PHOTOGRAPH THIS SHEET

AD A950060

DTIC ACCESSION NUMBER

LEVEL

INVENTORY

Wayne Engineering Research Inst., Detroit MI

"Statistical Manual Methods of Making Experimental Inferences" By C.W. Churchman

DOCUMENT IDENTIFICATION

Date: June 1951

Contract No. W20-018-ORD-13073 Second Revised Edition

DISTRIBUTION STATEMENT A
Approved for public release
Distribution Unlimited

DISTRIBUTION STATEMENT

ACCESSION FOR	
NTIS	GRA&I
DTIC	TAB
UNANNOUNCED	
JUSTIFICATION	
BY	(June 1951)
DISTRIBUTION /	Per ltr. on file
AVAILABILITY CODES	(FL-88/80-2250)
DIST	AVAIL AND/OR SPECIAL
A	

Released

DTIC
ELECTE
S NOV 3 1980 D
D

DATE ACCESSIONED

DISTRIBUTION STAMP UNANNOUNCED

80 10 27 029

DATE RECEIVED IN DTIC

PHOTOGRAPH THIS SHEET AND RETURN TO DTIC-DDA-2

DTIC FORM OCT 79 70A

DOCUMENT PROCESSING SHEET

AD A950060

PAGES _____

ARE

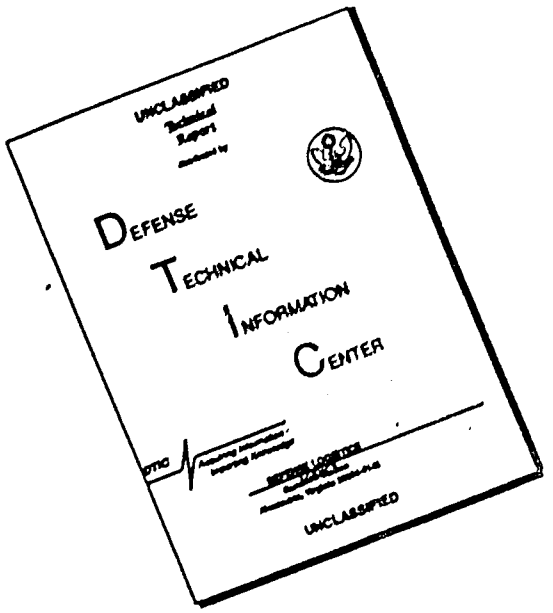
MISSING

IN

ORIGINAL

DOCUMENT

DISCLAIMER NOTICE



THIS DOCUMENT IS BEST QUALITY AVAILABLE. THE COPY FURNISHED TO DTIC CONTAINED A SIGNIFICANT NUMBER OF PAGES WHICH DO NOT REPRODUCE LEGIBLY.

STATISTICAL MANUAL
METHODS OF MAKING EXPERIMENTAL INFERENCES

by

C. W. CHURCHMAN

SECOND REVISED EDITION

PITMAN-DUNN LABORATORY
FRANKFORD ARSENAL
PHILADELPHIA, PA.

JUNE 1951

This manual was prepared by Dr. C. W. Churchman under Ordnance Contract W20-018-ORD-13073 with the Wayne Engineering Research Institute. Wayne University, Detroit, Michigan.

PITMAN-DUNN LABORATORY
FRANKFORD ARSENAL
JUNE 1951

CONTENTS

	Page
Preface	i, ii
Introduction	1
Glossary	33
PART I: Some Standard Procedures	
Problem 1. Test of a mean, variance known	37
Problem 2. Test of a mean, variance unknown	41
Problem 3. Test equality of two means, variance known	45
Problem 4. Test of equality of two means, variance unknown	48
Problem 5. Test of equality of means of paired differences	53
Problem 6. Test of equality of a set of means, variance known	55
Problem 7. Test of equality of a set of means, variance unknown	58
Problem 8. Test of a single variance	61
Problem 9. Test of equality of two variances	64
Problem 10. Test of equality of a set of variances	67
Problem 11. Test of equality of two probabilities (double dichotomies)	70
Problem 12. Factorially designed experiments	73
Problem 13. Test of equality of a set of means and variances	84
Problem 14. Test of correlation	88
Problem 15. Tests of normality	94
Problem 16. Test for erratic observations	99
Problem 17. Test for independence of observations	102
Problem 18. Test for proportion of a universe in the "tail"	105
PART II: Estimation	
Introduction	108
Common point estimates	109
Confidence intervals	111
Estimate of slope and intercept, both variables subject to error	114
Estimate of mean and variance for sensitivity data	115

	Page
PART III: Sequential Analysis	
Introduction	120
Test of a probability	124
Test of equality of two probabilities	127
Test of a mean, variance known	131
Test of a variance	136
PART IV: Simplified Procedures	
Introduction	138
Statistical Sign Test	139
Ranking test for differences	143
Test of runs of some property of the observations	147
Test of differences among a set of universes	150
TABLES and FIGURES	152

Page

..... 120
..... 124
..... 127
..... 131
..... 136

..... 138
..... 139
..... 143
servations . . 147
rses 150
..... 152

PREFACE TO SECOND REVISION

This manual grew from the need for a simple and direct treatment of statistical procedures commonly used to draw inferences from experimental data. Many of the procedures are to be found only in inaccessible texts and articles, so that the experimenter must undertake lengthy research to find the method he needs. And even when the methods are gathered in one place, there is a tendency to make technical references to the mathematical basis of the procedure, with a consequent confusion on the part of the experimenter who is not well equipped in statistical theory.

This manual attempts to emphasize clarity in the method of applying statistical tests, together with the logic of their application. The manual minimizes to the greatest possible extent the mathematical and technical foundations of the methods; it presents only those terms that are presumably essential in understanding the logic of statistical methods and the steps in their application. The reader interested in the mathematical foundations may wish to consult the books and articles cited in the References.

To carry out the purpose of the manual, it appeared that the problem-method of presentation would be most effective. The Introduction presents some of the necessary statistical terminology and theory, together with an outline of what is known at present about the logic of statistical procedures. The body of the manual presents each procedure in "cook-book" style, first in general outline, then in illustration. The illustrations are drawn in the main from the engineering sciences, though it is recognized that many of the methods originated in other fields.

This work was originally conceived and written by M. R. Stevens, S. S. Ensor, and C. W. Churchman of the Mathematics Section of the Frankford Arsenal Ordnance Laboratory, and published in 1944. A revision of the

text was prepared in 1946 by R. M. Sigmond. The present edition was prepared by C. W. Churchman, and combines the older edition with the material presented in Statistical Memorandum No. 5, "Risks Associated with Experimental Inferences from Data." Also, some additional tests that have recently appeared in statistical literature have been included. Also, some of the test procedures of the original editions have been changed in the light of more recent research. Finally, some of the tests proposed in the original editions have been omitted because it was found that they are seldom used.

igmond. The pre-
hurchman, and com-
ial presented in
Associated with
Also, some ad-
red in statistical
ome of the test
e been changed in
inally, some of the
have been omitted
dom used.

INTRODUCTION

1. GENERAL REMARKS In all experimental work in which we put a question to nature, the data have certain common properties, and the techniques of analyzing the results for the purpose of drawing a conclusion have certain similarities. The similarity of these techniques of analysis is the basis of extensive research that has been made in the last half-century in mathematical statistics. This research into the logic of drawing conclusions from observations is actually an extension of classical research in "inductive" methods. The first clear formulation of the inductive method is to be found in Aristotle's Posterior Analytics; here the "method of analyzing data" consisted simply in apprehending intuitively some common and essential property in a set of observations. With the birth of modern science in the Renaissance, there gradually developed the need for more precise methods of drawing inferences. Very significant steps were taken in this direction in the nineteenth century. First, there developed a general theory of probability which enabled the experimenter to take account of the disparity of observations. The method in general consisted in asserting that a set of observations were in agreement with theory if their mean did not deviate from the predicted quantity by more than the limits prescribed by the "probable error." This method actually constituted a test of an hypothesis, and modifications of the procedure are still used in the simpler forms of modern testing techniques. Second, certain logicians, especially J. S. Mill, turned their attention to formalizing more exactly the inductive method of checking hypotheses about causal laws. Mill's "Cannons of Induction" were nonstatistical, in the sense that they did not take into account errors in observations, but to Mill goes the credit for having suggested in definite form various types of "experimental designs." That is, Mill proposed in effect that the experimenter organize his data into certain sub-sets, so as to draw the most fruitful inferences. Such a suggestion marks an important turn in the logic of experimentation, for it

implies that the experimenter must "design" his experiment beforehand if he is to obtain the best possible results.

The turn of the century is characterized by renewed interest in the application of probability to experimental results; this interest occurred in the main in the biological sciences. Here for the first time the difference between experimentation with small samples, and experimentation with large samples, was studied thoroughly. It soon became apparent that the so-called "classical" methods commonly employed in the physical sciences were adequate only for large samples (over 30 for some procedures, over 100 or 1000 for others). With "Student" and Karl Pearson there began a long series of papers on the theoretically "best" methods of analyzing sets of data. Subsequently, primarily under the influence of R. A. Fisher, the technique of designing experiments was combined with modern statistical theory. Specifically, the classical design in which only one variable was altered, and everything else was held as constant as possible, was modified to include designs in which various factors are altered accoring to pre-assigned schemes. Finally, in the 1930's, J. Neyman and Egon Pearson initiated their important foundation-studies in the theory of testing hypotheses which put modern test techniques on a firmer logical basis.

As is common with new procedures, there has been a general confusion about the significance and importance of modern statistical methods: some claim that these methods are always preferable to any others, some claim that they can only be used when the data are very precise. The truth of the matter is that the new techniques are like any method in the process of problem-solving: they are to be used whenever they are most effective and can be employed without excessive time and cost. Most specific remarks about the application of the methods will be made after considering the general meaning of the theory that underlies their application.

2. SOME IMPORTANT CONCEPTS OF STATISTICAL THEORY

The basic notion of statistical theory is the process of drawing at random a "sample" from a "universe" (or, as it is sometimes called, a "population"). Each of these terms has mathematical meaning, but from the point of

view of application it is important to understand the terms on a common sense basis. For this purpose, think of a large bowl containing chips with numbers or other symbols written on them. This represents the notion of a "universe" of entities. A random sample from this bowl would be a sample that in the long run tends to represent adequately the contents of the bowl. More specifically, the random sample tends in the long run to include chips with a certain symbol on them about as frequently as they occur in the bowl itself. As an example of a nonrandom sample, suppose we were to sample from a mixture of several granulated compounds by taking material only from the top of the barrel. Such a procedure would not tend to select small grains in the proportion in which they occur in the barrel as a whole, and hence would be nonrepresentative, i.e., nonrandom. One is naturally anxious to know whether a given sample is random, and here there is a fundamental point of importance in the application of statistical methods. The description of random samples is given in terms of the "long-run" characteristics of the sample. But normally experimentation is short-run. There are, as we shall see, several common tests for randomness, even in the case of a small number of items; all such tests are negative, in the sense that they may establish (within limits of error) the existence of nonrandomness, but they do not in any case prove randomness. Actually, a random sample is an ideal term used to designate an unattainable (though approachable) objective of scientific inquiry. Like all such ideal terms of science, random sampling directs the kind of operations we perform; the tests for randomness enable us better to control our data.

There is one aspect of random sampling from a population that is often overlooked: that randomness depends on the purpose of the experiment. Actually, any set of data can be shown to satisfy the common tests of randomness relative to some universe. Thus, in the example cited above, the selection of granulated mixture may be a random selection from the top of the barrel if the purpose is only to use this portion of the barrel. But if the purpose of the experiment is to obtain information about how the entire barrel should be used, then the sample is not random. A random sample, then, consists of a set of

items that provide maximum information of a specific sort relative to a set of aims or purposes in which the universe of items is involved.

Typical examples of problems of inference that arise from the basic process of statistics are the following:

a) A series of samples have been drawn daily from a certain production machine over a period of time, say, one month. Each sample consists of five items, and measurements are made of the diameter (at a specified point) and length of each item. Here the universe is the production of the machine for a specific period of time (in this case, one day). The point of the experiment is to make inferences about the many daily universes based on the (supposedly random) samples of five items. That is, each sample of five is supposed to represent the day's production from which it was drawn. Each of these representative (random) samples is then used to make an inference about the universes.

b) (a special case of problem a.) Two sets of observations have been made on two samples from universes A and B. The sample results are

Sample drawn from A	Sample drawn from B
5.24	4.98
5.36	5.12
5.21	5.06
5.40	5.20

The question is: is there sufficient evidence to infer a real difference between universes A and B?

c) A certain improvement has been made in a product, with the purpose of reducing defectives. A sample of 15 is taken from the universe representing the old product, and a sample of 15 from the universe representing the new. In an increased severity test, eight failures are observed with the old, and four with the new. Is there sufficient evidence to infer that the new product is really better?

INTRODUCTION

In order that an understanding of the methods of statistical inference may be gained, it is essential to understand the following concepts.

DISTRIBUTION FUNCTION. The distribution function is a mathematical function which describes how the universe of items is distributed with respect to some scale or set of qualities. Thus, the universe may be a set of samples of the same metal, the scale may be tensile strength, and the distribution function of the universe of items would then describe how the items are grouped with respect to tensile strength. Specifically, the distribution function can be used to state the percentage of the universe that has tensile strengths lying between specified limits. Again, it may be advisable to describe the items of a universe in terms of a set of qualities; for example, the items of a production unit may be described as either defective or nondefective. The distribution function of such a universe would give the percentage of the universe that falls in each of the two parts. The number of qualities may be greater than two; for example, one might describe the manner in which the items of a universe are distributed with respect to several different colors.

MEASURES OF CENTRAL TENDENCY. When a universe of items is described by a distribution function, it is often convenient to concentrate attention on certain aspects of the function. Thus, we may be interested in the point along some scale where there is a concentration of the items of the universe. Such a point is called a central tendency measure of the universe. The measure of central tendency depends on what one means by "concentration." If concentration is defined in terms of the number of items that occur in a specified range, then the measure is called a "mode" of the distribution. In many, but by no means all, practical cases there is only one mode. Again, "concentration" may represent the "middle item" of the distribution, called the median. More commonly, concentration is defined in terms of the "center of gravity" of a distribution; the entire universe of items is thought of as so much "mass" distributed along some scale, and the center of gravity of this mass is called the "mean" of the universe. This is a very important

measure which we will use repeatedly in the sequel. It is symbolized by the Greek letter "mu," written μ.

MEASURES OF DISPERSION. The measures of dispersion characterize the amount of scatter in a universe with respect either to a scale or a set of qualities. In some cases, the items of a universe may be very much alike; for example, the samples of a chemical may be so carefully chosen that they nearly all have weights lying within a tenth of a milligram. On the other hand, the chamber pressures of a lot of ammunition may scatter from 40,000 p.s.i. to 60,000 p.s.i. Similarly, if the universe is described in terms of qualities, it may be found that most of the items are, for example, nondefective, so that the scatter is small; or, it may be found that the items are more or less evenly divided among the various qualities, in which case there would be a large dispersion. There are several methods of measuring dispersion, but the one now most commonly used is the "second moment" of the distribution of the items of the universe. Again, the distribution of items is conceived to be a distribution of "mass" along some scale. The second moment of such a distribution is called the "variance," and is symbolized by σ^2 the square of the Greek letter "sigma." The square root of the variance, σ, is called the "standard deviation" of the universe.

SKEWNESS AND KURTOSIS. In some cases, the scatter of items of a universe is such that more occur on one side of mean than occur on the other. For example, annual incomes are apt to follow such a pattern: the "center of gravity" of a social group may be as low as $3000, but many of the items may be scattered as high as $100,000, or more. In probability terms, we say that the probability of a certain amount of deviation on one side of the mean is greater than the probability of the same amount of deviation on the other side. One measure of this sort of scatter is "skewness," which is the third moment of the distribution divided by the cube of the standard deviation. If the items of a universe are scattered symmetrically about the mean, the skewness is zero. If the distribution curve does not descend as rapidly on the right side as on the left, the skewness is positive, and if the

distribution curve descends more rapidly on the right than on the left, the skewness is negative.

Another characteristic of distributions is measured in terms of the "flatness" of the distribution curve, by means of the fourth moment. If the curve is "flat" around the mean, the fourth moment tends to be large, while if it is "peaked" around the mean, the fourth moment is small. For "normal" distributions (see below), the fourth moment, divided by the fourth power of the standard deviation, should be three. This value is called the "kurtosis," and is used to test for normality.

Other moments of the distribution can also be calculated, but their use in practice is very restricted, principally because their estimates are subject to comparatively large error.

NORMAL DISTRIBUTIONS. A distribution that has importance in practical applications of statistical inference is the "normal." This distribution was discovered early in the nineteenth century, and even today it is common to assume that a universe is distributed normally unless the evidence is to the contrary. The classical conditions for a normal universe are as follows (the conditions are necessary, in the sense that all normal universes satisfy them, but are not sufficient, in the sense that other distributions also may have these properties):

> The scale-values of the items of the universe are symmetrically distributed about the mean.

This implies that the skewness of a normal universe is zero. It also implies that the mode, the median, and the mean are the same.

> The more an item deviates from the mean, the less frequent its occurrence.

That is, the larger the deviation from the mean, the less probable is the occurrence of an item.

> Very few "large" deviations from the mean occur. That is, as the deviation becomes larger

and larger, the probability of a corresponding item approaches zero.*

The following examples may help in gaining an intuitive idea of normal and non-normal distributions of items of a universe:

EXAMPLES OF NORMAL AND NON-NORMAL DATA

Normal Data

Weights of a Group of Men

Weights in Pounds	Percentage Frequency
100-109.9	0.2
110-119.9	0.9
120-129.9	2.8
130-139.9	6.6
140-149.9	12.1
150-159.9	17.5
160-169.9	19.3
170-179.9	17.5
180-189.9	12.1
190-199.9	6.6
200-209.9	2.8
210-219.9	0.9
220-229.9	0.2

(remaining percentage below 100 or above 230)

*More precise definitions of normality can be given mathematically; a set of items is said to be normally distributed if the probability of obtaining an observation in excess of a certain number ξ is given by

$$P(x > \xi) = \int_{\xi}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2} dx$$

where μ and σ^2 are the mean and variance respectively.

esponding

gaining an intui-
ributions of items

NORMAL DATA

n

tage Frequency

0.2
0.9
2.8
6.6
12.1
17.5
19.3
17.5
12.1
6.6
2.8
0.9
0.2

ining percentage
low 100 or
above 230)

n be given mathe-
normally distri-
observation in ex-
by

$\mu)^2/2\sigma^2 dx$

riance respectively.

Non-normal Data

Distribution of Part-time Employees in a Certain Industry According to Annual Incomes

Annual Earnings, dollars	Percentage of Employees
0-199	5.5
200-399	8.1
400-599	10.2
600-799	11.3
800-999	12.0
1000-1199	12.3
1200-1399	11.9
1400-1599	7.2
1600-1799	3.0
1800-1999	2.1
2000-2199	1.7
2200-2399	1.4
2400-2599	1.1
2600-2799	0.8
2800 and over	11.4

ESTIMATES OF THE MEASURES OF A UNIVERSE IN TERMS OF THE SAMPLE. Science proceeds by means of hypothetical constructs; even the observations that one makes on items of a universe are hypothetical in the sense that one accepts or "takes" certain readings as valid on the basis of assumptions about the reliability of instruments. It is important to see how the structure of the universe is related to the accepted data about the universe. If the sample of items (i.e., the set of observations) does not exhaust the universe, then in general the sample does not supply complete information about the measures of the universe, but presumably some inference can be made to connect the sample results with these measures. Such inferred connections are called "estimates." The most commonly used estimate of the mean of the universe is the "average" of the sample-item readings; that is, if the universe is described along some scale, one simply adds together all the measures of each item of the sample, and divides by the total number of items in the sample. Symbolically, we let $\sum_{1}^{n}$ stand for

the addition of n items, x_i stand for the ith item of a sample, and $\bar{x}$ stand for the average. The formula for $\bar{x}$ is

$$\bar{x} = \frac{\sum_{1}^{n} x_i}{n}$$

Note that $\bar{x}$ is only one possible estimate of the mean; for example, one might estimate the mean by taking the midpoint between the largest and the smallest observations of the sample.

One common method of estimating the variance is the following: the average, $\bar{x}$, is subtracted from each item of the sample, and the resulting deviation, $(x_i - \bar{x})$, is squared; all these squares are added together, and the result is divided by n-1, where n is the number of items in the sample; the estimated variance is symbolized as s^2, and its formula is

$$s^2 = \frac{\sum_{1}^{n} (x_i - \bar{x})^2}{n - 1}$$

Note that σ^2 is the symbol for the true variance, while s^2 is the symbol for one of the estimates of σ^2, an estimate that happens to have many fruitful uses. The calculation of s^2 may be simplified by using the equivalent formula

$$s^2 = \frac{n\sum_{1}^{n} x_i^2 - \left(\sum_{1}^{n} x_i\right)^2}{n(n-1)}$$

especially if an adding machine is available (the standard types will add sums and sums of squares simultaneously).

One method* of estimating the standard deviation, σ consists in taking the square root of s^2; that is,

*A better estimate of σ differs slightly from this estimate for small sample sizes.

$$s = \sqrt{\frac{\sum_{1}^{n} (x_i - x)^2}{n - 1}}$$

is an estimate of the standard deviation. Another estimate of σ is obtained by multiplying the "range" of the sample by a suitable constant. The range is simply the largest minus the smallest item of the sample, and is symbolized by R. Its formula is

$$R = x_{max} - x_{min}.$$

The constant by which R must be multiplied to obtain an estimate of σ depends on the sample size.

DEGREES OF FREEDOM. One important aspect of an estimate from a sample is the concept of the associated "degrees of freedom." This term indicates in general the number of factors which can be changed independently without causing a change in the estimate. Algebraically this property can be stated as follows; suppose there are two equations in 4 unknowns,

$$ax + by + cw + dz = k$$

$$ex + fy + gw + hz = l$$

In this case, once two of the unknowns are assigned values, the remainder of the unknowns are fixed; we say that there are two degrees of freedom. If another equation were added, there would only be one degree of freedom, since as soon as one of the unknowns was assigned a value, the remainder would be determined.

Suppose now that n independent observations are taken in an experiment, and s^2 is computed. The n observations are restricted by one equation, since their sum must add up to n times $\bar{x}$; that is

$$x_1 + x_2 + x_3 + \cdots + x_n = n\bar{x}.$$

Hence, in the computation of s^2, there are n-1 degrees of freedom. If the observations are restricted by other "linear" conditions (e.g., in considering the angles of a triangle, the restriction is usually imposed that the sum of the angles add up to 180°), then the degrees of freedom are lessened accordingly.

BINOMIAL DISTRIBUTIONS. It has already been noted that in many cases items of a universe are not described quantitatively. Suppose the items are of two kinds ("defective" or "nondefective," "failures" or "successes," etc.). Suppose too that several samples, each of size n, are drawn from such a universe. Each such sample may itself be considered as an item of a new universe; these new items will differ among each other with respect to the number of "failures" they contain. For example, one might toss a coin ten times; the quality of each item would be either a "head" or a "tail." Groups of ten tosses would differ with respect to the number of heads that occurred. The distribution of these groups is called the binomial distribution*.

The best method of estimating p, the proportion of items in the original population that have one of the two characteristics, is to take

$$\bar{p} = \frac{\text{number of successes in a sample}}{\text{number of items in sample}}$$

The variance of this estimate can be shown to be

$$\sigma^2_{\bar{p}} = \frac{p(1-p)}{n}$$

and its estimate is usually taken as

$$s^2_{\bar{p}} = \frac{\bar{p}(1-\bar{p})}{n}$$

JOINT DISTRIBUTIONS AND CORRELATIONS. It is common procedure in experimental work to sample simulataneously from more than one universe. Thus, one may examine more than one property of an individual; each property examined

*If p is the fraction of the original universe that has one of the two characteristics, then

$$\frac{n!}{r!(n-r)!} p^r (1-p)^{n-r}$$

gives the fraction of groups of n items that will contain the characteristic r times. E. g., if a coin is tossed over and over in groups of five tosses, then the proportion of groups that will contain exactly 2 heads is 5/16.

may be conceived as constituting an item from a universe, and the set of properties, therefore, constitute a set of universes. This set of universes may now be considered "jointly," in the sense that one may consider the proportion of items that have property A lying in some specified range and also have property B lying in another specified range; for example, one might be interested in the proportion of samples of a metal that have a certain tensile strength and also have a certain Rockwell hardness.

The joint distribution of items not only has the usual measures of dispersion, but also has measures of correlation. In some cases, it may be found that when an item falls above the mean with respect to property A it tends to fall above the mean with respect to property B as well; in such a case we say that there is a positive correlation between the properties. Or, the behavior of items with respect to one property may not have anything whatsoever to do with their behavior with respect to the other; in this case the correlation is zero, and the properties are independent. Finally, when an item is on one side of its mean with respect to property A it may tend to be the other side of the mean with respect to property B, and the correlation is negative.

Joint distributions have proved very useful in the scientific task of relating properties of objects. Often, one wants to say more than the mere statement that properties are correlated; the scientist may actually look for a functional relationship that tells, within the limits of error, just what sort of value one may expect for an item with respect to property B once one has determined the value of the item with respect to property A. The estimate of such relationships falls under the head of curve-fitting, and will be discussed in Parts I and II.

3. STATISTICAL THEORY OF TESTING HYPOTHESES. Because of the mechanical ease and apparent precision of the application of statistical procedures to data, there is apt to be a misconception about the logic which underlies the methods. The experimenter is prone to think that once he succeeds in getting his data into a suitable form for statistical analysis, he must accept the results of the analysis as scientifically valid. Actually, statistical methods share the common property of all instruments:

they are good only in so far as they serve the purposes of the investigation. For this reason, it is essential that the experimenter have some notion of the logical foundations of statistical theory, if he is to learn how to apply statistical methods in a fruitful manner. The description of statistical analyses in the sequel will be given in "cook-book" style; but, as in good cooking, so in good statistics, one must know how the process works, and what function it serves.

In all experimental work where the objective is to come to some decision, the scientist is faced with the problem of selecting one hypothesis from among two or more alternatives. Each hypothesis represents a possible answer to the question which the experimenter raises, and his task consists of an attempt to marshall his evidence so that he chooses the correct hypothesis with a minimum risk of error. Modern science has come to realize that no matter how carefully an experiment is conducted, the experimenter inevitably runs certain risks. But though these risks always appear in every aspect of experimental work, it is possible to choose procedures for drawing conclusions which run less risk than any other possible procedure. In certain cases, statistical methods have been provided which, under explicit conditions, guarantee a minimization of risk.

In this section, the consideration of risks associated with tests of hypotheses is restricted to statistical terminology; the more general problem of over-all risks in experimentation will be discussed in the next section. Attention here, therefore, is concentrated on that aspect of experimentation where the hypotheses are concerned with unknown properties of one or more universes; the experimenter wants to use a sample to determine whether a universe has certain characteristics. For example, he may want to test the hypothesis that the mean of a universe has some value, or lies in some range of values; or, he may want to determine whether the items of one universe are more widely scattered about their mean than are the items of some other universe.

The following principles are pertinent to all the tests which will be discussed in the body of this memorandum:

they serve the purposes
reason, it is essential that
tion of the logical founda-
is to learn how to apply
manner. The description
quel will be given in
ood cooking, so in good
he process works, and

here the objective is to
ntist is faced with the
esis from among two or
esis represents a possible
e experimenter raises, and
to marshall his evidence
ypothesis with a minimum
has come to realize that
riment is conducted, the
tain risks. But though
ery aspect of experimental
rocedures for drawing con-
n any other possible pro-
tical methods have been
nditions, guarantee a

ation of risks associated
icted to statistical termi-
n of over-all risks in ex-
n the next section. Atten-
ted on that aspect of
eses are concerned with
re universes; the experi-
determine whether a
ics. For example, he may
he mean of a universe has
e of values; or, he may
ms of one universe are
mean than are the items

pertinent to all the tests
y of this memorandum:

SOME ASSUMPTIONS MUST BE MADE ABOUT THE UNIVERSES AND THE METHOD OF DRAWING THE SAMPLE.

Modern statistical techniques require some prior information about the origin of the sample for their proper application. In some cases, the assumption will be made that the universe is normal, in the sense defined above. This stipulation sometimes causes confusion on the part of the nonstatistician in the application of statistical techniques, since he may not be sure that he has made the right assumption concerning normality. For this reason, there has been some extensive research on the part of statisticians to determine how serious is an incorrect assumption of normality. For most cases, they have found that it is not a very serious error to assume non-normal data to be normally distributed, provided a very large skewness does not occur; a large skewness is often discernable by an examination of the data to determine whether there is a more pronounced "tail" on one side of the distribution function than on the other.

It is also commonly assumed that the data are random draws from the same universe. In particular, it is usually assumed (and will be assumed in most tests in the sequel), that there are no "erratic" members of the sample. This is an assumption concerning which experimenters are apt to be too cautious, rather than too negligent. It is common practice to "throw out" an observation that seems to be peculiar; such a practice, if employed blindly, leads to biased selection of hypotheses. In particular, the practice tends to make the estimated variance much smaller than the true variance, with the consequence that the experimenter discerns differences which do not really exist. There is no fool-proof statistical method for excluding observations, and in the case of small samples, there is no universally good method for exclusion. It is best always to have some sound scientific explanation for the exclusion of data, and not to rely on statistical methods alone.

