)
$$\begin{cases} \int_0^{2\pi} f(Q\cos(\tau-\varepsilon)) \cos\tau \, d\tau = \cos\varepsilon \int_0^{2\pi} f(Q\cos\sigma) \cos\sigma \, d\sigma \\ \int_0^{2\pi} f(Q\cos(\tau-\varepsilon)) \sin\tau \, d\tau = \sin\varepsilon \int_0^{2\pi} f(Q\cos\sigma) \cos\sigma \, d\sigma \\ \int_0^{2\pi} g(\Omega Q \sin(\tau-\varepsilon)) \cos\tau \, d\tau = -\sin\varepsilon \int_0^{2\pi} g(\Omega Q \sin\sigma) \sin\sigma \, d\sigma \\ \int_0^{2\pi} g(\Omega Q \sin(\tau-\varepsilon)) \sin\tau \, d\tau = \cos\varepsilon \int_0^{2\pi} g(\Omega Q \sin\sigma) \sin\sigma \, d\sigma \end{cases}$$

From the foregoing, and with the further notation

$$(12.10) \qquad \eta^2 = \frac{\Omega^2}{\kappa^2} \quad , \quad \Delta = \frac{p}{\kappa^2} - \frac{P}{c}$$

we may finally write the two Ritz conditions (12.7) as

$$(12.11a) \qquad -\eta^2 \cos\varepsilon + 2DG \sin\varepsilon + F\cos\varepsilon = \frac{\Delta}{Q}$$

$$(12.11b) \qquad -\eta^2 \sin\varepsilon - 2DG\cos\varepsilon + F\sin\varepsilon = 0$$

or, in an equivalent form,

$$(12.12a) \qquad -\eta^2 + F(Q) = \frac{\Delta}{Q}\cos\varepsilon$$

$$(12.12b) \qquad 2DG(\Omega,Q) = \frac{\Delta}{Q}\sin\varepsilon \quad .$$

From these last two equations we may obtain independent expressions for Q and ε :

$$(12.13a) \qquad \left[F(Q) - \eta^2\right]^2 + 4D^2G^2(\Omega,Q) = \frac{\Delta^2}{Q^2}$$

$$(12.13b) \qquad \tan\varepsilon = \frac{2DG(\Omega,Q)}{F(Q) - \eta^2}$$

These equations give us the amplitude Q and the phase angle ε, for the general case of arbitrary restoring and damping forces, in a very concise form.

13. Undamped Systems: Response Curves

From the general results (12.13) it is very easy to study a number of special cases. In the present section, and the one following, we shall do so in order to give some idea of the abundance of information which these equations contain.

If damping forces are absent, i.e., $b=0$ (or $D=0$) we may write immediately [from (12.12)]

$$F(Q) - \eta^2 = \pm \frac{A}{Q} \quad , \tag{13.1}$$

where the upper sign corresponds to $\varepsilon=0$ and the lower to $\varepsilon=\pi$. This equation describes a family of response curves, in an amplitude-frequency diagram (Q - η^2), with A (a measure of the intensity of the driving force) as a parameter. These curves lie to either side (at equal values of η^2) of a particular curve known as the "backbone" curve. This "backbone" curve represents the free ($A=0$) undamped vibrations of the system and has the simple amplitude-frequency relationship

$$\eta^2 = F(Q) \quad . \tag{13.2}$$

The response curves lying to the side of the "backbone" for smaller values of η^2 go with $\varepsilon=0$; those lying to the side

for larger values of η^2 go with $\varepsilon = \pi$.

As a matter of course we note that the linear case is included in the foregoing results. For $f(\varphi) = \varphi$ the function $F(Q)$ is simply unity, and, therefore, eq. (13.1) yields the well known result

$$\eta^2 = 1 \pm \frac{\varepsilon}{\varphi} \tag{13.1a}$$

Eq. (13.2) furthermore may be interpreted as describing an equivalent linear restoring coefficient $\hat{c}$, which is the coefficient of the displacement φ in a linear differential equation leading to the same frequency ω of the non-linear vibration (of amplitude Q), so that $\hat{c} = a\omega^2$. Putting $\eta^2 = \frac{\omega^2}{\kappa^2}$, eq. (13.2) states

$$\omega^2 = \kappa^2 F(Q) \tag{13.2a}$$

or, after multiplication with mass a ,

$$\hat{c} = c F(Q) . \tag{13.2b}$$

This equivalent linear restoring coefficient $\hat{c}$ depends, of course, on the amplitude Q in a way described by (13.2b).

For a more elaborate example, we now treat what will be called the "model case"; this system has a restoring force

given by

$$f(q) = q + \mu^2 q^3 . \tag{13.3a}$$

From eq. (13.8a) then, follows

$$F(Q) = 1 + \frac{3}{4}\mu^2 Q^2 .$$

With the dimensionless quantities

$$\bar{Q}^2 = \frac{3}{4}\mu^2 Q^2 \quad , \quad \bar{A}^2 = \frac{3}{4}\mu^2 A^2 \tag{13.4}$$

we obtain from (13.1)

$$\eta^2 = (1 + \bar{Q}^2) \mp \frac{\bar{A}}{\bar{Q}} \tag{13.5}$$

where again the upper sign goes with $\epsilon = 0$ and the lower with $\epsilon = \pi$.

Figure 13/1 shows the response curves, as given by equation (13.5), in a $\bar{Q} - \eta^2$ diagram. The curves are drawn for the case of a "stiffening" spring; i.e., it is assumed that $\mu^2 > 0$.

If the spring is "softening",

$$f(q) = q - \mu^2 q^3 \quad , \quad \mu^2 > 0 \tag{13.3b}$$

we have, instead of (13.5)

$$\eta^2 = (1 - \bar{Q}^2) \mp \frac{\bar{A}}{\bar{Q}} . \tag{13.5a}$$

Figure 13/2 shows the response curves which correspond to this case.

The equations for the " backbone" are

$$(13.6) \qquad \begin{cases} \gamma^2 = 1 + \bar{Q}^2 & \text{(hardening spring)} \\ \gamma^2 = 1 - \bar{Q}^2 & \text{(softening spring)} \end{cases}$$

The curves on the right-hand side of the " backbone" in fig. 13/1, and on the left-hand side in fig. 13/2, indicate the familiar " looped" feature which gives rise to the well-known " jump phenomenon" that occurs as the driving frequency is varied.

Speaking now of the hardening spring, fig. 13/1, the region on the right-hand side of the " backbone" may be divided into two domains by a line representing the locus of vertical tangents. For the softening spring, fig. 13/2, we must look at the region on the left-hand side of the " backbone". For our " model" case (13.3a or b) the locus curve is given by the equation

$$(13.7) \qquad \gamma^2 = 1 \pm 3\bar{Q}^2 \ .$$

The upper sign is for the hardening spring and the lower for the softening spring.

We may find a general expression for the locus curve, i.e. for an arbitrary restoring force $f(q)$, from (13.1) by applying the condition $\frac{\partial(\gamma^2)}{\partial Q} = 0$. The expression is

(13.8) $\gamma^2 = F(Q) + QF'(Q)$

if F' denotes the derivative $\frac{\partial F(Q)}{\partial Q}$.

It is important to note that, no matter what the function for the restoring force might be [and therefore whatever function enters into eq. (13.1)], the domain where the sequence of response curves with increasing values of A is such that Q increases for a fixed value of γ^2 (so-called "natural behavior") contains representative points of stable motion (regions I, III). The domain where the sequence of response curves with increasing values of A is such that Q decreases for a fixed value of γ^2 (so-called "un-natural behavior") contains representative points of unstable motions (region II).

The foregoing "criterion" for stability will be proved, and thoroughly discussed, in Part III of this report.

Here it is sufficient to note that the curves described by (13.8) and (13.2) form the boundary lines between the stable and unstable domains.

14. Damped Systems; Response Curves

When damping forces are present we must consider the function $G(\Omega, Q)$ as well as $F(Q)$ and, therefore, we are required to employ the equation (12.12) or (12.13).

For linear damping forces,

(14.1) $g(\dot{q}) = \dot{q}$;

the function $G(\Omega, Q)$ simply reduces to

$$(14.2) \qquad G(\Omega, Q) = \frac{1}{\kappa Q \pi} \int_0^{2\pi} \Omega Q \sin^2 \sigma \, d\sigma = \frac{\Omega}{\kappa} = \eta \, .$$

The formulae (12.13) thus become

$$(14.3a) \qquad \left[F(Q) - \eta^2\right]^2 + 4D^2\eta^2 = \frac{A^2}{Q^2}$$

$$(14.3b) \qquad \tan \epsilon = \frac{2D\eta}{F(Q) - \eta^2} \, .$$

The first of these equations determines the amplitude; the second determines the phase angle, of the resulting motion. We now solve (14.3a) for η^2 :

$$(14.3c) \qquad \eta^2 = \left[F(Q) - 2D^2\right] \mp \sqrt{\left(\frac{A}{Q}\right)^2 - 4D^2\left[F(Q) - D^2\right]} \, .$$

From this equation, which describes the response curves, we see that the curves do not lie at equal horizontal distances from the "backbone" curve $\eta^2 = F(Q)$; they do lie, however, at equal horizontal distances from a curve, parallel to the "backbone", which is given by the relation

$$\eta^2 = F(Q) - 2D^2 \, .$$

As other examples, we see that for a system having a Coulomb damping force

$$(14.4) \qquad q\left(\frac{\dot{x}}{b}\right) = \nu_0\left(\frac{\dot{x}}{b}\right) \, ;$$

the function $G(\Omega, Q)$ becomes

$$(14.4a) \qquad G(\Omega, Q) = \frac{4}{\pi} \frac{\gamma_0}{\kappa Q} \quad ;$$

and, for a system having a damping force proportional to the square of the velocity

$$(14.5) \qquad g(\dot{q}) = \gamma_2 (\operatorname{sgn} \dot{q})\, \dot{q}^2$$

the function G becomes

$$(14.5a) \qquad G(\Omega, Q) = \frac{8}{3\pi} \gamma_2 Q \Omega \eta \quad .$$

We may now generalize these results somewhat. For

$$(14.6) \qquad \begin{cases} g(\dot{q}) = \gamma_n \dot{q}^n & (n \text{ odd}) \\ g(\dot{q}) = \gamma_n (\operatorname{sgn} \dot{q})\, \dot{q}^n & (n \text{ even}) \end{cases}$$

we find

$$(14.6a) \qquad G(\Omega, Q) = G_n(\Omega, Q) = \gamma_n Q^{n-1} \Omega^{n-1} \eta \left\{ \frac{4}{\pi} \int_0^{\frac{\pi}{2}} \sin^{n+1} \sigma \, d\sigma \right\} .$$

The expression within the braces is denoted by $\varphi(n)$ in (5.6); (12.6a), therefore, becomes

$$(14.6b) \qquad G_n(\Omega, Q) = \gamma_n Q^{n-1} \Omega^{n-1} \eta \, \varphi(n) \quad .$$

It is worthwhile noting that this result agrees exactly with that found by L. S. Jacobsen [1]. Use was made, then of the concept of an equivalent linear damping force, where the non-linear and the equivalent linear damping forces were to dissipate the same amount of energy per cycle. The Averaging Method and the method of Jacobsen both lead to the same results which means that the underlying assumptions amount to the same thing.

Let us again study our "model" case, which is defined by

$$f(y) = y + \mu^2 y^3 - \qquad (\mu^2 > 0)$$

and which includes now a damping term

$$g(\dot{y}) = \dot{y}$$

Figure 14/1 shows the family of response curves for this system when $D = 0.1$ and Λ is the parameter. It is seen that the amplitudes are now finite, because of the damping; i.e. the response curves are continuous and therefore cross the "backbone". This crossing, or limiting of amplitude, would occur regardless of what the damping function $g(\dot{y})$ may be; this is easily seen from eq. (12.13a).

The point at which a response curve crosses the "backbone" curve is termed the "resonance point", and it readily follows that by introducing the condition for resonance [i.e. free, undamped vibrations: $F(Q) = \gamma^2$] into eq. (12.13a) we obtain

$$2DQG(\Omega, Q) = \Lambda \qquad (14.7)$$

---

1) "Steady Forced Vibrations as Influenced by Damping", Trans. of American Society of Mechanical Engineers, 52 (1930) p. 169 (APM-52-15).

For linear damping eq. (14.7) reduces, because of (14.2), to

$$\gamma Q = \frac{A}{2D} \quad . \tag{14.7a}$$

This equation, which is the locus of crossing points, describes a family of hyperbola-like curves in the response diagram with $\frac{A}{2D}$ as a parameter. In the most general case [i.e. with damping forces such as to give (14.6)] this locus becomes, from eqs. (12.12), (13.2), and (14.6b),

$$Q^n \gamma^n = \frac{A}{2D\varphi(n)\nu_n \kappa^{n-1}} \quad . \tag{14.8}$$

It is interesting to note that these equations for the locus of crossing points do not contain the nonlinearity of the restoring force; however, the " backbone" curve does, and therefore, affects the location of the crossing point. Also, it may be seen from eq. (12.13b) that at the crossing point (where $F(Q) = \gamma^2$) the phase angle always has the value $\epsilon = \frac{\pi}{2}$.

