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A CLASS OF GAMES WITH UNIQUE SOLUTIONS

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# A CLASS OF GAMES WITH UNIQUE SOLUTIONS

I. Glicksberg and O. Gross

SUMMARY

In a game with payoff $M(x,y) = \phi(xy) + \rho(x) + \tau(y)$ played over the unit square (such that $\rho, \tau$ are continuous and $\phi$ is analytic and with sufficiently many non-vanishing coefficients in its power series expansion about zero) if either player has a non-step function[1] optimal strategy, the opposing player has a unique optimal strategy. Examples are included which illustrate the fact that games with well-behaved payoffs can have unique solutions[2] which are more or less pathological.

§1. For any distribution f (which we may consider as a measure) we may define the spectrum of f, $\sigma(f)$, as the complement of all open sets of f-measure zero. The set $\sigma(f)$ is a closed Borel set, since we may obtain $\sigma(f)$ by deleting the intervals of f-measure zero which have rational end points. If one is given a constant, v, strategies f and g, and functions $\phi$, H, K, such that

(1) $\phi$ is continuous on the unit square

(2) $H(x) \leq v$, and $H(x) = v$ on $\sigma(f)$, H continuous

(3) $K(y) \geq v$, and $K(y) = v$ on $\sigma(g)$, K continuous

then by setting

$$(4) \quad M(x,y) = \phi(x,y) - \int \phi \, df(x) - \int \phi \, dg(y) + \iint \phi \, df(x) \, dg(y) - v + H(x) + K(y),$$

one obtains the payoff M of a game which has value v, and (f,g) as a solution.

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[1] By a step function we mean a distribution based on a finite set of points.

[2] By a solution we mean a pair (f,g) consisting of an optimal strategy f for player I, g for player II.

For

$$\int M df(x) = \int \phi df(x) - \int \phi df(x) - \iint \phi dfdg + \iint \phi dfdg$$

$$- v + v + K(y) = K(y) \geq v ,$$

since $H(x) = v$ on $\sigma(f)$, and similarly

$$\int M dg(y) = H(x) \leq v.$$

The representation (4) of the payoff holds trivially in the case of any payoff M, if we select for (f,g) any solution of the game with payoff M, since we may then set v equal to the value and

$$\phi = M , \quad H(x) = \int M dg(y) , \quad K(y) = \int M df(x) ,$$

and obtain (4) as a trivial identity. (4) has, however, some non-trivial consequences if we replace $\phi(x,y)$ by a function of the product xy.

<u>Theorem 1</u>: <u>Let</u> f <u>and</u> g <u>be non-step functions, and let</u> H , K , <u>and</u> v <u>satisfy</u> (2) <u>and</u> (3) (above). <u>Let</u> $\phi$ <u>be an analytic function such that</u>

$$\phi(t) = \sum_{j=0}^{\infty} a_j t^{n_j} \quad \text{for } |t| \leq r , \ r > 1 ,$$

$$a_j \neq 0 \quad \text{and} \quad \sum_{j=1}^{\infty} \frac{1}{n_j} = \infty .$$

<u>Then the game with payoff</u> M <u>defined by</u>

$$M(x,y) = \phi(xy) - \int \phi(xy) df(x) - \int \phi(xy) dg(y) + \iint \phi(xy) df(x) dg(y)$$

$$- v + H(x) + K(y)$$

<u>has</u> (f,g) <u>as its unique solution</u>.

Proof: Because of the uniform convergence we have assumed for $\phi$ we have

$$\phi(xy) - \int \phi(xy)df(x) - \int \phi(xy)dg(y) + \iint \phi(xy)df(x)dg(y)$$

$$= \sum a_j(x^{n_j} y^{n_j} - f_{n_j} y^{n_j} - x^{n_j} g_{n_j} + f_{n_j} g_{n_j})$$

$$= \sum a_j(x^{n_j} - f_{n_j})(y^{n_j} - g_{n_j})$$

where $f_n$ is the n-th moment of f. Hence we may write

$$M(x,y) = \sum a_j(x^{n_j} - f_{n_j})(y^{n_j} - g_{n_j}) - v + H(x) + K(y) .$$

Of course (f,g) is a solution of the game, and we only have to show uniqueness. Let f' be an optimal strategy for player I. Then

$$\int Mdf'(x) = \sum a_j(f'_{n_j} - f_{n_j})(y^{n_j} - g_{n_j}) - v + \int H(x)df'(x) + K(y)$$

$$= \sum a_j(f'_{n_j} - f_{n_j})(y^{n_j} - g_{n_j}) - v + v + K(y) \geq v .$$

But $K(y) = v$ on $\sigma(g)$ so that

$$\sum a_j(f'_{n_j} - f_{n_j})(y^{n_j} - g_{n_j}) \geq 0 \quad \text{for } y \varepsilon \sigma(g) .$$

Actually we must have equality on $\sigma(g)$, since otherwise there exists a $y_0 \varepsilon \sigma(g)$ such that

$$\sum a_j(f'_{n_j} - f_{n_j})(y_0^{n_j} - g_{n_j}) > 0 ,$$

and hence an interval containing $y_0$ in which this is true;

however since $\sum a_j(f'_{n_j} - f_{n_j})(y^{n_j} - g_{n_j})$ is non-negative on

$\sigma(g)$, from

$$\int \sum a_j(f'_{n_j} - f_{n_j})(y^{n_j} - g_{n_j})dg(y) = \sum a_j(f'_{n_j} - f_{n_j})(g_{n_j} - g_{n_j}) = 0$$

we conclude that this interval is of g-measure zero, hence that $y_0 \notin \sigma(g)$ which is a contradiction. Thus

$$\sum a_j(f'_{n_j} - f_{n_j})(y^{n_j} - g_{n_j}) = 0 \quad \text{on } \sigma(g)$$

and since $\sigma(g)$ is not a finite set of points the analytic function on the left is identically zero, whence

$$f'_{n_j} = f_{n_j} \qquad j = 1, 2, \ldots \qquad .$$

However, this implies f' = f , as is shown in [1] say, since

$\sum \frac{1}{n_j} = \infty$ . A similar argument suffices to show g is unique.