Finally, it is usual to assume that the items of a sample are statistically independent, in the sense that the result in the case of one item does not influence what happens on another item. For example, it might happen that when a machine is used to make a measurement, the

operation affects the machine so that on the next try it will tend to record a slightly lower result; or, an observer may be influenced by the preceeding reading. In such cases the items of the sample would be correlated. The assumption of independence of sample items is usually a valid one in the physical sciences, though in every case some care should be taken to check its validity as far as possible.

Statistical methods are not all predicated on the assumption of normality, or randomness. In the body of this memorandum (Part IV) there are procedures outlined that are to be used whenever the distribution of the universe deviates very radically from the normal distribution. But the assumption of randomness will be made throughout this book for specific samples.

EACH ALTERNATIVE HYPOTHESIS MAKES AN ASSERTION ABOUT ONE OR MORE PROPERTIES OF THE UNIVERSES.

This principle states the obvious fact that to apply statistical methods, the problem must be translated into statistical terms.

ANY TWO OF THE ALTERNATIVE HYPOTHESES CANNOT SIMULTANEOUSLY BE TRUE, AND AT LEAST ONE MUST BE TRUE, RELATIVE TO THE ASSUMPTIONS.

This principle guarantees first that there is a problem (if both the hypotheses could be true simultaneously, there would be no real issue involved), and second that all possible solutions to the problem are being considered.

THE ACCEPTANCE OF ANY HYPOTHESIS ON THE BASIS OF SAMPLES FROM A UNIVERSE IS SUBJECT TO A PROBABILITY OF ERROR.

This principle represents the core of modern statistical theory. It is therefore essential that its implications be fully understood if statistical methods are to be properly applied. To simplify the theory, consider first the case where there are only two hypotheses, which can be represented as H_0 and H_1. In statistical theory one commonly recognizes two possible errors associated with a test of these hypotheses: TYPE I ERROR - the error of rejecting H_0 (i.e., accepting H_1) when H_0 is actually true.

TYPE II ERROR - the error of accepting H_0 when it is false (i.e., when H_1 is true).

In practically all the well known statistical tests, the probability of a type I error can be set by the experimenter. That is, the experimenter can decide just what chance he is willing to take of committing a type I error. It is to be noted in passing that for any method of treating data, statistical or otherwise, if the probability of a type I error is set at zero, then the probability of a type II error will be at a maximum. It is common practice to set the probability of a type I error at 0.05 (1 chance in twenty) or 0.01 (one chance in one hundred), though, as we shall see, this practice should not be arbitrary, but should rather be based on the seriousness of the error.

The main reason why the probability of a type I error can be set explicitly is that H_0 normally is itself very explicit; it usually asserts that the mean of a universe has some specific value, or that there is no difference between two means, or that the variance of one universe equals the variance of another. The type II error, however, is another matter, because its meaning depends on the fact that the hypothesis H_0 is false. Now a very specific hypothesis can be false in many ways. If a man tells you that he owes you $3.00, when he actually owes you $3.05, he is telling a falsehood, but most of us would admit that his falsehood was not a very serious one. But if he should say he owed you three dollars, when he actually owed twenty, he is telling a "bad" lie. And so it is in scientific matters. If one predicts that a ball will take exactly 2 seconds to travel a distance down an inclined plane, and it actually takes 2.0001 seconds, then the original hypothesis is wrong, but not "very" wrong. The actual difference will probably not worry anyone, since it might be due, for example, to peculiarities of the incline. But if the true time is 2.10 seconds, then the falsity of the hypothesis may be serious.

One may put the discussion of errors in another way. Actually in practice one rarely if ever expects that the true mean of a universe is exactly some specified number; it is certainly unrealistic to think, for example, that the mean velocity of a lot of ammunition will be exactly 1800 ft./sec. The real point is that a specific hypothesis

with respect to the mean guides one in setting up the types of action that are to be taken as a result of the experiment. The hypotheses have no meaning by themselves; they merely act in the context of an entire experiment, and are understandable only in terms of the actions to be taken as a result of the experiment and the associated risks of wrong actions.

The procedures to be described in the body of this text are in general poor in detecting small degrees of falsity of the hypothesis, but good in detecting large degrees. This can be put exactly by saying that, if one uses these prodedures, he is apt to accept the specific hypothesis H_0 when it happens to be false but not false in any great degree. On the other hand, he is not apt to accept H_0 when it is false to a large degree.

The situation with respect to the type II error can best be made explicit by means of an "Operating Characteristic" (OC) Curve. Figure II is such a curve for the test described in Problem 1. in the body of the text. In this problem, hypothesis H_0 asserts that the mean μ of a universe has some specific value, let us call it a. Now H_0 is false if the quantity μ -a is different from zero, and the "degree" of its falsity can be conveniently measured by μ-a. Actually, a somewhat more convenient measure of this degree of falsity is obtained by dividing μ -a by the standard deviation of the universe, which in the case of Problem 1. is assumed to be known. Thus, one unit away from the zero point on the base axis of Fig. II represents the event that the true mean μ of the population is actually one standard deviation away from the value, a, which H_0 asserts to be the true value of μ. To make the example explicit, suppose that the problem is to test the hypothesis that a ball takes two seconds to travel a specific distance down an inclined plane. Suppose that the standard deviation of a universe of readings on the times is 8 milliseconds. In this case, H_0 asserts that μ is 2, and H_1 asserts that μ is not 2. The degree of falsity of H_0 is measured in terms of $(\mu\text{-a})/0.008$. One unit of this degree of falsity is therefore 0 008, and this represents the event that the true time is either exactly 1.992 or exactly 2.008 seconds. Two units of falsity would be 0.016, which would represent the event that the true time is either 1.984 or 2.016.

The curves drawn in Fig. II represent the probability that the hypothesis H_0 will be accepted in the event that any given degree of falsity actually pertains. When the hypothesis is actually true, i.e., the degree of falsity is zero, then the probability of a type I error can be assigned, and in the case of Fig. II it has been made 0.05; this means that one time in twenty H_0 will be rejected when it is true, and hence, it will be accepted when true 95% of the time. The ordinate of the curve shown in Fig. II is therefore 0.95 when H_0 is true. As we move to the right, and consider possibilities where the hypothesis H_0 fails, the curves of the probability of acceptance naturally begin to drop. It will be noted that the behavior of this curve depends in part on the size of the sample. If, for example, we have only three observations in the sample, the probability of acceptance is about 0.58 when we move out one unit, whereas if the sample size is ten, the probability of acceptance is about 0.12.

Thus Fig. II not only tells us the chances of making certain degrees of error in reaching a decision by the methods of Problem II, but it also explains the advantages of increased sample sizes. The larger the sample, the less probable it is that one will accept H_0 when the degree of falsity is large. Fig. II shows clearly how the "OC" curve drops much more sharply for larger samples. For the very small samples of one and two items, the probability of an erroneous acceptance of H_0 is still quite high even when we go out two units, and hence are relatively far away from what H_0 accepts as true.

Consider again the experiment of the ball on the inclined plane; if the true and known standard deviation of the universe of readings is 0.008 seconds, if the chance of a type I error is set at 0.05, and if ten readings are taken, then the curve in Fig. II tells us:

a) If the ball actually takes 2.004 seconds (one-half unit along the base line), the hypothesis H_0 (which states that the ball takes exactly 2 seconds) will be accepted about 65% of the time. In frequency terms, this means that if we set a number of experimenters to work, and tell each one to make ten

measurements, and if the true mean time in each case were 2.004 seconds, then in about sixty-five percent of the cases (using the procedures to be described in the body of this text) the experimenters would accept the hypothesis that the true time is 2 seconds.

b) If the true time is 2.008 seconds (one unit along the base line), the probability of accepting H_0 is about 0.12.

c) If the true time is 2.012 seconds, the probability of accepting H_0 is virtually (though not exactly) zero.

The OC curve is a very useful and important piece of information relative to experimental work. By means of it we can actually judge how confident we really are in a given hypothesis on the basis of experimental results. Suppose, for example, that an experimenter has taken three readings of the time required for the ball to travel a specified distance, and his results are 2.001, 1.997, and 2.006; do these results confirm H_0, the hypothesis that the true mean is exactly 2 seconds? The average, $\bar{x}$, of this sample of readings is a little greater than 2.001, and there would be a strong inclination to say that the results "checked very well" with the hypothesis. Actually, if the procedures of Problem 1. are carried out, H_0 would normally be accepted (e.g., if the probability of a type I error is set at 0.05). But the OC curve shows us that with a sample of only three items, we could accept H_0 even though the true time was actually 2.020 seconds, and that we would be more apt than not to accept H_0 if the true time were 2.008 seconds. This means that one's confidence in the reliability of the result is not very great unless an error of 0.02 seconds is taken to be practically insignificant.

This last point may be generalized by recalling the perennial question of experimenters: "How large a sample should be taken in determining the answer to a certain question?" The reply to this question depends first of all on the degree of falsity which can be tolerated. That is, one must first decide how significant are the possible errors; then one can use the OC curve to

determine just what sample size will virtually preclude the significant error.

To illustrate in terms of the simple experiment of the inclined plane:

a) if an error greater than 0.040 seconds is significant, but anything less is not, use only one sample (i.e., the OC curve virtually goes to zero for a sample size of one at about five units from the origin).

b) use a sample size of at least five, if 0.016 seconds or more is a serious error.

c) use a sample size of at least twenty-five if 0.008 seconds is serious.

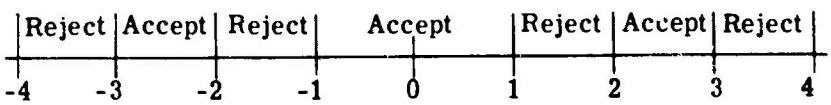
It may seem awkward to have to make judgments about the seriousness of errors in order to set the proper sample size, but the truth is that there is no purely statistical method for determining the proper sample to use; at best, the statistician can determine the risks entailed in the use of a given sample size, but he clearly cannot determine whether or not such risks can legitimately be taken. Such a determination is to be made by the "ethics of science."

At this point, the new and very useful techniques of sequential analysis should be noted. Sequential analysis is a statistical method in which the experimenter decides beforehand just what kind of OC curve is suitable, and proceeds to collect sufficient data to guarantee the kind of assurance he wants. In effect, the experimenter sets both the probability of the type I error, and (for one value) the probability of a type II error. That is, the experimenter decides what is a large degree of error, and determines the risk he is willing to take of making this size error. He then proceeds to take observations, and the sequential method at each step either tells him that there is not enough evidence for the kind of assurance desired, or else tells him that there is enough evidence and which hypothesis should be accepted. Examples of sequential analysis are given in the text.

The OC curve also assists in setting the type I error. It may happen that because of expense, or time, or some other consideration, one cannot make the number of readings that would be required to avoid a serious error. One possible way out of the difficulty is to make an adjustment in the type I error. The effect of such an adjustment is to raise or lower the ordinate of the OC curve at zero. If this ordinate is lowered, then in general there is a lowering of all ordinates of the curve (though not in equal amounts). Thus, one can lower the chance of an erroneous acceptance of H_0 by increasing the chance of an erroneous rejection. Put otherwise, one can lower the chance of a type II error by increasing the chance of a type I error (e.g., one may increase this chance to 0.5 or higher). Just what adjustment one makes in the chance of a type I error also depends on the seriousness of this error. Thus the experimenter, like any other decision maker is forced to weigh the cost of gathering evidence against the seriousness of a possible mistake. Given the method of conducting the experiment and the method of analyzing results, he can choose either to modify the sample size or the probability of either type of error, or both.

So much has been said about the risks of error in connection with statistical tests that the reader is apt to come to the fallacious conclusion that some other method of analyzing data must surely be preferable. But in point of fact, the OC curve represents the logical justification for the use of statistical tests, for in most cases it can be shown that any other method of analyzing data would incur a greater risk than the risk entailed in the use of the procedures outlined in the text. This does not at all imply that statistical procedures ought to be used in all cases. Indeed, in some experiments the additional cost entailed in the computation required for making a statistical analysis is a pure waste, since the decision may be quite clear from a mere examination of the data. But at present statistical criteria are essential in making any precise judgment of the superiority of one method of analyzing data compared to another. To illustrate this point, consider an alternative procedure to the one given in Problem 1 for testing whether a mean of a universe has some

specific value. This alternative is not to be recommended, of course; but the reasons why it is unsatisfactory are the important concern. Suppose, then, that the true variance of a universe is known and that it is desired to test the hypothesis H_0 that the true mean, μ, is equal to some value a. Instead of the procedure or test recommended in Problem 1., this alternative procedure tells us to accept H_0 if and only if $\bar{x}$, the estimated mean of n observations, lies within $\sigma/\sqrt{n}$ of a, or else lies within two to three $\sigma/\sqrt{n}$ of a. (The value $\sigma/\sqrt{n}$ is actually the standard deviation of a set of averages of n observations). If we consider a scale whose unit is $\sigma/\sqrt{n}$, this procedure means that we accept H_0 when $\bar{x}$ is within one unit of a, or else when it lies between two and three units; we reject H_0 when $\bar{x}$ lies between one and two units of a, or lies beyond three units. Graphically,



If we used this procedure, we would obtain the OC curve shown in Fig. VIII (which is the same for all sample sizes).

Now the procedure suggested seems very illogical, and the OC curve helps to indicate why. As one can see, the probability of accepting the hypothesis does not go down continuously the further the actual situation is from the hypothetical. The probability of accepting H_0 is less when the true mean is one and one-half units from a than it is when the true mean is two units from a. This does not seem to be a particularly fortunate result; we don't want the risk of making a mistake to increase the further our hypothesis departs from the truth.

From this example, one can begin to formulate the criteria which modern statistical test procedures have used in deciding on adequate methods for analyzing evidence:

a) The probability of accepting an hypothesis should be greatest when the hypothesis is actually true; that is, one would hardly like to have a procedure of analysis

lead him to accept the hypothesis more frequently when it is false then when it is true. Graphically, this criterion states that the OC curve of a test procedure should reach a maximum when the hypothesis is true. A procedure satisfying this criterion is commonly called (statistically) unbiased.

b) The probability of accepting an hypothesis should decrease (or, at least, not increase) the further the hypothesis is from the truth. That is, the OC curve should not rise and fall, as in the example given above.

c) The probability of acceptance should approach zero as the hypothesis deviates more and more from the actual state of affairs. This means that the OC curve should approach zero as the value along the abcissa increases.

d) The OC curve should descend more and more rapidly as the sample size is increased. This is the demand of economy, for it seems reasonable to require that the observations one makes, i.e., the more effort one puts into an experiment, the less risk of a mistake one should incur.

These criteria are by no means sufficient to characterize the best method of drawing inferences. In the next section, some of the more general criteria will be briefly considered.

Before leaving the theory of statistical tests, several specific points should be noted.

It does not always happen that there are just two hypotheses to be tested, or that any one of the hypotheses is specific. But the remarks made above can easily be applied to the more general cases. There is one differentiation in hypothesis testing that is important, however, for a proper understanding of the sequel. This is the difference between "one-sided" and "two-sided" tests. Suppose one wished to test whether a piece of metal was hard enough for some manufacturing purpose. In such a case, one is only interested in assuring oneself that the metal does not have a hardness less than some value a. Hence, one of the hypotheses states that the hardness is at least as great as $\underline{a}$ ($\mu \geq \underline{a}$), while the other states that it is less than a. This example is called a one-sided test. On the other hand, one might be comparing the

hypothesis more frequently when
t is true. Graphically, this cri-
C curve of a test procedure should
the hypothesis is true. A pro-
riterion is commonly called (sta-

cepting an hypothesis should de-
increase) the further the hy-
th. That is, the OC curve should
the example given above.

ceptance should approach zero as
more and more from the actual
eans that the OC curve should
lue along the abcissa increases.

descend more and more rapidly
creased. This is the demand of
easonable to require that the ob-
e., the more effort one puts into
risk of a mistake one should

y no means sufficient to charac-
drawing inferences. In the next
e general criteria will be briefly

eory of statistical tests, several
noted.

appen that there are just two
or that any one of the hypotheses
arks made above can easily be
ral cases. There is one differ-
sting that is important, however,
g of the sequel. This is the
sided" and "two-sided" tests.
t whether a piece of metal was
nufacturing purpose. In such a
ed in assuring oneself that the
rdness less than some value a.
ses states that the hardness is
≥a), while the other states that
ample is called a one-sided
one might be comparing the

metal against some standard; in such a case, one hypothesis would assert that the hardness was equal to a, while the other would assert that it was not equal to a. This case would be a two-sided test, since the alternative hypothesis is valid if the hardness deviates from the standard on either side. The distinction is important, since it determines how the tables are to be used once the probability of a type I error has been set.

It is often desirable to present an "estimate" of certain quantities, and not to reach any final decision. Estimation, or the reporting of quantities in tabular or graphical form, is actually an incomplete analysis of data. Instead of deciding between alternative hypotheses, the experimenter may merely want to report a result, since he is uncertain how the results are to be used. One convenient report of an estimate of a universe-measure is the confidence interval. Vaguely expressed, the confidence interval represents an interval within which there is some assigned degree of likelihood that the true value of the universe-measure will lie. Specifically, the meaning of the confidence interval can be expressed as follows: suppose θ is some measure of central tendency, or dispersion, or skewness etc., of a universe. Suppose n observations are taken for the purpose of estimating θ. The methods of computing a "95% confidence interval" to be described in the text guarantee that, under the specified assumptions, 95% of all such computed intervals will, in the long run, include the true value of θ. That is, if a large number of experimenters were to follow the procedure of taking n observations and computing the interval as directed, about 95% of them would "catch" the true value of θ in the intervals they computed. Confidence intervals are the modern refinement of the classical "mean plus or minus the probable error."

It often happens that the statistical method which gives the "best" OC curve also requires costly computations, while there may be another method of analysis, not quite as good with respect to risks, but much more economical to use. Some of these "less efficient" but useful techniques will be mentioned in Part IV of the text.

4. GENERAL ASPECTS OF EXPERIMENTATION It is clearly impossible within the scope of this text to cover all the general aspects of experimental work. The main purpose of this section is to emphasize, by way of specific questions, that statistical procedures per se have no necessary validity, and that their validity is only to be judged in the general context of experimental design.

It is recommended that before conducting analyses, or, indeed, before collecting evidence, the following questions be kept in mind:

a) Are you confident that the data are really pertinent with respect to the problem?

The point here is that statistical methods will give definite answers, once certain types of data are fed into the formulae; but if the original data are not pertinent to the real issue involved, or are unreliable in one way or another, then no amount of precise thinking and mathematics will improve the situation. It is often very difficult or impossible to be sure that data are pertinent, but the important thing is to have some explicit rationale which can serve as a basis for adequate critical examination by others. One good way to formulate this rationale is to ask whether a different outcome in the data would make any real difference in the course of action adopted as a result of the experiment.

b) Has all pertinent information been applied to the problem?

Statistical methods are, of course, applicable to any given set of data; but if an experimenter ignores previous work, he is in effect choosing a biased sample as the basis of his conclusion. Again, it is well-nigh impossible to be sure that all available evidence has been marshalled, but it is as important to tap all known sources of information as it is to take as many observations as possible.

c) Are the alternative hypotheses real with respect to action?

Statistical results as such are meaningless; the intellectual acceptance of an hypothesis means nothing unless such acceptance can be translated into some kind

of definite action. The action may consist of additional experiments, or of a modification of a production procedure, or the adoption of a new drawing for an article. The point is: does each alternative hypothesis represent a distinct type of action to be followed? If the acceptance of any two of the hypotheses would lead to essentially the same course of action, then the hypotheses are the same, no matter how different they may look on paper or in statistical terminology. To illustrate, suppose the statistical test consists of comparing two variances. The first variance σ_1^2 represents dispersion in the length of rods manufactured by one type of machine, and the second, σ_2^2, represents dispersion in the lengths coming off another machine. One might be tempted to say that there are three hypotheses here: σ_1^2 is less than σ_2^2, σ_1^2 is equal to σ_2^2, and σ_1^2 is greater than σ_2^2. But suppose the point of the experiment is to see whether the second machine should be used in place of the first, which is now used in regular production. Then there are only two actions involved: keep using the old machine, or replace it by the new. And therefore there are only two hypotheses: σ_1^2 is less than or equal to σ_2^2 (in which case the corresponding action consists in keeping the old machine since it gives as uniform products as the new), and σ_1^2 is greater than σ_2^2 (in which case the new machine would be adopted). It is to be noted that the statement of courses of action in connection with hypotheses minimizes the wasteful procedure of testing first and deciding what to use the results for later.

d) Do the data suggest any new avenues of inquiry?

The experimenter who has become familiar with statistical techniques frequently wants to hurry into the procedures without carefully examining the data. Very often the data present patterns which may not only indicate erratic results, but also open up new avenues for investigation. It is always advisable to plot data on a chart whenever feasible, so that the visual sense can more readily detect patterns. If the observations have been made by different observers, or on different days, or with different machines, for example, graphical plotting may reveal some unsuspected results. In the text, some

useful preliminary tests on data are suggested which are also very helpful in detecting unsuspected patterns.

e) What statistical assumptions can legitimately be made about the data?

Since all procedures of analysis (statistical or otherwise) demand certain assumptions about the data, the experimenter is often puzzled as to what he can legitimately assume. No one can ever be completely confident that the data are random, or come from a normal universe. The real point of this question is to emphasize again the risks entailed in experimentation: if the issues are not serious ones, then considerably greater leniency can be taken in making assumptions. Also, certain assumptions greatly simplify the method of analysis and hence reduce costs. The relative seriousness of certain mistakes in assumption-making is discussed above. Perhaps the most important point under the head of this question is that the assumptions be made explicit, for only in this way can the most fruitful criticism be made of the experimental method.

f) Is a statistical analysis necessary?

This question is intended to offset the tendency to rush into statistical computations, simply because they have the appearance of lending precision to one's results. Quite often a tabular or graphical presentation will reveal all the information that is needed, without the necessity of any elaborate analysis. It must be emphasized, however, that the opposite tendency of relying wholly on graphs or tables is perhaps a more grievous sin against sound scientific method.

g) How should the probability of error be set?

This is a crucial point in the adequate application of statistical methods. The common tendency to use "0.05" or "0.01" as the values of the probability of a type I error is certainly not to be recommended as a sound practice in general. At the present time there is no very precise method of setting the risk of either the type I or type II errors. But the following considerations may be of assistance. Since each hypothesis leads to some specific course of action intended to accomplish

certain objectives, the experimenter should try to decide (with the aid of consultation) just how serious a consequence is entailed in acting as though H_0 is true when actually it is not true. Thus, H_0 may be the hypothesis that a certain machine is producing satisfactory material, with the corresponding action of maintaining present production without any check-up. To accept H_0 erroneously means to continue making imperfect material, while to reject H_0 erroneously means to make unnecessary adjustments. In such a case, the erroneous acceptance of H_0 may be much more serious than the erroneous rejection (depending, of course, on the value of the items made and their importance for the completed article). Hence, the type I error is not as serious as the type II error, and the probability of a type I error may be set high (e.g., as high as 0.5).* On the other hand, H_0 may be the hypothesis that a certain standard is at least as good as a new experimental model; the action of accepting H_0 erroneously means missing something that may improve quality but the action of rejecting it erroneously means an expensive change in machinery and industrial designing without any improvement in quality. Hence, the type I error is in this case quite serious, and its probability should be very low (say 0.01 or less).

In general, the principles which guide one in the selection of the proper OC curve are based not only on the probability of making an error, but on the importance of the error relative to some specific set of purposes. The criteria that must be employed in evaluating any method of reaching a conclusion depend on the criteria of efficiency and inefficiency with respect to the most important objectives. The "loss" involved in the application of any method of experimental inference is measured in terms of the chance of making an error and the seriousness of the error. Sometimes, this loss is measured in terms of "cost," since monetary scales provide seemingly precise measures. But the use of cost measures alone is not adequate in general, for there is no practical way to measure losses of health or life or security in terms of cost. But the fact that one

*Unfortunately, in some cases tables are not available for values other than 0.05 and 0.01.

cannot always find precise scales for estimating loss ought not to prevent the scientist from making as sensible an estimate as possible, rather than none at all.

The discussion of the logic of statistical tests can best be summarized by two fallacies which are apt to misguide the experimenter in the use of statistical instruments. The one is the Fallacy of Statistical Parsimony, and consists in such inaccurate recommendations as: "Don't run a statistical analysis until you are sure of all the factors," "Don't analyze data until you are certain they are sound (random, from the same universe, etc.)," "Don't assume a normal universe unless you are morally certain," "Don't apply statistical methods to small samples, or to widely scattered data, or to data that are very precise, etc., etc." The opposite Fallacy of Statistical Profligacy insists that one should gather data first, analyze second; or, one should use statistics in all possible cases in order to be "precise"; or, if a new statistical test comes out, use it; etc., etc. Perhaps the remarks included in this Introduction will help the experimenter to choose the wise mean between these two fallacious extremes.

REFERENCES

1. For the logic of statistical methods, see

   1a) A. Wald, On the Principles of Statistical Inference, Notre Dame University, 1941.

   1b) J. Neyman and E. S. Pearson, "On the Problem of the Most Efficient Tests of Statistical Hypotheses," Philos. Trans. of the Royal Soc., vol. 231, pp. 289-337.

   1c) C. W. Churchman, Theory of Experimental Inference, Macmillan, 1948.

   1d) W. A. Shewhart, Statistical Method from the Viewpoint of Quality Control, U. S. Dept. of Agriculture, 1939.

2. For the mathematics of statistical methods, see:

2a) P. Hoel, Introduction to Mathematical Statistics, John Wiley, 1947.

2b) M. Kendall, Advanced Theory of Statistics, vol. I and II, Charles Griffin and Co., Ltd., London, 1947.

2c) C. Eisenhart, "The Assumptions Underlying the Analysis of Variance," Biometrics, vol. 3, 1947, pp. 1-21.

2d) W. G. Cochran, "Some Consequences When the Assumptions for the Analysis of Variance are not Satisfied," Biometrics, vol. 3, 1947, pp. 22-38.

2e) A. Wald, Sequential Analysis, John Wiley, 1947.

3. For practical applications, see

3a) R. A. Fisher, Statistical Methods for Research Workers, Oliver and Boyd Ltd., Edinburg, 1941.

3b) H. Freeman, Industrial Statistics, John Wiley, 1942.

3c) Techniques of Statistical Analysis, by the Statistical Research Group, Columbia University; McGraw-Hill, 1946.

3d) K. A. Brownlee, Industrial Experimentation, Chemical Publishing Co., Inc., 1947.

3e) K. Mather, Statistical Analysis in Biology, 2nd Ed., Interscience Publishers, Inc., 1947.

3f) W. G. Cochran and G. M. Cox, Experimental Designs, John Wiley, 1950.

3g) P. O. Johnson, Statistical Methods in Research, Prentice-Hall, 1949.

4. Some journals in which statistical methods are frequently discussed and new techniques explained are:

Journal of the American Statistical Association (mostly non-technical)

Biometrika (technical and nontechnical; excellent tables)

Annals of Mathematical Statistics (mostly technical)

Biometrics Bulletin (mostly nontechnical; has a good "Query" section)

Journal of the Royal Statistical Society and its Supplement (technical and nontechnical)

ION

ics (mostly technical)

ontechnical; has a

l Society and its
echnical)

GLOSSARY OF TERMS AND SYMBOLS

A. Mathematical Symbols

1. "$x < y$" means "x is less than y."
2. "$x > y$" means "x is greater than y."
3. "$x \leq y$" means "x is less than or equal to y," i.e., "x does not exceed y."
4. "$x \geq y$" means "x is greater than or equal to y," i.e., "x is not less than y."
5. "$x \neq y$" means "x does not equal y."
6. $|x|$, which stands for the absolute value of an expression, means that the value inside $|\ |$ is to be taken as plus, regardless of its actual sign. This $|-4| = 4$, $|25| = 25$, etc.
7. "log x" means "the logarithm of x to the base 10." This will be found in regular "log tables." The meaning of "log x" is as follows: if $\log x = y$, then $10^y = x$.
8. "ln x" means "the logarithm of x to the 'natural' base e," where $e = 2.7183\cdots$ That is, if $\ln x = y$, then $e^y = x$.
9. "∞" means "infinity."
10. "Σ" means "sum of." Thus $\sum^{n} x_i$ = sum of n numbers, x_1, x_2, etc. $\sum^{n} x_i^2$ = sum of the squares of the numbers. In general, "n" will be omitted from "Σ" if the limits of sumation are clear in the context.
11. "$\pm$" means "plus or minus" and is used to denote a range of values. Thus "10 ± 2" means "10 plus or minus 2," i.e., the range from 8 to 12.

B. Statistical Terms and Symbols.

1. x_1, x_2, x_3, etc., refer to a set of data presented in some order.

2. n in general will represent the number of observations (items) in a sample.

3. k in general will represent the number of samples, when more than one or two are discussed.

4. A "universe" is a collection of items characterized by some common property, which property may be expressed quantitatively or qualitatively.