An observation of a practical nature may be made at this time. Equations (12.13) for the general case are often rather tedious to compute; however, if one computes the response curves for the undamped system from (13.1), which is rather simple, and then finds the crossing point as described above, sufficient information will then be available for describing with a great deal of accuracy the behavior of the system. Only rarely will it be found necessary to evaluate eqs. (12.13).

Fig. 14/2 shows the response curves for our "model" case but with "softening" spring. Note that the hyperbola-like curve which describes the resonance points may intersect the "backbone" curve in two places - this leads to the double set of response curves which are shown in the figure. Fig. 14/3 gives an enlarged view of this interesting region in order to show some of the details more clearly.

We see that again in all the figures referring to damped systems we have regions of "natural" and "unnatural" sequence of the response curves with increasing values of the parameter A (see section 13 for the undamped system). Again, these regions are separated by the loci of vertical tangents.

We find the equation of the locus of vertical tangents (for simplicity we consider only linear damping) in the following manner: The equation of the family of response curves is [from eqs. (12.13) and (14.2)]

$$(14.9) \qquad H \equiv \left[F(Q) - \eta^2\right]^2 + 4D^2\eta^2 - \left(\frac{A}{Q}\right)^2 = 0 \,.$$

Applying to this equation the condition $H_Q = 0$ we find (by eliminating the parameter A)

$$(14.10) \qquad M \equiv \left[F(Q) - \eta^2\right]\left[F(Q) + QF'(Q) - \eta^2\right] + 4D^2\eta^2 = 0$$

as the equation of the locus.

One easily recognizes that if $D=0$, eq. (14.10) reduces to the two equations

$$F(Q) - \bar{\gamma}^2 = 0$$

$$F(Q) + QF'(Q) - \bar{\gamma}^2 = 0$$

which agree with the results of section 13.

For our "model" case, eq. (14.10) becomes

$$(14.10a) \qquad \left[1 + \bar{Q}^2 - \bar{\gamma}^2\right]\left[1 + 3\bar{Q}^2 - \bar{\gamma}^2\right] + 4D^2\bar{\gamma}^2 = 0 \ .$$

Fig. 14/1 shows the response curves and the locus of vertical tangents (14.10a) for $D = 0.1$.

Now, imagine similar graphs to be drawn for other values of D ; then imagine that all of the loci of vertical tangents are drawn in a single plot — this is shown in fig. 14/2. Here we have a new family of curves having D as the parameter. This family of curves also has a locus of vertical tangents; the equation is found by eliminating D from eq. (14.10) and the equation $M_Q = 0$. The result is

$$(14.11) \qquad \left\{3F(Q)F'(Q) + Q\left[F(Q)F''(Q) + F'^2(Q)\right]\right\}$$

$$-\bar{\gamma}^2\left[3F'(Q) + QF''(Q)\right] = 0$$

for the general case, and

$$\rho^2 = 1 + \frac{3}{2}\bar{Q}^2 \tag{14.11a}$$

for the " model" case.

Fig. 14/5 shows a family of response curves for a fixed value of $\bar{\lambda} = (\bar{r} - 0.5)$, D being the parameter. This set of curves contains all of the features of the previous family, and we could discuss them in a similar manner; however, we shall not do so here.

It seems worthwhile to lay emphasis, again, on the fact that all information, within the limits of accuracy of the Averaging Method, regarding the response curves for a single degree of freedom system having the equation of motion (12.1), or (12.3), is contained in the equations (12.12) or (12.13).

D. OTHER APPLICATIONS OF THE AVERAGING METHOD AND CONCLUDING REMARKS

15. Self-sustained vibrations.

So far we have considered the steady state of free or forced vibrations in either conservative or dissipative systems and have attempted to approximate them by a harmonic motion.

Systems which allow self-sustained vibrations are of such nature that for small oscillations there is an energy input (from some source) and for large oscillations energy is consumed. It is well known that, in systems of this type, periodic motions are possible -- the so-called "limit cycles". The displacement-time curves are more or less harmonic, depending, of course, upon the values of certain parameters in the differential equations. We shall show here that the Averaging Method again allows us to obtain an approximate solution for the steady state motion.

As an example, let us study the " van der Pol equation"

$$E \equiv \ddot{q} - 2\delta\dot{q}(1-q^2) + \kappa^2 q = 0 \quad . \tag{15.1}$$

Here q denotes a dimensionless quantity, and κ and δ are constants whose dimensions are reciprocals of time. We assume

$$q = Q\cos(\tau - \varepsilon) \quad , \quad \tau = \omega t \quad . \tag{15.2}$$

The Ritz method then requires that

$$(15.3)\qquad \begin{cases} \int_0^{2\pi} E[\tilde{q}] \cos\tau \, d\tau = 0 \\ \int_0^{2\pi} E[\tilde{q}] \sin\tau \, d\tau = 0 \end{cases} .$$

After introducing (15.2) into (15.1) and performing the integrations (15.3) we get

$$(15.4)\qquad \begin{cases} \left(1-\eta^2\right)\cos\varepsilon + 2D\left(\frac{Q^2}{4}-1\right)\sin\varepsilon = 0 \\ \left(1-\eta^2\right)\sin\varepsilon - 2D\left(\frac{Q^2}{4}-1\right)\cos\varepsilon = 0 \end{cases}$$

where $\eta = \frac{\omega}{K}$, $D = \frac{\delta}{K}$. These two equations are then equivalent to

$$(15.5)\qquad \begin{cases} 1-\eta^2 = 0 \\ \frac{Q^2}{4}-1 = 0 \end{cases} .$$

Thus we have

$$(15.6a)\qquad \omega = K$$

and

$$(15.6b)\qquad Q = \pm 2$$

so that the solution which is provided by the Ritz method has the form

$$(15.6)\qquad q = 2\cos(Kt - \varepsilon)$$

with ϵ being arbitrary. It is worth noting that this result coincides with that which is obtained as the first approximation from the Kryloff-Bogoliuboff procedure.

The differential equation (15.1) is known to have solutions which approach harmonic ones as δ becomes smaller, and which become definitely non-harmonic for larger values of δ ("relaxation" oscillations). In any case, the Ritz method gives the best approximation (in the sense described earlier) which is possible under the assumption (15.2).

A second example for a self-sustained vibration is described by the following differential equation (so-called "Rayleigh's equation")

$$\ddot{q} - (\alpha^2 - \beta^2 \dot{q}^2 - \gamma^2 q^2)\dot{q} + \kappa^2 q = 0 \qquad (15.7)$$

Here the same procedure, as described above, leads to the steady state solution

$$q = \frac{4\alpha^2}{\beta^2 + 3\gamma^2\kappa^2} \cos(\kappa t - \epsilon) \qquad (15.8)$$

again in agreement with the result by the Kryloff-Bogoliuboff method.

16. Concluding Remarks.

The systems treated in this Part II of the report do not exhaust the possibilities. They are intended to be looked upon merely as examples. The number of such examples can be increased

in many ways.

Here one-term approximations (in the sense described in section 1) were treated. Higher order approximations generally lead to a set of two or more (coupled) algebraic equations for the two or more coefficients which are to be determined. If non-linear expressions are present in the differential equation then algebraic equations are also non-linear. One realizes at once that the complexity of the problem increases rapidly as the number of terms in the assumed form of the solution (1.2) increases. However, the difficulties are of "purely" algebraic nature, the integration of the differential equation having been performed. Furthermore there is no inherent difficulty either in applying the Averaging Method to systems of more than one degree of freedom, where the majority of other methods become extremely complicated, or fail altogether. There again the remaining difficulties are of purely algebraic character.

The evidence presented so far, it is hoped, will be sufficient to support the statement that the Averaging Method presents an excellent tool for treating a large variety of vibration problems described by non-linear differential equations.

TABLE 1

Values of the Coefficients $\psi(n)$, $\chi(n)$, and $\varphi(n)$ and the Percentage Error

n	$\psi(n)$	$\chi(n)$	$\varphi(n)$	$\frac{\varphi(n)}{\chi(n)}$	% Error
0	1.4142	1.2337	1.2732	1.0320	3.2
1	1.5708	1.0000	1.0000	1.0000	0
2	1.7157	0.8373	0.8488	1.0138	1.4
3	1.8541	0.7185	0.7500	1.0439	4.4
4	1.9819	0.6282	0.6791	1.0809	8.1
5	2.1035	0.5577	0.6250	1.1208	12.1
6	2.2186	0.5013	0.5820	1.1612	16.1
7	2.3282	0.4552	0.5469	1.2013	20.1

$\psi(n)$: see eq. (4.12)

$\chi(n)$: see eq. (4.14a)

$\varphi(n)$: see eq. (5.6)

$$\% \text{ error} = \left[\frac{\varphi(n)}{\chi(n)} - 1\right] 100$$

TABLE 2

	$\gamma = 2$			$\gamma = \frac{1}{2}$			$\gamma = 0$ $(c_2 = 0)$		
$\frac{y}{\delta}$	APPROX. $\frac{u}{u_0}$	EXACT $\frac{u}{u_0}$	DIFF.	APPROX. $\frac{u}{u_0}$	EXACT $\frac{u}{u_0}$	DIFF.	APPROX. $\frac{u}{u_0}$	EXACT $\frac{u}{u_0}$	DIFF.
1.00	1.0000	1.0000	0	1.0000	1.0000	0	1.00000	1.00000	0
1.25									
1.50	1.10419	1.10216	0.00203	0.94359	0.94285	0.00074	0.88360	0.87979	0.00381
1.75									
2.00	1.17942	1.17750	0.00192	0.89693	0.89586	0.00107	0.78038	0.77409	0.00629
2.25									
2.50	1.22654	1.22514	0.00140	0.86475	0.86365	0.00110	0.70399	0.69638	0.00761
2.75									
3.00	1.25830	1.25729	0.00101	0.84162	0.84064	0.00098	0.64550	0.63759	0.00791
3.25									
3.50	1.28113	1.28028	0.00085	0.82424	0.82346	0.00078	0.59894	0.59128	0.00766
3.75									
4.00	1.29798	1.29739	0.00059	0.81094	0.81018	0.00076	0.56147	0.55366	0.0078
4.25									
4.50	1.31128	1.31079	0.00049	0.80018	0.79963	0.00055	0.52967	0.52235	0.007
4.75									
5.00	1.32179	1.32129	0.00050	0.79147	0.79100	0.00047	0.50285	0.49579	0.007