As is evident from the above proof Theorem 1 may be stated in the following one-sided form:

Corollary 1: Let M, $\phi$, H, K, v, satisfy the requirements of Theorem 1. Then if either player has a non-step function optimal strategy, his opponent has a unique optimal strategy.

§2. Theorem 1 may be simplified to

Theorem 2: Let

$$M(x,y) = \phi(xy) + \rho(x) + \tau(y)$$

(where $\rho$ and $\tau$ are continuous on [0,1] and

$$\phi(t) = \sum_{j=0}^{\infty} a_j t^{n_j} \quad \text{for } |t| \le r \text{ , } r > 1$$

$$\text{and} \quad a_j \neq 0 \text{ , } \sum_{j=1}^{\infty} \frac{1}{n_j} = \infty \text{ )}$$ be the payoff of a game in which each player has a non-step function optimal strategy. Then the optimal strategies are unique.

Proof: Let f and g be the non-step function strategies for I and II. Then

$$K(y) = \int M df = \int \phi(xy) df(x) + \int \rho(x) dt(x) + \tau(y) \geq v$$

$$H(x) = \int M dg = \int \phi(xy) dg(y) + \rho(x) + \int \tau(y) dg(y) \leq v$$

$$v = \int H(x) df(x) = \iint \phi(xy) df dg + \int \rho(x) df + \int \sigma(y) dg$$

and H, K and v obviously satisfy (2) and (3). Moreover writing

$$M(x,y) = (M(x,y) - H(x) - K(y) + v) - v + H(x) + K(y)$$

and replacing the terms in the parentheses we obtain

$$\begin{aligned} M(x,y) = \phi(xy) + \rho(x) + \tau(y) &- \int \phi(xy) df(x) - \int \rho(x) df(x) - \tau(y) \\ &- \int \phi(xy) dg(y) - \rho(x) - \int \tau(y) dg(y) \\ &- \iint \phi(xy) df(x) dg(y) - \int \rho(x) df(x) - \int \tau(y) dg(y) \\ &\quad - v + H(x) + K(y) \end{aligned}$$

or

$$\begin{aligned} M(x,y) = \phi(xy) &- \int \phi(xy) df(x) - \int \phi(xy) dg(y) \\ &+ \iint \phi(xy) df(x) dg(y) - v + H(x) + K(y) \end{aligned}$$

so that Theorem 1 immediately applies.

Theorem 2 may also be put in a one-sided form

<u>Corollary</u> 2: Let $M(x,y) = \phi(xy) + \rho(x) + \sigma(y)$ , $\rho, \sigma$ continuous, $\phi(t) = \sum_{j=1}^{\infty} a_j t^{n_j}$ , for $|t| \leq r$, $r > 1$ and $a_j \neq 0$, $\sum_{j=1}^{\infty} \frac{1}{n_j} = \infty$ . If either player in the game with payoff M has a non-step function optimal strategy then his opponent has a unique optimal strategy.

## 3. Examples

The first example is a game with a rational payoff function with a unique solution consisting of distributions with countable spectra. Set

$$\phi(xy) = \frac{2}{2-xy} - \frac{2}{4-xy}$$

and

$$f(x) = g(x) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \; I_{2^{-n}}(x)$$

Then

$$\int \phi(xy)\,df(x) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \left( \frac{2}{2-2^{-n}y} - \frac{2}{4-2^{-n}y} \right)$$

$$= \sum_{n=0}^{\infty} \left( \frac{1}{2^{n+1}-y} - \frac{1}{2^{n+2}-y} \right) = \frac{1}{2-y}$$

and by symmetry

$$\int \phi(xy)\,dg(y) = \frac{1}{2-x} \; .$$

Setting $H(x) \equiv v \equiv K(y)$, and forming the function M given by (4) (omitting constants) we obtain

$$M(x,y) = \frac{2}{2-xy} - \frac{2}{4-xy} - \frac{1}{2-y} - \frac{1}{2-x}$$

as the payoff of a game having $(f,g)$ as a solution (the strategy $f = g$ is not a step function in our terminology!), and since M is of the form required by Theorem 2, the solution is indeed unique.

Our second example is

$$M(x,y) = e^{xy} - \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \, e^{x \cos^2 n} - \sum_{n=0}^{\infty} \frac{1}{n!} \, e^{y \sin^2 n - 1}$$

which is formed from

$$\phi(xy) = e^{xy}$$

$$H(x) \equiv v \equiv K(y)$$

$$f(x) = \frac{1}{e} \sum_{n=0}^{\infty} \frac{1}{n!} \, I_{\sin^2 n}(x)$$

$$g(y) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \, I_{\cos^2 n}(y)$$

(again omitting constants). The strategies f and g have jumps at a dense set of points in [0,1], and are the unique strategies by Theorem 2. We note that the payoff in this example is the sum of two payoffs which have saddle points

$$e^{xy} \quad \text{and} \; -\frac{1}{e} \sum \frac{1}{n!} e^{y} \sin^2 n - \sum \frac{1}{2^{n+1}} e^{x} \cos^2 n$$

Reference

[1]. I.Glicksberg and O. Gross, A Class of Games with Unique Density Function Solutions, RM-501

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