5. A "sample" is a subset of items (finite in number) of a universe.

6. "Random" is a term used to describe a method of selecting a sample from the universe; practically, a random sample gives representative information about the universe relative to certain purposes.

7. "Distribution function" is a mathematical expression used for describing the manner in which the quantitative or qualitative properties of a universe are distributed.

8. "Measure of central tendency" is a mathematical expression used to describe a quantity or quality where items of the universe are concentrated.

9. "Measure of dispersion" is a mathematical expression used to describe the scatter of the items of a universe with respect to the properties of the items.

10. "μ" represents the mean of a universe, which is a measure of central tendency that describes the concentration of the items of the universe in terms of the center of gravity of the distribution.

11. "σ^2" represents the variance of the universe, which is a measure of dispersion describing the scatter of items of the universe in terms of the second moment of the distribution.

12. "σ", the standard deviation of the universe, is the square root of the variance.

GLOSSARY

13. "Statistic" is any measure (quantitative) computed from the properties of one or more items of the sample.

14. "$\bar{x}$" is a statistic used to estimate the mean of a universe. Its formula is $\Sigma\, x_i/n$.

15. "s^2" is a statistic used to estimate the variance of a universe. Its formula is $\Sigma\, (x_i-\bar{x})^2/n-1$ or, for purposes of computation, $[n\Sigma x_i^2 - (\Sigma x_i)^2]/n(n-1)$.

16. $\sigma_{\bar{x}}^2$ represents the variance of a universe of $\bar{x}$ values; $s_{\bar{x}}^2$ is an estimate of $\sigma_{\bar{x}}^2$, obtained by dividing s^2 by n.

17. R represents the range of a sample, i.e., the maximum reading minus the minimum.

18. p is a measure of central tendency applied to universes described in terms of certain qualities; it is the proportion of items of the universe that have a certain property. If the universe is described in terms of a property (e.g., defective item) and its absence then "1-p" represents the proportion of items that do not have the property; in such cases, "1-p" is sometimes symbolized by "q."

19. $\bar{p}$ is an estimate of p based on a sample; its formula is the number of occurrences of the property in the sample, divided by n, the total number of items in a sample.

20. "y = a + bx" is a formula used to express a linear realtionship between x and y which is estimated from a sample of a joint distribution. "a" will be called the "intercept" and "b" the "slope," or, alternatively, the "coefficient of linear regression."

21. "Normal universe" describes a universe where items are distributed symmetrically about the mean and have certain other properties described in the introduction; "has normality" means "comes from some normal universe."

22. "independence" is applied to two or more observations, and means roughly that the observations do

not influence each other. Observations that are not correlated are "independent."

23. "Degrees of Freedom," symbolized DF, is a term associated with a statistic and is used to describe the number of ways in which the statistic can vary.

24. "Type error" is the error of rejecting an hypothesis when it is true. If there are two hypotheses, H_0 and H_1 , the type I error is the error of rejecting H_0 when it is true, and the type II error is the error of rejecting H_1 (i.e., accepting H_0) when H_1 is true. The probability of a type I error is a constant if the hypothesis H_0 refers to some specific value, but otherwise the probability varies.

25. "Operating Characteristic Curve" (OC curve) is the curve which shows the probability of accepting an hypothesis when a certain method of analysis is used; that is, the OC curve shows the probability of accepting an hypothesis under all pertinent conditions, both when the hypothesis is true, and when it is false.

er. Observations that are independent."

," symbolized DF, is a term istic and is used to describe n which the statistic can vary.

error of rejecting an hypothe- If there are two hypotheses, I error is the error of re- true, and the type II error is H_1 (i.e., accepting H_0) when bility of a type I error is a esis H_0 refers to some spe- wise the probability varies.

istic Curve" (OC curve) is s the probability of accepting certain method of analysis is curve shows the probability esis under all pertinent con- hypothesis is true, and when

PART 1
SOME STANDARD PROCEDURES FOR TESTING HYPOTHESES

PROBLEM 1. Test whether a single normal universe mean has a certain value, when the true variance is known.

This test is usually employed to determine whether a theory is correct, or some experimental value checks with a standard or expected value. The experimenter presupposes that the variance σ^2, of the universe is known, e.g., on the basis of previous experiments or theoretical considerations.

a) TWO-SIDED TEST

ASSUMPTIONS: All observations come from the same normal universe, are independent, and the universe variance σ^2 is known.

HYPOTHESES: H_0: $\mu = a$

H_1: $\mu \neq a$

METHOD:

1. Compute $\bar{x}$.
2. Compute

$$T = \frac{|\bar{x}-a|\sqrt{n}}{\sigma}$$

where n is the number of items tested, and σ is the known standard deviation.

3. If the probability of a type I error (chance of rejecting H_0 when it is true) is set at 0.5, accept H_0 when $T \leq 0.67$; otherwise reject (i.e., otherwise accept H_1). If the probability of a type I error is set at 0.05, accept H_0 when $T \leq 1.96$; otherwise reject. If the probability of a type I error is 0.01, accept H_0 when $T \leq 2.58$, otherwise reject.

For any other assigned probability of a type I error, consult Table I. First take one-half of the assigned

probability of a type I error. Then look for the value in the "Q" column of Table I which lies closest to this halved value. Let T_0 be the corresponding value in the "T" column; accept H_0 if and only if $T \leq T_0$.

The Operating Characteristic Curve for this test is shown in Fig. II.

EXAMPLE: To determine whether the distance between the centers of gravity of two bodies as predicted by theory is in agreement with the observations. The predicted distance is 97 inches (a = 97). The assumed variance is 25 ($\sigma^2 = 25$). The observations are 92, 99, 90, and 95 inches.

ASSUMPTIONS: Observations are independent and come from the same normal universe. The variance $\sigma^2 = 25$.

HYPOTHESES: H_0: $\mu = 97$

H_1: $\mu \neq 97$

METHOD: 1. $\bar{x} = 94$

2. $T = \dfrac{|94-97|\sqrt{4}}{5} = 1.20$

3. If the probability of a type I error is 0.5 accept H_1 (since $1.20 > 0.67$). If the probability of the type I error is 0.05, accept H_0 (since $1.20 < 1.96$). Similarly, if the probability of the type I error is 0.01, accept H_0 (since $1.20 < 2.58$).

As an example of the use of the Operating Characteristic Curve for this test, Fig.II shows that if the probability of the type I error is 0.05, H_0 would be accepted about 50% of the time when the true distance is one standard deviation anway from 97 inches, i.e., when the true distance is 102 or 92.

b) ONE-SIDED TEST

Here the assumptions and the method are the same as in a) but the experimenter is only interested in testing whether the mean is greater than some value (or,

Then look for the value in
ich lies closest to this
orresponding value in the
only if $T \leq T_0$.

tic Curve for this test is

ther the distance between
odies as predicted by
observations. The pre-
= 97). The assumed
observations are 92, 99,

re independent and come
. The variance $\sigma^2 = 25$.

$\frac{}{\sqrt{4}} = 1.20$

bility of a type I error is
7). If the probability of
H_0 (since 1.20 < 1.96).
he type I error is 0.01,

the Operating Character-
shows that if the proba-
, H_0 would be accepted
true distance is one
7 inches, i.e., when the

TEST

e method are the same
only interested in testing
some value (or,

alternatively, is only interested in testing whether the mean is less than some value). The probability of a type I error varies, but the experimenter sets the maximum probability he will allow.

ASSUMPTIONS: Normality, independence, and the value of σ^2.

HYPOTHESES: H_0: $\mu \leq a$

H_1: $\mu > a$

METHOD: 1. Compute $\bar{x}$.

2. Compute $T = \frac{(\bar{x}-a)\sqrt{n}}{\sigma}$ [Note: do not use absolute value of $(\bar{x}-a)$.]

3. If the probability of the type I error is set at 0.5, accept H_0 when $T \leq 0$; otherwise reject. If the probability of the type I error is set at 0.05, accept H_0 when $T \leq 1.65$; otherwise reject. If the probability of the type I error is set at 0.01, accept H_0 when $T \leq 2.32$; otherwise reject.

If the probability of the type I error is set at some other value, use Table I; look for the value in the "Q" column that lies closest to the assigned probability of the type I error and let T_0 be the corresponding "T" value; accept H_0 if and only if $T \leq T_0$.*

NOTE: If H_0 is the hypothesis that $\mu \geq a$, then the same procedures are followed, except that one accepts H_0 if and only if $T \geq 0$, or $T \geq -1.65$, or $T \geq -2.32$, etc.

EXAMPLE: Suppose one obtains a mean breaking strength ($\bar{x}$) of 6.52 ounces for nine items of a certain type of cotton thread, and from past experience it is known that the average breaking strength for this particular thread is 7.50 ounces while the standard deviation is 1.20 (i.e., the variance is 1.44) ounces. Assuming the observations are independent draws from a normal universe, one is to

*No Operating Characteristic Curve is known for this test. Hereafter, the OC curves will be mentioned only if they are known to be available.

test whether the consignment from which the sample was taken is really below standard.

ASSUMPTIONS: Normality, independence, $\sigma^2 = 1.44$.

HYPOTHESES: H_0: $\mu \geq 7.50$ (The consignment is at least as good as the average)

H_1: $\mu < 7.50$ (The consignment is worse than average)

METHOD:

1. $\bar{x} = 6.52$

2. $T = \dfrac{(6.52 - 7.50)\sqrt{9}}{1.20} = -2.45$

3. H_0 is rejected for probabilities of type I error of 0.5, 0.05, and 0.01.

PROBLEM 2. Test whether a single mean of a normal universe has a certain value, when the true variance is unknown.

This test will be employed when the experimenter does not feel justified in assuming any particular value, or small range of values, for the variance.

a) TWO-SIDED TEST

ASSUMPTIONS: The observations all come from the same normal universe, and are independent.

HYPOTHESES: H_0: $\mu = a$

H_1: $\mu \neq a$

METHOD: 1. Compute $\bar{x}$ and s, the estimated mean and the square root of the estimated variance.

2. Compute

$$t = \frac{|\bar{x} - a| \sqrt{n}}{s}$$

3. Compute DF = n - 1.

4. Accept H_0 according to Table IV, as follows: if t is less than or equal to the value shown in the "0.5 probability" column for the appropriate degrees of freedom, and the probability of a type I error is set at 0.5, accept H_0. If the probability of the type I error is 0.05, and t is less than or equal to the value shown in the appropriate "0.05 probability" column, accept H_0. If the probability of the type I error is set at 0.01, and t is less than or equal to the value shown on the "0.01 probability" column, accept H_0. Otherwise in each case accept H_1. Similarly for other assigned probabilities for the type I error.

The Operating Characteristic Curve for this test is shown in Fig. I.

EXAMPLE: Suppose a machine is set to produce mica washers having a thickness of 10 mils. A sample of 10

washers gives an average thickness ($\bar{x}$) of 9.52 mils with an estimated standard deviation (s) of 0.60 mils. Is a deviation of this magnitude to be expected due to chance fluctuations?

ASSUMPTIONS: Normality and independence.

HYPOTHESES: H_0: $\mu = 10$

H_1: $\mu \neq 10$

METHOD: 1. $\bar{x} = 9.52$; $s = 0.60$

2. $t = \dfrac{|9.52-10|\ \sqrt{10}}{0.60} = 2.53$

3. DF = 10-1 = 9

4. The value shown opposite "Degrees of Freedom" of 9 in Table IV is 0.703 for the "0.5" column; hence if the probability of a type I error has been set at 0.5, reject H_0; the value in the "0.05" column is 2.262, and again for a probability of a type I error of 0.05, H_0 should be rejected. But the value for the "0.01" column is 3.250, and hence for an assigned probability of a type I error of 0.01, H_0 should be accepted.

NOTE: The following simplified method can also be used for this problem. The method avoids the computation of s^2. In general, it is less "efficient" than the method given above, in the sense that the chances of a type II error are greater, though the efficiency is about the same as that of the method described above if the sample size is five or less. The method can often be completed in a matter of a minute or so, and is very useful for quick checks. Further, if this method rejects H_0 for a given probability of the type I error, then so will the method given above, so that often no further computations need be made. But if this method does <u>not</u> reject H_0, the more efficient method may still do so.

The method is as follows (the assumptions and hypotheses are the same):

1. Compute $\bar{x}$.

2. Compute R = difference between largest and smallest observation in sample.

3. Compute

$$t' = \frac{|\bar{x} - a|}{R}$$

4. Consult Table XI. If the value of t' is less than or equal to the value opposite the appropriate sample size (n) and under the appropriate probability of a type I error, accept H_0. Otherwise reject H_0.

Thus suppose the data are: 1.21, 1.25, 1.19, 1.27, 1.15, 1.24, 1.30. Can one accept the hypothesis that these observations come from a normal universe with a mean, μ, of 1.20? Here $\bar{x} = 1.23$, $R = 0.15$, $t' = 0.03/0.15 = 0.20$. We enter Table XI, with $n = 7$. For a probability of the type I error of 0.05 or less, H_0 is accepted.

b) ONE SIDED-TEST

ASSUMPTIONS: The observations all come from the same normal universe, and are independent.

HYPOTHESES: H_0: $\mu \leq a$

H_1: $\mu > a$

METHOD: 1. Compute $\bar{x}$ and s, the estimated mean and square root of the estimated variance.

2. Compute $t = \frac{(\bar{x} - a)\sqrt{n}}{s}$ [NB: do not use absolute value of $(x - a)$.]

3. Compute DF = n - 1.

4. Accept H_0 as follows: if the probability of a type I error is set at 0.5, accept H_0 if and only if $t \leq 0$. If the probability of a type I error is set at 0.05, then consult Table IV; first double 0.05 to obtain 0.10, and then determine the value in Table IV that appears under the 0.10 column opposite the number of Degrees of Freedom. Call this value t_0. Accept H_0 if and

PROBLEM 3. Test whether two means from normal universes are equal, variances known.

This test consists of a comparison of two universes, i.e., a comparison of the origins of two samples, assuming the variance of each universe in known from previous experience.

a) TWO-SIDED TEST

ASSUMPTIONS: Each sample is drawn from some normal universe; the observations are independent, and the variances are known.

HYPOTHESES: H_0: $\mu_1 = \mu_2$

H_1: $\mu_1 \neq \mu_2$

METHOD: 1. Compute $\bar{x}_1$ and $\bar{x}_2$, the averages for each sample.

2. Compute

$$T = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

where n_1 is the number of items in the first sample, and n_2 is the number in the second, and σ_1^2 and σ_2^2 are the known variances.

3. Use the same method for accepting H_0 as is described in Problem 1, two-sided test.

EXAMPLE: Two machines are used for testing hardness in a laboratory. It is desired to determine whether the machines are consistent with each other. A standard block is used, and eight determinations are made with each machine. The variance of readings from each machine is known, and in both cases is 0.16.

ASSUMPTIONS: Normality of each sample, independence of observations, and $\sigma_1^2 = \sigma_2^2 = 0.16$.

HYPOTHESES: H_0: $\mu_1 = \mu_2$

H_1: $\mu_1 \neq \mu_2$

METHOD: 1. Suppose $\bar{x}_1 = 67.3$ and $\bar{x}_2 = 66.7$

2. $$T = \frac{|67.3 - 66.7|}{\sqrt{\frac{0.16}{8} + \frac{0.16}{8}}} = 3.00$$

3. Since T = 3 is greater than any one of the numbers 0.67, 1.96, and 2.58, H_0 is rejected for a probability of the type I error of 0.5, 0.05, or 0.01; i.e., the hypothesis, H_1, that there is a real difference in the machines, is to be accepted.

The Operating Characteristic Curve for this test is shown in Fig. II, where $\mu = \mu_1 - \mu_2$, $\sigma^2 = \sigma_1^2 + \sigma_2^2$.

b) ONE-SIDED TEST

ASSUMPTIONS: Normal universes, independence of observations, and variance of each sample is known to have some specific value.

HYPOTHESES: H_0: $\mu_1 \leq \mu_2$

H_1: $\mu_1 > \mu_2$

METHOD: 1. Compute $\bar{x}_1$ and $\bar{x}_2$

2. Compute

$$T = \frac{(\bar{x}_1 - \bar{x}_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

[NB: do not use the absolute value of $(\bar{x}_1 - \bar{x}_2)$].

3. Use the same procedure for accepting H_0 as is described in Problem 1, one-sided test.

NOTE: If the hypotheses are H_0: $\mu_1 \geq \mu_2$ and H_1: $\mu_1 < \mu_2$, then use the same procedure for accepting H_0 as is described in the Note to the one-sided test of Problem 1.

EXAMPLE: If the standard deviation of measurements on the breaking strength of a certain type of thread is known to be 1.20 ounces from considerable past experience, and a sample of 9 from one factory gives an average breaking strength of 6.52 ounces, while a sample of 16 from another factory gives an average breaking strength of 7.20 ounces, can we regard the general average output of the second factory to be superior?

ASSUMPTIONS: Normality of the universe of items from each factory, independence of observations and $\sigma_1 = \sigma_2 = 1.20$

HYPOTHESES: H_0: $\mu_1 \geq \mu_2$ (First factory at least as good as second)

H_1: $\mu_1 < \mu_2$ (Second factory superior)

METHOD:

1. $\bar{x}_1 = 6.52$ and $\bar{x}_2 = 7.20$

2. $$T = \frac{(6.52 - 7.20)}{1.20\sqrt{\frac{1}{9} + \frac{1}{16}}} = -1.36$$

3. If the probability of a type I error is 0.5, reject H_0 (since $T < 0$). If the probability of a type I error is set at 0.05 or 0.01, accept H_0 (no difference in average output of factories), since $T > -1.65$ and $T > -2.32$.

PROBLEM 4. Test whether two means from normal universes are equal, variances unknown but equal.

This test is like that described in Problem 3, except that the variance of each universe is unknown; but it is assumed that whatever the variances may be, they are equal.

a) TWO-SIDED TEST

ASSUMPTIONS: The items of each sample come from the same normal universe and are independent; the variance of one universe is equal to the variance of the other universe. $(\sigma_1^2 = \sigma_2^2)$.

HYPOTHESES: H_0: $\mu_1 = \mu_2$

H_1: $\mu_1 \neq \mu_2$

METHOD:

1. Compute $\bar{x}_1$, $\bar{x}_2$, s_1^2, and s_2^2.

2. Compute

$$s^2 = \frac{(n_1-1)\, s_1^2 + (n_2-1)\, s_2^2}{n_1 + n_2 - 2} = \frac{\sum^{n_1} (x_1-\bar{x}_1)^2 + \sum^{n_2} (x_2-\bar{x}_2)^2}{n_1 + n_2 - 2}$$

where n_1 and n_2 are the respective sample sizes.

3. Compute

$$t = \frac{|\bar{x}_1 - \bar{x}_2|}{s\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

4. Compute DF $= n_1 + n_2 - 2$

5. Accept H_c according to the same procedure as is described in Step 4 of the method of Problem 2, two-sided test.

The Operating Characteristic Curve for this test is shown in Fig. I, where $\mu = \mu_1 - \mu_2$, $\sigma^2 = \sigma_1^2 + \sigma_2^2$.

EXAMPLE: Two manufacturers each submit 10 samples of their product (a weld) to determine if the breaking strength of the manufactured product of one is better

than the product of the other. The sample of 10 were submitted to test with the following results:

	$\bar{x}$	s^2
Manufacturer A	8.44 lbs.	1.59
Manufacturer B	8.37	1.69

Is there a real difference in the mean breaking strengths of the universes?

ASSUMPTIONS: Normality of each sample, independence, and equal variances.

HYPOTHESES: H_0: $\mu_1 = \mu_2$

H_1: $\mu_1 \neq \mu_2$

METHOD:

1. $\bar{x}_1 = 8.44$, $\bar{x}_2 = 8.37$, $s_1^2 = 1.59$, $s_2^2 = 1.69$.

2. $$s^2 = \frac{(9)(1.59) + 9\,(1.69)}{10 + 10 - 2} = 1.64$$

3. $$t = \frac{|8.44 - 8.37|}{\sqrt{(1.64)\left(\frac{1}{9} + \frac{1}{9}\right)}} = 0.12$$

(NOTE: When $n_1 = n_2 = n$, it can easily be shown that the formula for t reduces to $\dfrac{|\bar{x}_1 - \bar{x}_2|}{\sqrt{\frac{s_1^2 + s_2^2}{n}}}$).

4. DF = 10 + 10 - 2 = 18

5. The values given in Table IV opposite 18 Degrees of Freedom are 0.688, 2.101, and 2.878 for the "0.5", "0.05," and "0.01" columns respectively. Hence, H_0 (no difference in manufacturers' products) is accepted for all three of these probabilities of the type I error.

NOTE: If the sample sizes (n_1 and n_2) are equal, then the following simplified (but less efficient) method can be used. The remarks contained in the Note to Problem 2 apply here as well. The method is (assumptions and hypotheses same as above):

1. Compute $\bar{x}_1$ and $\bar{x}_2$

2. Compute R_1 and R_2, the ranges of each sample (thus R_1 is the difference between the largest and smallest observation in the first sample).

3. Compute

$$t' = \frac{|\bar{x}_1 - \bar{x}_2|}{1/2\,(R_1 + R_2)}$$

4. Consult Table XI. If t' is less than or equal to the value shown opposite the appropriate n (where $n = n_1 = n_2$), and under the assigned chance of a type I error, then accept H_0. Otherwise accept H_1. For example, suppose the data are

1	2
47	38
35	42
42	37
50	48
38	39
46	42
52	35

Is there a real difference in the means of universe A and universe B, assuming normality and independence? Here $\bar{x}_1 = 44.3$, $\bar{x}_2 = 40.1$, $R_1 = 17$, $R_2 = 13$, $t' = (4.2)/15 = 0.28$. Hence (entering Table XI with $n = 7$), H_0 is accepted for a chance of a type I error of 0.05 or less.

b) ONE-SIDED TEST

ASSUMPTION: Normality of each sample, independence, and equal variances.

HYPOTHESES: H_0: $\mu_1 \leq \mu_2$

H_1: $\mu_1 > \mu_2$

METHOD: 1. Compute $\bar{x}_1$, $\bar{x}_2$, s_1^2, and s_2^2.

2. Compute

$$s^2 = \frac{(n_1 - 1)\, s_1^2 + (n_2 - 1)\, s_2^2}{n_1 + n_2 - 2} = \frac{\sum^{n_1} (x_1 - \bar{x}_1)^2 + \sum^{n_2} (x_2 - \bar{x}_2)^2}{n_1 + n_2 - 2}$$

3. Compute

$$t = \frac{(\bar{x}_1 - \bar{x}_2)}{s\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

[NB: do not use absolute value of $(\bar{x}_1 - \bar{x}_2)$.]

4. Compute DF = $n_1 + n_2 - 2$

5. Accept H_0 according to the same procedure as is described in Step 4 of the method of Problem 2, one-sided test.

Similarly, if the hypotheses are H_0: $\mu_1 \geq \mu_2$ and H_1: $\mu_1 < \mu_2$, follow the procedure of acceptance described in the Note to the one-sided test of Problem 2.

EXAMPLE: Twenty rounds of standard ammunition and 10 rounds of test ammunition have been fired in random sequence. The average barrel pressure of the standard was 52,400 p.s.i., while the average pressure of the test was 55,300 p.s.i. The variance of the sample of the standard was 4,230,000, while the variance of the sample of test ammunition was 3,800,000. Can one infer that the test ammunition gave excessive pressures?

ASSUMPTIONS: Normality, independence.

HYPOTHESES: H_1: $\mu_1 \geq \mu_2$ (the standard ammunition gives pressures at least as large as the test).

H_2: $\mu_1 < \mu_2$ (the test ammunition gives pressures that are definitely in excess of the standard).

METHOD: 1. $\bar{x}_1 = 52{,}400$; $\bar{x}_2 = 55{,}300$; $s_1^2 = 4{,}230{,}000$; $s_2^2 = 3{,}800{,}000$.

2.

$$s^2 = \frac{(19)(4{,}230{,}000) + (9)(3{,}800{,}000)}{28} = 4{,}091{,}786$$

3. $$t = \frac{(52{,}400 - 55{,}300)}{(2023)\sqrt{\frac{1}{19} + \frac{1}{9}}} = -3.54$$

4. DF = 20 + 10 - 2 = 28.

5. For any of the assigned probabilities of the type I error of 0.5, 0.05, or 0.01, H_0 is rejected (the test ammunition gives definitely higher pressures than the standard.)

NOTE: In this test it has been assumed that the variances of the two universes are equal. If the experimenter is virtually certain that this assumption is invalid, he should use the Behrens-Fisher method, or the Cox-Cochran, both of which are discussed in reference (3g) of the Introduction. However, it is <u>not</u> recommended that the test for equality of variances be run first (according to the method of Problem 9). Such a procedure compounds the probabilities, so that the probability of a type I error is no longer the same as previously assigned; in particular, the test of equality of variances is subject to error, and this chance of error affects the choice of test. Any precision that may be gained from the use of the Behrens-Fisher or Cox-Cochran method is probably offset by this compounding of tests. It is therefore recommended that the present test be used unless there is some sound basis for rejecting equality of variances, which basis is independent of the test-data.

$\frac{800,000)}{} = 4,091,786$

$\frac{55,300)}{\frac{1}{5}+\frac{1}{9}} = -3.54$

$0 - 2 = 28.$

he assigned probabilities
or 0.01, H_0 is rejected
tely higher pressures

assumed that the vari-
equal. If the experi-
this assumption is in-
s-Fisher method, or the
discussed in reference
er, it is not recom-
of variances be run
Problem 9). Such a
bilities, so that the
no longer the same as
ar, the test of equality of
nd this chance of error
precision that may be
rens-Fisher or Cox-
set by this compounding
mended that the present
me sound basis for re-
hich basis is independent

PROBLEM 5. Test whether two means of normal universes are equal, when the sample items from each are paired.

This test is used when one compares two measurements on the same individual. Thus the experimental design consists in taking n individuals, and first constructing a series of measurements on each one, and then (after some treatment) conducting another series. The point is to see whether any real difference has occurred as a result of the treatment. In this case, the two samples are set in a one-to-one correspondence, since an observation in the first sample is paired with an observation in the second.

The test method is really a special case of Problem 1 or Problem 2 (depending on whether the variance is known or unknown). The point of the test is to determine whether the difference between the paired items is really different from zero. The test can be one-sided or two-sided. Instead of treating all possibilities, only one will be illustrated here.

EXAMPLE: Five items are first weighed, and then subjected to a low temperature treatment; they are then weighed after the treatment. Is there any difference due to the low temperature? The results are:

	Weight before treatment (in grams)	Weight after treatment (in grams)
SAMPLE 1.	2.53	2.54
2.	2.63	2.59
3.	2.54	2.51
4.	2.58	2.55
5.	2.61	2.59

This problem is first translated into a problem concerning the differences, D, for each sample item:

	Difference between weight before and after treatment
SAMPLE 1.	- 0.01
2.	+ 0.04
3.	+ 0.03
4.	+ 0.03
5.	+ 0.02

The test now becomes one of determining whether or not the mean, μ_D, of the universe of differences is equal to zero.

ASSUMPTION: Normal distribution of D-values, independence of D-values (and a known or unknown variance of D-values).

HYPOTHESES: H_0: $\mu_D = 0$

H_1: $\mu_D \neq 0$

METHOD:

1. Compute $D = \Sigma D_i/n = + 0.022$

2. Assuming variance is unknown, compute $s_D^2 = 0.00037$

3. Compute $t = \dfrac{|\bar{D}|\sqrt{n}}{s_D} = 2.55$

4. Compute DF = number of differences minus one = 4.

5. Proceed as in Problem 1 or 2 (depending on whether variance is known or unknown) to determine whether H_0 is to be accepted. In this case (where the variance is unknown) H_0 is rejected (according to Table IV, DF = 4) for a probability of the type I error of 0.5 ($2.55 > 0.741$), but is accepted for a probability of the type I error of 0.05 ($2.55 < 2.776$) and 0.01 ($2.55 < 4.604$).

The Operating Characteristic Curve of this test is shown in Figs. I and II, where μ is the true value of the average paired differences, and σ^2 is the variance of the paired differences.

PROBLEM 6 Test whether a set of normal population means are the same, variance known.

This test is a generalization of the tests described in Problems 1-4. Instead of two means, the test is concerned with a set of k sample means from k universes. In this and the next problem, the illustrated method will assume that the number of items (n) in each sample is the same, and that the variances of all the universes are equal.

This test is actually divisible into two parts; one might be testing only to determine whether there is any real difference in the means, or one might not only test to determine whether there is difference but also whether the estimated averages are suspiciously close together. The second alternative rarely occurs as a practical problem, though for the sake of completeness, it is illustrated below.

It is to be noted that the test of differences among a set of means may not be significant, though a test of two of the means from the set by means of the method of Problem 1 or Problem 2 may be significant. This is because the present test is essentially doing a much bigger job than a single comparison.

ASSUMPTIONS: Each sample is drawn from some normal universe; the observations are all independent; the variances of each universe are known and are all equal.

HYPOTHESES: H_0: $\mu_1 = \mu_2 = \cdots = \mu_k$ (i.e., the means of the universes are all equal)

H_1: $\mu_i \neq \mu_j$ for at least one i and j (i.e., there is a difference between at least two of the means)

METHOD: 1. Compute $\bar{x}$ for each sample.