$\frac{u}{u_0} = \left|\frac{\bar{q}}{\bar{q}}\right|$ — approx

$\frac{u}{u_0} = \left(\frac{q}{q}\right)_{max}$ exact

Fig. 11/4b: Displacement-Time Curves for Bilinear Systems

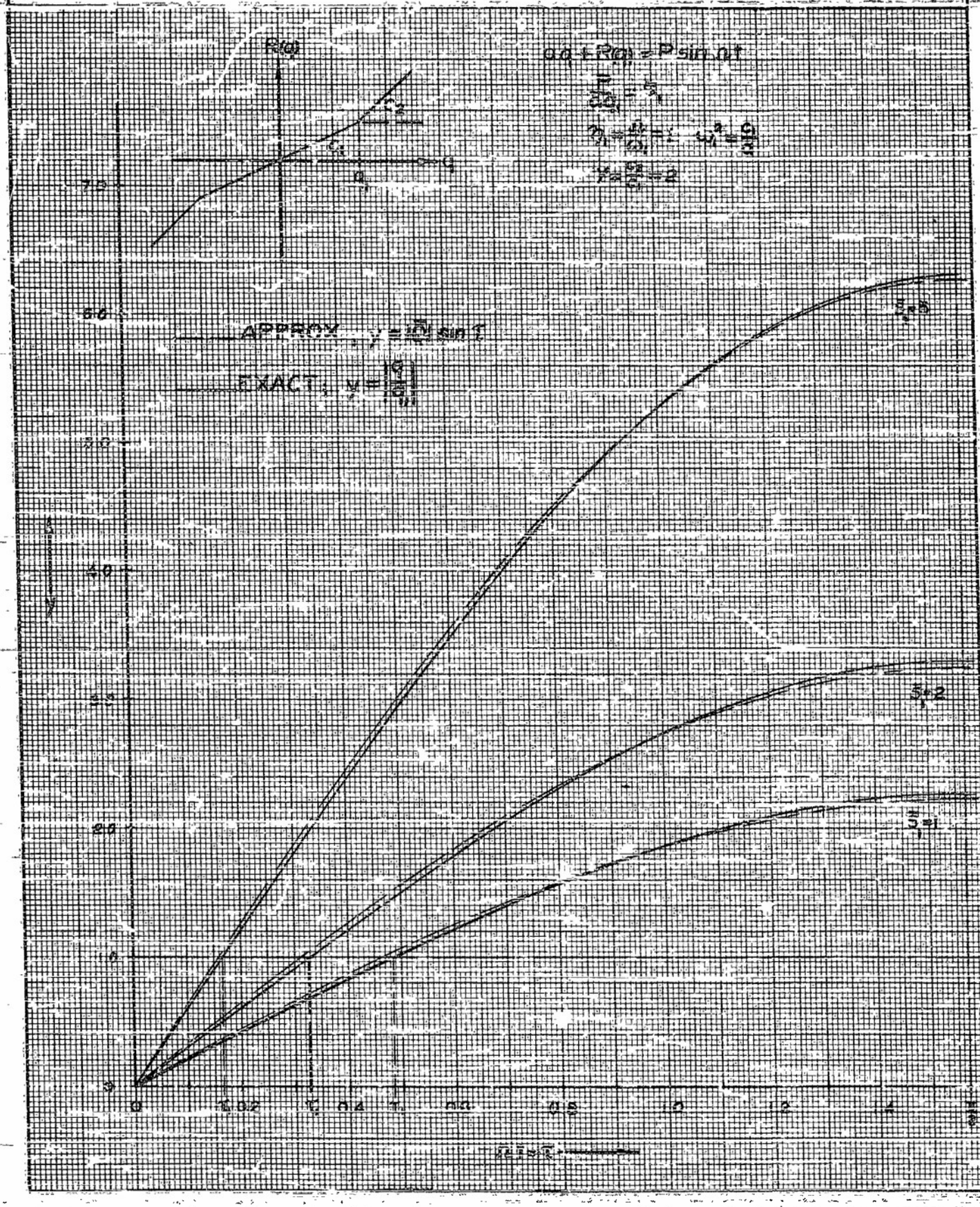


Fig. 11/4c: Displacement-Time Curves for Bilinear Systems

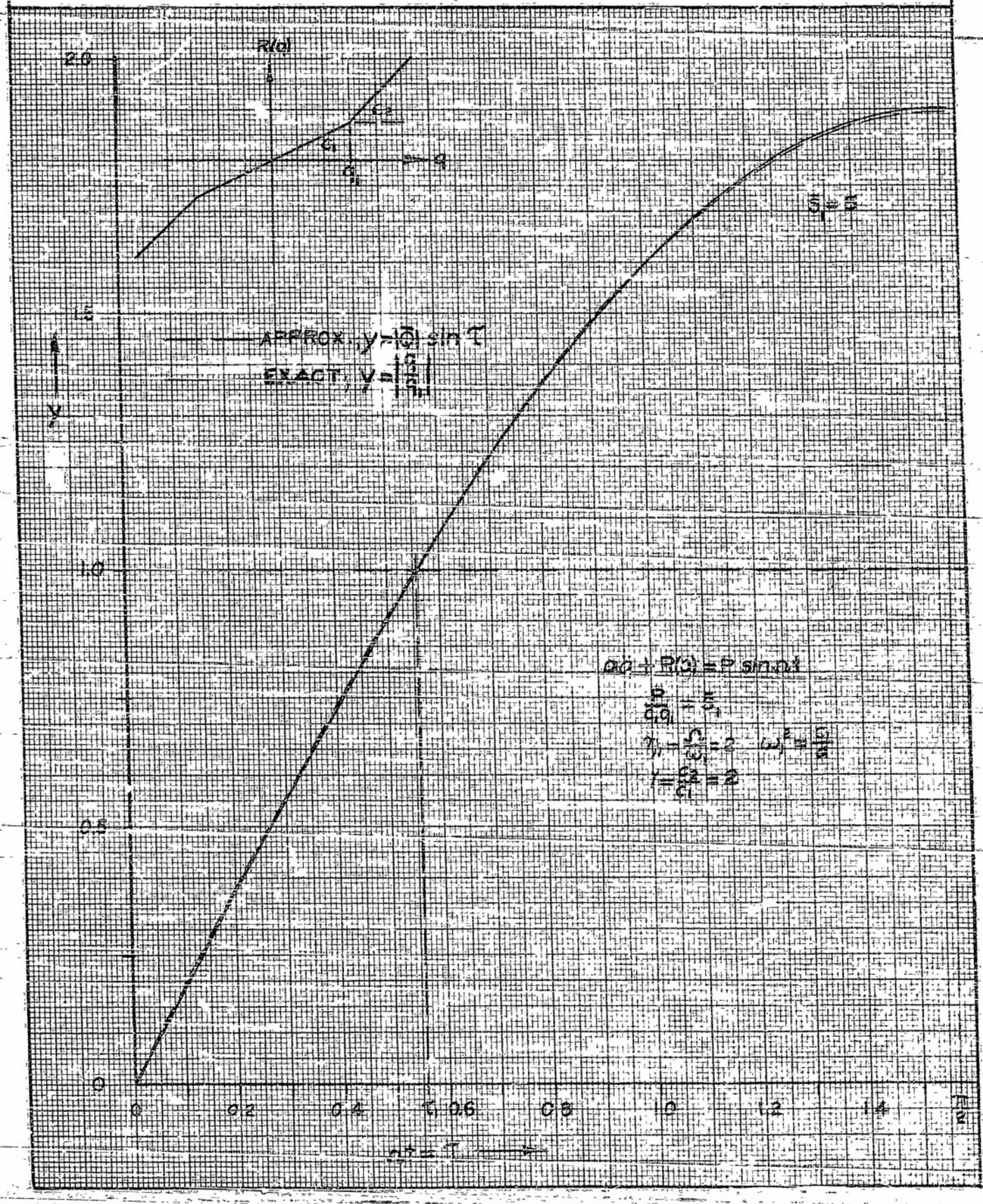


Fig. 11/5a: Displacement-Time Curves for Bilinear Systems

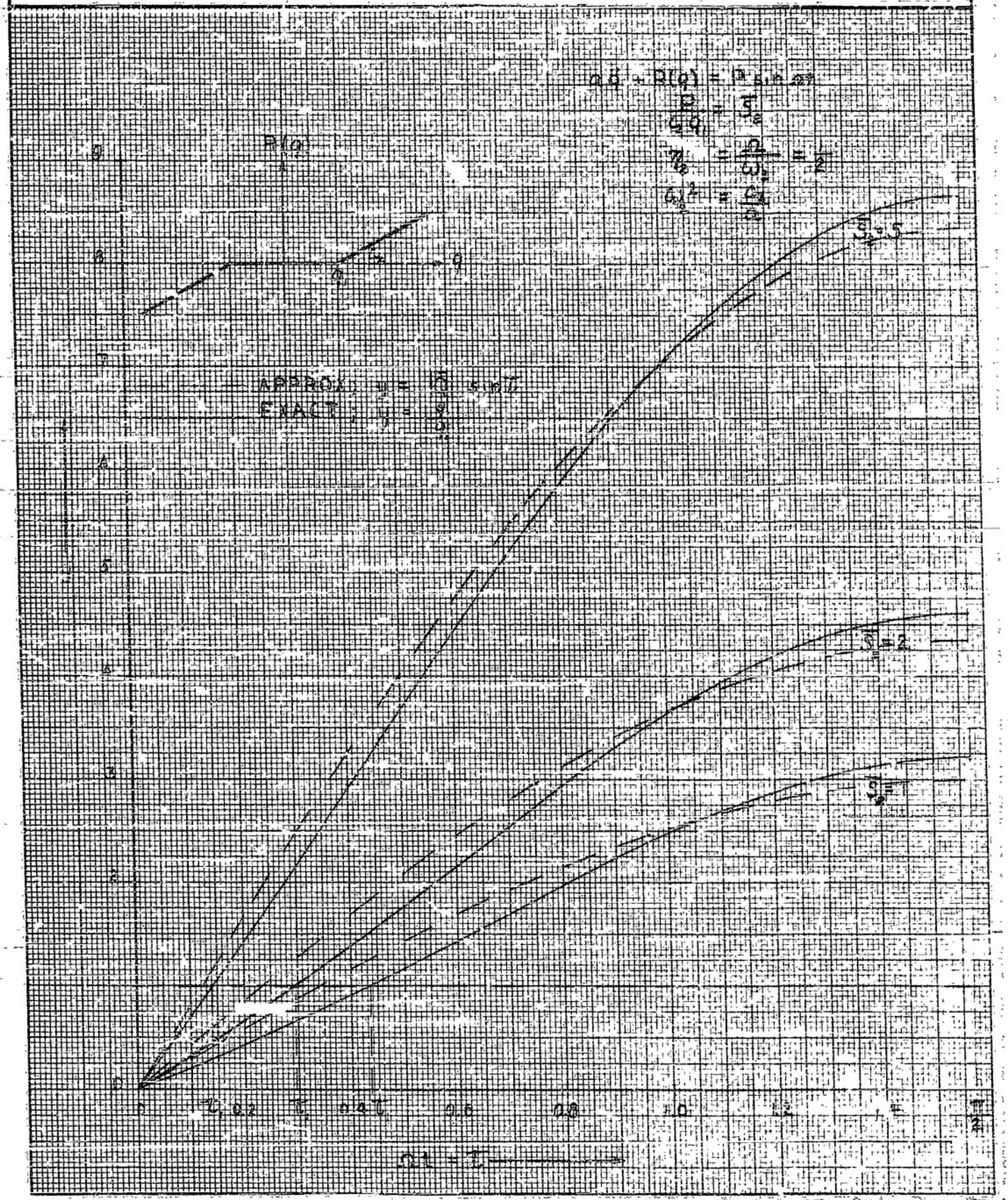


Fig. 11/5b: Displacement-Time Curves for Bilinear Systems

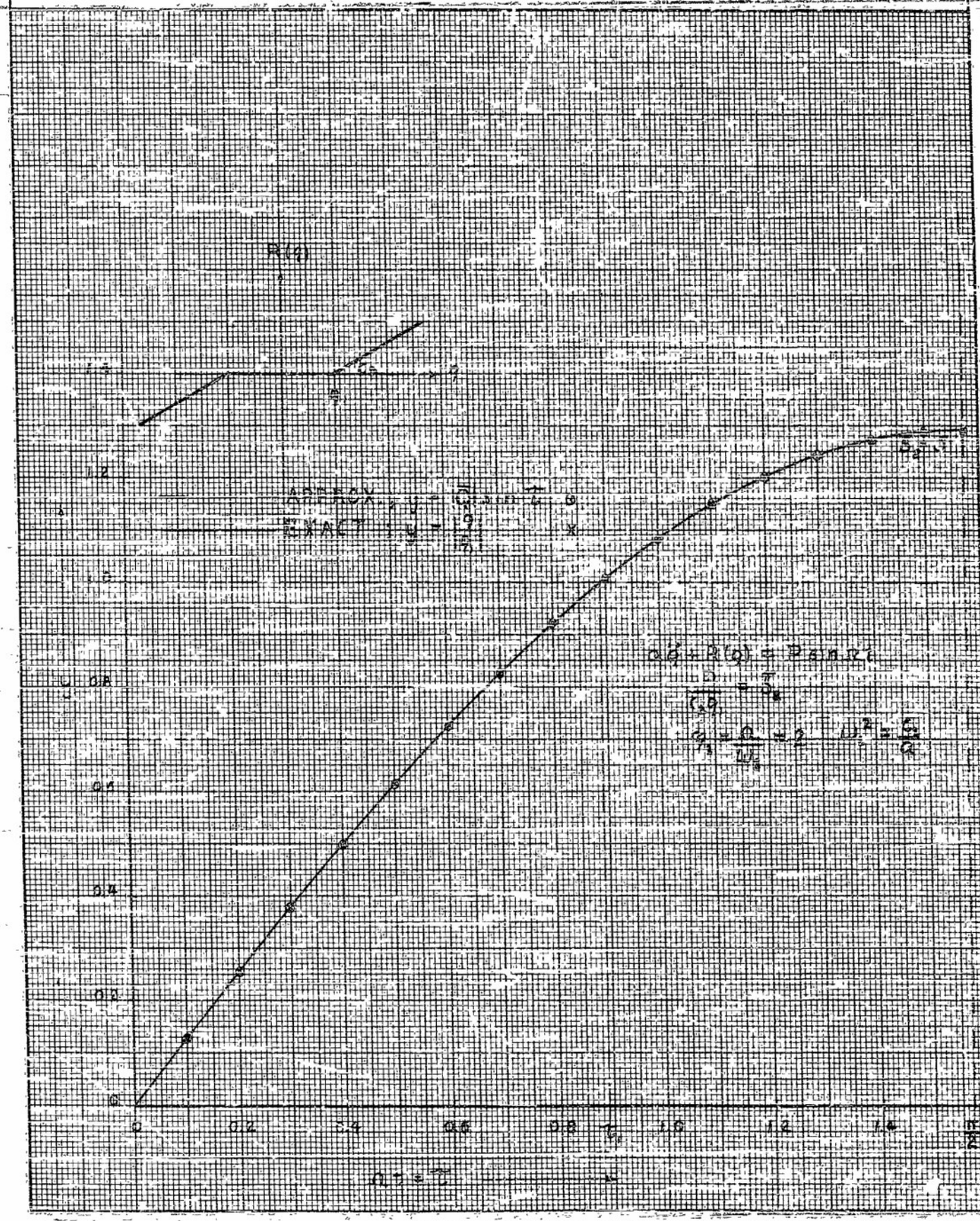


Fig. 11/6a: Displacement-Time Curves for Bilinear Systems

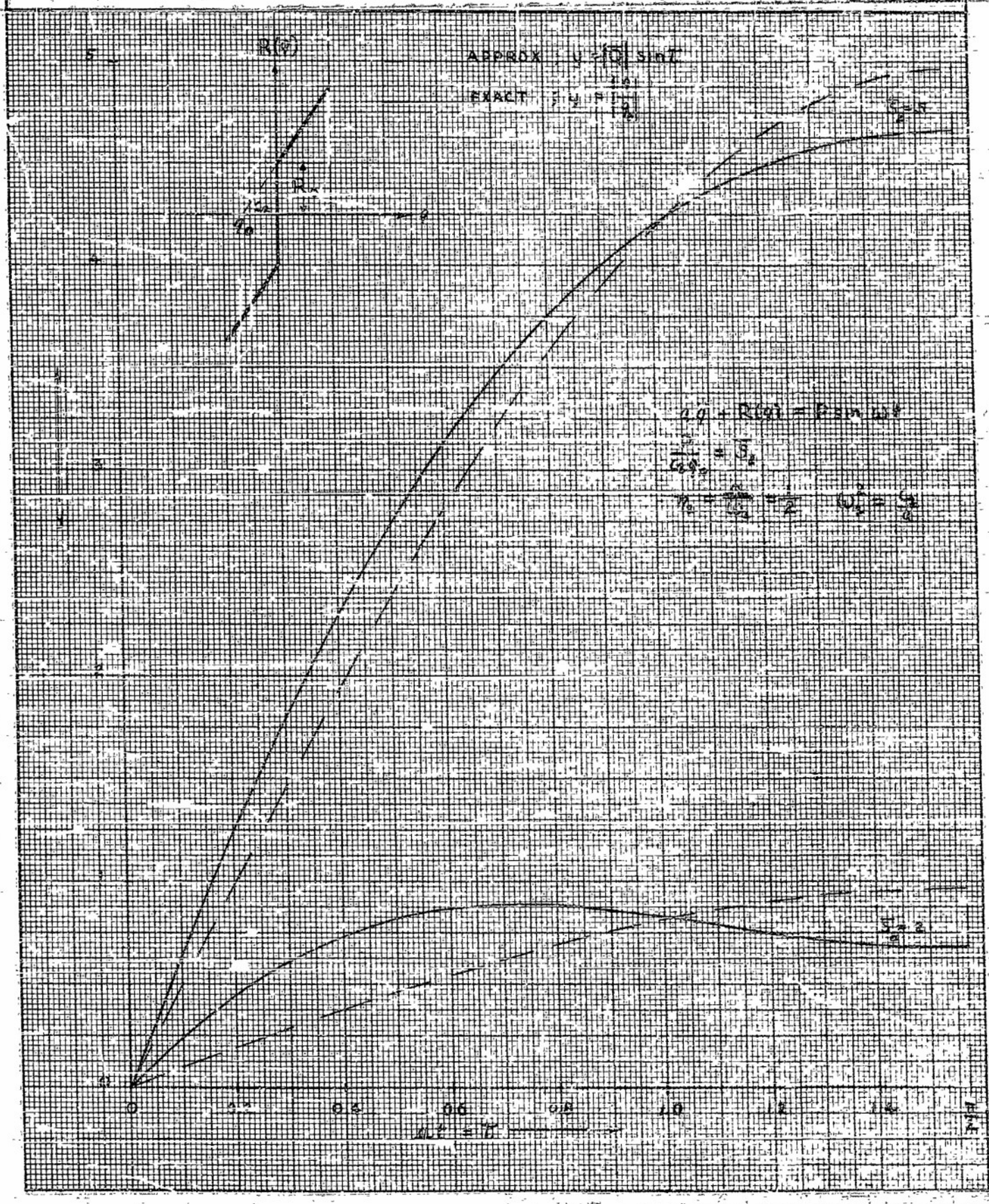


Fig. 11/6b: Displacement-Time Curves for Bilinear Systems