2. Compute $\sum^{k} (\bar{x} - \bar{\bar{x}})^2$, where $\bar{\bar{x}}$ is the grand average of all the observations (i.e., compute the sum of the squares of the deviations of each average

from the grand average). For convenience, use the formula $\sum_{k}^{k} (\bar{x} - \bar{\bar{x}})^2 = \sum \bar{x}^2 - \frac{(\sum \bar{x})^2}{k}$ where k = number of samples.

3. Compute DF = k - 1

4. Compute χ^2 (chi-square) = $\frac{n \sum (\bar{x} - \bar{\bar{x}})^2}{\sigma^2}$ where n is the number of items in each sample.

5. Use Table V as the basis for accepting or rejecting H_0. Specifically, if the value of χ^2 obtained in step 4 is less than or equal to the value shown opposite the appropriate Degrees of Freedom, in the appropriate column for the assigned probability of a type I error, accept H_0; otherwise reject.

EXAMPLE: Through long experience, it has been shown that the variance, σ^2, for chemical analysis for percentage content of manganese in a sample of 0.8% carbon steel, is 0.0001. Five analysts are given samples of the steel and run analyses in duplicate with the following results:

	A	B	C	D	E
	0.460	0.461	0.470	0.468	0.455
	0.470	0.465	0.456	0.454	0.471
$\bar{x}$ =	0.465	0.463	0.463	0.461	0.463

Is there any real difference due to the analysts?

ASSUMPTIONS: Normality of each sample, independence.

HYPOTHESES: H_0: $\mu_1 = \mu_2 = \cdots = \mu_k$ (no difference due to analysts)

H_1: $\mu_i \neq \mu_j$ (at least two of the analysts differ)

METHOD:

1. $\bar{x}$ - values computed as shown above.
2. $\sum_{k} (\bar{x} - \bar{\bar{x}})^2 = 0.000008$
3. DF = 4

4. $X^2 = \frac{2(0.000008)}{0.0001} = 0.16$

5. The value of X_0^2 in table V for 4 Degrees of Freedom, and a 0.5 probability of the type I error is 3.357, which is greater than 0.16, and hence H_0 will be accepted. Similary, H_0 will be accepted for an assigned 0.05 or 0.01 probability of a type I error.

NOTE: Actually, the $\bar{x}$ values are suspiciously close (perhaps due to a common understanding among the analysts). This can be determined by using both ends of Table V. If the probability of a type I error is set at 0.02, then one rejects equality of means if the X^2 of step 4 is greater than the value shown in the "0.01" column of Table V (opposite the appropriate Degrees of Freedom), and one accepts suspiciously close means if the value of X^2 in step 4 is <u>less</u> than the value shown in the "0.99" column of Table V (opposite the appropriate Degrees of Freedom). Such a test is really a test of H_0 against an H_1 made up of two alternatives (the means are not equal or else the means are too close).

PROBLEM 7. Test whether a set of normal universe means are equal, variances unknown but equal.

ASSUMPTIONS: Each sample is drawn from some normal universe; the observations are all independent; the variance of each universe is unknown, but $\sigma_1^2 = \sigma_2^2 = \cdots = \sigma_k^2$.

HYPOTHESES: H_0: $\mu_1 = \mu_2 = \cdots = \mu_k$ (i.e., the means of the universes are all equal)

H_1: $\mu_i \neq \mu_j$, for some i and j (i.e., there is a difference between at least two of the means)

METHOD: (throughout n is the number of items in each sample, k is the number of samples)

1. Compute $\bar{x}$ and s^2 for each sample.

2. Compute $\sum^{k} (\bar{x} - \bar{\bar{x}})^2 = \sum^{k} \bar{x}^2 - \frac{(\sum^{k} \bar{x})^2}{k}$

3. Compute the average variance =

$$\bar{s}^2 = \frac{\sum^{k} s_i^2}{k}$$

4. Compute $DF_2 = k - 1$ and $DF_1 = k(n-1)$

5. Compute $F = \frac{n \sum (\bar{x} - \bar{\bar{x}})^2}{(k-1)\,\bar{s}^2}$

6. Use Table III as the basis of acceptance. DF_1 refers to the values along the top of the table (indicated by ν_1) and DF_2 refers to the values along the side of the table (designated by ν_2). Be sure to use the top numbers only for DF_1, regardless of the comparative values of the denumerator and denominator in F. If the value of F in step 5 is less than or equal to the number shown in the appropriate block in Table III, under the selected probability of a type I error ("percentage point"), accept H_0; otherwise reject.

EXAMPLE: Six representative samples of four detonative mixtures were stored at 40°C and 90% RH and examined periodically until excessive oxidation had taken place. The problem is to determine whether there is a real difference in the mean storage life among the several mixtures from which the samples were chosen. Here n = 6, k = 4.

ASSUMPTIONS: Normality of each sample with respect to time of survival; independent observations; equal (but unknown) variances of the times of survival for all samples.

HYPOTHESES: H_0: $\mu_1 = \mu_2 = \mu_3 = \mu_4$ (i.e., the mean time of survival is the same for all mixtures)

H_1: $\mu_1 \neq \mu_2$ or $\mu_1 \neq \mu_3$ or $\mu_1 \neq \mu_4$ or or $\mu_2 \neq \mu_3$, etc. (i.e., at least two of the mixtures differ in stability)

METHOD: 1. The computed averages and variances were as follows:

$\bar{x}_1 = 105$ (days)	$s_1^2 = 8.3$
$\bar{x}_2 = 100$	$s_2^2 = 11.2$
$\bar{x}_3 = 107$	$s_3^2 = 12.0$
$\bar{x}_4 = 102$	$s_4^2 = 8.5$

2. $\Sigma (\bar{x} - \bar{\bar{x}})^2 = 78 - \frac{196}{4} = 29$

3. $\bar{s}^2 = 10.0$

4. $DF_1 = 4 \times 5 = 20$; $DF_2 = 3$

5. $F = \frac{(6)(29)}{(3)(10)} = 5.80$

6. Suppose the probability of a type I error is set at 0.05. The number in the block corresponding to $DF_1 = 20$ and $DF_2 = 3$ in Table III is 8.66. Since F in step 5 is less than this value, H_0 is accepted for a probability of a type I error of 0.05. On the other hand, H_0 would be rejected for a probability of the type I error of 0.10, since the appropriate value, 5.18, is less than 5.80.

NOTE: This test can also be used to determine whether the means are suspiciously close together. The procedure is the same, except that in step 5 one computes

$$F = \frac{(k - 1)\ \bar{s}^2}{n\ \Sigma\ (\bar{x} - \bar{\bar{x}})^2}$$

and in using Table III, DF_1 refers to the numbers along the side and DF_2 to the numbers at the top. One accepts H_0 if F is less than or equal to the number shown in the appropriate block. Otherwise, one accepts H_1 (the sample means are suspiciously close).

PROBLEM 8. Test whether a normal universe variance has a certain value.

This test is used to determine whether the scatter or variation in a sample is an indication of either "excessive" scatter, or else high precision.

a) TWO-SIDED TEST

Here the objective is to determine whether the variance is larger than, or smaller than the standard. Actually, there are three hypotheses: the variance is less than the standard, the variance is equal to the standard, the variance is greater than the standard. It is customary to take the second hypothesis (equality of variance with standard) as H_0, and to divide the probability of the type II error among the other two hypotheses.

ASSUMPTION: The sample is drawn from a normal universe; the observations are independent.

HYPOTHESES: H_0: $\sigma^2 = a^2$

H_1: $\sigma^2 > a^2$

H_2: $\sigma^2 < a^2$

METHOD:

1. Compute s^2.
2. Compute $DF = n - 1$
3. Compute $X^2 = \dfrac{DF\ s^2}{a^2}$
4. Use Table V as the basis of acceptance. If the probability of the type I error is 0.02, then use the "0.99" and "0.01" columns (i.e., split the probability to cover the two ends of the distribution). If the value of X^2 in step 3 is less than the value shown in the "0.99" column opposite the appropriate Degrees of Freedom, accept H_2. If the value of X^2 in step 3 is greater than the value shown in the "0.01" column opposite the appropriate degrees of freedom, accept H_1. Otherwise accept H_0. Similar procedures are to be followed for other assigned probabilities of the type I error.

EXAMPLE: A machine is to be compared with a standard for the precision with which it cuts off pieces. The variance for the standard is 0.030. A sample of ten items from the machine gives a variance of 0.058. Is there a real difference between the variances of the new machine and the standard?

ASSUMPTIONS: Normality and independence.

HYPOTHESES: H_0: $\sigma^2 = 0.030$

H_1: $\sigma^2 > 0.030$

H_2: $\sigma^2 < 0.030$

METHOD: 1. $s^2 = 0.058$

2. DF = 9

3. $X^2 = \frac{(9)(0.058)}{0.030} = 17.4$

4. If the probability of the type I error is taken to be 0.02, consult Table V for the values in the "0.01" and "0.99" columns for 9 degrees of freedom. The values are 21.666 and 2.088 respectively. Since 17.4 lies between these values, H_0 is accepted.

b) ONE-SIDED TEST; FIRST CASE

ASSUMPTIONS: Normality and independence.

HYPOTHESES: H_0: $\sigma^2 \leq a^2$

H_1: $\sigma^2 > a^2$

METHOD: 1. Compute s^2

2. Compute DF = n - 1.

3. Compute $X^2 = \frac{DFs^2}{a^2}$

4. Consult Table V, where the numbers along the top are to be taken as the probability of a type I error. If the value of X^2 in step 3 is less than or

compared with a stand-
it cuts off pieces. The
0. A sample of ten items
e of 0.058. Is there a
nces of the new machine

ndependence.

) = 17.4

lity of the type I error
V for the values in the
degrees of freedom.
respectively. Since 17.4
accepted.

FIRST CASE

ndependence.

= n - 1.

$\frac{DFs^2}{a^2}$

V, where the numbers
the probability of a type
ep 3 is less than or

equal to the value shown in Table V opposite the appropriate Degrees of Freedom and under the assigned probability of a type I error, accept H_0; otherwise accept H_1.

The Operating Characteristic Curve for this test is shown in Fig. III.

b) ONE-SIDED TEST; SECOND CASE

ASSUMPTIONS: Normality and independence.

HYPOTHESES: $H_0: \sigma^2 \geq a^2$

$H_1: \sigma^2 < a^2$

METHOD: Steps 1-3 same as above for the first case.

4. Consult Table V. Let P_0 be the probability of a type I error; then $1-P_0$ is the value to be used in Table V. If the value of X^2 in step 3 is greater than or equal to the value shown in Table V opposite the appropriate Degrees of Freedom, and under $1-P_0$, accept H_0; otherwise accept H_1.

The Operating Characteristic Curve for this test is shown in Fig. IV.

EXAMPLE: Does a value of s^2 of 58 in a sample of 11 items indicate a true variance less than 100?

ASSUMPTIONS: Independence and normality.

HYPOTHESES: $H_0: \sigma^2 \geq 100$

$H_1: \sigma^2 < 100$

METHOD: 1. $s^2 = 58$

2. DF = 10

3. $X^2 = \frac{(10)(58)}{100} = 5.80$

4. Suppose the type I error is set at 0.05. Then 0.95 is the value to be used in Table V. The value in the "0.95" column opposite 10 degrees of freedom is 3.940. Hence H_0 is accepted (one cannot infer a true variance less than 100).

PROBLEM 9. Test whether two normal universe variances are equal.

This test is to be used in comparing the dispersion of two samples, both of which are supposed to have come from normal universes. For example, one might be interested in comparing the dispersions of sets of observations made by two observers, or by the same observer on different days, or in two different laboratories.

a) ONE-SIDED TEST

ASSUMPTION: Each sample comes from a normal universe; the observations are independent.

HYPOTHESES: H_0: $\sigma_1^2 \leq \sigma_2^2$

H_1: $\sigma_1^2 > \sigma_2^2$

METHOD: 1. Compute s_1^2 on the basis of the first sample of n_1 items, and s_2^2 on the basis of the second sample of n_2 items.

2. Compute $DF_1 = n_1 - 1$ and $DF_2 = n_2 - 1$

3. Compute $F = \frac{s_1^2}{s_2^2}$

4. Use Table III as the basis of acceptance. DF_1 refers to the values along the top of the table (labeled ν_1) and DF_2 refers to the values along the side of the table (labeled ν_2). If the value of F in step 3 is less than or equal to the value shown in the appropriate block in Table III for the assigned probability of a type I error ("percentage point"), accept H_0. Otherwise reject H_0.

EXAMPLE: Observer A takes 8 readings, and the estimated variance of the resulting set, s_1^2, is 0.0026. Observer B takes 8 readings also, and the estimated variance of his set is $s_2^2 = 0.0015$. Can A be regarded as precise as B in his readings?

ASSUMPTION: Each observer's readings are independent and come from some normal universe.

HYPOTHESES: H_0: $\sigma_1^2 \leq \sigma_2^2$ (observer A is at least as precise as observer B)

H_1: $\sigma_1^2 > \sigma_2^2$ (observer A is less precise than observer B)

METHOD:

1. $s_1^2 = 0.0026$ and $s_2^2 = 0.0015$
2. $DF_1 = DF_2 = 7$
3. $F = \frac{0.0026}{0.0015} = 1.73$

4. The value in the block in Table III corresponding to $DF_1 = 7$ and $DF_2 = 7$ and a type I error chance of 0.05, is 3.7870. Since the F in step 3 is less than this value, H_0 is accepted; i.e., for a probability of a type I error = 0.05, it is inferred that observer A is at least as precise as observer B.

The Operating Characteristic Curves for this test are shown in Figs. V - VII.

b) TWO-SIDED TEST

ASSUMPTION: Each sample comes from some normal universe, with independent observations in the samples.

HYPOTHESES: H_0: $\sigma_1^2 = \sigma_2^2$

H_1: $\sigma_1^2 \neq \sigma_2^2$

METHOD:

1. Compute s_1^2 and s_2^2.
2. Compute $DF_1 = n_1 - 1$ and $DF_2 = n_2 - 1$.
3. Compute F, the ratio of s_1^2 and s_2^2, where the larger of the two is placed in the numerator.
4. Table III is used as the basis of acceptance, in a manner similar to that described for the one-sided test with the following difference: the DF of the numerator of F (larger s^2) is used to enter the top row of Table III, and the DF of the denominator

(smaller s^2) is used to enter the figures along the side. And for a probability of the type I error of 0.05, one consults the table of 2.5 percent points; for a probability of a type I error of 0.01, one consults the 0.5 percentage point table, and so on.

EXAMPLE: Suppose samples have been drawn from two lots of material, A and B, 5 from A and 9 from B. Each item is weighed. The estimated variance of the A-items is 8.9 and the estimated variance of the B-items is 4.6. Can one infer equal uniformity in the two lots with respect to weight?

ASSUMPTIONS: Samples from each lot are taken from a normal universe; weight-observations are independent.

HYPOTHESES: H_0: $\sigma_A^2 = \sigma_B^2$

H_1: $\sigma_A^2 \neq \sigma_B^2$

METHOD:

1. $s_A^2 = 8.9$ and $s_B^2 = 4.6$

2. $DF_A = 4$ and $DF_B = 8$

3. $F = \frac{8.9}{4.6} = 1.93$

4. The value in Table III for a ν_1 of 4 and a ν_2 of 8 in the table of 0.5 percentage points is 21.352. Since the F of step 3 is less than this value, H_0 is accepted for assigned probability of the type I error of 0.01 (i.e., the lots are inferred to be equally uniform). On the other hand, if the probability of a type I error is set at 0.5, one consults the table of 25 percent points, and the appropriate value, 1.6642, is less than the F obtained in step 3. Hence for this probability of a type I error, H_0 is rejected.

PROBLEM 10. Test whether a set of normal universe variances are equal.

This test is similar to the test for "homogeneity" of variances described in Problem 9, except that now there are k samples ($k > 2$). It is to be noted that the present test may fail to show a real difference between variances, while the method of Problem 9 may show a real difference between a certain pair of the variances from the total set. This is because the present test is designed to do a much bigger job than the test in Problem 9.

ASSUMPTIONS: The observations <u>within</u> each sample all come from the same normal universe and are independent.

HYPOTHESES: H_0: $\sigma_1^2 = \sigma_2^2 = \cdots = \sigma_k^2$ (variances are all equal)

H_1: $\sigma_i^2 \neq \sigma_j^2$ for at least some i and j (at least two of the variances differ).

METHOD: 1. Calculate $\sum^{n_i} (x_i - \bar{x})^2$ for each sample; i.e., calculate $\sum^{n_i} x_i^2 - \left(\sum^{n_i} x_i\right)^2 / n_i$ for each sample, where n_i is the number of items in the sample

2. Add all the results of (1). I.e., calculate $\sum^{k} [\sum^{n_i} x_i - \bar{x}]^2$.

3. Calculate $DF_i = n_i - 1$ for each sample

4. Calculate ΣDF_i

5. Calculate

$$\Sigma DF_i \ln \left\{ \left[\sum^{k} (\Sigma x_i - \bar{x})^2 \right] / \Sigma DF_i \right\}.$$

That is, divide (2) by (4) and take the natural log of the result; then multiply this result by (4).

6. Calculate s_i^2 for each sample (i.e., divide (1) by (3) for each sample).

7. Calculate $\ln s_i^2$ for each sample.

8. Calculate $(DF_i \ln s_i^2)$ for each sample.

9. Add all the results of (8); i.e., calculate $\sum^{k} (DF_i \ln s_i^2)$.

10. Subtract (9) from (5). Let M = (5) - (9). That is, in sum, calculate

$$M = \sum^{k} DF_i \ln \left\{ \sum^{k} [\sum (x_i - \bar{x})^2] / \sum DF_i \right\} - \sum^{k} (DF_i \ln s_i^2).$$

11. Consult Table XII. If M is less than the first (left-hand) entry opposite the appropriate k, then accept H_0. If M is greater than the largest entry, accept H_1. If M lies between these two entries, calculate

$$C_1 = \sum^{k} \left(\frac{1}{DF_i}\right) - \frac{1}{\sum DF_i}$$

and consult the appropriate C_1 column; if M is less than or equal to the value shown in the column, accept H_0. Otherwise accept H_1.

NOTE: If the DF_i are very widely dispersed so that some DF_i are very small (1 or 2) and some very large (12 or more), and great accuracy is needed, consult C. M. Thompson and M. Merrington, "Tables for Testing the Homogeneity of a Set of Variances," Biometrika, vol. 33 (1943-46). pp. 296-304. Table XII was taken (and adopted) from this article.

EXAMPLE: The set of 10 variances in column (2) and (3) below were obtained on the weights of items in lots of material received from a manufacturer*. Can one assume equality of the universe variances?

ASSUMPTIONS: All sample items normally distributed and independent.

HYPOTHESES: H_0: $\sigma_1^2 = \sigma_2^2 = \cdots = \sigma_{10}^2$

H_1: $\sigma_i^2 \neq \sigma_j^2$ for some i and j.

*This example is adapted from one give in the article of Thompson and Merrington cited above.

s_i^2) for each sample.
ts of (8); i.e., calcu-

(5). Let M = (5) - (9).
calculate

$$\left.\begin{matrix}\\ \\ f_i\end{matrix}\right\} - \sum^{k} (DF_i \ln s_i^2).$$

If M is less than
e appropriate k, then
largest entry, accept
ries, calculate

$$\frac{1}{DF_i}$$

n; if M is less than
olumn, accept H_0.

ispersed so that some
e very large (12 or
onsult C. M. Thomp-
sting the Homoge-
rika, vol. 33 (1943-46).
d adopted) from this

in column (2) and
s of items in lots of
r*. Can one assume

rmally distributed

σ_{10}^2
e i and j.

ve in the article of
e.

METHOD: The computations are best made in the following kind of table.

(1) Sample (No.)	(2) Number of items in sample = n_i	(3) DF_i	(4) $\Sigma(x_i - \bar{x})^2$	(5) s_i^2	(6) $\ln s_i^2$	(7) $DF_i \ln s_i^2$	(8) $1/DF_i$
1	10	9	459	51	3.93	35.4	0.111
2	15	14	1093	78	4.36	61.0	0.071
3	21	20	1818	91	4.51	90.2	0.050
4	23	22	1140	52	3.95	86.9	0.045
5	15	14	1415	101	4.62	64.7	0.071
6	11	10	361	36	3.58	35.8	0.100
7	31	30	1230	41	3.71	111.3	0.033
8	15	14	1066	76	4.33	60.6	0.071
9	3	2	129	64	4.16	8.3	0.500
10	6	5	455	93	4.53	22.6	0.200
	150	140				576.8	1.252

The steps of the METHOD can then be followed readily; as a result,

$$M = (140)(4.183) - 576.8 = 8.8$$

Suppose the probability of a type I error is set at 0.05. The value of M opposite k = 10 is consulted in Table XII. This is less than the left-hand entry; hence H_0 (equality of variances) is accepted. If M = 18.50, say, then one computes $C_1 = 1.25$. Since 18.50 is in excess of the value for $C_1 = 1.0$ and 1.50, it can be inferred (by interpolation) that M exceeds the value for $C_1 = 1.25$. Such a value of M would therefore lead to the rejection of H_0.

PROBLEM 11. To determine whether two probabilities are equal.

This test is used to compare two samples of equal size drawn from two universes, where each item is examined as to whether it does, or does not have some particular property. The property may be "survival," or "non-defectiveness," or the like. Thus the observations are not quantitative, and in general may be regarded as assignments of "success" or "failure," depending on whether or not the item has the property in question. The object is to determine whether the "successes" of one type occur in different proportion from the "successes" of the other type. The method to be described is called the method of "double dichotomy."

a) TWO-SIDED TEST

ASSUMPTIONS: The observations are independent; further, whatever the probability of a "success" may be for one type, this probability remains constant throughout the experiment (i.e., the types do not change with respect to the chance of a "success" or "failure").*

HYPOTHESES: H_0: $p_1 = p_2$ (the two probabilities are equal)

H_1: $p_1 \neq p_2$ (the two probabilities are unequal).

METHOD: 1. Construct the following "double dichotomy" table:

	Number of Successes	Number of Failures	Totals
Sample drawn from first universe	a	b	a + b
Sample drawn from second universe	c	d	c + d
Totals	a + c	b + d	a + b + c + d = 2n

*If the samples are drawn from a universe with a small number of items and are not replaced after each try, this condition will fail to hold.

In this table a represents the number of occurrences in the first sample, b the number of non-occurrences, etc. Set the table up so that $a < c$. This can always be done by reversing the order of the samples if necessary. (If $a = c$, accept H_0).

2. Compute

$$X = \sqrt{\frac{2n\,(ad - bc + n)^2}{(a + c)(b + d)(a + b)(c + d)}},$$

where n is the number of observations in each sample.

3. Accept H_0 if $X \leq 0.67$ and the type I risk is 0.5; otherwise accept H_1. Accept H_0 if $X \leq 1.96$ when the type I risk is 0.05; otherwise accept H_1. Accept H_0 if $X \leq 2.58$ if the type I risk is 0.01; otherwise accept H_1. In general use the same procedure of acceptance as in Problem 1, two-sided test, of this Part.

EXAMPLE: A test has been run to determine whether treating a weld changes the chance of breakage. Ten samples of untreated and ten samples of treated are subjected to a severe blow. Five of the untreated break, while only one of the treated breaks. Can one infer a real difference?

ASSUMPTIONS: The observations are independent and the probability of a breakage remains constant for each type.

HYPOTHESES: H_0: $p_1 = p_2$ (the treated and untreated are alike with respect to the chance of breakage)

H_1: $p_1 \neq p_2$ (the two types really differ).

METHOD: 1. Here a "success" is a nonbreakage. The table then is as follows:

	Survival (nonbreakage)	Nonsurvival	
Nontreated	5	5	10
Treated	9	1	10
	14	6	20

2.

$$X = \sqrt{\frac{20\,[(5)(1) - (9)(5) + (10)]^2}{(14)(6)(10)(10)}} = \sqrt{2.14} = 1.46$$

3. If the type I risk is 0.5, accept H_1; if it is 0.05 or 0.01, accept H_0 (no real difference is apparent on the basis of the sample).

b) ONE-SIDED TEST

The method is the same as the two-sided test, except that one uses the method of acceptance described in Problem 1, one-sided test, in this Part.

PROBLEM 12. To determine whether a set of normal universe means differ and to assign the origin of the differences.

A large number of experiments are concerned with the effect of certain variables on a given dimension or property. We are concerned with the most efficient method of design of experiments of this type. Very often, such experiments consist of varying one aspect of the environment and holding everything else constant. For example, we may vary cup thickness in primers, and manufacture the lots of primers on the same day, using the same anvils, on the same machine, etc., or again we may vary powder charge, but keep the cases, primers, bullets, loading operations, etc., as nearly the same as possible.

The disadvantage of this type of simple design is clear: if one observes a significant relationship between the variable under study and the dimension measured (e.g., primer cup thickness and sensitivity), this relationship might not hold in the same manner if the other components were changed. For example, a result obtained with one gun barrel might not be true for another; thus it would be inaccurate to base a velocity-wear curve on the results of one test alone. Again, pellet weight is known to influence primer sensivity, but the amount of such influence is dependent on anvil height; as a consequence, if one designs an experiment the only variation in which is pellet weight, then a considerable amount of useful information is lost because the results apply only for a fixed type of anvil. In sum, as R. A. Fisher puts it, it is always preferable to present nature with a questionnaire rather than a single question, wherever feasible.

This idea is no new one, but what is comparatively new is a method of designing experiments in which more than one component varies, in such a way that small sample sizes can be used and a large amount of information gained. The statistical treatment of such experiments falls under the head of "analysis of variance" (ANOVA).

A number of different methods of designing experiments have been developed, but only one of the most commonly used will be discussed here. This is called a "factorial" design. Suppose that one is interested in the possible effect of 3 variables on a certain dimension, say igniter composition, tracer composition, and type of coning on tracer functioning; or type of primer composition, vent hole size, and type of powder on velocity. For convenience, symbolize these variables as X, Y, and Z. The experiment is designed to change these variables in certain ways. Suppose, for example, that Y is varied in two ways (Y_1 and Y_2) and X and Z are varied in three ways (X_1, X_2, and X_3; Z_1, Z_2, and Z_3). For example, we may take a sharp anvil (X_1), a normal anvil (X_2), and a blunt anvil (X_3), a high crown (Y_1) and a low crown (Y_2), and a light pellet weight (Z_1), a medium weight (Z_2), and a heavy weight (Z_3). A factorially designed experiment consists in combining the changes in variables in all possible ways. Thus we will make up a lot with X_1, Y_1, and Z_1 (i.e., the lot will contain primers with sharp anvils, high cup crowns, and light pellet weights); we will also have lots made up of X_1, Y_1, and Z_2; X_1, Y_1, and Z_3; ---; X_3, Y_2, Z_3. The total number of possible combinations is the product of the number of variations, in this case, $3 \times 2 \times 3 = 18$. For each of the 18 lots, a set of determinations is made. Thus, if the number of determinations for each lot is k, then a total of 18 times k tests will be run in the complete experiment. Sometimes in the physical sciences, and more often in the biological and social sciences, only one determination can be made for each experimental type, but this restriction does not in general preclude the application of the method to be described.

It is worthwhile having some notion of the reasoning that lies behind the computations to be described.* The method of analyzing the data consists essentially of (1) determining the amount of dispersion that exists among all the data, as measured by the variance around the grand mean; (2) allocating the dispersion due to uncontrolled sources of error; (3) allocating the dispersion

*The reader is referred to References 2c) and 2d) in the Introduction for a more detailed discussion of these reasonings.

possibly due to one of the controlled variables (the X, or Y, or Z in the examples cited above). If (2) and (3) are of the same magnitude, within the limits of expected error, then the variable in question is inferred to have no influence. But if (3) is significantly larger than (2), a real difference is inferred.

A very important feature of the factorial design of experiments is that one is able to determine certain "interaction" influences. For example, in being able to determine the dispersion introduced by X and Y acting together, one is able to determine whether the amount of influence of X depends on Y (or vice versa).

ASSUMPTIONS: 1. To understand the first assumption, consider the simple experiment in which there are only two variables, X and Y. The method to be followed in analyzing the data assumes that any observation x_i can be explained in terms of the addition of four components: (1) some constant value μ, which is not changed as one changes the X and the Y, (2) a value due to the influence of X, say, τ_X, (3) a value due to the influence of Y, say τ_Y, and (4) the error due to uncontrolled factors. In other words, any x_i can be conceived in terms of the following formula:

$$x_i = \mu + \tau_X + \tau_Y + \epsilon$$

Now of course the values μ, τ_X, τ_Y, and ϵ are all unknown, but the important thing is that one assumes each observation to be a sum of these unknown components. Note that μ, τ_X, or τ_Y may be zero, without affecting the validity of the assumption.

2. Let us call the contribution to an observation that is not due to one of the controlled variables or to the constant μ, the "residual effect." Thus, in the discussion above, ϵ stands for the residual effect of x_i. The residual effects are the errors of the observations produced by uncontrolled aspects of nature. It is assumed that these errors are independent and normally distributed, and have the same variance for all observations (i.e., one assumes that all observations have the same accuracy). It is further assumed that the mean of these residual effects is zero (i.e., one expects the positive and negative residuals to balance each other in the long run).