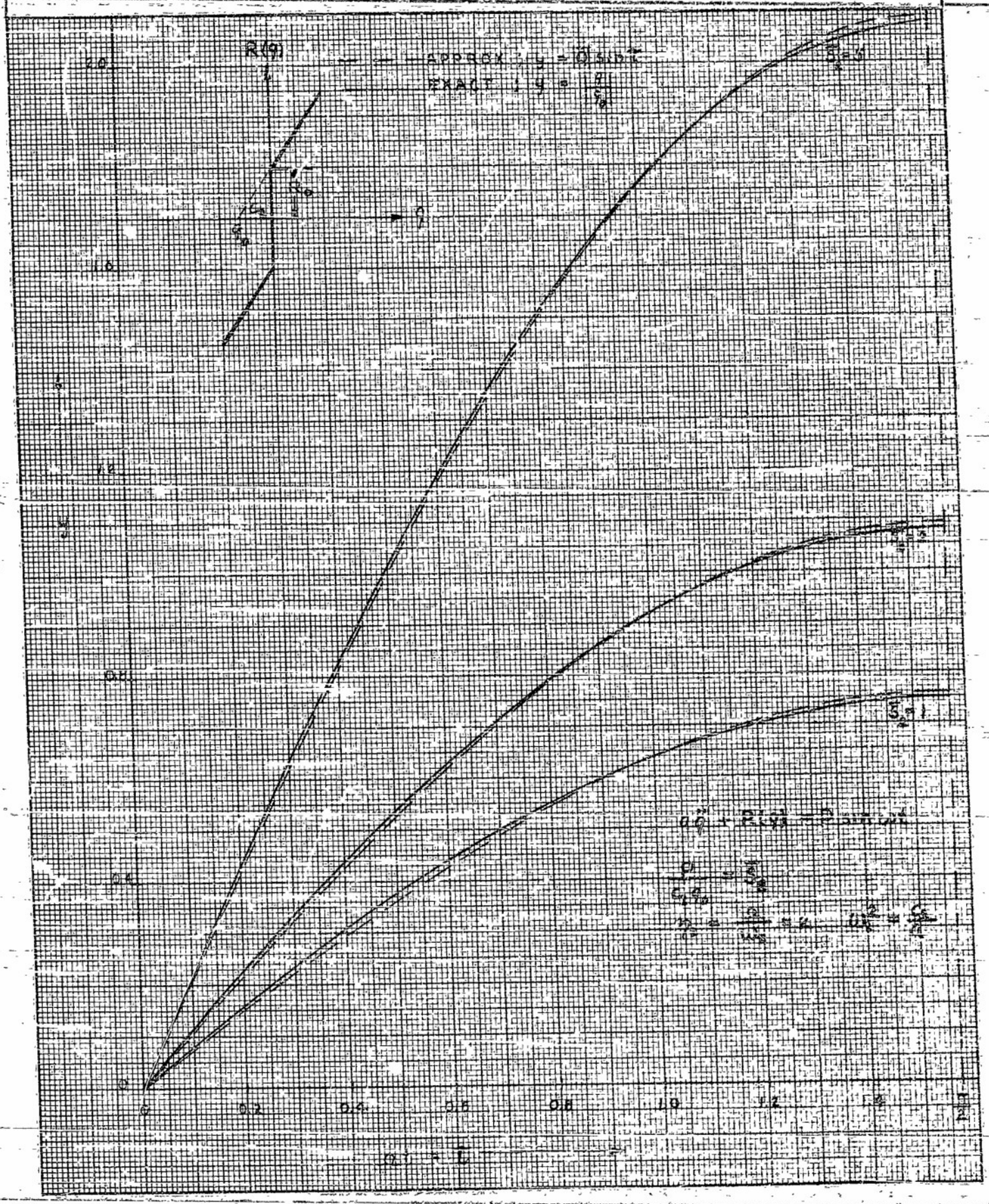


Fig. 11/7: Response Curves

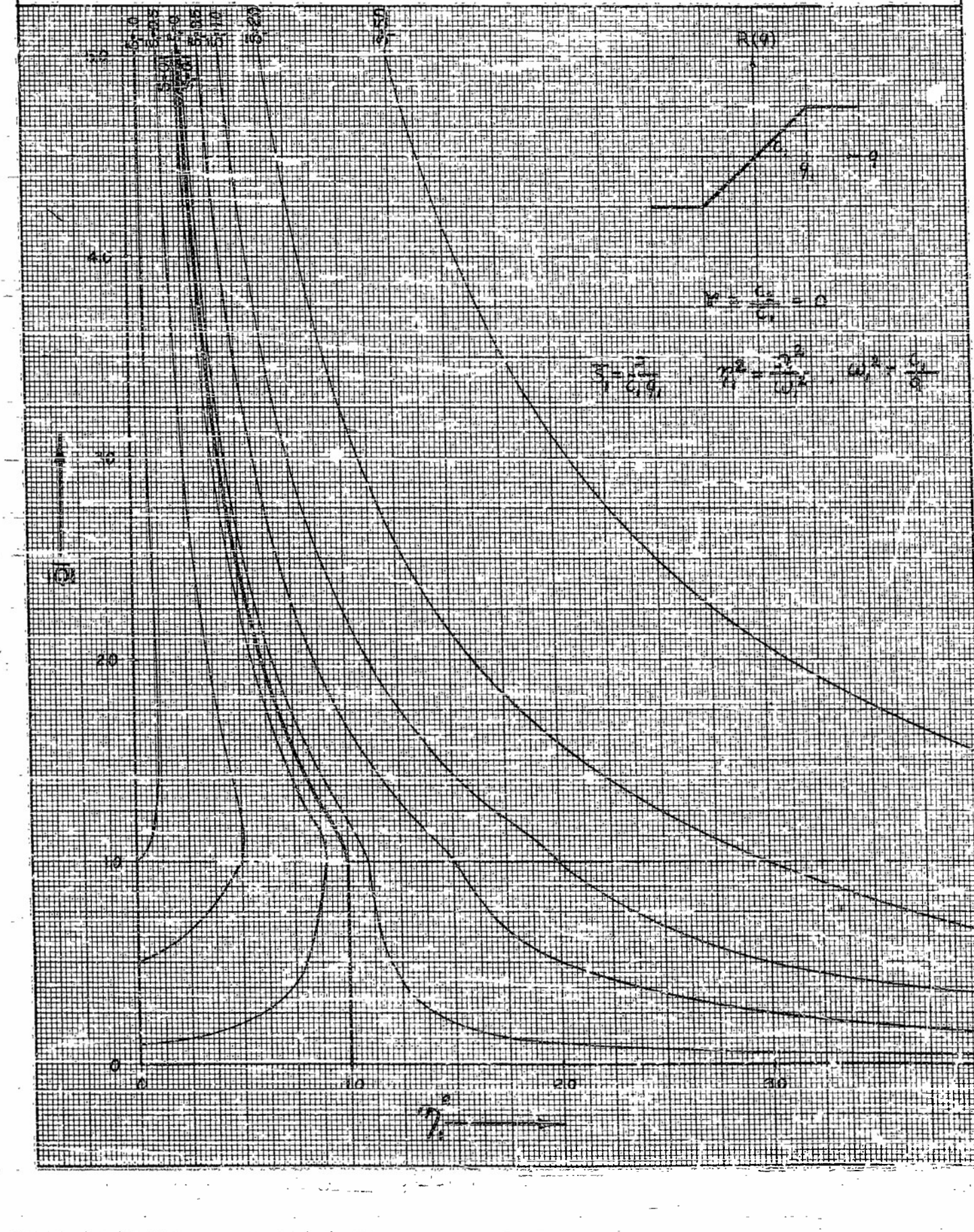


Fig. 11/8: Response Curves

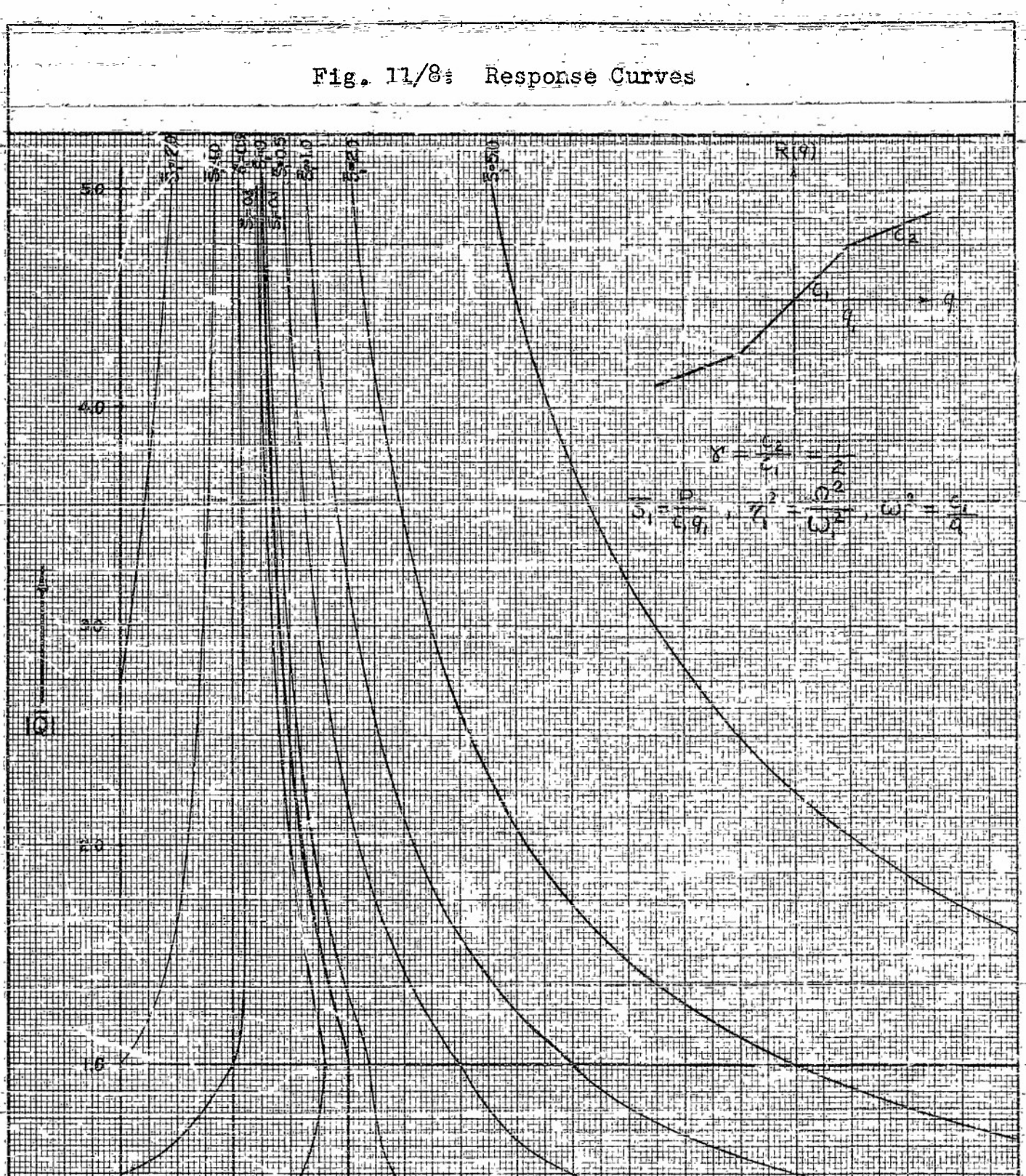


Fig. 11/9: Response Curves

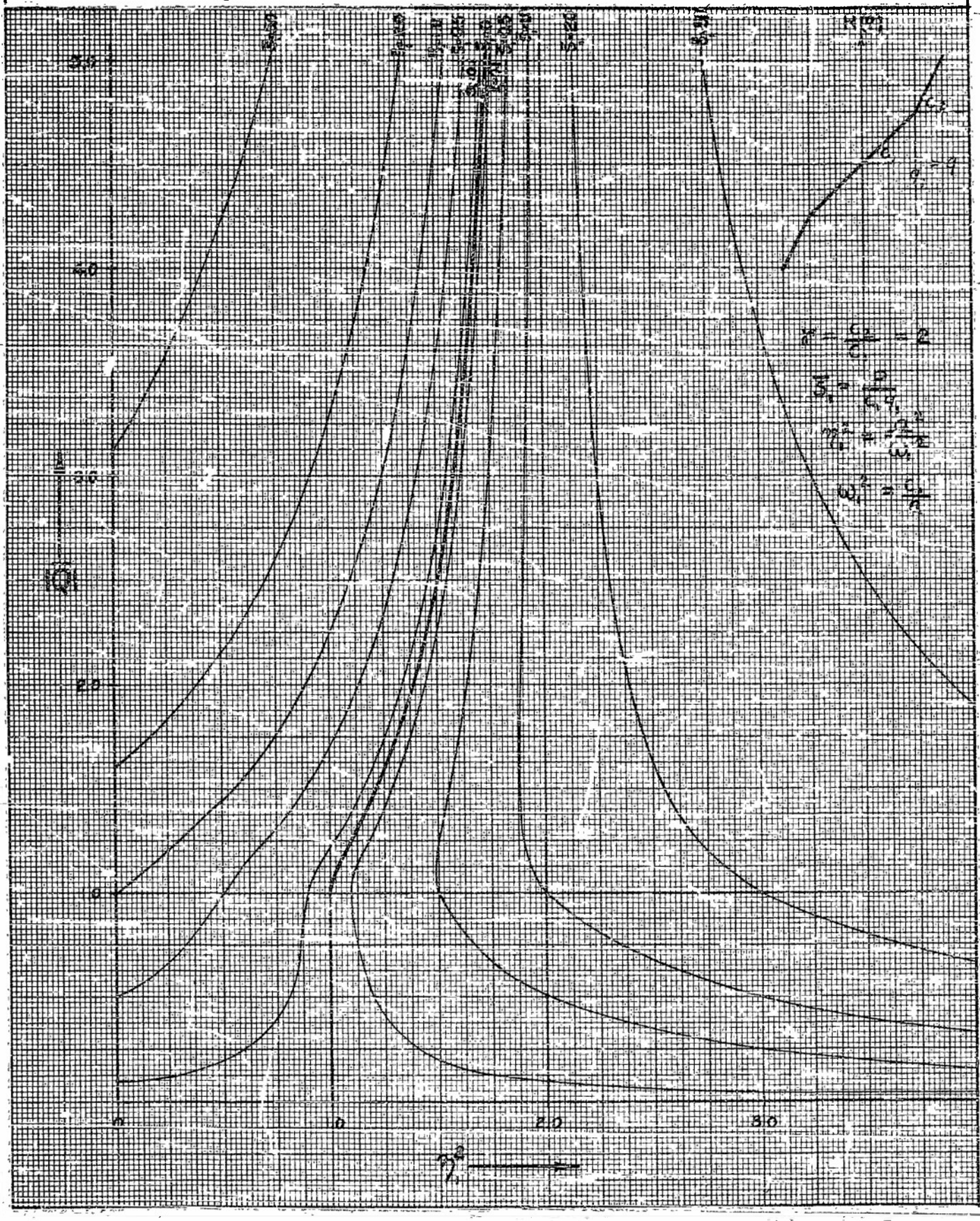


Fig. 11/10: Response Curves

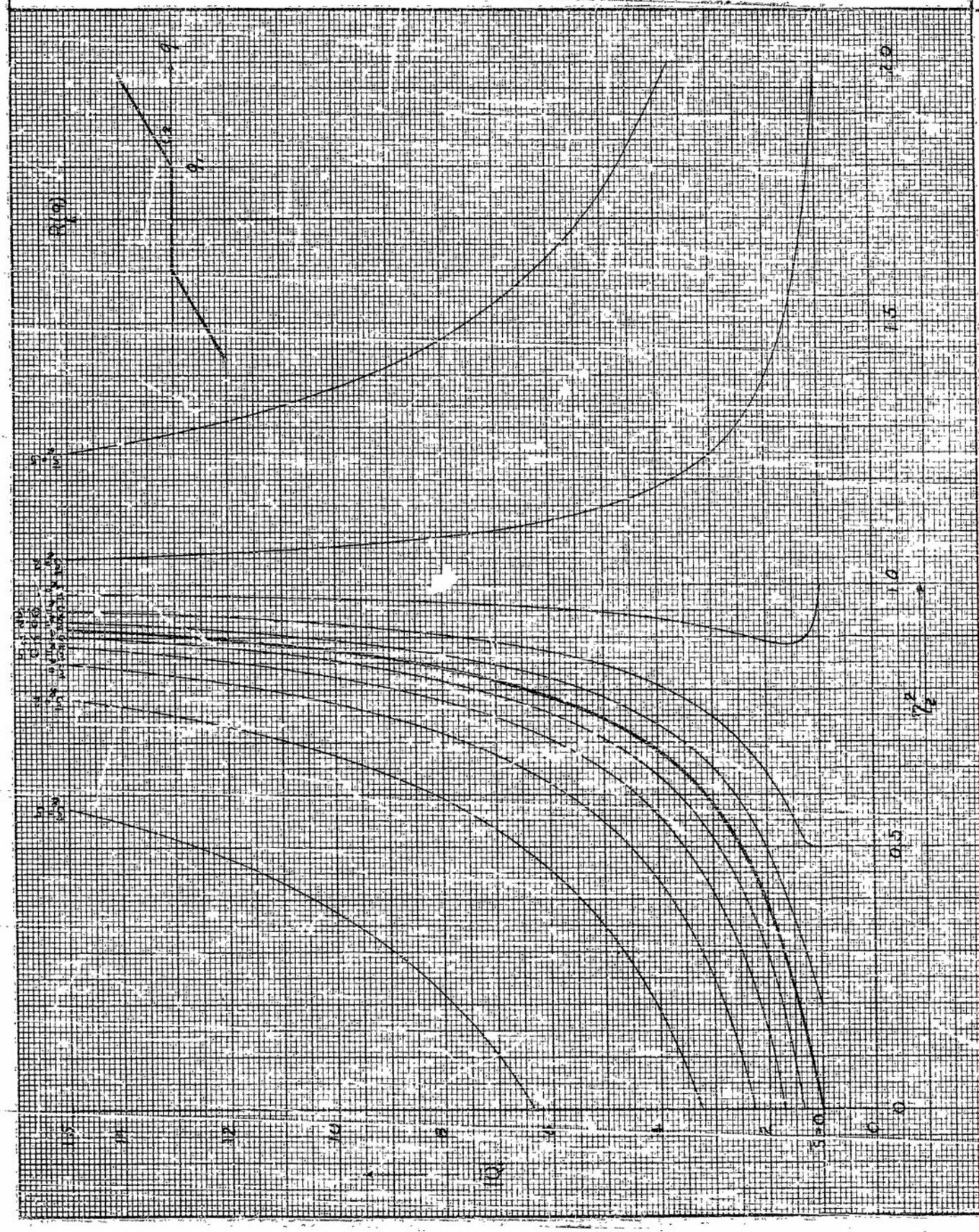


Fig. 11/11: Response Curves

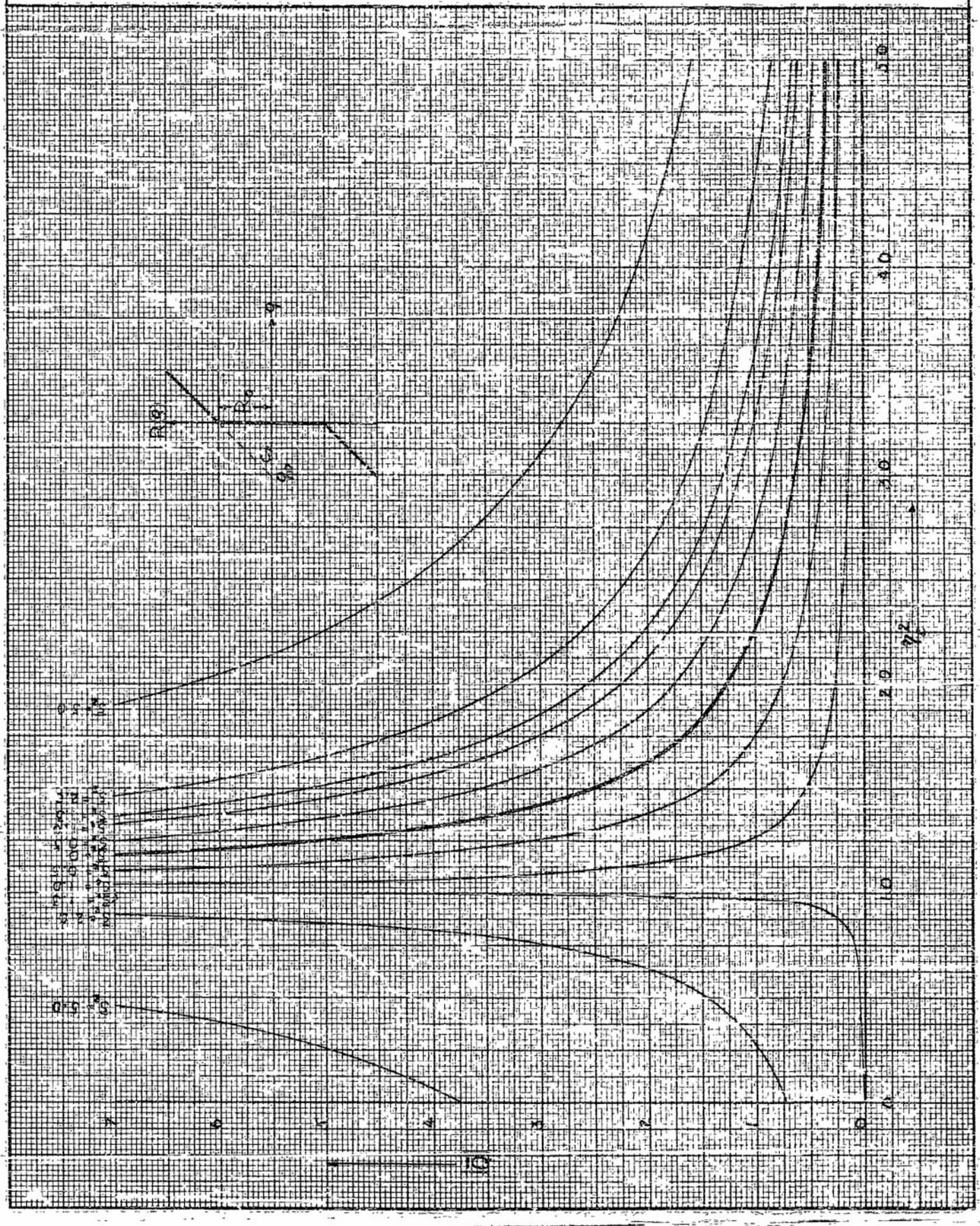


Fig. 11/12: Response Curves

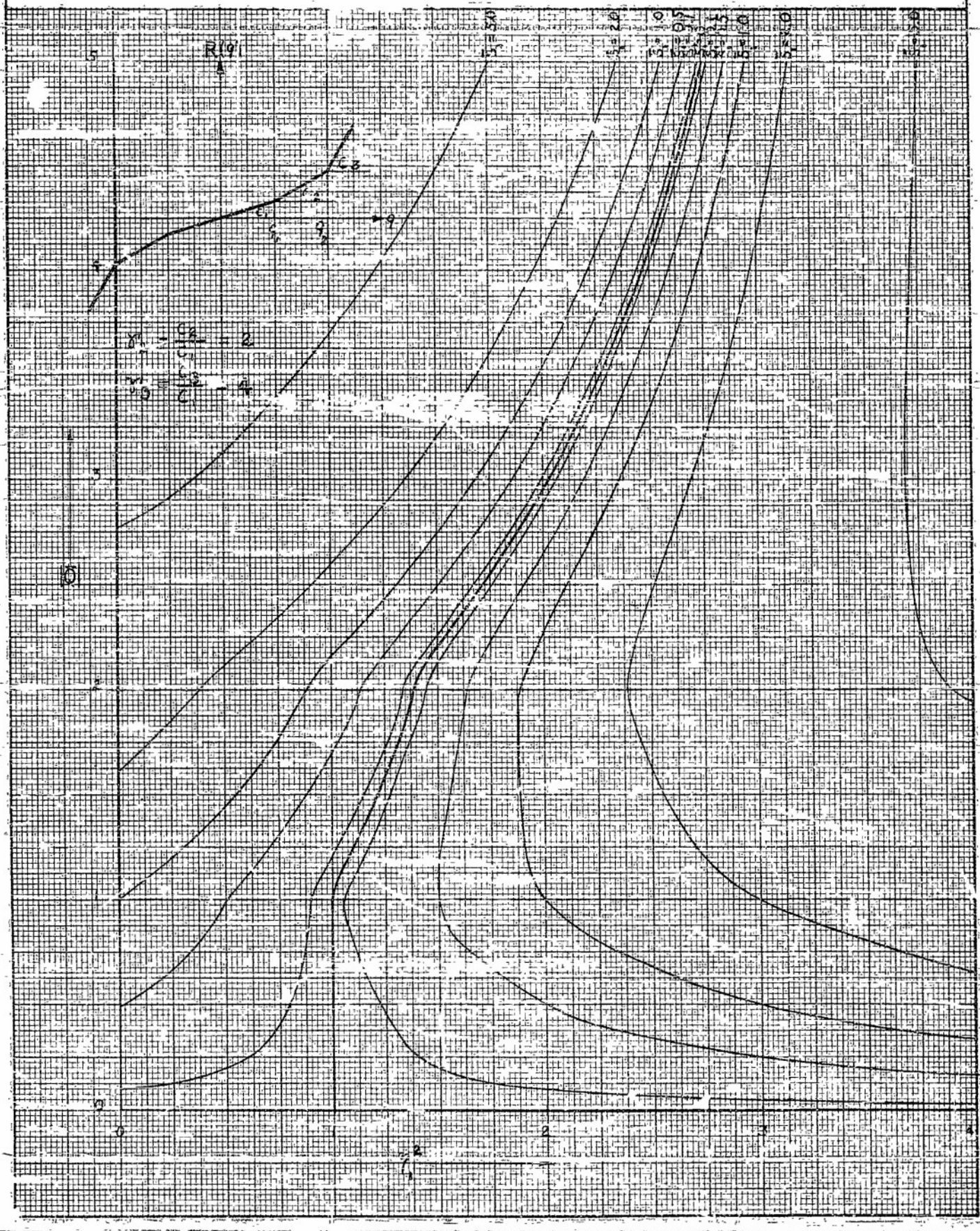


Fig. 11/13: Displacement-Time Curves for Bilinear Systems

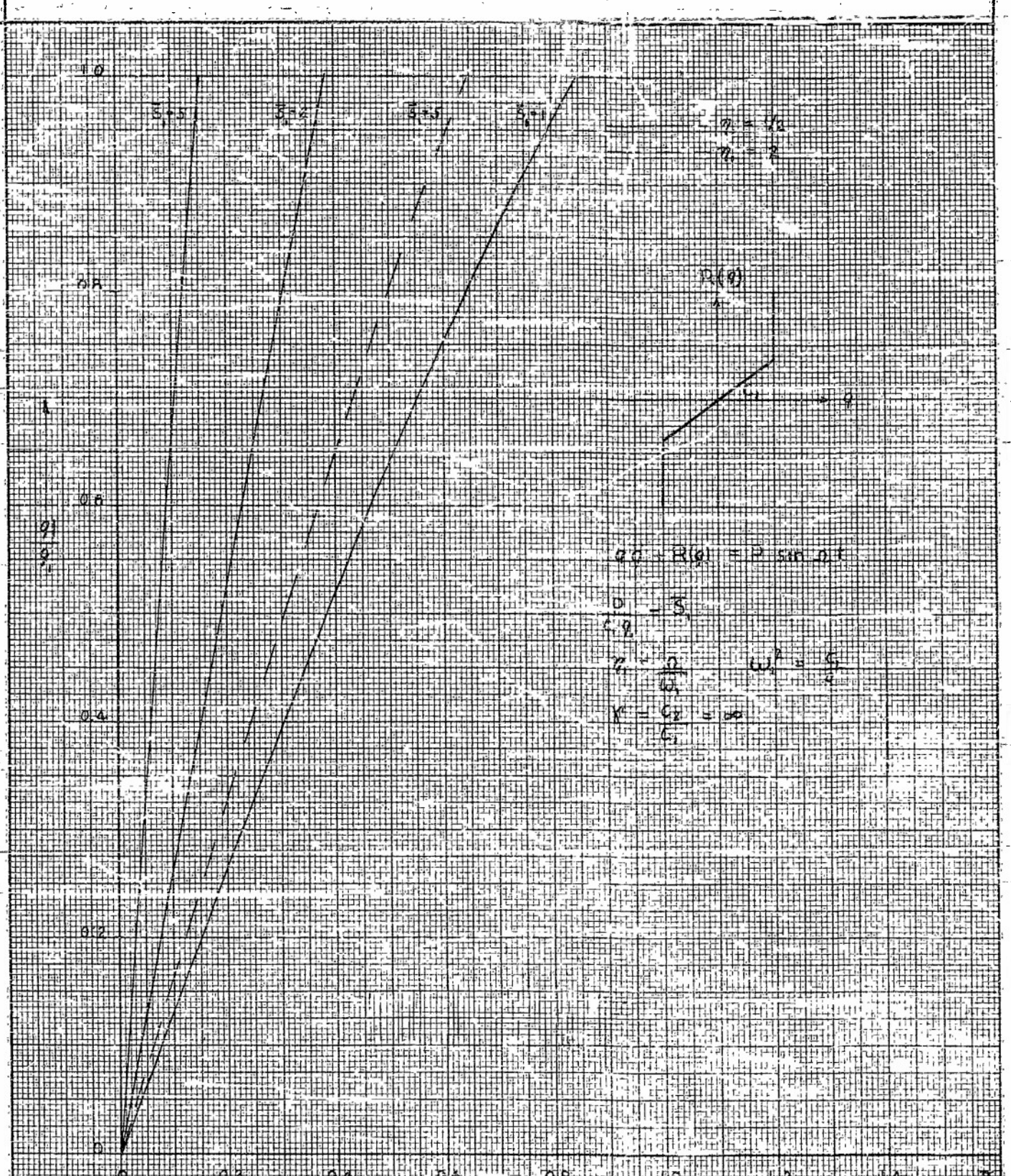


Fig. 11/14: Backbone Curves for Increasing γ

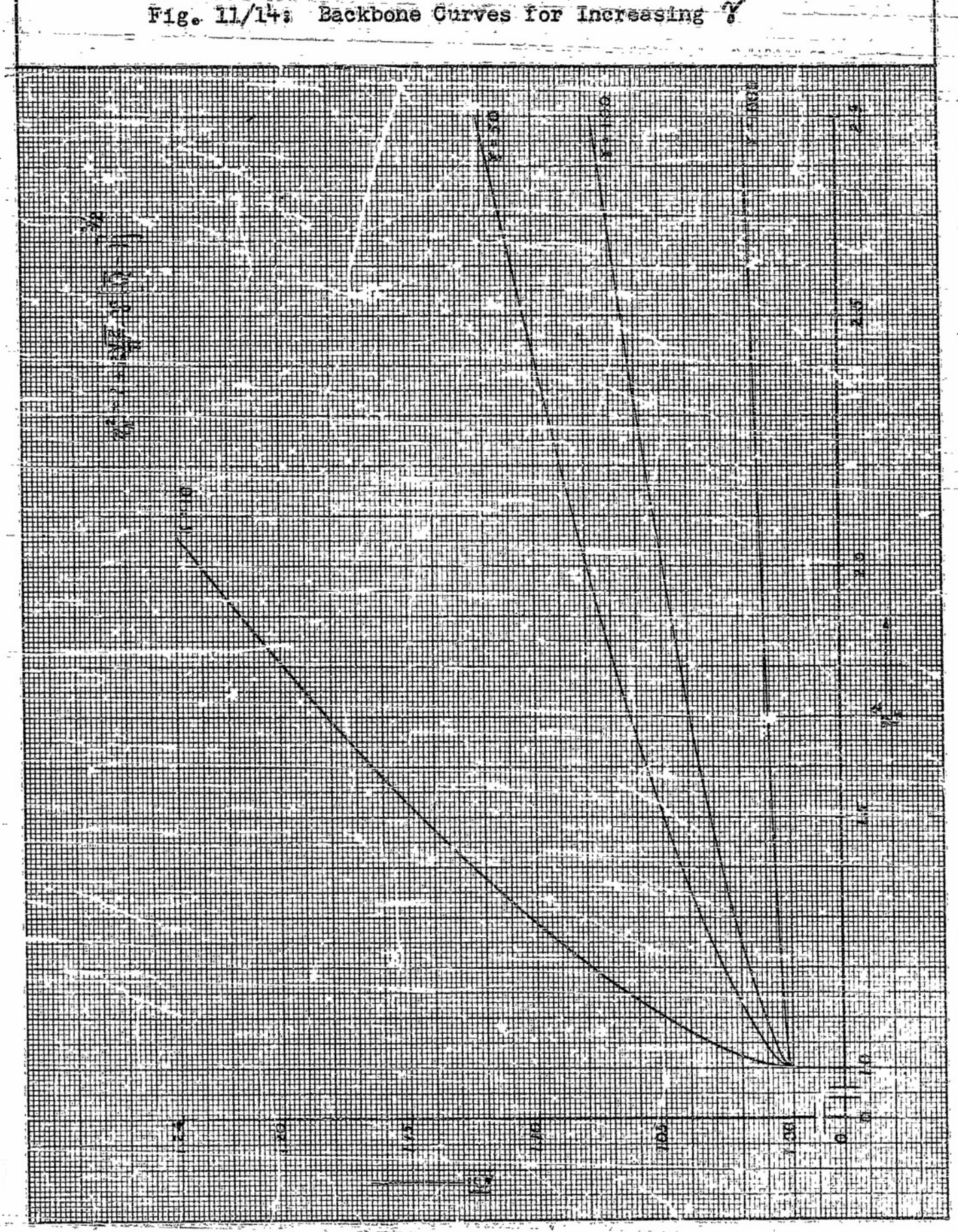