Of these assumptions, that concerning equality of error variance appears to be the most important, in the sense that if the assumption is incorrect, the probability of the type I error will deviate significantly from that indicated by the tables. Of course, the seriousness of an incorrect assumption depends on the magnitude of the deviation of reality from what is assumed.

The reader interested in details concerning the assumptions should consult Cochran and Cox, Design of Experiments, cited in the References.

In the explanation of the method, the regular procedure of this manual will be abandoned, since the step-by-step description is rather lengthy. Instead a typical problem will be worked out. In this example, there is only one reading for each setting of the variables. In the note at the end, the procedure for testing when there are several readings for each setting is explained.

EXAMPLE: Consider the case where we study the effect of 3 variables* on a given dimension. We suppose also that on each test a certain size sample was used, and observations made.** In general, the same sample size should be used for each test. Suppose, as above, that variable X is varied from X_1 to X_2 to X_3, variable Y from Y_1 to Y_2, and variable Z from Z_1 to Z_2 to Z_3 to Z_4.

As examples, these variables might occur in the following experiments:

A X_1, X_2, X_3 = 3 different types of exposure of cartridge cases.
Y_1, Y_2, = 2 types of weapon
Z_1, Z_2, Z_3, Z_4 = 4 finishes on the cartridge case

Measurement or Determination: number of malfunctions.

* It should be noted that though the illustration considers only 3 variables, the general technique can be applied to 4, 5, or any number of variables.

** The observations may be the number of defectives, or may be actual measurements in terms of inches, pounds, etc., or averages of a set of measurements or any other variates normally distributed.

B X_1, X_2, X_3 = 3 different lots of copper pressure cylinders.
Y_1, Y_2 = 2 different heat treatments
Z_1, Z_2, Z_3, Z_4 = 4 different aging times

Measurement or Determination: pressure readings, or indent readings.

The results of the experiment are tabulated as shown below. In many cases it will be convenient to change the scale of measuremdnt so that the absolute magnitude of the numbers is reduced; this can be done conveniently by setting the smallest reading at 0, and adjusting the other readings accordingly. The resulting computations are simpler, and the final analysis of the results remains unchanged. The table also includes the totals as shown; these totals are essential for the subsequent analysis.

Table I

Table of Determinations

		X_1	X_2	X_3	Total
Y_1	Z_1	2	0	0	2
	Z_2	22	1	0	23
	Z_3	31	14	3	48
	Z_4	54	18	3	75
Total		109	33	6	148
Y_2	Z_1	0	1	0	1
	Z_2	18	24	15	57
	Z_3	26	35	27	88
	Z_4	41	46	35	122
Total		85	106	77	268
Y_1 and Y_2	Z_1	2	1	0	3
	Z_2	40	25	15	80
	Z_3	57	49	30	136
	Z_4	95	64	38	197
Total		194	139	83	416

We first compute the sum of the squared deviations of each determination from the average of all determinations,

416/24; this calculation can be simplified by using the algebraic equivalent $\Sigma X_i^2 - (\Sigma X_i)^2/n$, where X is any determination, and n = 24. Then we have

$$(2)^2 + (22)^2 + __ + (35)^2 - 1/24\ (416)^2 = 13682 - 7210.67 = 6471.33$$

This sum, 6471.33, represents the total variation, and will be used as a basis of reference in the final analysis.

We now consider the contributions of X and Y to the total variation, 6471.33. To this end, we consider only the totals of the X and Y variables over all the Z-tests; this procedure averages out the effect of Z. The following table is constructed:

Table IA

Table of X's and Y's

	X_1	X_2	X_3	Total
Y_1	109	33	6	148
Y_2	85	106	77	268
Total	194	139	83	416

The total variation contributed by X and Y will be, as before, the sum of the squared deviations of each item in this table from the average:

$$1/4[(109)^2 + (33)^2 + ____ + (77)^2] - 1/24\ (416)^2 = 9349.00 - 7210.67 = 2138.33$$

(Sum of Sqs. of Deviations for X's and Y's)

The reason that we take one-fourth of the sum of the squares is that each value such as 109, is the total of four items (the four Z determinations for X).

The contribution of X alone is

$$1/8[(194)^2 + (139)^2 + (83)^2] - 1/24\ (416)^2 = 7980.75 - 7210.67 = 770.08,$$

(Sum of Sqs. of Deviations for X's)

and the contribution of Y alone is

$$1/12[(148)^2 + (268)^2] - 1/24\ (416)^2 = 7810.67 - 7210.67 = 600.0$$

(Sum of Sqs. of Deviations for Y's)

The contribution of the interaction of X and Y (i.e., variations caused by Y's having an influence on the

difference between X_1, X_2, and X_3, or vice versa) will be the remainder of the total X and Y variation after the individual X and Y contributions have been subtracted:

Interaction term between X's and Y's

2138.33 - (770.08 + 600.0) = 768.25

A similar table is made for X and Z (by totaling the Y determinations):

Table IB

Table of X's and Z's

	X_1	X_2	X_3	Total
Z_1	2	1	0	3
Z_2	40	25	15	80
Z_3	57	49	30	136
Z_4	95	64	38	197
Total	194	139	83	416

The total variation in this table is

Sums of Sqs. of Deviations for X's and Z's

$1/2[(2)^2 + (1)^2 + ___ + (38)^2] - 1/24\ (416)^2 = 11{,}785 - 7210.67 = 4574.33$

The X-contribution is the same as before = 770.08.

The Z-contribution is

Sums of Sqs. of Deviations for Z's

$1/6[(3)^2 + (80)^2 + (136)^2 + (197)^2] - 1/24\ (416)^2 = 10{,}619.0 - 7210.67 = 3408.33$

The Z and X interaction contribution is then

Interaction term for X's and Z's

4574.33 - (770.08 + 3408.33) = 395.92

The Y and Z table is given by

Table IC

Table of Y's and Z's

	Y_1	Y_2	Total
Z_1	2	1	3
Z_2	23	57	80
Z_3	48	88	136
Z_4	75	122	197
Total	148	268	416

The total Y and Z contribution is

Sum of Sqs. of Deviations for Y's and Z's

$$1/3\,[(2)^2 + (1)^2 + ____ + (122)^2] - 1/24\,(416)^2 = 11{,}446.67 - 7210.67 = 4236.0$$

The Y-contribution, as before, is 600.0, and the Z-contribution is 3408.33; the interaction of Y and Z is then

Interaction term between Y's and Z's

$$4236.0 - (600.0 + 3408.33) = 227.67$$

The contribution of the total interaction of X, Y, and Z acting together will be the remainder of the total interaction after the contributions of X, Y, and Z alone, and their interactions, have been subtracted:

Interaction term between X's, Y's and Z's

$$6471.33 - (770.08 + 600.0 + 3408.33 + 768.25 + 395.92 + 227.67) = 301.08$$

We now construct the following table on the basis of these calculations.

Table ID

Contribution	Sums of Sqs. of Deviations	Degrees of Freedom	Mean Sq. Deviation	F	Significance Test
X's	770.08	3-1 = 2	385.04	7.67	Significant (0.05)
Y's	600.0	2-1 = 1	600.0	11.96	Significant (0.05)
Z's	3408.33	4-1 = 3	1136.11	22.64	Significant (0.01)
X's and Y's	768.25	2x1 = 2	384.13	7.66	Significant (0.05)
X's and Z's	395.92	2x3 = 6	65.99	1.32	Not significant
Y's and Z's	227.67	1x3 = 3	75.89	1.51	Not significant
X's, Y's and Z's	301.08	2x3x1 = 6	50.18	1.00	
Total	6471.33	3x2x4-1 = 23			

The "degrees of freedom" referred to in this table are obtained by subtracting 1 from the total number of ways we have varied the variable under consideration.

It can be shown that the best estimate of the true variance of a set of readings is the sum of the squared deviations divided by the degrees of freedom. It can also be shown that the degrees of freedom for an interaction term of a set of variates is the product of the separate degrees of freedom of the variates.

The fourth column of Table ID, headed "Mean Square Deviation," is obtained by dividing the sum of the squares of deviations by the corresponding degrees of freedom. Thus the mean square deviation for the Y's in 770.08/2 = 385.04. Each mean square is regarded as an "estimate" of the true variation in readings based on the data at hand. Some of these estimates are too large, and the task is to find which ones deviate from the "expected" or best estimate of the true variation.

The final mean square, 50.18, is taken to be the standard of judgment; i.e., 50.18 is the best estimate of variations in readings caused by factors other than the three under consideration; 50.18 is thus the basis by which we judge whether the other mean squares are too large. If the other mean squares are too large, then the variable under consideration does play a significant role in its effect on the final observations.

We compute F by putting 50.18 in the denominator, and the appropriate Mean Square Deviation in the numerator. In this test, the Mean Square Deviation of X, Y, Z is always put in the denominator of F. If this value is greater than the Mean Square Deviation associated with some other component in Table ID, one infers no significance due to the component. One must not put the Mean Square Deviation in the numerator, to test for the significance of the X, Y, Z term relative to another term, for then the probabilities of a type I error are no longer the same as those given in Table III.

As an example, we test the hypotheses (H) that the X-variable has no significant effect on the result. The appropriate F is obtained by dividing the Mean Square Deviation for X (385.04) by the Mean Square Deviation of X, Y, Z (50.18). The result is 7.67. If the probability of a type I error is set at 0.05, one consults the table of five percent points in Table III. Here ν_1 is the Degrees of Freedom of X, and ν_2 is the Degrees of Freedom of X, Y, Z. Hence, in this case, $\nu_1 = 2$ and $\nu_2 = 6$. The appropriate value in Table III is 5.1433, which is less than 7.67. Hence, one rejects the hypothesis that X had no effect; that is, one infers that the variable X did affect the result.

The F-values in Table ID show that all the variables singly had an influence on the determinations. What is interesting in addition is the significance of the X and Y interaction term. This shows that the amount of the influence of X depends on Y. This can be seen by a reference to the original Table I; the difference between X_1 and X_2 is large for Y_1, but small for Y_2. Such a result as this may be highly important, for it may show the engineer the correct value to choose for the Y-value in order to obtain better uniformity if the X-value cannot be controlled beyond certain limits. Such a result is difficult to assert with any degree of precision without an analysis of variance of the sort described here.

NOTE: In some case, the items shown in Table I will be averages of samples of n items, and not merely one reading. In this case, we can obtain additional information, provided the n observations of any sample are random and independent. This additional information

he denominator, and
ion in the numerator.
tion of X, Y, Z is
If this value is
tion associated with
ne infers no signifi-
ust not put the
ator, to test for the
lative to another
type I error are
n Table III.

otheses (H) that the
on the result. The
g the Mean Square
n Square Deviation
7.67. If the proba-
05, one consults the
e III. Here ν_1 is the
is the Degrees of
is case, $\nu_1 = 2$ and
Table III is 5.1433,
e rejects the hypothe-
ne infers that the

that all the varia-
determinations. What
nificance of the X and
at the amount of the
can be seen by a
he difference between
ll for Y_2. Such a
rtant, for it may show
choose for the Y-value
if the X-value can-
mits. Such a result
ec of precision with-
sort described here.

hown in Table I will
, and not merely one
in additional informa-
f any sample are ran-
nal information

consists in testing the significance of the last contribution shown in Table ID, the combined contribution of X and Y and Z. The analysis above takes this contribution to be an unbiased estimate of error, but a better estimate of the error can be obtained if the readings in Table I are averages. For each sample of n items, compute the variances s^2. Average all the variances, calling the result $\bar{s}^2$. The degrees of freedom of $\bar{s}^2$ are k (n-1), where k is the total number of items in Table I. Then the best estimate of the error of the items in Table I is $\bar{s}^2/n$. This error may now be used to test the significance of any of the contributions in Table ID. If the last contribution (X and Y and Z) is significant, this may usually (but not always) be taken as a sign of lack of proper control in the experiment. Its significance means that the manner in which Y affects variations due to X, depends on Z. In general, if this term is significant, the experiment should be repeated.

PROBLEM 13. To determine whether a set of normal means and/or the corresponding variances are equal.

This test is used when the experimenter merely wishes to compare two or more normal universes without specifically basing the comparison on a difference of means or a difference of variances. Thus the present method is designed to detect any of the possible reasons for differences in universes.

Some remarks are necessary with respect to the three tests 1) of difference in means (F-test), 2) of difference in variances (M-test), and 3) of difference in both means and variances (L_0-test). It may happen that the F-test of means is significant for a certain chance of a type I error, and yet the L_0-test run on the same data is not significant for the same chance of a type I error. The reason for this is that the L_0 test is doing a much bigger job than the F-test, since L_0 hunts for any kind of difference, while the F-test assumes equality of variances. Similarly, the M-test may detect a difference where L_0 does not. The natural question is: why not run the F- and M-tests instead of L_0? The answer is that if one is interested in any kind of difference, then the combined use of the F- and M-tests will be deceptive. For suppose one sets the chance of a type I error at 0.05 in order to test for any difference. This means that one time in twenty one is willing to say a real difference of some sort exists when as a matter of fact no difference exists. Now if one uses a combined F- and M-test with a 0.05 type I error, his method will be incorrect, for the F-test will reject a true equality of means one time in twenty, and the M-test will reject a true equality of variances one time in twenty. The combined test will therefore not reject the true hypothesis of no difference one time in twenty, but far more often. In other words, the combined F- and M-test, each run at a 0.05 level, does not provide a test of no difference at the 0.05 level, but at a much smaller level. One could, of course, make the proper adjustments for the use of the combined test, but this would be more complicated than the simple use of the L_0-test.

ether a set of normal
variances are equal.

experimenter merely
normal universes with-
arison on a difference of
nces. Thus the present
y of the possible reasons

ry with respect to the
means (F-test), 2) of
), and 3) of difference in
est). It may happen that
nt for a certain chance
L_0-test run on the same
ame chance of a type I
that the L_0 test is doing
est, since L_0 hunts for
e F-test assumes equality
-test may detect a differ-
natural question is: why
tead of L_0? The answer
ny kind of difference,
- and M-tests will be
ts the chance of a type I
for any difference. This
one is willing to say a
xists when as a matter of
if one uses a combined
I error, his method will
ll reject a true equality
nd the M-test will reject
e time in twenty. The
t reject the true hypothe-
n twenty, but far more
nbined F- and M-test,
not provide a test of no
t at a much smaller level.
proper adjustments for
ut this would be more
e of the L_0-test.

NOTE: In this test the sample sizes should be equal or nearly so. Let n be the number of items in each sample.

ASSUMPTIONS: All universes to be compared are normal, and the observations are independent.

HYPOTHESES: H_0: $\mu_1 = \mu_2 = \cdots = \mu_k$

and $\sigma_1^2 = \sigma_2^2 = \cdots = \sigma_k^2$

(i.e., all means and variances are equal).

H_1: For some i and j, either $\mu_i \neq \mu_j$ or $\sigma_i^2 \neq \sigma_j^2$. (i.e., not all means and/or variances are equal)

METHOD: 1. Compute all averages, $\bar{x}_1$, $\bar{x}_2$, etc., and all estimated variances, s_1^2, s_2^2, etc.

2. Compute the grand variance, s_0^2, i.e., the sum of the deviations of all observations from the grand mean; the following formula will prove useful:

$$s_0^2 = \frac{\sum^{nk}(x - \bar{\bar{x}})^2}{nk - 1} = \frac{\sum^{k} s_i^2}{k} + \frac{\sum^{k}(\bar{x}_j - \bar{\bar{x}})^2}{k} = \bar{s}^2 + s_{\bar{x}}^2$$

3. Look up in log tables the logs of all estimated variances, s_1^2, s_2^2, etc., and the log of s_0^2.

4. Let k be the number of samples. Compute the average of the logs of the s_i^2, i.e., compute

$$\frac{\Sigma \log s_i^2}{k}$$

5. Subtract the log of s_0^2 from the result in 4. This is the log of L_0. That is

$$\log L_0 = \frac{\Sigma \log s_i^2}{k} - \log s_0^2$$

Determine L_0.

6. Consult Table VI. If the L_0 value is greater than or equal to the appropriate value shown in the column headed "Five percent value" and the chance of a type I error is set at 0.05, accept H_0; otherwise accept H_1. Similarly, if L_0 is greater than or equal to the

appropriate value shown in the column headed "One percent value," and the chance of a type I error is set at 0.01, accept H_0; otherwise accept H_1.

EXAMPLE: A lot of ammunition is stored at room temperature for the purpose of determining whether there is a change in bullet pull over a period of time. Here the various universes consist of possible bullet pull observations at specific times. The problem is to determine whether the means or the variances or both have changed over the period.

ASSUMPTIONS: The bullet pull data are normally distributed for any time, and are independent.

HYPOTHESES: H_0: All the means are equal, and all the variances are equal.

H_1: Not all the means and/or variances are equal.

METHOD: 1. The averages and variances of samples of 20 were as follows:

EXPOSURE PERIOD	$\bar{x}$	s^2
controls	206.30	580.81
24 hrs.	193.90	839.26
1 wk.	204.30	577.44
2 wks.	203.35	569.78
3 wks.	178.90	824.26
4 wks.	182.65	1171.69
8 wks.	199.25	578.88
12 wks.	194.70	717.70
16 wks.	178.25	670.29
20 wks.	165.40	436.39
24 wks.	172.00	876.16

2. $s_0^2 = 712.97 + 182.25 = 895.22$

3. and 4.

$$\frac{\Sigma \log s_i^2}{k} = \frac{2.76403 + 2.92392 + \text{----}}{11} = 2.83795$$

PROBLEM 14. To test for a type of functional relationship between two variables, where the determinations of one are not subject to error.

In much of experimental work it is desirable to establish some sort of relationship between two variable aspects of nature. In this problem, some of the simpler methods are explained. In the first case, it is assumed that the experimenter can fix values of one of the variables (the so-called "independent" variable) and that the fixed values are known with comparatively little or no error. This assumption is valid in much of laboratory testing where the scientist can set the environmental conditions at specific and known values, and then observe the results. If settings can be made, then it is comparatively easy to use equally spaced intervals. The test is designed to determine whether the relationship between the independent and dependent variables is non-existent, linear, quadratic, etc. In the second example, the independent variable is not subject to error but does not occur at equally spaced intervals. In this example, only a test of linearity is explained, since the tests for higher-order relationships are very complicated. The reader is referred to Part II for a method of curve-fitting to be applied when both variables are subject to error.

a) EQUALLY SPACED INTERVALS

ASSUMPTIONS: The x-values (independent variable) are not subject to error (or virtually are error-less), and the errors of the y-values are independent and normally distributed, with a zero mean (i.e., the y-errors balance in the positive and negative directions) and equal variance (the y-errors are of the same magnitude).

HYPOTHESES: In this method there are actually an indefinite number of separate tests.* Suppose the relationship between x and y is expressed as follows:

*It is to be noted that the method below tells one how to test any one of these sets of hypotheses. If more than one set is tested, the probability of a (Continued on next page)

$$y = A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \cdots$$

where the coefficients A_0, A_1, etc., are those defined in the method below (they are called "orthogonal coefficients"). The first set of hypotheses are

$$H_{01}: \quad A_1 = 0$$

$$H_1: \quad A_1 \neq 0$$

The second set of hypotheses are

$$H_{02}: \quad A_2 = 0$$

$$H_2: \quad A_2 \neq 0$$

In general,

$$H_{0k}: \quad A_k = 0$$

$$H_k: \quad A_k \neq 0$$

METHOD: (to be illustrated by a specific example)

1. Suppose the x and y values are those given in the table below and that the interval between each x is one unit. Opposite each x and y value write the appropriate values shown in Table VII. Since there are 17 pairs of observations, we use the table for n = 17. We start with the central value where x = 12, and reproduce opposite to this value the first values shown under ξ_1', ξ_2' and ξ_3' in the table. (The values ξ_1', ξ_2' and ξ_3' are called values of "orthogonal polynominals.") We proceed to write the next values of ξ_1', ξ_2' and ξ_3' opposite 13, and the next opposite 14, etc. We then fill in the values below 12 by using the reverse order with an opposite sign for odd-numbered ξ -values, and with the same sign for even

type I error must be computed as follows: Let 0.05 be the probability of a type I error in each sub-test. Then if there are k sub-tests, the type I error consists of saying that at least some $A_i \neq 0$ when actually every $A_i = 0$ $(i > 0)$. The probability of such an error is $1-(.95)^k$. Thus is there are four tests, the probability of an overall type I error would be .186. It is also to be noted that A_3, for example, may be zero, and yet a quadratic term will occur in the final equation if A_3 is not zero.

numbered ξ-values. (The figures shown at the botton of the tables have the following meaning: $\Sigma [\xi_1'(x_i)]^2 = 408$; $\Sigma [\xi_2'(x_i)]^2 = 7752$; $\Sigma [\xi_3'(x_i)]^2 = 3876$; these will be used in the test procedure). The data then appear as follows:

x_i	y_i	ξ_1'	ξ_2'	ξ_3'
4	85	-8	40	-28
5	114	-7	25	-7
6	134	-6	12	7
7	152	-5	1	15
8	176	-4	-8	18
9	202	-3	-15	17
10	224	-2	-20	13
11	243	-1	-23	7
12	260	0	-24	0
13	293	1	-23	-7
14	300	2	-20	-13
15	318	3	-15	-17
16	341	4	-8	-18
17	356	5	1	-15
18	384	6	12	-7
19	409	7	25	7
20	431	8	40	28

2. Compute the coefficients of the functional relationship according to the following formulae:

$$A_0 = \bar{y} = \frac{4422}{17} = 260.12$$

$$A_1 = \frac{\Sigma[y_i\, \xi_1'(x_i)]}{\Sigma[\xi_1'(x_i)]^2} = \frac{8563}{408} = 20.99$$

$$A_2 = \frac{\Sigma[y_i\, \xi_2'(x_i)]}{\Sigma[\xi_2'(x_i)]^2} = \frac{-545}{7752} = -0.07$$

$$A_3 = \frac{\Sigma[y_i\, \xi_3'(x_i)]}{\Sigma[\xi_3'(x_i)]^2} = \frac{663}{3876} = 0.17$$

$$A_k = \frac{\Sigma[y_i\, \xi_k'(x_i)]}{\Sigma[\xi_k'(x_i)]^2}$$

3. Set up a "significance test" table as follows:

	Sum of Squares	Degrees of Freedom (DF)	Mean Square = Sum of Squares/DF
(1) Σy^2	1,330,374	17	
(2) Correction for mean = $\frac{(\Sigma y)^2}{n}$	1,150,240	1	
(3) Deviation from mean = (1) - (2)	180,134	16	
(4) Linear regression $= \frac{\{\Sigma[y_1 \xi_1'(x_1)]\}^2}{\Sigma\{\xi_1'x_1]^2}$ $= \frac{73,324,969}{408} =$	179,718	1	179,718
(5) Deviation from Linear Regression = (3) - (4) =	416	15	27.7
(6) 2nd Degree regression = $\frac{\{\Sigma[y_1 \xi_2'(x_1)]\}^2}{\Sigma\{\xi_2'(x_1)]^2} = \frac{297,025}{7752} =$	38	1	38
(7) Deviation from 2nd Degree Regression = (5) - (6)	378	14	27
(8) 3rd Degree regression $\frac{[\Sigma y_1 \xi_3'(x_1)]\}^2}{\Sigma\{\xi_3'(x_1)]^2} = \frac{439,569}{3876} =$	113	1	113
(9) Deviation from 3rd Degree regression = (7) - (8) =	265	13	20.4

4. To test H_1 against H_{01}, consider the ratio, F, of the mean square in (4) to the mean square in (5), 179,718/27.7 = 650. Enter Table III with $\nu_1 = 1$ and $\nu_2 = 15$. Accept H_1 if the ratio exceeds the number in the appropriate block in Table III for a given probability of a type I error ("percentage point"); otherwise reject (i.e. accept H_{01}). In this case, H_1 (linearity) is accepted for both 0.05 and 0.01 probabilities of the type I error.

5. Similarly, to test H_2 against H_{02}, use the ratio of the mean square of (6) to the mean square of (7) = 38/27, with $\nu_1 = 1$ and $\nu_2 = 14$. In this case the ratio is less than the number shown in the appropriate block for a type I error probability of 0.05 and hence H_{02} is accepted at this level.

6. To test H_3, use the ratio of the mean square of (8) to the mean square of (9) with $\nu_1 = 1$ and $\nu_2 = 13$. Here H_3 is accepted at 0.05 probability of a type I error, rejected at 0.01 probability.

NOTE: If it is desirable to express the functional relationship of y to x, the following procedure should be followed. One computes

$$y = \bar{y} + A_1 \xi_1'(x) + A_2 \xi_2'(x) + A_3 \xi_3'(x) + \text{---}$$

Where

$$\xi_1'(x) = \lambda_1 (x - \bar{x})$$

$$\xi_2'(x) = \lambda_2 (x - \bar{x})^2 - \frac{n^2 - 1}{12}$$

$$\xi_3'(x) = \lambda_3 \left[(x - \bar{x})^3 - (x - \bar{x}) \left(\frac{3n^2 - 7}{20} \right) \right]$$

$$\xi_4'(x) = \lambda_4 \left[(x - \bar{x})^4 - (x - \bar{x})^2 \left(\frac{3n^2 - 13}{14} \right) + \frac{3(n^2 - 1)(n^2 - 9)}{560} \right]$$

where the λ_i are constants found from the last row of the table, in Table VII. In the particular example given above, for a third degree relationship we have

$$y = 260.12 + 20.99 \lambda_1 (\bar{x} - \bar{x}) + .17 \lambda_3 \left[(x - \bar{x})^3 - (x - \bar{x}) \frac{(3n^2 - 7)}{20} \right]$$

where $\lambda_1 = 1$, $\lambda_3 = 1/6$, and where the second degree term is taken to be zero. Simplifying,

$$y = 0.028\ x^3 - 1.019\ x^2 + 31.999\ x - 26.060.$$

It is to be noted that the inclusion or exclusion of an A_i in the computations is subject to error. In particular, if A_i is zero in reality, and one uses the method described above for testing $A_i = 0$ against $A_i \neq 0$ with a probability of a type I error of 0.05, then five percent of the time one will include A_i in the calculations when actually A_i should not be included. This probability becomes compounded when more than one A_i is used (see footnote to the HYPOTHESES for more details on this remark).

b) UNEQUAL INTERVALS

ASSUMPTIONS: The x-values are not subject to error, and the errors of the y-values are independent, normally distributed with zero mean and the same variance.

HYPOTHESES: H_0: The relationship between x and y is not linear.

H_1: The relationship is linear (i.e., the slope of the x, y line is greater than zero).

METHOD: 1. Compute b, the slope of the line (see p. 110, in Part II).

2. Compute s_b^2, the variance of the estimate of the slope (see p. 110 in Part II).

3. Compute

$$t = \frac{b}{s_b}$$

4. Consult Table IV, with DF = n - 2, n being the number of pairs of observations. If the value of t computed in step 3 is greater than the value shown in Table IV opposite the appropriate DF, under the assigned probability of a type I error, accept H_1; otherwise accept H_0.

EXAMPLE: Suppose one wishes to test whether a slope b equal to 0.38 is significantly different from zero, when $s_b = 0.0553$. Here t = 0.38/0.0553 and since t is therefore greater than 6, one would accept H_1 (a real positive slope) for a probability of a type I error of 0.05, 0.01 or 0.001.

PROBLEM 15. To test whether a set of data come from a normal universe.

The emphasis on the assumption of normality of the observations will naturally raise in the mind of the experimenter the question whether all sets of data should not be checked for this assumption. For this reason, the following points should be considered. (1) There is no adequate test of normality for small sample sizes; that is, the confidence in normality in such cases must come from past experience or theoretical bases. (2) For many of the tests considered in this manual, the behavior of the data around the mean is the crucial point. Very often, an adequate notion of normality or lack of normality can be obtained by plotting the observations on normal probability paper in order to detect visually any real deviations from a straight line (deviations at either end of the line are not to be considered as seriously as deviations in the center). Further, for most of the tests of this Part, there may occur large deviations from normality without any serious effect on the validity of the method. (3) The test of "goodness of fit" which is discussed in many statistics tests and which makes use of X^2, suffers from several serious defects, and is not recommended in general. (4) The tests to be described here assume a large sample size. Further, it is recommended that some caution be exercised in following the procedure of first testing for normality and then selecting the test for analysis of the data. Such a procedure is apt to bias the probability of the type I or type II errors. In general, the use of tests of normality should be restricted either to cases where some information about possible sources of erratic behavior of the observations may be useful, or to a check before testing for the proportion of the universe exceeding (or less than) some value ξ (where the assumption of normality is important; see Problem 18).