Fig. 13/1: Response Curves

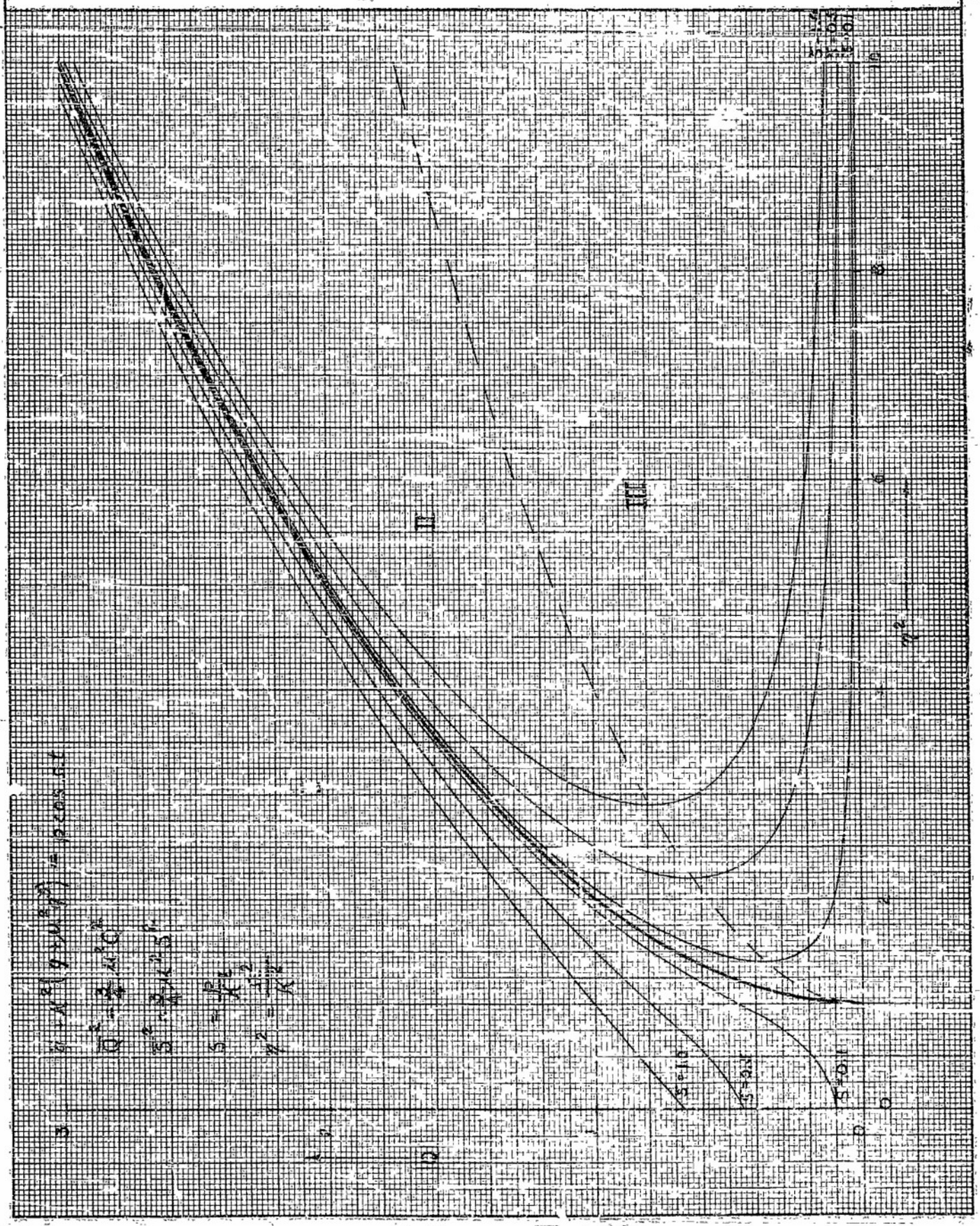


Fig. 13/2: Response Curves

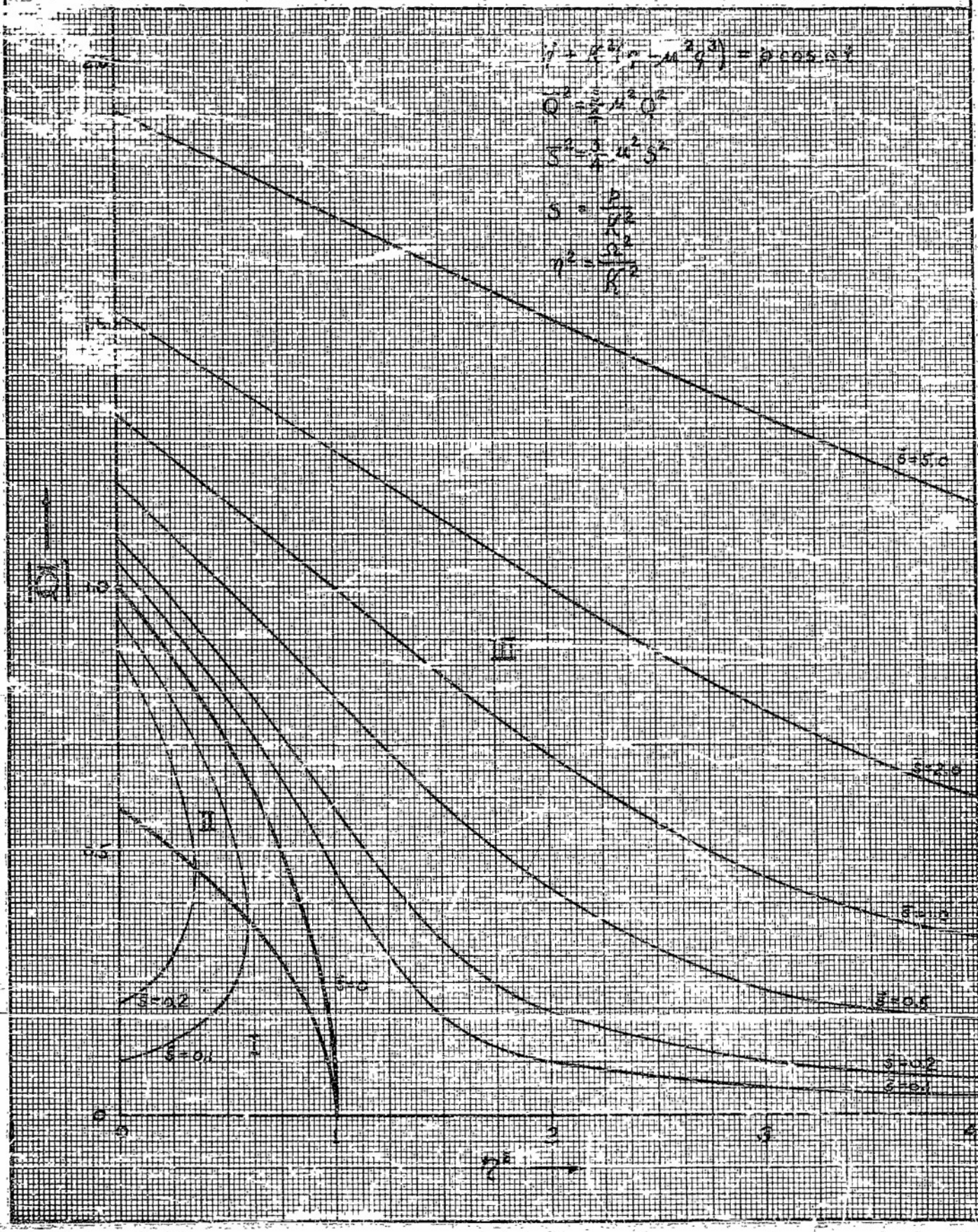


Fig. 14/1: Response Curves

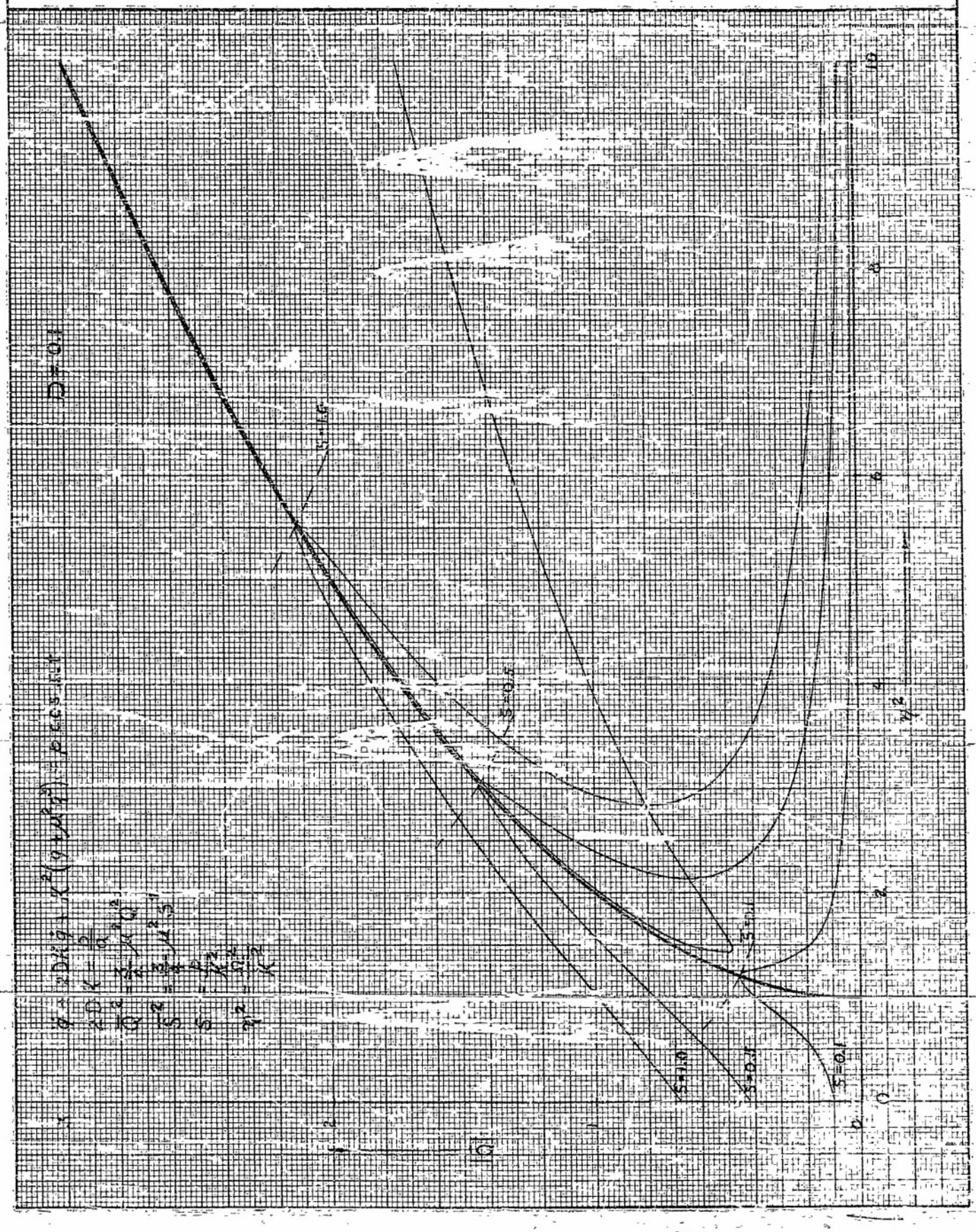


Fig. 14/2: Response Curves

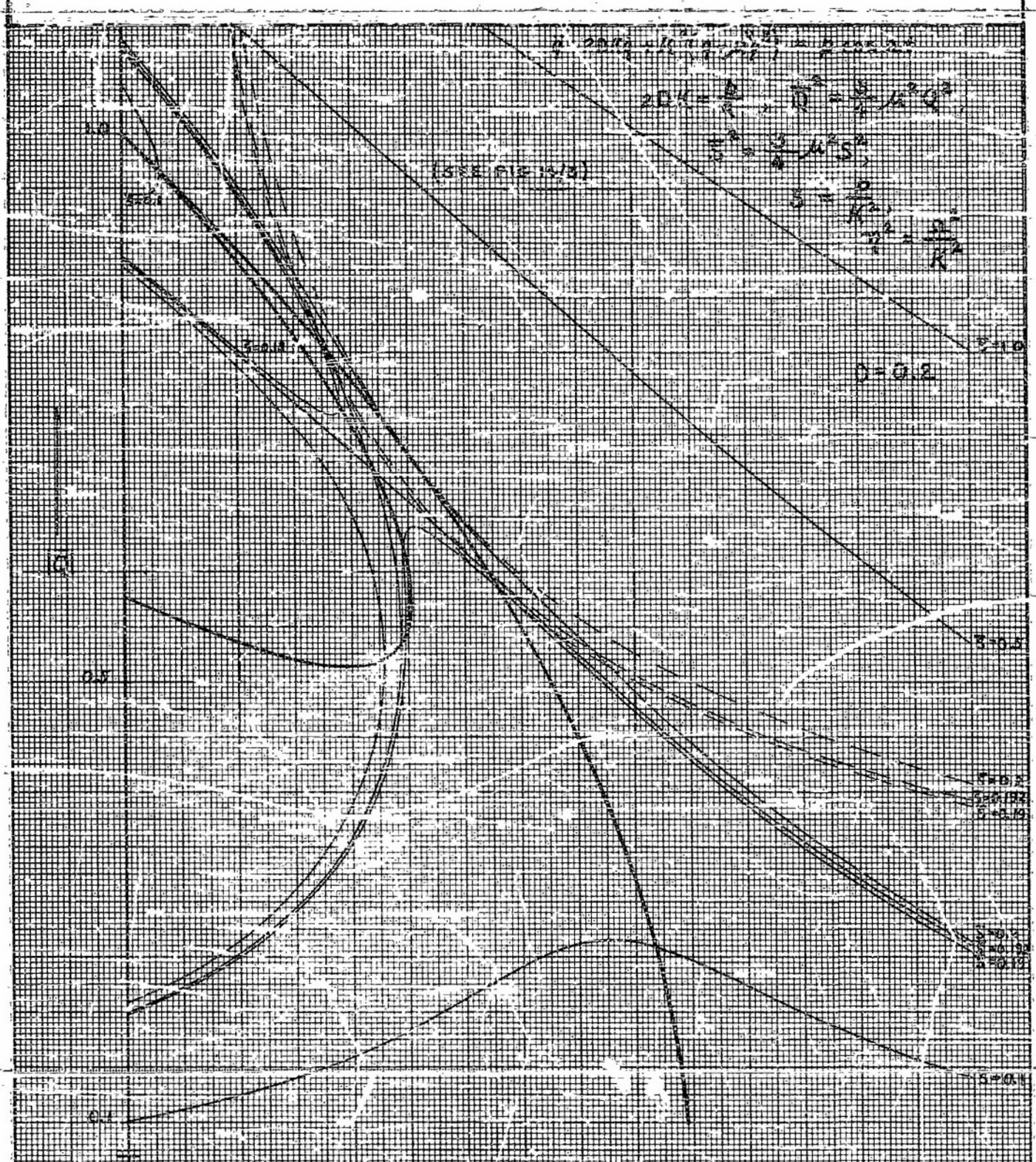


Fig. 14/3: Response Curves
(See Fig. 14/2)

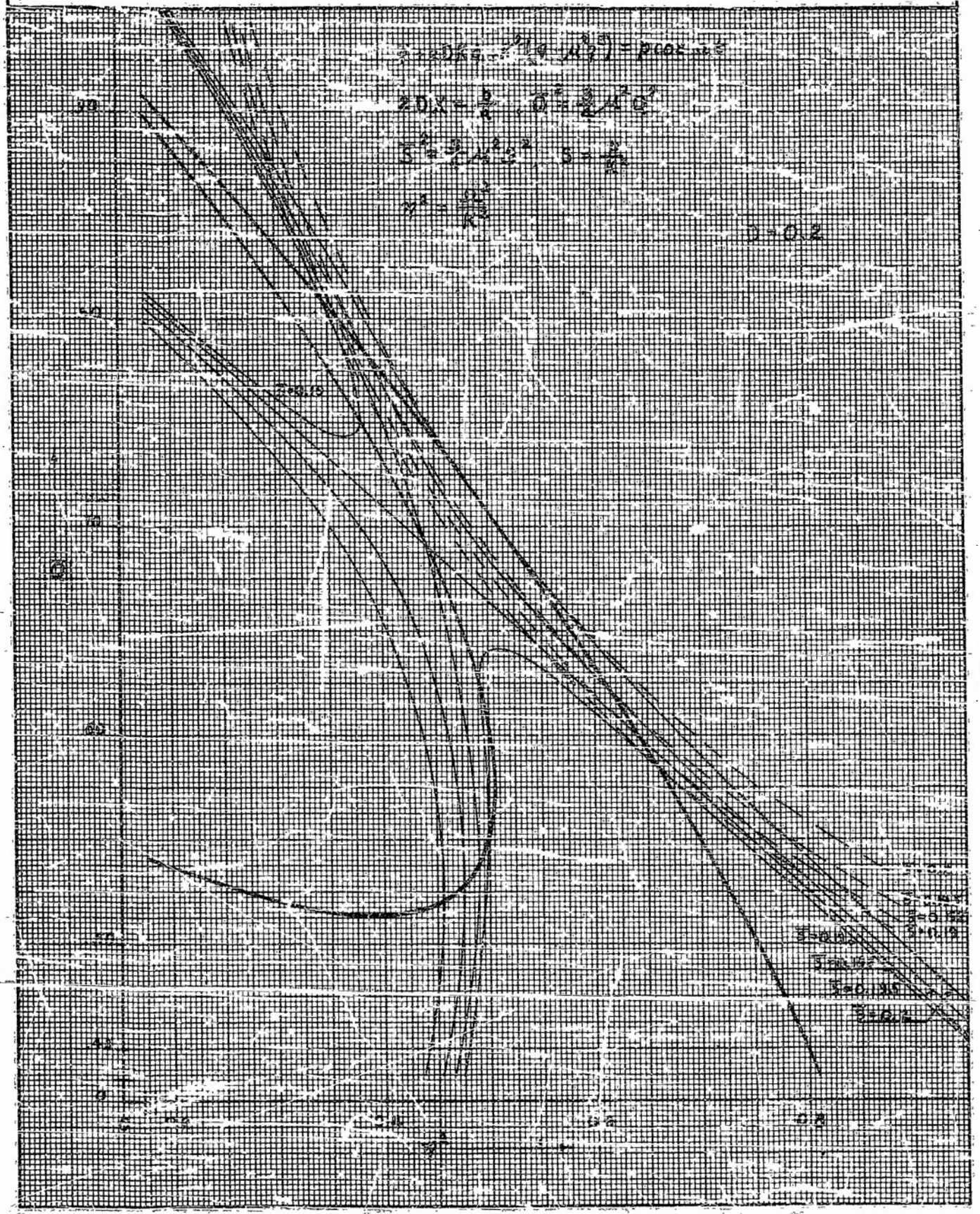


Fig. 1.4: Family of Loci of Vertical Tangents

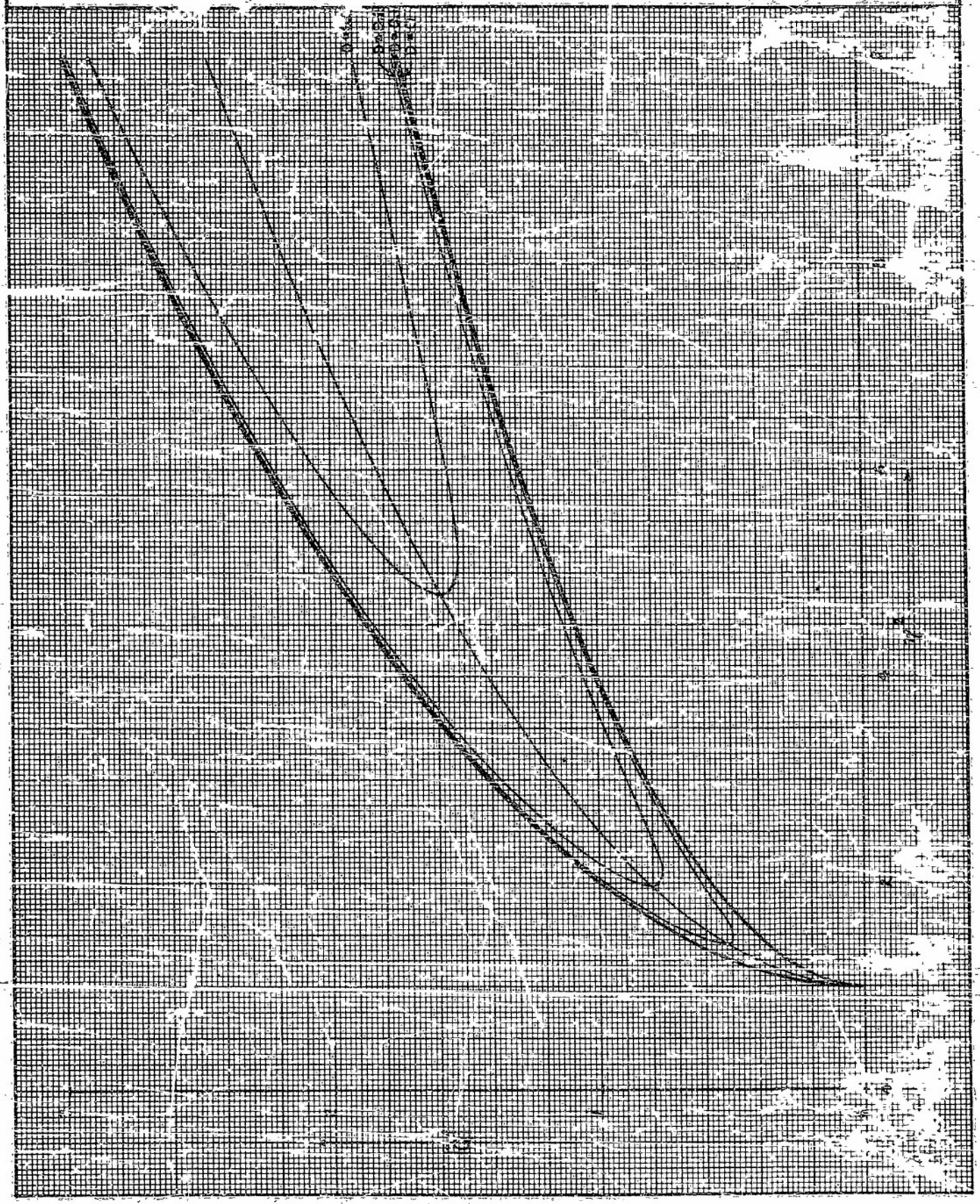
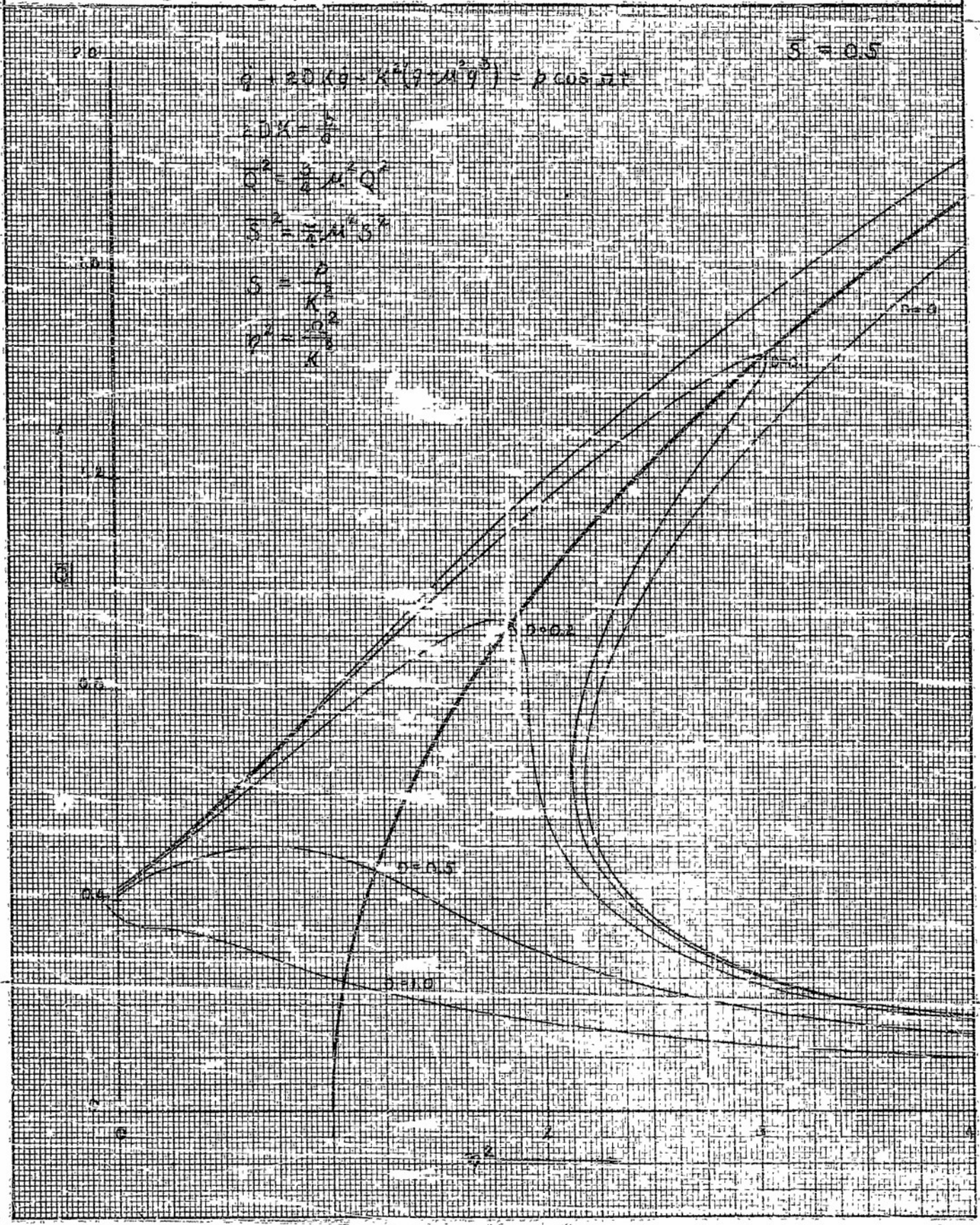


Fig. 14/5: Response Curves



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