The two tests described here are based on two properties of a normal distribution. We think of a normal distribution curve as a description of the scatter of "mass" along some line; as has been pointed out in the Introduction the mean of such a distribution is the first moment, i.e., the "center of gravity." The variance is

the second moment. Our interest here is concentrated on the third and fourth moments. Since the size of the moments depends on the scale, we first divide the third moment by the cube of the standard deviation, which gives us an index of "skewness." If the distribution is normal, the skewness should be zero. If the skewness is really positive, then the distribution curve will tend to show a "lag" to the right, while if it is negative, the curve will lag to the left. Similarly, the fourth moment is made an index of "kurtosis" by dividing it by the fourth power of the standard deviation. If the distribution is normal, the kurtosis should be three; an excess of three indicates a "flat-top" distribution curve, and a value less than three indicates a "peaked" curve. In the first test, we assume that the only possible grounds for suspecting non-normality is a positive or negative skewness; in the second test, the only suspect is the kurtosis. If both values are suspected, and a probability of a type I error of 0.05 is assigned for each test, the probability of the type I error for the combined test must be taken as about 0.10.*

a) TEST OF SKEWNESS

ASSUMPTIONS: The observations are independent, and all come from one universe which has the properties of normality except perhaps for skewness.

HYPOTHESES: H_0: The universe is normal (i.e., the skewness is zero).

*The reasoning is as follows. Suppose the universe is normal; then the probability of rejecting this correct hypothesis is 0.05 using the test of skewness, and hence the probability of accepting the correct hypothesis is 0.95. Similarly if the test of kurtosis is run, the probability of acceptance is 0.95. Now the two tests happen to be independent. Hence, the probability of acceptance using the combined tests is 0.95 times 0.95 or 0.9025, and hence the probability of rejection (type I error) is one minus 0.9025 or about 0.10. If the type error probabilities are 0.01, then the probability for the combined test is one minus (0.99) times (0.99), or 0.02.

H_1: The universe is non-normal (the skewness is not zero).

METHOD: 1. Compute $\sqrt{b_1}$ the estimate of skewness:

$$\sqrt{b_1} = \frac{\sum (x - \bar{x})^3/n}{[\sum (x - \bar{x})^2/n]^{3/2}} \quad \text{(take } \sqrt{b_1} \text{ positive)}$$

(See work sheet below.)

2. Enter Table IX, and accept H_0 if $\sqrt{b_1}$ is less than or equal to the value shown under the chosen probability of a type I error and opposite the appropriate sample size; otherwise reject.

b) TEST OF KURTOSIS

ASSUMPTIONS: The observations are independent and all come from one universe which has the properties of normality except possibly for kurtosis.

HYPOTHESES: H_0: The universe is normal (the true value of kurtosis is three).

H_1: The universe is not normal (the kurtosis is not equal to three).

METHOD: 1. Compute b_2, the estimate of kurtosis:

$$b_2 = \frac{\sum (x - \bar{x})^4/n}{[\sum (x - \bar{x})^2/n]^2}$$

(See work sheet below.)

2. Enter Table IX, and accept H_0 if b_2 lies between the values shown for the assigned probability of the type I error, opposite the appropriate sample size.

EXAMPLE: The method of computation for either test is shown below for a set of data that have been grouped in convenient intervals. Suppose x represents the weights of a group of men.

We construct the following table to simplify the necessary computations,

where $x = a + c\,z$

a = an arbitrary constant

c = class interval

If we sum the above expression over all observations and divide by n, we get

$$\bar{x} = a + c\,D$$

where $D = f\,z/n$

f = number of men having weights in a given class interval. Substituting these relations into the equation for m_k, the kth moment,

$$m_k = \frac{\sum^{n}(x - \bar{x})^k}{n} ,$$

we obtain the equations for the moments shown at the bottom of the table. The last column, suggested by Charlier, is for check purposes for

$$\Sigma f(z + 1)^4 = \Sigma fz^4 + 4\Sigma fz^3 + 6\Sigma fz^2 + 4\Sigma fz + n$$

Weights of Group of Men

x	f	z	fz	fz^2	fz^3	fz^4	$f(z + 1)^4$
105	1	-6	-6	36	-216	1296	625
115	5	-5	-25	125	-625	3125	1280
125	14	-4	-56	224	-896	3584	1134
135	33	-3	-99	297	-891	2673	528
145	61	-2	-122	244	-488	976	61
155	87	-1	-87	87	-87	87	0
165	99	0	-	-	-	-	99
175	88	1	88	88	88	88	1408
185	60	2	120	240	480	960	4860
195	33	3	99	297	891	2673	8448
205	14	4	56	224	896	3584	8750
215	4	5	20	100	500	2500	5184
225	1	6	6	36	216	1296	2401
Σ	500	-	-6	1998	-132	22842	34,778

$a = 165$ $\quad D = \frac{fz}{n}$ $\quad f(z+1)^4 = 34{,}778$

$\bar{x} = a + cD$ $\quad D = -.01$ $\quad D^2 = 0.0001$

$\bar{x} = 164.9$ $\quad c = 10$ $\quad D^3 = -0.000001$

$D^4 = 0.00000001$

$$m_2 = \frac{c^2(\Sigma fz^2 - nD^2)}{n} \qquad m_3 = \frac{c^3(\Sigma fz^3 - 3D\Sigma fz^2 + 2nD^3)}{n}$$

$$= \frac{100(1998 - .05)}{500} \qquad = \frac{1000(-132 + 59.94 - .001)}{500}$$

$$= \frac{199795}{500} \qquad = \frac{-72061}{500}$$

$$= 399.59 \qquad = -144.12$$

$$m_4 = \frac{c^4(\Sigma fz^4 - 4D\Sigma fz^3 + 6D^2\Sigma fz^2 - 3nD^4)}{n}$$

$$= \frac{10000\,(22{,}842 - 5.28 + 11988 - .000015)}{500}$$

$$= \frac{228379187.85}{500} = 456{,}758$$

Charlier Check

$$34{,}778 = 22{,}842 - 528 + 11988 - 24 + 500$$
$$34{,}778 = 34{,}778$$

From the moments we get

$$\sqrt{b_1} = \frac{m_3}{(m_2)^{3/2}} = \frac{-144.12}{(399.59)^{3/2}} = -0.018$$

$$b_2 = \frac{m_4}{(m_2)^2} = \frac{456{,}758}{(399.59)^2} = 2.86$$

If the tests were used separately, then H_0 would be accepted in both cases according to Table IX, both for a probability of a type I error of 0.05 and 0.01.

PROBLEM 16. Determine whether to discard "excessively high" or "excessively low" observations.

The problem of discarding observations is, at the present stage of science, one of the most difficult. In particular, the practice of discarding observations that "seem" erratic is apt to introduce a definite bias in the test procedures, especially if the sample is small.

Nevertheless, there will be cases where there is some good reason to suspect one or more of the observations as being excessively high or low, and the test suggested here can be used to assist the experimenter in coming to a reasonable decision.

ASSUMPTIONS: Observations are independent and come from some normal universe, except possibly the highest.

HYPOTHESES: H_0: The largest observation comes from the same universe as the rest of the sample.

H_1: The largest observation does not come from the same universe as the rest.

METHOD: 1. Compute

$$\sum_{1}^{n-1} (x_i - \bar{x}_n)^2 = \sum_{1}^{n-1} x_i^2 - \left(\sum_{1}^{n-1} x_i\right)^2/n - 1$$

where $\bar{x}_n$ is the mean of all the observations except the highest.

2. Compute

$$\sum_{1}^{n} (x_i - \bar{x})^2 = \sum_{1}^{n} x_i^2 - \left(\sum_{1}^{n} x_i\right)^2/n$$

where $\bar{x}$ is the mean of all the observations.

3. Compute

$$\frac{s_n^2}{s^2} = \frac{\sum_{1}^{n-1} (x_i - \bar{x}_n)^2}{\sum (x_i - \bar{x})^2}$$

4. Consult Table XIII. The columns refer to the type I error (thus the "1%" column refers to the probability of 0.01 for the type one error). If the value of $\frac{s^2_m}{s^2}$ in step 3 is greater than the value shown in the column opposite the appropriate n, then accept H_0; otherwise accept H_1.

NOTE: Exactly the same procedure is to be followed if the suspected observation is the lowest, except that $\bar{x}_n$ is the mean of all but the lowest observation.

EXAMPLE: The following set of data was obtained from a chemical analysis (scale is one of percentage of a certain element):

6.9	6.9
7.2	7.3
7.6	7.2
7.0	7.4
7.5	

Can one infer that the observation 7.6 is excessively high?

ASSUMPTIONS: The chemical determinations are normally distributed, and come from the same universe except possibly the highest. The observations are independent.

HYPOTHESES: H_0: 7.6 is not an excessively high observation; i.e., this observation did come from the same universe as the rest of the observations.

H_1: 7.6 is excessively high (does not come from the same universe as the rest).

METHOD:

1. $\sum^{8} (x_i - \bar{x}_n)^2 = 0.355$

2. $\sum^{9} (x_i - \bar{x})^2 = 0.516$

3. $\frac{s_n^2}{s^2} = \frac{0.355}{0.520} = 0.688$

4. If the probability of a type I error is 0.01, 0.025, 0.05, or 0.10, accept H_0, since 0.688 is greater than any value shown opposite n = 9 in table XIII.

PROBLEM 17. To determine whether there is a correlation between consecutive observations.

This test is used to determine whether the observations of a sample are independent. The test is actually used to test only one type of non-independence, namely, the case where one observation has a possible effect on the one just succeeding it. Thus, observers often tend to make a reading high if the preceeding reading has been low; in such a case there will be a tendency towards a negative correlation between consecutive observations. Or, a testing apparatus may begin to show a trend towards higher readings as it goes out of adjustment, in which case there will be a positive correlation between consecutive readings. On the other hand, if the even numbered observations are correlated, the present method will not detect this fact. The present method is offered here because of the frequency with which the correlation of consecutive observations occurs in practice.

ASSUMPTIONS: The data all come from some normal universe.

HYPOTHESES: H_0: Consecutive observations are not correlated.

H_1: Consecutive observations are correlated.

METHOD:

1. Compute $\sum x_i^2$.

2. Compute $\sum_{1}^{n-1} x_{i+1}\, x_i$. That is, multiply the first observation by the second, the second by the third, etc., up to the next-to-last by the last, and add the results.

3. Compute $\left(\sum_{1}^{n} x_i\right)^2$.

4.

$$\frac{\delta^2}{s^2} = \frac{\sum (x_{i+1} - x_i)^2}{\sum (x_i - \bar{x})^2} = \frac{2\sum_{1}^{n} x_i^2 - x_1^2 - x_n^2 - 2\sum_{1}^{n-1} x_{i+1}\, x_i}{\sum_{1}^{n} x_i^2 - \left[\sum_{1}^{n} x_i\right]^2/n}$$

(where x_1 = first observation taken, and x_n = last observation taken).

5. Consult Table VIII. If the computed value of δ^2/s^2 in step 4 lies between the values of 0.05 and 0.95 opposite the appropriate n, and if the probability of the type I error is 0.10, accept H_0; otherwise accept H_1. Similarly, if the probability of the type I error is set at 0.02, then the computed value should lie between the values shown in the 0.01 and 0.99 columns opposite the appropriate n if H_0 is to be accepted; and so also for a probability of 0.002, the value should lie between those shown in the 0.001 and 0.999 columns if H_0 is to be accepted.

EXAMPLE: The following laboratory observations were made on consecutive days:

Day	Observation
1	206.30
2	193.90
3	204.30
4	203.35
5	178.90
6	182.65
7	199.25
8	194.70
9	178.25
10	165.40
11	172.00

Are the consecutive observations correlated?

ASSUMPTIONS: All the data come from some normal universe.

HYPOTHESES: H_0: The consecutive observations taken from day to day are not correlated.

H_1: The consecutive observations are correlated.

METHOD: 1. $\sum_1^n x_i^2 = 394{,}935.7200$

2. $\sum_{1}^{n-1} x_i x_{i+1} = 358{,}038.7575$

3. $\left(\sum_{1}^{n} x_i\right)^2 = 4{,}322{,}241.0000$

4. $\frac{\delta^2}{s^2} = \frac{1650.24}{2004.72} = 0.8232$

5. Reference to Table VIII shows that for n = 11, the computed value of δ^2/s^2 falls outside the values for 0.05 and 0.95; hence for a type I error of 0.10, H_0 is rejected. But δ^2/s^2 falls between the values for 0.01 and 0.99; hence for a type I error of 0.02, H_0 is accepted. Similarly H_0 is accepted for a type I error of 0.002.

358,038.7575

,241.0000

0.8232

able VIII shows that for
falls outside the
a type I error of
lls between the values
I error of 0.02, H_0
ted for a type I error

PROBLEM 18. Test whether the proportion of a normal universe that exceeds some assigned value ξ, is less than or equal to p_1.

This test is to be used when one wants to ascertain whether there is a certain proportion of a universe that exceeds some value that is taken to be "dangerous." Thus, one may not want to use a lot of ammunition having too many high-pressure rounds. It is to be noted that in this case the type II error is usually much more important than the type I. The type II error consists in believing that a universe does not have many items with excessive values, when actually it does, and such a mistake may be quite serious. For this reason, the chance of a type I error is usually set quite high, so that the chance of a type II error can be minimized.

ASSUMPTIONS: The items of the sample are all drawn from a normal universe and are independent.

HYPOTHESES: Let p be the true proportion of the universe that lies above the value ξ.

$$H_0: \quad p \leq p_1$$

$$H_1: \quad p > p_1$$

METHOD: 1. Assign the probability of a type I error at p_1; call this γ. Let $\alpha = 1 - \gamma$. As mentioned above, γ will ordinarily be high, so that α will be a low value.

2. Determine the values K_{p_1} and K_α from the following table:

p_1 or α	K_{p_1} or K_α
0.001	3.09023
0.005	2.57583
0.01	2.32635
0.02	2.05375
0.05	1.64485
0.10	1.28155
0.15	1.03643
0.20	0.84162
0.25	0.67449

3. Compute

$$a = 1 - \frac{K_{\alpha}^2}{2(N-1)}$$

where N = number of observations

$$b = K_{p_1}^2 - \frac{K_{\alpha}^2}{N}$$

and

$$k = \frac{K_{p_1} + \sqrt{K_{p_1}^2 - a\,b}}{a}$$

4. Compute $\bar{x}$ and s

5. Compute

$$U = \bar{x} + k\,s$$

6. If $\mu \leq \xi$, accept H_0. Otherwise accept H_1 (for the assigned probability of a type I error, which is γ).

NOTE: If it is desired to test whether the proportion less than some assigned value exceeds p_1, one proceeds as above, except that in Step 5 the value $\mu = \bar{x} - k\,s$ is computed, and H_0 is accepted if and only if $\mu \geq \xi$.*

EXAMPLE: Suppose a sample of 13 items is chosen, where $\bar{x} = 30$ and $s = 5$. Can one assume that no more than 0.1% of the population lies above 50? The type I error is set at 0.95 (i.e., if exactly 0.1% of the population lies above 50, then one rejects H_0 95% of the time; hence, H_0 cannot be erroneously accepted more than at most 5% of the time).

ASSUMPTIONS: Normal population and independent observations.

HYPOTHESES: H_1: $p \leq 0.001$

H_2: $p > 0.001$

*For a test to determine the proportion outside of both a lower and upper limit, see Chapter 2 of Reference 3 c) of the Introduction.

ARMAMENT RESEARCH

METHOD:

1. $\gamma = 0.95$; $\alpha = 0.05$

2. $K_{p_1} = 3.09023$

   $K_{\alpha} = 1.64485$

3. $a = 1 - \frac{(1.64485)^2}{(2)(12)} = 0.8873$

   $b = (3.09023)^2 - \frac{(1.64485)^2}{13} = 9.3414$

   $k = \frac{3.09023 + \sqrt{(3.09023)^2 - (0.8873)(9.3414)}}{0.8873} = 4.7483$

4. $\bar{x} = 30$, $s = 5$.

5. $U = 30 + (4.748)(5) = 53.74$

6. Since $53.74 > 50$, H_1 is accepted; that is, one rejects the hypothesis that no more than 0.1% of the population lies above 50.

PART II
ESTIMATION

1. **INTRODUCTION** Estimation of values is a convenient way to present experimental results when the experimenter is not primarily interested in the direct use of his conclusions. In effect, estimations are "incomplete" decisions that are to be supplemented by other information before they are translated into action. This incompleteness makes it difficult to specify the best estimation process. In this Part, some of the commonly practiced methods will be described. The reader is referred to the Introduction for the meaning of an "estimate" and the relation of estimates to true values.

First, a distinction is usually made between "point" estimation and "interval" estimation. In point estimation, the experimenter would like to present on the basis of a sample some definite quantity which is a "good" guess about a measure of the universe. The meaning of "goodness" in this connection varies, but at least two considerations are important in selecting an adequate method of point estimation: (1) the error of the estimate (i.e., the amount the estimate probably differs from what it is supposed to estimate) should be small relative to all other possible methods of estimation, and (2) as the sample size increases, the error should tend to approach zero.

The methods described below for estimating certain measures of normal and binomial universes satisfy these conditions. Most of the methods are "maximum liklihood estimates"; see Reference 3 a) at the end of the Introduction.

Interval estimation is based on the desire to present an interval along some scale or set of scales, within which there is some expectation that the true measure lies. The methods described here are those developed by Neyman and Pearson and lead to confidence intervals. A confidence interval is an interval that has some preassigned chance of "catching" the true value. Thus, a

95% confidence interval is an interval which will catch the true value in the long run 95% of the time. This means that if a number of experimenters made independent observations, and computed the confidence interval as indicated below, the probability that any one of them will include the true value in the interval he computes is 0.95. Similar remarks apply to a 50% or 99% interval. Clearly, the size of the interval and the method of computing it depend on what the experimenter can assume about the data.

2. SOME COMMON POINT ESTIMATES (ASSUMING OBSERVATIONS ARE INDEPENDENT):

a) of μ, the mean of a normal universe:

$$\bar{x} = \frac{\Sigma x_i}{n}$$

b) of σ^2, the variance of a normal universe:

$$s^2 = \frac{\Sigma(x_i - \bar{x})^2}{n} = \frac{n \Sigma x_i^2 - (\Sigma x_i)^2}{n(n-1)}$$

c) of σ, the standard deviation of a normal universe:

$$s = \sqrt{s^2}$$

(Note: Some correction is needed for very small samples).

d) of p, the probability of an event in a defined series of events:

$$\bar{p} = \frac{\text{number of occurrences of event in sample}}{\text{number of items in sample}}$$

e) of the sum of k normal universe means:

$$\bar{x}_1 + \bar{x}_2 + \bar{x}_3 + \cdots + \bar{x}_k$$

f) of the difference between two normal universe means:

$$\bar{x}_1 - \bar{x}_2$$

g) of the variance of the estimate of the mean given in a):

$$s^2_{\bar{x}} = \frac{s^2}{n}$$

of the variance

of the probability

of a sum of means

$$+\frac{s_3^2}{n_3}+ \text{---} + \frac{s_k^2}{n_k}$$

of the difference

ion:

$\Sigma y_i)$

ns, n the number of

ession:

(Σx_i)

$)^2$

of the slope of a

$x_i y_i$

$^2/n]$

intercept of a

$\Sigma\ x_i y_i)$

$x_i)^2]$

3. SOME CONFIDENCE INTERVALS (OBSERVATIONS ASSUMED TO BE INDEPENDENT):

a) of the mean of a normal universe, variance known:

$$95\%: \quad \bar{x} \pm 1.96 \quad s/\sqrt{n}$$

$$99\%: \quad \bar{x} \pm 2.58 \quad s/\sqrt{n}$$

b) of the mean of a normal universe, variance unknown:

$$95\%: \quad \bar{x} \pm t_{0.05} \frac{s}{\sqrt{n}}$$

where $t_{0.05}$ is the value found in the "0.05" column of Table IV opposite the appropriate Degrees of Freedom, which are equal to n - 1.

$$99\%: \quad \bar{x} \pm t_{0.01} \frac{s}{\sqrt{n}}$$

where $t_{0.01}$ is the value in the "0.01" column of Table IV opposite the appropriate Degrees of Freedom.

c) of the variance of a normal universe:

$$90\%: \frac{(DF)s^2}{\chi^2_{0.05}} \text{ to } \frac{(DF)s^2}{\chi^2_{0.95}}$$

where $\chi^2_{0.95}$ is the value found in Table V under the "0.95" column opposite the appropriate DF (Degrees of Freedom = n - 1) and $\chi^2_{0.05}$ is the value found in the "0.05" column in Table V opposite the appropriate DF (Degrees of Freedom).

$$98\%: \frac{DFs^2}{\chi^2_{0.01}} \text{ to } \frac{(DF)s^2}{\chi^2_{0.99}}$$

where $\chi^2_{0.99}$ and $\chi^2_{0.01}$ are values found in Table V opposite the appropriate Degrees of Freedom in the "0.99" and "0.01" columns respectively.

d) of a probability of an event, large sample ($n > 30$) and $p > 0.10$ or $p < 0.90$.

$$95\%: \quad \bar{p} \pm 1.96 \sqrt{\frac{\bar{p}(1-\bar{p})}{n}}$$

$$99\%: \quad \bar{p} \pm 2.58 \sqrt{\frac{\bar{p}\,(1 - \bar{p})}{n}}$$

e) of the difference between two (independent) means, variances unknown (but equal):

$$95\%: \quad (\bar{x}_1 - \bar{x}_2) \pm t_{0.05} \sqrt{\left(\frac{n_1 + n_2}{n_1 + n_2 - 2}\right) \left[\frac{(n_1 - 1)\, s_1^2 + (n_2 - 1)\, s_2^2}{n_1\, n_2}\right]}$$

$$99\%: \quad (\bar{x}_1 - \bar{x}_2) \pm t_{0.01} \sqrt{\left(\frac{n_1 + n_2}{n_1 + n_2 - 2}\right) \left[\frac{(n_1 - 1)\, s_1^2 + (n_2 - 1)\, s_2^2}{n_1\, n_2}\right]}$$

where $t_{0.05}$ and $t_{0.01}$ are values in the "0.05" and "0.01" columns of Table IV, opposite Degrees of Freedom $= n_1 + n_2 - 2$, where n_1 is the size of the first sample and n_2 is the size of the second.

f) of the slope of a linear regression:

$$95\%: \quad b \pm t_{0.05}\, s_b$$

$$99\%: \quad b \pm t_{0.01}\, s_b$$

where $t_{0.05}$ and $t_{0.01}$ are values in the "0.05" and "0.01" columns of Table IV, opposite Degrees of Freedom = n - 2, and where n is the number of paired items used to calculate the regression.

EXAMPLES: a) If five readings of length are taken, with $\bar{x}$ = 65.31 ft., and the variance of a single reading is known to be 0.12, then the 99% confidence interval for the mean is $65.31 + (2.58)\left(\frac{0.12}{\sqrt{5}}\right) = 65.31 + 0.138$.

b) A sample of 12 items (DF = 11) is weighed, and the results are $\bar{x}$ = 1.230 gr. and s^2 = 0.0144 gr.^{2}; hence s = 0.12. The 95% confidence interval for the mean is

$$1.230 \pm \frac{(2.201)(0.12)}{\sqrt{12}} = 1.230 + 0.076.$$

c) The 98% confidence interval for the variance in the example in b) is $\frac{(11)(0.0144)}{24.725}$ to $\frac{(11)(0.0144)}{3.053}$; i.e., 0.0064 to 0.052.

d) On a firing range, 250 rounds of a test incendiary ammunition were shown to have 185 successes and 65 failures. The 95% confidence interval for success is:

$$\frac{185}{250} \pm 1.96 \sqrt{\left(\frac{185}{250}\right)\left(1 - \frac{185}{250}\right)/250} = 0.740 \pm 0.054$$

e) A sample of nine standard items was shown to have an average tensile strength of 3.5 lbs., with an estimated variance, s^2 of 1.2. A sample of nine items of experimental material had an $\bar{x} = 4.4$ and an estimated variance of 1.5. What is the smallest and what is the largest likely magnitude of the true difference between the standard and test materials? The 95% confidence interval for the difference is:

$$(3.5 - 4.4) \pm 2.120 \sqrt{\frac{18}{16}\left[\frac{(8)(1.2) + (8)(1.5)}{81}\right]} = -0.9 \pm 1.2.$$

The difference may range from - 2.1 to + 0.3.

f) Samples of ammunition are stored under high temperature and humidity, and periodically rounds are removed and examined for bullet pull. The results are

x (weeks of storage)	y (bullet pull) (average of 25, suitably reduced)
0	1
1	3
3	2
6	5
8	4

The estimated slope is

$$b = \frac{(5)(71) - (18)(15)}{(5)(110) - (18)^2} = 0.38$$

The variance of b is:

$$s^2_b = \left[\frac{(5)(55) - (15)^2}{4}\right] \left[\frac{4}{(5)(110) - (18)^2}\right] = 0.0553$$

The 95% confidence interval for b is:

$$0.38 \pm (3.162)(.235) = 0.38 \pm 0.75$$

4. POINT ESTIMATION OF THE SLOPE AND INTERCEPT OF A LINEAR REGRESSION WHEN x AND y ARE BOTH SUBJECT TO ERROR. In fitting a straight line to data, it sometimes happens that both of the paired measurements are subject to error. In this case, there is a relatively simple method which can be employed. The method can also be employed when the x_i are not subject to error, but is less efficient than the method described above.

ASSUMPTIONS: Both the x-values and y-values are subject to error; the errors of the x values are independent and normally distributed with the same variance over the entire range; similarly the errors of the y-values are independent and normally distributed with the same variance.

METHOD: 1. Compute the grand mean of all x_i values, $\bar{x}$, and the grand mean of all y_i values, $\bar{y}$.

2. Divide the n pairs of items ordered in the x-direction into three groups, by placing the same number of items in the extreme groups and making the number as close to 1/3 n as possible. (If the groups overlap, order the items in the y-direction.)

3. Compute $\bar{x}_1$ and $\bar{y}_1$, the averages of the lowest group; similarly compute $\bar{x}_3$ and $\bar{y}_3$, the averages for the highest group. Compute:

$$b = \frac{\bar{y}_3 - \bar{y}_1}{\bar{x}_3 - \bar{x}_1}$$

4. Compute

$$a = \bar{y} - b\bar{x}$$

5. The estimated linear regression is then

$$y = a + bx$$

EXAMPLE: Suppose the following pairs of observations are collected on velocities and pressures of a certain type of ammunition (velocities reduced by 1800 ft./sec. and pressures reduced by 1000 p.s.i.):

velocity x
50
52
54
54
56
57
62
63
64
68
70
74

ASSUMPTIONS: The
mally distributed with
are independent; simi

METHOD: 1. $\bar{x}$

2. $\bar{x}_1$

$$\bar{x}_3 = \frac{64 + 68 + 70 +}{4}$$

$$\bar{y}_3 = \frac{61 + 62 + 69 +}{4}$$

3. b

4. a

5. T

y

5. ESTIMATES OF THE SENSITIVITY DATA. "
applied to that type
act of taking a mea
destroys the object
object that it can no

velocity x	pressure y
50	45
52	48
54	50
54	53
56	54
57	58
62	60
63	62
64	61
68	62
70	69
74	68

ASSUMPTIONS: The errors of the x-readings are normally distributed with the same variance throughout, and are independent; similarly for the y-values.

METHOD: 1. $\bar{x} = 60.3$; $\bar{y} = 57.5$

2. $\bar{x}_1 = \dfrac{50 + 52 + 54 + 54}{4} = 52.5$;

$\bar{x}_3 = \dfrac{64 + 68 + 70 + 74}{4} = 69$; $\bar{y}_1 = \dfrac{45 + 48 + 50 + 53}{4} = 49$;

$\bar{y}_3 = \dfrac{61 + 62 + 69 + 68}{4} = 65.$

3. $b = \dfrac{69 - 52.5}{65 - 49} = \dfrac{16.5}{16} = 1.03$

4. $a = 57.5 - \left(\dfrac{16.5}{16}\right)(60.3) = -4.61$

5. The linear regression is

$$y = 4.61 + 1.03\,x$$

5. ESTIMATES OF THE MEAN AND VARIANCE FOR SENSITIVITY DATA. "Sensitivity data" is a general term applied to that type of experimental data for which the act of taking a measurement on the object tested either destroys the object or so changes the properties of the object that it can no longer be used in further testing.

ESTIMATION

We shall call the "stimulus" that condition to which we subject the samples, and we shall call the stimulus at which a given object just fails the "critical stimulus"; that is, for a stimulus of less intensity than the critical stimulus, the object will not fail, and for a stimulus of greater intensity than the critical stimulus, the object will fail. These terms are illustrated in the following examples:

Objects Tested	Type of Stimulus	Critical Stimulus
Primer Caps	Blow of a certain energy	Energy of blow just necessary to explode
Welds	Shear stress (in pounds)	Stress just necessary to shear
Rope	Load (in pounds)	Load just necessary to break
Insulation	Voltage	Voltage just necessary to give breakdown
Organisms	Dosage of drug	Amount just necessary to kill

The aim is to estimate the mean critical stimulus and the variance of the critical stimuli, as well as the errors of these estimates. Only one of the available methods will be discussed here*. This method assumes that samples of the universe have been subjected to a

*Two other methods are in common use. The one, due in part to C. I. Bliss and R. A. Fisher, involves a least squares fitting procedure, and is discussed in some detail in D. J. Finney's Probit Analysis, Cambridge University Press, 1947. The advantage of the Bliss-Fisher method is that it can be applied even when a complete "run down" of the data from all-success to all-failure is not available; further, the method gives "best" estimates if the underlying distribution of critical stimuli is normal, though this second advantage is relatively unimportant and has been greatly over-emphasized. (See J. Cornfield and N. Mantel, "Some New Applications of Maximum Likelihood to the Calculation of the Dosage Response Curve," Journal of the American Statistical Assoc., Vol. 45, 1950, p. 181-210). The disadvantage of the Bliss-Fisher method is the complexity of computation as compared with the method given here. The other method of analyzing sensitivity data consists of varying the stimulus throughout the test (Continued on next page)

series of stimuli in such a manner that all of the sample fails to respond for one stimulus, while all of the sample responds for a stronger stimulus. Further, at least three stimuli are used where some of the sample responds and some does not. The magnitudes of the stimuli are all known with relatively little error, and are equally spaced**.

The method will be explained by an example. An insecticide is being tested.

The test is carried out by testing 20 insects at each of the following dosages: 5, 10, 15, 20, 25, 30, and 35 milligrams. The results are as follows:

Dosage	Number Tested	Number Surviving	Fraction Surviving
x_1 = 5 mg.	20	20	p^1 = 1.0
x_2 = 10 mg.	20	19	p^2 = .95
x_3 = 15 mg.	20	18	p^3 = .90
x_4 = 20 mg.	20	14	p^4 = .70
x_5 = 25 mg.	20	4	p^5 = .20
x_6 = 30 mg.	20	1	p^6 = .05
x_7 = 35 mg.	20	0	p^7 = 0

If we denote the dosages by (x_i), it is convenient to transform the set of dosages (x_i) into a set (x_i') where

so as to obtain maximum information concerning the mean critical stimulus. This method is called the "Staircase" method of sensitivity testing, and is discussed in the Navy Bureau of Ordnance Navord Report 65-46 by T. W. Anderson, P. J. McCarthy, and J. W. Tukey. The advantage of the method is simplicity and minimization of sample size. Its possible disadvantage lies in the necessity of constantly changing the stimulus, and in the fact that the estimate of the variance of the critical stimuli is presumably not as good as that provided by the method discussed here.

**For a complete account of this method, together with its generalization for unequally spaced intervals, see C. W. Churchman and B. Epstein, "Tests of Increased Severity," Journal of the American Statistical Assoc., Vol. 41, 1946, pp. 567-590.

$$x_i' = \frac{x_i - 5}{5}, \quad x_i' = (0, 1, 2, 3, \cdots, 6).$$

The x_i' now range from 0 to 6. Only the results of stimuli where some survive and some do not are used in the computations. Let p_i be the fraction surviving at stimulus x_i, where $0 < p_i < 1$. Then the estimate of the mean critical stimulus in units of x' is given by

$$\bar{x}' = x_1' + .5 + \sum_1^6 p_i.$$

The estimate of the variance in units of x' is given by

$$s_{x'}^2 = \sum_1^6 k_i p_i - \left(\sum_1^6 p_i\right)^2 + \frac{\sum_1^6 p_i (1 - p_i)}{n - 1} - 0.08,$$

where k_i are the odd numbers in sequence (thus $k_1 = 1$, $k_2 = 3$, $k_3 = 5$, etc.), and n is the number (20) tested at each stimulus.

To estimate the mean in the original units, use the formula

$$\bar{x}' = \frac{\bar{x} - 5}{5},$$

or

$$\bar{x} = 5\,\bar{x}' + 5.$$

Note that if the original stimulus interval had been made up of unit intervals starting with $x_1 = 1$, we would merely have set $x_i' = x_i - 1$, and hence

$$\bar{x} = \bar{x}' + 1.$$

The estimate of the variance in the original units is

$$s^2 = (5)^2\, s_{x'}^2.$$

The estimate of the variance of $\bar{x}$ is

$$s_{\bar{x}}^2 = (5)^2 \left(\frac{\Sigma p_i (1 - p_i)}{n - 1}\right)$$

The estimate of the va

$$s^2_{s^2} = (5)^2$$

The computations are

x_i'	p_i	k_i	$1 - p_i$	$p_i(1 - p_i$
1	.95	1	.05	.048
2	.90	3	.10	.09
3	.70	5	.30	.21
4	.20	7	.80	.16
5	.05	9	.95	.048

It will be found that

$\Sigma p_i = 2.8$; $\Sigma k_i p_i = 9.$

Hence

$$\bar{x}' = x_1' +$$

$$s^2_{x'} = \Sigma k_i p_i$$

$$= 9.00 - 7$$

$$s_{x'} = 1.05$$

Hence

$$\bar{x}$$

$$s^2$$

$$s$$

$$s^2_{\bar{x}'}$$

$$s^2_{\bar{x}}$$

$$s^2_{s^2}$$

The estimate of the variance of s^2 is

$$s^2_{s^2} = (5)^2 \left[\frac{\Sigma(k_i - 2\Sigma p_i)^2 p_i (1 - p_i)}{n - 1}\right]$$

The computations are illustrated below:

x_i'	p_i	k_i	$1 - p_i$	$p_i(1 - p_i)$	$k_i p_i$	$(k_i - 2\bar{x}_i')$	$(k_i - 2\bar{x}_i')^2$	$(k_i - 2\bar{x}_i')^2 p_i(1 - p_i)$
1	.95	1	.05	.048	.95	-4.6	21.18	1.02
2	.90	3	.10	.09	2.70	-2.6	6.76	.61
3	.70	5	.30	.21	3.50	- .6	.38	.08
4	.20	7	.80	.16	1.40	1.1	1.96	.31
5	.05	9	.95	.048	.45	3.4	11.56	.55

It will be found that (where $q_i = 1 - p_i$):

$\Sigma p_i = 2.8$; $\Sigma k_i p_i = 9.0$; $\Sigma p_i q_i = .56$; $\Sigma(k_i - 2\Sigma p_i)^2 p_i q_i = 2.57$.

Hence

$$\bar{x}' = x_i' + .5 + \Sigma p_i = .5 + 2.8 = 3.3$$

$$s^2_{x'} = \Sigma k_i p_i - (\Sigma p_i)^2 + \frac{\Sigma p_i q_i}{n - 1} - .08$$

$$= 9.00 - 7.84 + .03 - .08 = 1.11$$

$$s_{x'} = 1.05$$

Hence

$$\bar{x} = 5\,(3.3) + 5 = 21.50$$

$$s^2 = (25)(1.11) = 27.75$$

$$s = 5.25$$

$$s^2_{\bar{x}'} = \frac{0.56}{19} = 0.029$$

$$s^2_{\bar{x}} = (25)(0.029) = 0.725$$

$$s^2_{s^2} = 25\left(\frac{2.57}{19}\right) = 3.4$$

PART III
SEQUENTIAL ANALYSIS

INTRODUCTION In Part I we have discussed methods of making inferences from observations when the experiment has been designed in such a manner that a fixed number of observations, or fixed sample size, is to be taken. It must appear obvious, though, that fixing the sample size prior to conducting the experiment is not always the most efficient method, for no one can predict beforehand exactly how much information will be needed to reach a decision. There will be some cases where one collects too much data, in the sense that a smaller number of observations would give all the assurance that is required, and there will be other cases where not enough observations are made, in the sense that there is not sufficient evidence upon which to act with the assurance that is required.

As an example, suppose that one wishes to determine whether the modification of a certain manufacturing procedure produces material with fewer defectives than normal; in the classical theory of experimental design, one selects a certain sample size suggested by economy and past experience in these matters, and tests the two products concurrently. This procedure might turn out to be very inefficient, especially if the experimental situation does not at all conform to past experience; the new method might be very much better than the old, and only a few observations would make this matter apparent, or the two might be so nearly the same in quality that the sample size chosen still leaves one in doubt in the end.

The procedures described in this Part are designed to enable one to choose a sample size in an efficient manner by letting the evidence of the observations determine when the experiment should be stopped. The sample size itself is now no longer fixed; it is, in statistical terms, a "random variate" which is conditioned by chance events, the kind of question that is asked, the assumptions that are made, and the kind of assurance that is desired.

SEQUENTIAL ANALYSIS

It should be emphasized at the outset that the method to be described in this Part cannot be applied in all cases; there are many instances where one must decide at the outset how many items to select for the sample, and where it is extremely difficult to manufacture new items once the test has started. The general conditions on the application of the method described are the following:

a) random, i.e., representative sampling can be obtained with any sample size.

b) the number of sample items (observations) can be increased indefinitely at any stage of the experiment, and on the other hand it is very desirable to use the smallest possible sample.

Condition a) fails to apply where some kind of organized sampling is used: e.g., stratified or area. For example, there are many instances in the inspection of lots of manufactured articles where the inspection procedure demands an examination of a number of boxes, and where a test of only a few does not give reliable information about the entire lot. In such cases, it may be possible to run sequential tests by taking representative groups of data. In the case of b), it will not be very efficient to use the procedures of this Part if the number of samples to be used must be fixed beforehand. The biologist is often so restricted: he must start with a certain number of "subjects", and cannot increase or decrease his sample size in the middle of his experiment. Similarly, if test ammunition is manufactured, the lot size must be specified beforehand, and it is impossible to manufacture new material that will have the same properties as the original.

But despite these restrictions, there are many cases in which the methods of this section can be applied with little or no difficulty: if the "samples" are observations made on the same object within a laboratory, for example, then in general all observations are equally reliable, and there will be no difficulty in varying the number of observations at will; or if the samples come off some production machine, the two conditions named above will usually hold.

The general procedure here is almost the same as that described in Part I. One first translates the question asked into two or more hypotheses, and the purpose of the

experiment is then defined to be one of selecting from among these hypotheses the one we take to represent the "true" state of affairs. As before, there are, in the case of two hypotheses, two types of error: type I, rejecting H_0 when it is true, and type II accepting H_0 (i.e., rejecting H_1) when it is false. As before, one determines the allowable chance of committing the type I error if H_0 refers to some explicit value, or else one sets the maximum allowable probability of the type I error if H_0 refers to a range of values. But to apply the methods in this Part, one must also set the probability of the type II error at one point. This procedure is best clarified by an example. Suppose we wish to determine whether a certain lot of a manufactured article is acceptable in the sense that it does not contain too many defectives, say 2% or less. The hypotheses then are:

H_0: The lot contains 2% defectives or less.

H_1: The lot contains more than 2% defectives.

As in Part I, one assigns a maximum probability for the type I error, say 0.05. However, one also assigns for the method of this Part the probability of one type II error. This is usually done by deciding what condition is "very bad". Thus, if a lot contains 2.05% defectives, the hypothesis H_0 is incorrect, but not intolerably so. Thus, one might decide that 4% defectives was very serious, and that very few lots (say no more than one percent) having 4% defectives should be accepted. This amounts to saying that the probability of accepting H_0 (i.e., accepting the lot) shall be 0.01 or less when 4% is the true percentage of defectives.

In effect, the experimenter decides on two points of the Operating Characteristic Curve (see Introduction). In general, these points represent the two "intolerable" errors: that of accepting H_1 when H_0 is true, and that of accepting H_0 when H_1 is true. Actually, in Part I the OC curve is also set by the experimental design, but the experimenter may not be aware of the fact. One educational value of sequential methods is that they make the experimenter conscious of the dual or multiple risks of error that must always be taken in any research work.

To date, the meth
been investigated for
particular, two practic
general extension of s
able that (lacking a hi
culation is required a
(2) the number of hyp
number in a three-wa
excessive).

The procedures t
are: (1) set up the
assign the probability
(3) start taking obser
servation there will
graphical inspection
ance of H_0, or the ac
that an additional obs

The following tab
the application of the

Number
.5
.05
.01
.001
.95
.99
.999

To date, the methods of sequential analysis have not been investigated for all the tests described in Part I. In particular, two practical conditions have restricted the general extension of sequential methods: (1) it is desirable that (lacking a high-speed computer) very little calculation is required at the end of each observation; and (2) the number of hypotheses should be small (e.g., the number in a three-way factorial design is usually excessive).

The procedures to be followed in general in this Part are: (1) set up the ASSUMPTIONS and HYPOTHESES; (2) assign the probability of one type I and one type II error; (3) start taking observations; (4) at the end of each observation there will be a METHOD of computation or graphical inspection which will lead either to the acceptance of H_0, or the acceptance of H_1, or to the requirement that an additional observation be taken.

The following table of logarithms will prove useful for the application of the methods of this Part:

Number	log	ln
.5	-.30103	-.69315
.05	-1.30103	-2.99573
.01	-2.00000	-4.60517
.001	-3.00000	-6.90776
.95	-.02228	-.05129
.99	-.00436	-.01005
.999	-.00043	-.00100

PROBLEM 1. Test whether the probability of an event has a certain value (one-sided test).

ASSUMPTIONS: The probability of the event remains constant throughout the experiment, and the observations are independent.

HYPOTHESES: H_0: $p \leq a$

H_1: $p > a$

METHOD: 1. Set the maximum chance of the type I error for $p = a$; call this α.

2. Set one value of the type II risk, at some value b (where, obviously, $b > a$); call this β.

3. Compute

$$h_1 = \frac{\log (1 - \alpha) - \log \beta}{\log b - \log a + \log (1 - a) - \log (1 - b)}$$

$$h_2 = \frac{\log (1 - \beta) - \log \alpha}{\log b - \log a + \log (1 - a) - \log (1 - b)}$$

$$s = \frac{\log (1 - a) - \log (1 - b)}{\log b - \log a + \log (1 - a) - \log (1 - b)}$$

4. Accept H_0 as soon as the number of occurrences of the event is less than $-h_1 + sn$ where n is the number of items tested. Accept H_1 as soon as the number of occurrences of the event is greater than $h_2 + sn$. Otherwise continue to take observations.

EXAMPLE: To determine whether the proportion of defective articles in a lot of manufactured material is less than or equal to 0.02.

ASSUMPTIONS: Random observations: i.e., the probability of a defective article is constant throughout inspection, and the observations are independent.

HYPOTHESES: H_0: $p \leq 0.02$

H_1: $p > 0.02$

METHOD: 1. Suppose $\alpha = 0.05$ when $p = 0.02 = a$.

2. Suppose $\beta = 0.05$ when $p = 0.10 = b$.

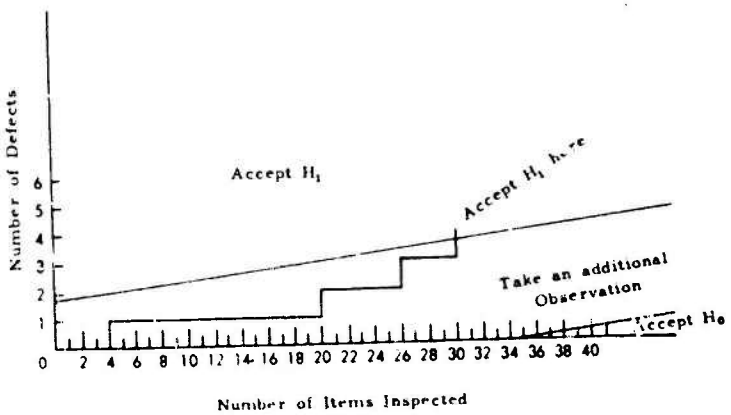
3. $h_1 = 1.74$

$h_2 = 1.74$

$s = 0.05$

(NOTE: When $\alpha = \beta$, $h_1 = h_2$. The table of logs and lns at the end of the Introduction to this Part will prove useful in computations.)

4. The procedure can be carried out graphically as follows (in this example, defects occurred at the 4th, 20th, 26th, and 30th items; H_1 was accepted at the 30th item; that is, the lot was judged to contain an excessive number of defectives at the 30th item.)



Each time an observation is made, move one unit along the base. If a defect occurs, move one unit up at the same time. As soon as the "zig-zag" line reaches the "Accept H_0" area, or the "Accept H_1" area, stop the inspection.

NOTE: Although the sequential method will always end with a finite number of samples, it is possible that the number may prove to be excessive. For this reason, it is usually advisable to "truncate" the plan at some point. Such truncation slightly alters the chances of error, but in general not seriously. The extreme sample size* should normally be chosen as follows:

*The average maximum sample size to be expected is about one-third the extreme sample size; it is the average size if the fraction defective in the lot is exactly s. If this value appears excessive, the values α and/or β can be made larger.

$$\text{Extreme sample size} = 3\left(\frac{h_1 h_2}{s(1-s)}\right)$$

If no decision has been reached by the time the extreme sample size is reached, then the decision regarding the hypotheses should be based on the following criterion (letting n_e = Extreme Sample Size and d = number of occurrences of the event):

If the quantity

$$d\,[\log b - \log a + \log (1-a) - \log (1-b)]$$

is less than or equal to the quantity

$$n_e\,[\log (1-a) - \log (1-b)]$$

accept H_0; if the first is greater than the second, accept H_1.

PROBLEM 2. Test whether the probability of one event is greater than that of another event.

To apply sequential methods to this problem, we do not consider the probabilities themselves, but a certain ratio, k, of the occurrences to the nonoccurrences. That is

$$k = \frac{p}{1 - p}$$

where p is the probability of an occurrence. Thus, if $p = .98$, then $1 - p = .02$, and $k = 49$. If the probabilities p_1 and p_2 of the two events are equal, then the respective k-values, k_1 and k_2, are equal. If $p_2 > p_1$, then $k_2 > k_1$. The hypotheses are stated in terms of k_1 and k_2.

ASSUMPTIONS: The probability of occurrence remains constant within each sample from each universe, and the observations are independent.

HYPOTHESES: H_0: $k_1 \geq k_2$ (i.e., the probability p_1 is greater than or equal to the probability p_2)

H_1: $k_1 < k_2$

METHOD: 1. Set the probability of the type I risk for the case where $k_1 = k_2$ ($p_1 = p_2$). Call this α.

2. Set the probability of a type II error for the case where $k_2 = uk_1$, where u, some value greater than 1, is selected by the experimenter. Call this β.

3. Compute

$$h_1 = \frac{\log (1 - \alpha) - \log \beta}{\log 2u - \log (1 + u)}$$

$$h_2 = \frac{\log (1 - \beta) - \log \alpha}{\log 2u - \log (1 + u)}$$

$$s = \frac{\log 1/2 (1 + u)}{\log 2u - \log (1 + u)}$$

4. Take observations from each universe alternately. Thus the observations come in pairs. Only those cases in which an occurrence is obtained from one

type and a nonoccurrence from the other are of interest. Call t_1 the number of cases at any time where the first universe (p_1) gives an occurrence, and the second a nonoccurrence. Call t_2 the corresponding number of cases where the second universe gives an occurrence, and the first a nonoccurrence. Accept H_0 as soon as

$$t_2 < -h_1 + st_1$$

and accept H_1 as soon as

$$t_2 > h_2 + st_1$$

Otherwise continue to take pairs of observations.

The following table of values of $2u - \log(1 + u)$ and of s will prove useful:

u	$\log 2u - \log(1 + u)$	s
1.5	0.07918	0.8171
2	0.12494	0.7095
3	0.17609	0.5840
4	0.20412	0.5129
5	0.22185	0.4650
10	0.25964	0.3507

EXAMPLE: It is desired to choose a certain method of sealing a time mechanism. Two possibilities are open, the standard, and a new development, and samples of each are subjected to high pressure conditions for a short period of time. If an article shows no leakage during the trial period, it is called effective; otherwise it is called defective. The problem is then to determine whether the new type has a smaller chance of having defectives than the standard. Here p_1 refers to the probability of a defective with the new type, p_2 the probability of a defective with the standard.

ASSUMPTION: Random sampling from each type: i.e., the probability of a defective remains the same for all items selected from one type. The observations are independent.

HYPOTHESES: H_0: $k_1 \geq k_2$ (i.e., the probability of a defective from the new type is at least as great as from the standard)

METHOD:

0.10, this would

following kind o
occurrence.

Sample	New Type
1	1
2	1
3	1
4	1
5	1
6	1
7	0
8	1

NOTE: This
First an appro
This guess ca
perience or of
taken as the l

$$3\bar{n}_1 = 3\left[\frac{(1-\alpha)}{}\right.$$

$$3\bar{n}_2 = 3\left[\frac{(1-\beta)}{\left[k_2\right.}\right.$$

(the value $\bar{n}_1$
average sampl
ber of items

H_1: $k_1 < k_2$ (the new type is an improvement with respect to the probability of a defective)

METHOD: 1. Suppose $\alpha = 0.05$

2. Suppose $\beta = 0.01$ for $u = 3$. (If p_1 were 0.10, this would mean that p_2 would be 0.25 for $u = 3$)

3. Then $h_1 = 11.23$, $h_2 = 7.36$, $s = 3.42$

4. The work may be organized by the following kind of table, where 1 = occurrence, 0 = non-occurrence.

Sample	New Type	Standard Type	t_1	t_2	$-h_1 + st_1$	$h_2 + st_1$	Action
1	1	1	0	0	---	---	Take another pair of observations
2	1	0	1	0	minus	10.78	
3	1	1	1	0	---	---	
4	1	0	2	0	minus	14.20	
5	1	1	2	0	---	---	"
6	1	0	3	0	minus	17.58	"
7	0	1	3	1	2.45	17.58	"
8	1	0	4	1	2.45	21.04	Accept H_0

NOTE: This plan can be "truncated" (see Problem 1). First an approximate guess is made as to the size of k_1. This guess can usually be made on the basis of past experience or of theory. The extreme sample size is to be taken as the larger of the two quantities:

$$3\bar{n}_1 = 3\left[\frac{(1-\alpha)[\log(1-\alpha) - \log\beta] - \alpha[\log(1-\beta) - \log\alpha]}{k_1\left[\log\frac{u+1}{2} - \log\frac{2u}{1+u}\right]} \Big/ (1+k_1)^2\right]$$

$$3\bar{n}_2 = 3\left[\frac{(1-\beta)[\log(1-\beta) - \log\alpha] - \beta[\log(1-\alpha) - \log\beta]}{\left[k_2\left(\log\frac{2u}{1+u}\right) - k_1\left(\log\frac{u+1}{2}\right)\right]} \Big/ (1+k_1)(1+k_2)\right]$$

(the value $\bar{n}_1$ or $\bar{n}_2$, whichever is larger, is the maximum average sample size, and gives some idea as to the number of items that will probably have to be tested under the

most unfavorable conditions). If the test has run on to the extreme sample size, and no decision has been reached, then the following truncating scheme can be used without seriously altering the probabilities of the errors:

Accept H_0 if the quantity

$$t_1 \log \frac{1 + u}{2}$$

is greater than or equal to the quantity

$$t_2 \log \frac{2u}{1 + u}$$

Otherwise accept H_1.

the test has run on to
decision has been reached,
me can be used without
s of the errors:

u

uantity

u

PROBLEM 3. Test whether the mean of a normal universe is equal to a certain value, variance known.

a) ONE-SIDED TEST

ASSUMPTIONS: All observations are drawn independently from some normal universe, with a known variance.

HYPOTHESES: H_0: $\mu \leq a$

H_1: $\mu > a$

METHOD: 1. Set the probability of the type I error for a; call this α.

2. Set one value of the probability of the type II error; i.e., decide for one possible value of μ, say b, the chance you are willing to take of accepting H_0 (erroneously) when actually $\mu = b$. Call this probability β. The value of b will usually represent an "outside limit" of the error one is willing to make.

3. Compute

$$h_1 = \frac{[\ln (1 - \alpha) - \ln \beta]\sigma^2}{(b - a)}$$

$$h_2 = \frac{[\ln (1 - \beta) - \ln \alpha]\sigma^2}{(b - a)}$$

$$s = \frac{a + b}{2}$$

4. Keep taking observations, x_i; accept H_0 as soon as

$$\sum^{n} x_i \leq -h_1 + sn$$

where n is the number of observations taken. Accept H_1 as soon as

$$\sum^{n} x_i \geq h_2 + sn$$

Otherwise take an additional observation.

EXAMPLE: To determine whether the predicted amount of a chemical in a solution is 0.09%; σ is given as 0.01, and hence $\sigma^2 = 0.0001$.

ASSUMPTIONS: Random observations from a normal universe; $\sigma^2 = 0.0001$.

HYPOTHESES: H_0: $\mu \leq 0.09$

H_1: $\mu > 0.09$

METHOD:

1. Suppose $\alpha = 0.05$ (for a = 0.09)
2. Suppose $\beta = 0.10$ for b = 0.10.
3. Then $h_1 = 0.023$, $h_2 = 0.029$, $s = 0.095$.
4. The method can be followed on a table:

n = number of observations	observation = x_i	Σx_i	$-h_1 + sn$	$h_2 + sn$	Action
1	0.095	0.095	0.072	0.124	take another reading
2	0.112	0.207	0.167	0.219	"
3	0.108	0.315	0.262	0.314	Accept H_1

NOTE: The extreme sample size for this method will in general be the larger of the two quantities:

$$3\bar{n}_1 = 3\,\frac{(1-\alpha)[\ln(1-\alpha) - \ln\beta] - \alpha[\ln(1-\beta) - \ln\alpha]}{(b-a)^2/2\sigma^2}$$

$$3\bar{n}_2 = 3\,\frac{(1-\beta)[\ln(1-\beta) - \ln\alpha] - \beta[\ln(1-\alpha) - \ln\beta]}{(b-a)^2/2\sigma^2}$$

where $\bar{n}_1$ or $\bar{n}_2$, whichever is larger, is the average maximum sample size for the most unfavorable conditions. If the test is run to the extreme sample size, then the following truncating plan can be used:

If $\Sigma x_i \leq \frac{(a+b)n}{2}$, H_0 is to be accepted.

If $\Sigma x_i > \frac{(a+b)n}{2}$, H_1 is to be accepted.

ine whether the predicted amount
tion is 0.09%; σ is given as 0.01,

observations from a normal uni-

$\leq$ 0.09

$>$ 0.09

se α = 0.05 (for a = 0.09)

se β = 0.10 for b = 0.10.

h_1 = 0.023, h_2 = 0.029, s = 0.095.

ethod can be followed on a table:

Σx_i	$-h_1 + sn$	$h_2 + sn$	Action
0.095	0.072	0.124	take another reading
0.207	0.167	0.219	"
0.315	0.262	0.314	Accept H_1

le size for this method will
the two quantities:

$$\frac{\ln\beta] - \alpha[\ln(1-\beta) - \ln\alpha]}{(b-a)^2/2\sigma^2}$$

$$\frac{\ln\alpha] - \beta[\ln(1-\alpha) - \ln\beta]}{(b-a)^2/2\sigma^2}$$

s larger, is the average maxi-
ost unfavorable conditions. If
me sample size, then the fol-
e used:

H_0 is to be accepted.

H_1 is to be accepted.

2

b) TWO-SIDED TEST

ASSUMPTIONS: All observations taken from a normal universe, are independent (hence random) and σ^2 is known.

HYPOTHESES: H_1: $\mu < a_1$

H_2: $a_1 \leq \mu \leq a_2$

H_3: $\mu > a_2$

METHOD: 1. The important values for this method are to be assigned as follows. First draw a line representing the possible values of μ. Mark off on the line the position of a_1 and a_2. At some position to the left of a_1, mark the value θ_1, below which it would be serious to accept any hypothesis other than H_1. Set the probability of any such erroneous acceptance and call it γ_1. Next, mark off to the right of a_1 a point θ_2 so that the distance from θ_1 to a_1 = the distance from a_1 to θ_2. Go through the same procedure for the upper part of the line: above a_2 mark off the value θ_4 above which it would be serious to accept any hypothesis other than H_3. Set the probability of such a serious acceptance as γ_3. Mark off to the left of a_2 a point θ_3 such that the distance from θ_3 to a_2 = distance from a_2 to θ_4. Make sure that θ_2 is to the left of θ_3. The interval θ_2 to θ_3 is now considered to be one where it would be serious to accept any hypothesis other than H_2. Assign the probability of such a mistake; call it γ_2. The scheme appears as follows:

Below θ_1 only H_1 should be accepted. Probability of not accepting H_1 is γ_1.

Between θ_2 and θ_3 only H_2 should be accepted. Probability of not accepting H_2 is γ_2.

Above θ_4 only H_3 should be accepted. Probability of not accepting H_3 is γ_3.

2. Compute

$$h_1 = \frac{[\ln (1 - \gamma_1) - \ln \gamma_2]\sigma^2}{\theta_2 - \theta_1}$$

$$h_2 = \frac{[\ln (1 - \gamma_2) - \ln \gamma_1]\sigma^2}{\theta_2 - \theta_1}$$

$$s_1 = \frac{\theta_1 + \theta_2}{2}$$

$$h_3 = \frac{[\ln (1 - \gamma_2) - \ln \gamma_3]\sigma^2}{\theta_4 - \theta_3}$$

$$h_4 = \frac{[\ln (1 - \gamma_3) - \ln \gamma_2]\sigma^2}{\theta_4 - \theta_3}$$

$$s_2 = \frac{\theta_3 + \theta_4}{2}$$

3. Keep taking observations. Accept H_1 as soon as

$$\Sigma x_i \leq - h_1 + s_1 n \quad \underline{\text{and}} \quad \Sigma x_i \leq - h_3 + s_2 n$$

Accept H_2 as soon as

$$\Sigma x_i > h_2 + s_1 n \quad \underline{\text{and}} \quad \Sigma x_i \leq - h_3 + s_2 n$$

Accept H_3 as soon as

$$\Sigma x_i > h_2 + s_1 n \quad \underline{\text{and}} \quad \Sigma x_i > h_4 + s_2 n$$

EXAMPLE: It is desired to run a quick test on a lot of ammunition in order to determine whether its velocity should be classified as "low," "standard," or "high." "Low" velocity ammunition in this case is ammunition having a mean velocity less than 1750 ft./sec. "Standard" ammunition has a mean velocity ranging from 1750 ft./sec. to 1775 ft./sec., while "high" velocity ammuntion has a mean velocity in excess of 1775 ft./sec. The standard deviation of velocities is known to be 10 ft./sec.

ASSUMPTIONS: All observations are random draws from a normal universe.

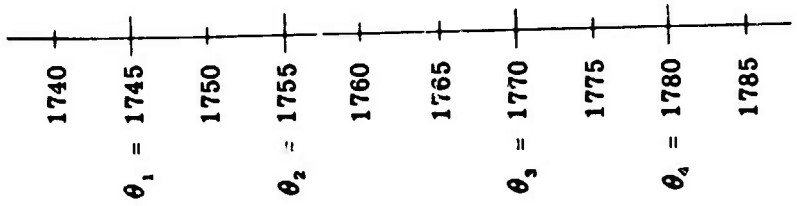
SEQUENTIAL ANALYSIS

HYPOTHESES: H_1: $\mu < 1750$

H_2: $1750 \leq \mu \leq 1775$

H_3: $\mu > 1775$

METHOD: 1. It is decided that it would be serious to accept ammunition as anything but "low" if the true mean velocity is less than 1745 ft./sec. The probability of such an error is set at 0.01. Similarly, ammunition above 1780 ft./sec. should not be accepted as anything but "high"; the probability of an erroneous acceptance is set at 0.01. Ammunition ranging between 1755 and 1770 should be taken as "standard" and the probability of a wrong decision is set at 0.05. The scheme then appears as follows:



2. $h_1 = 2.99$; $h_2 = 4.60$; $h_3 = 4.60$; $h_4 = 2.99$; $s_1 = 1750$; $s_2 = 1775$.

3. The work is outlined as follows:

Sample size - n	x_i (reduced by 1700)	$\sum x_i$	$-h_1 + s_1 n$	$+h_2 + s_1 n$	$-h_3 + s_2 n$	$+h_4 + s_2 n$	Action
1	50	50	47	55	70	78	Take another reading
2	52	102	97	105	145	153	- - -
3	58	160	147	155	215	228	Accept H_2

PROBLEM 4. Test where a normal universe variance is less than a standard.

ASSUMPTIONS: Observations are all from the same normal universe and are independent.

HYPOTHESES: H_0: $\sigma^2 \leq a^2$

H_1: $\sigma^2 > a^2$

METHOD: 1. Set the maximum probability of the type I error, where $\sigma^2 = a^2$. Call this α.

2. Set one probability of a type II error, where $\sigma^2 = b^2$ $(b^2 > a^2)$. Call this β.

3. Compute

$$h_1 = \frac{2[\ln(1-\alpha) - \ln\beta][a^2b^2]}{b^2 - a^2}$$

$$h_2 = \frac{2[\ln(1-\beta) - \ln\alpha][a^2b^2]}{b^2 - a^2}$$

$$s = \frac{[\ln b^2 - \ln a^2][a^2b^2]}{b^2 - a^2}$$

4. After each observation compute

$$\sum^{n} (x_i - \bar{x})^2$$

where $\bar{x}$ is the estimated mean up to that point. Accept H_0 as soon as

$$\sum^{n} (x - \bar{x})^2 < -h_1 + s(n-1);$$

accept H_1 as soon as

$$\sum^{n} (x - \bar{x})^2 > h_2 + s(n-1)$$

Otherwise take an additional observation.

EXAMPLE: To determine whether there is too large a dispersion in times for a lot of time fuzes. It is considered acceptable for the standard deviation to be 0.2 seconds, while 0.5 seconds is excessive.

ASSUMPTIONS: Random sampling from a normal universe.

HYPOTHESES: H_0: $\sigma \leq 0.2$ (or $\sigma^2 \leq 0.04$)

H_1: $\sigma > 0.2$ (or $\sigma^2 > 0.04$)

METHOD:

1. Suppose $\alpha = 0.02$ for $\sigma^2 = 0.04$.
2. Suppose $\beta = 0.02$ for $\sigma^2 = 0.25$ ($\sigma = 0.5$).
3. $h_1 = 0.371$; $h_2 = 0.371$; $s = 0.087$.
4. The procedure may be schematized as follows (reducing the observations to simplify computations does not affect the result):

(1) Sample number = n	(2) Observation	(3) Reduced observation (by 30) = x_i	(4) x_i^2	(5) Σx_i^2	(6) Σx_i	(7) $(\Sigma x)^2$	(8) $(\Sigma x)^2/n$	$\Sigma(x_i - \bar{x})^2$ (5) - (8)	$-h_1 + s(n-1)$	$h_2 + s(n-1)$
1	29.7	-.3	0.09	#	-.3	#	#	#	#	#
2	30.5	0.5	0.25	0.34	0.2	0.04	0.02	0.32	-	.46
3	30.2	0.2	0.04	0.38	0.4	0.16	0.05	0.33	-	.54
4	30.0	0.0	0.00	0.38	0.4	0.16	0.04	0.34	-	.63
5	31.1	1.1	1.21	1.59	1.5	2.25	0.45	1.14	0.02	.72††

#- No computation needed for these values; no decision can possibly be made at this point.
†† Accept H_1 at this point (too large a dispersion).

NOTE: The extreme sample size, n, will be the larger of the two quantities:

$$3\bar{n}_1 = 3\left[\frac{(1-\alpha)[\ln(1-\alpha) - \ln\beta] - \alpha[\ln(1-\beta) - \ln\alpha]}{1/2\,[\ln b^2 - \ln a^2 - 1 + a^2/b^2]} + 1\right]$$

$$3\bar{n}_2 = 3\left[\frac{(1-\beta)[\ln(1-\beta - \ln\alpha] - \beta[\ln(1-\alpha) - \ln\beta]}{1/2\,[\ln a^2 - \ln b^2 - 1 + a^2/b^2]} + 1\right]$$

($\bar{n}_1$ or $\bar{n}_2$, whichever is larger, is the maximum average sample size).

If the test is run to the extreme sample size the following truncation plan can be used. If

$$\Sigma(x_i - \bar{x})^2 < \frac{(\ln b^2 - \ln a^2)\,a^2 b^2}{b^2 - a^2}(n-1)$$

then accept H_0; otherwise accept H_1.

PART IV
SIMPLIFIED PROCEDURES

INTRODUCTION In this part there are presented some procedures that require less calculation than those discussed previously. This gain in economy is made with some loss in "efficiency." Specifically, in the language of the Introduction, the OC curves for these procedures are not as good as those of Parts I and III, so that the chances of making a wrong decision do not as rapidly decrease for increasing magnitude of error. In many practical cases this loss in efficiency is not serious. Moreover, if the procedures of this Part reject H_0 for an assigned probability of the type I error, then in general the procedures of Parts I and III will also reject H_0. Hence, one can often run a quick statistical "pre-test" using the methods of this Part; if H_0 is rejected, then no more work need be done. However, if H_0 is accepted by the methods of this Part, the more efficient tests of Parts I and III might reject instead.

Some of the procedures to be described in this Part are simplified in another sense: they do not require the assumption of normality, nor, indeed, of any other specific distribution. Such tests are often called "nonparametric," because they do not test for the value of a "parameter" of the universe (like μ or σ^2), but rather are concerned with the property of the universe as a whole. Thus, a nonparametric test will be used to determine whether two universes of things, A and B, are the same, but if the hypothesis of dissimilarity is accepted, the nonparametric test does not tell us in what exact respect the dissimilarity exists. Again, the gain in reducing assumptions is compensated by some loss in efficiency.

In general, the nonparametric tests will consist of a measure of the relative positions of various observations, and the significant calculations consist of counting. In some cases, more than one method will be suggested, the choice depending on the experimental set-up.

PROBLEM 1. (Statistical Sign Test). Test whether two universes whose items are observed under different conditions, are the same.

This test is most frequently used when comparing two types, where the test conditions are known to vary, perhaps by design or perhaps because the experimenter cannot exert control. The usual "t-test" (Problems 3 and 4 of Part I) is not advisable in such cases. This test can be used even if the conditions do not change, though other tests may then be preferable.

a) TWO-SIDED TEST

ASSUMPTIONS: The observations are made in pairs, one from type A, one from type B; each pair is made under essentially the same conditions; the pairs of observations are independent (the outcome of one test does not influence the outcome of any other).

HYPOTHESES: H_o: Universe A and Universe B are the same.

H_1: Universe A and Universe B are different.

METHOD: 1. Let x_i be the i th observation on A, and y_i the i th observation on B. For each pair determine whether $x_i - y_i$ is plus or minus, and count the number of pluses and minuses. Let r denote the number of times the less frequent sign appears. (If the number of pluses = the number of minuses, accept H_o).

2. Let n be the number of pairs.

3. Enter Table XIV; if r exceeds the value shown opposite the appropriate n, and under the assigned chance of the type I error, then accept H_o. If r is less than or equal to the number shown, accept H_1.

NOTE: It may happen that there are some "ties," so that $x_i - y_i = o$. In such cases, the number of ties may be divided equally among the pluses and minuses (using halves if necessary). It is also to be noted that the values

shown in Table XIV are approximate; actually, the chance of the type I error is usually slightly less than that shown. This is because in the mathematics of the procedure, r does not come out to be a whole number and must be rounded off.

Table XV gives information as to the sample size that must be taken if a type II error of 0.05 exists at various points. For example, suppose that the true proportion of minuses is 0.40 (so that the true proportion of pluses is 0.60). Suppose, too, that the chance of a type I error is set at 0.01 (1%). Then about 442 pairs would have to be examined to make sure (with a chance of error of 0.05) that H_0 is rejected. Table XV shows, then, that except for large differences, this test will not reject H_0 unless the sample size is very large. The experimenter must attempt to decide what makes a large difference in practical terms.

EXAMPLE: Two types of weapon are to be tested for accuracy in normal firing on the range at 100 yds. Samples of the weapons are used by different persons, the same person firing five shots with each type. The "center of gravity" of the five shots is computed, and the pertinent measure is the distance of this center from the true center of the target. The data are

Observer No.	Distance of C.G. from bull's eye		Sign of $(x_i - y_i)$
	A	B	
1.	1.56	1.40	+
2.	1.64	1.60	+
3.	1.05	1.00	+
4.	1.83	1.60	+
5.	1.57	1.63	-
6.	2.03	1.87	+
7.	1.42	1.36	+
8.	1.40	1.33	+
9.	1.07	1.13	-
10.	1.98	1.80	+
11.	2.14	2.07	+
12.	1.40	1.20	+
13.	1.31	1.27	+
14.	1.83	1.70	+
15.	1.06	.99	+

ASSUMPTIONS: The conditions of firing are the same for A and for B, at each try; the score of one person does not depend on the score of any other person.

HYPOTHESES: H_0: Gun type A and gun type B are equivalent with respect to accuracy.

H_1: The types are not equivalent.

METHOD 1. r (the number of minuses) = 2.

2. n = 15

3. According to Table XIV, H_0 is rejected for a probability of a type I error of 0.01 or greater (for example, at 0.01, the value shown is 2, and since r is not greater than this amount, H_0 is rejected). Thus, for these chances of a type I error, one infers a difference between the guns (B is better for accuracy, assuming that the persons firing the guns are representative).

Note that by Table XV, with a sample of 15, one might have accepted H_0 even though the true ratio of plus to minus was 0.95 to 0.05, with a chance of a type I error of 0.01.

b) ONE-SIDED TEST

Here the aim is merely to see whether the number of pluses exceeds the minuses. Thus, type A may be known to be no better than B, and hence there is no interest in the "A better than B" side.

ASSUMPTIONS: Same as above for the two-sided test.

HYPOTHESES. H_0: A does not give scores exceeding the scores of B.

H_1: A does give scores exceeding the scores of B.

METHOD. 1. Count the number of minus values for $(x_i - y_i)$. Call this r.

2. Let n = number of pairs.

3. Proceed as in step 3 above, except that in Table XIV the values of the chance of a type I error are one-half those for the two-sided test.

3 above, except
nce of a type I
ded test.

PROBLEM 2. To test for a difference between universes, using ranking.

This, and the remaining problems of Part IV, make use of a method of ranking the observations. Instead of the actual quantitative value of a reading along some scale, one uses a number that indicates the comparative position of the observation with respect to the other readings. Thus, if one were comparing two samples, where the observations of one all ranked above those of the other, one would be inclined to infer a difference between the universes from which the samples were drawn. One might also be inclined to make such an inference if all but one of the observations of the first sample were larger than those of the second. The method described here is designed to introduce the necessary refinements in making such inferences.

It should be noted that the method of ranking has one very distinct advantage: very often it is impossible to give quantitative estimates with the equipment that is available; in such cases, however, one may be able to rank objects with respect to some quality. Thus, suppose the experimenter is comparing metal sheets that have been treated in various ways. The sheets are subjected to storage conditions and compared with respect to erosion. The experimenter may not have any convenient way of assigning quantitative measures of erosion, but he may be able to find a consistent method of ranking the sheets with respect to the amount of erosion. In such cases, the present methods will prove useful.

Finally, it should be noted that there are actually three distinct methods of comparing universes discussed in this manual: (1) the T- or t-tests, which assume quantitative data and normality, (2) the sign-test (Problem I of this Part), and (3) the ranking test. In general, if time of computation is not important, and normality can be assumed, use the T- or t-tests (or their generalization, the F- and L_0- tests). If normality cannot be assumed, use the ranking method; also use the ranking method if the data are not presented quantitatively. Finally, if economy of time is essential, use the sign test (which is the fastest method to apply but is in general somewhat less efficient than the others).

SIMPLIFIED PROCEDURES

First, the simple case of the comparison of two samples will be considered. This case will be explained in the two sub-cases, the one where the observations are not paired, and the other where they are (e.g., taken in pairs by different observers, or on different days, etc.)

a) UNPAIRED OBSERVATIONS

The method assumes that the same number of observations occur in each sample.

ASSUMPTIONS: The observations in each sample are independent and each sample comes from one universe (not necessarily normal).

HYPOTHESES: H_0: The universes from which the samples are drawn are the same.

H_1: The universes from which the samples are drawn are different.

METHOD: 1. Arrange all the data of both samples in ascending order, assigning to the lowest the rank value 1, and to the highest the rank value 2n (where n is the number of observations in each sample). In case of ties, assign the mean rank; thus if two observations have the same ranking after the tenth (i.e., occupy the 11th and 12th positions) assign to each the rank 11.5. Note that this ranking can be performed even though the observations are not quantitatively expressed, as long as some ranking occurs.

2. Add the ranks of each sample.

3. Enter Table XVI with the lower ranking score of the two samples. If this value is less than the value shown opposite the appropriate n and under the assigned probability (P) of a type I error, accept H_1; otherwise accept H_0.

EXAMPLE: Two samples of metal sheet are to be compared for effect of high humidity on amount of corrosion. Samples of fifteen are taken of each type, and after storage at high humidity, are ranked as follows, where lowest ranking indicates least corrosion in the judgment of the

AAABAABBAAAABBAABBBAABAABBBBBB

RES

arison of two
will be explained
observations are
e (e.g., taken in
erent days, etc.)

IONS

ber of observations

ch sample are in-
one universe (not

which the sam-
the same.

which the sam-
different.

of both samples
est the rank value
(where n is the
In case of ties,
vations have the
the 11th and
1.5. Note that
h the observa-
long as some

sample.

the lower rank-
lue is less than
n and under the
, accept H_1;

are to be com-
nt of corrosion.
, and after stor-
s, where lowest
dgment of the
BAABBBBBB

SIMPLIFIED PROCEDURES

ASSUMPTIONS: The judgments on corrosion are independent, and all of sample A come from one universe, as do all of sample B.

HYPOTHESES: 1. The A- and B-universe are the same.

2. The A- and B-universe differ.

METHOD: 1. The rank numbers of the observations can readily be determined from the data given above.

2. Clearly the rank score of A will be lower than that of B. We need only concern ourselves with the A-sample therefore. Its ranks are 1, 2, 3, 5, 6, 9, 10, 11, 12, 15, 16, 20, 21, 23, 24. The sum of these values is 178.

3. If the probability of the type I error is set at 0.05, one accepts H_1 (since 178 is less than 185), and hence one infers that the A-type has definitely less corrosion than the B-type. But if the probability of the type I error is 0.02 or 0.01, one accepts H_0 (no real difference in corrosion of the two types).

b) PAIRED DATA

ASSUMPTIONS: The data are independent, except that pairs of observations from each of the samples are associated (either by time, locale, observer, or some other property). It is assumed that the data of each sample come from the same universe (not necessarily normal).

HYPOTHESES: H_0: The universes are the same.

H_1: The universes differ.

METHOD: 1. Compute the differences between each of the pairs of observations, paying attention to the signs. It does not matter which sample is taken as "first."

2. Rank the differences in order of magnitude (independent of sign).

3. Add the ranks of the minus differences and the ranks of the plus differences.

4. Enter Table XVI with the lower of the two values obtained in (3). If this value is less than the value shown opposite the appropriate number of pairs (n), and the assigned probability (P) of a type I error, accept H_1; otherwise accept H_0.

NOTE: This method cannot be used unless the data are assigned quantitative values, or unless the observer can at least assign a ranking to the differences between the paired items.

EXAMPLE: Samples of each of two types of material are compared in several different climates for durability. The data are given in terms of the life of each item in weeks. The results follow, together with the computations outlined in the METHOD (the Assumptions and Hypotheses need not be repeated here):

LOCATION	Life of Sample A	Life of Sample B	Difference A-B	Rank
Nova Scotia	36	42	-6	4.5
South Carolina	32	39	-7	6.0
Florida	29	46	-17	10.0
Pennsylvania	41	40	+1	1.0
Illinois	36	42	-6	4.5
Alberta	34	32	+2	2.0
New Mexico	49	41	+8	7.0
California	43	59	-16	9.0
Texas	41	50	-9	8.0
Panama	29	32	-3	3.0

METHOD: 1-2. The computations are shown in the table above.

3. Only the ranks of the positive differences concern us here, and these add up to 10. The number of pairs is 10, and Table XVI tells us that H_0 is to be accepted for probabilities of type I error of 0.05, 0.02 and 0.01. That is, the data cannot be used (for these assigned probabilities of error) to infer that the life of type-A articles is shorter than that of type-B articles.

RES

ith the lower of the
e is less than the
umber of pairs (n),
ype I error, accept

nless the data are
the observer can at
es between the pair-

ypes of material are
for durability. The
each item in weeks.
computations outlined
Hypotheses need not

Difference A-B	Rank
-6	4.5
-7	6.0
-17	10.0
+1	1.0
-6	4.5
+2	2.0
+8	7.0
-16	9.0
-9	8.0
-3	3.0

s are shown in the

the positive differ-
up to 10. The num-
ls us that H_0 is to be
ror of 0.05, 0.02 and
d (for these assigned
e life of type-A
B articles.

PROBLEM 3. To test for the possible excessive occurrence of some feature of a set of observations, in terms of the repetitions of this feature.

The problem here is one of testing "runs" of some aspect of a set of data, where it is natural to suppose that the feature should occur in a random manner throughout. Thus, one may become suspicious concerning runs of observations that end in an even digit. Or, one may expect in quality control inspection that the observations should occur above and below some general average line in a random manner; if runs are observed above or below the line, one wants to know whether the clusters are accidental or can be taken as an indication of lack of control.

To apply the method, it is assumed that the data can be arranged in some sequence, e.g., in terms of the time at which the observation was made, or the position of the object measured, or the magnitude of the observation.

ASSUMPTIONS: The observations all come from one universe.*

HYPOTHESES: H_0: The repetitions (runs) of some feature of the observations occur in a random fashion.

H_1: The runs do not occur in a random fashion.

METHOD: 1. Be sure the data are ordered, either in terms of time of occurrence, or position, or some other characteristic.

2. Count the number of runs of feature A and feature B, where by runs we mean the total number of subsequences entirely made up of A or B. Call this number of runs u'. (In general, if the features are not distributed randomly, the number of runs will tend to be too small.)

*Technically, a universe whose distribution function has at most a finite number of discontinuities.

3. Count the total number of occurrences of A and the total number of occurrences of B; call the lower of these two numbers m, and the larger n. If m = n, then it is indifferent which is selected to represent A and which to represent B.

4. Enter Table X with m, n and u'. The number shown in the appropriate block represents the probability of obtaining a u' as low or lower than the one observed. If the probability of a type I error is set at 0.05, and the number shown in the block is less than 0.05, then reject H_0 (i.e., infer nonrandomness in the data). Similarly, if the number shown in the block is less than 0.01 and the probability of a type I error is set at 0.01, reject H_0.

EXAMPLE: The following set of data were obtained by a certain observer using a physical testing apparatus:

Observation no.	Reading		Observation no.	Reading	
1.	32		16.	36	7
2.	34	1	17.	36	
3.	36		18.	27	
4.	30		19.	31	8
5.	35	2	20.	31	
6.	32	3	21.	35	
7.	30		22.	34	9
8.	29	4	23.	37	10
9.	36	5	24.	34	
10.	34		25.	38	11
11.	31		26.	34	
12.	29	6	27.	32	
13.	31		28.	31	
14.	35		29.	35	12
15.	32	7	30.	37	

Are the runs of odd and even readings random?

ASSUMPTIONS: The data all come from the same universe.

umber of occurrences
ences of B; call the
the larger n. If
s selected to represent

th m, n and u′. The
k represents the
or lower than the one
e I error is set at
lock is less than 0.05,
ness in the data).
e block is less than
rror is set at 0.01,

a were obtained by a
ting apparatus:

6.	36	7
7.	36	
8.	27	8
9.	31	
0.	31	
1.	35	
2.	34	9
3.	37	10
4.	34	11
5.	38	
6.	34	
7.	32	
8.	31	12
9.	35	
0.	37	

random?

om the same

SIMPLIFIED PROCEDURES

HYPOTHESES: H_0: The data are random with respect to runs of odd and even readings.

H_1: The data are not random with respect to runs.

METHOD: 1. The runs are bracketed off in the table above. Here u′ = 12.

2. The total number of odd readings, m, is 14, and the total number of even readings, n, is 16. We consult Table X. The value shown in the appropriate is 0.1007161. Hence, if the probability of a type I error has been set less than 0.10, H_0 will be accepted; i.e., one will infer no lack of randomness due to runs of odd and even observations.

PROBLEM 4. To determine whether there is a real difference existing between two or more universes.

This test is comparable to the test of significant difference among several means or several variances discussed in Part I, except that here no assumption of normality is made, and the data need not be expressed numerically, so long as some method of ranking is available. The data are arranged so that the readings from each universe are paired, i.e., each set of k readings from the k universes were made under essentially the same conditions.

ASSUMPTIONS: The observations from each sample come from one universe and are independent.

HYPOTHESES: H_0: The universes are all the same.

H_1: The universes are not all the same.

METHOD: (This can best be explained by example).

1. Let I, II, III, etc., be the universes to be examined, and A, B, C, etc., the conditions under which the data were obtained. Rank the readings for A, B, C, etc., for each universe. For example, samples from three lots of material (I, II, III) were ranked for performance under five conditions (A, B, C, D, E), and we wish to know whether there is agreement in performance.

	Condition				
	A	B	C	D	E
Lot I	1	2	3	4	5
Lot II	1	2	4	3	5
Lot III	2	3	1	4	5
Totals	4	7	8	11	15

2. Let p be the number of conditions (here p = 5), and let n. the number of lots (n = 3 in the example).

3. Add up the rank numbers for each condition, as is shown in the table above (thus the ranks for condition A were 1, 1, and 2, with a total of 4).

r there is a real
nore universes.

test of significant
several variances dis-
o assumption of nor-
ot be expressed nu-
of ranking is available.
eadings from each uni-
readings from the k
lly the same conditions.

om each sample come
nt.

are all the same.

are not all the same.

ed by example).

, be the universes to
conditions under which
adings for A, B, C,
e, samples from three
ed for performance
), and we wish to
performance.

dition

	D	E
	4	5
	3	5
	4	5
	11	15

ber of conditions
f lots (n = 3 in the

umbers for each con-
(thus the ranks for
total of 4).

4. Compute

$$\chi^2 = \frac{12}{np(p+1)} \times \text{sum of the squared rank totals} - 3n(p+1)$$

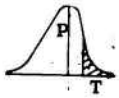
$$= \frac{12}{3 \times 5 \times 6} \times (4^2 + 7^2 + 8^2 + 11^2 + 15^2) - 3 \times 3 \times 6 = 9.3$$

5. Enter Table V with Degrees of Freedom equal to p - 1, i.e., in this case to 4, and examine under the appropriate P, the set probability of a type I error. If the calculated χ^2 in Step 4 is greater than that shown opposite the appropriate Degrees of Freedom and under the appropriate P, then reject H_0; otherwise accept. In this case, the value shown under P = 0.05 is greater than 9.3, and hence H_0 is accepted for a probability of a type I error of 0.05 or less.

TABLE I

AREAS AND THEIR STANDARD SAMPLING ERRORS FOR THE NORMAL CURVE

$$Q = \int_T^\infty z\,dx, \qquad P = 1 - Q$$

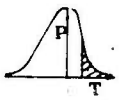


T	Q	PQ	T	Q	PQ
.00	.5000000	.2500000	.60	.2742531	.1990363
.01	.4980106	.2499641	.81	.2709309	.1975273
.02	.4920217	.2499363	.62	.2678269	.1960037
.03	.4860335	.2496566	.63	.2643473	.1944676
.04	.4840466	.2497455	.64	.2610663	.1929202
.05	.4600612	.2496024	.65	.2576461	.1913615
.06	.4760778	.2494277	.86	.2546269	.1897920
.07	.4720966	.2492214	.67	.2514269	.1882124
.08	.4661166	.2469836	.68	.2482522	.1666230
.09	.4641436	.2487143	.69	.2450971	.1650245
.10	.4601722	.2484137	.70	.2419637	.1634173
.11	.4562047	.2460820	.71	.2386521	.1616016
.12	.4522416	.2477191	.72	.2357625	.1601765
.13	.4482832	.2473254	.73	.2328951	.1765481
.14	.4443300	.2469009	.74	.2296500	.1769109
.15	.4403623	.2464457	.75	.2266274	.1752674
.16	.4364405	.2459802	.78	.2238273	.1736181
.17	.4325051	.2454444	.77	.2206499	.1719635
.16	.4285763	.2446987	.76	.2176954	.1703041
.19	.4246546	.2443231	.79	.2147639	.1686404
.20	.42(7403	.2437179	.80	.2116554	.1669727
.21	.4168338	.2430834	.81	.2069701	.1653016
.22	.4129358	.2424198	.62	.2081081	.1636276
.23	.4090459	.2417274	.63	.2032694	.1619510
.24	.4051651	.2410063	.84	.2004542	.1602723
.25	.4012937	.2402571	.85	.1978625	.1585920
.26	.3974319	.2394796	.88	.1948945	.1569106
.27	.3935601	.2366748	.67	.1921502	.1552285
.28	.3897368	.2378425	.88	.1694297	.1535481
.29	.3659081	.2389830	.69	.1667329	.1516637
.30	.3820888	.2380969	.90	.1840601	.1501620
.31	.3782805	.2351844	.91	.1814113	.1485012
.32	.3744842	.2342458	.92	.1787664	.1466218
.33	.3707000	.2332815	.93	.1781655	.1451442
.34	.3669283	.2322919	.94	.1736068	.1434686
.35	.3631893	.2312774	.95	.1710561	.1417959
.38	.3594238	.2302383	.96	.1685276	.1401260
.37	.3556912	.2291750	.97	.1660203	.1384595
.38	.3519727	.2280879	.98	.1635431	.1367968
.39	.3482683	.2269775	.99	.1610671	.1351380
.40	.3445783	.2258441	1.00	.1586553	.1334636
.41	.3409030	.2246881	1.01	.1562476	.1318343
.42	.3372427	.2235101	1.02	.1536642	.1301900
.43	.3335978	.2223103	1.03	.1515050	.1265512
.44	.3299688	.2210893	1.04	.1491700	.1269183
.45	.3263552	.2198475	1.05	.1468591	.1252915
.46	.3227581	.2185853	1.06	.1445723	.1236712
.47	.3191775	.2173032	1.07	.1423097	.1220576
.46	.3156137	.2160017	1.08	.1400711	.1204512
.49	.3120689	.2146611	1.09	.1378566	.1166522
.50	.3085375	.2133421	1.10	.1356661	.1172600
.51	.3050257	.2119850	1.11	.1334995	.1156774
.52	.3015318	.2106104	1.12	.1313569	.1141023
.53	.2980560	.2092186	1.13	.1292381	.1125356
.54	.2945985	.2078102	1.14	.1271432	.1109776
.55	.2911597	.2063857	1.15	.1250719	.1094289
.56	.2877397	.2049456	1.16	.1230244	.1078894
.57	.2843388	.2034902	1.17	.1210005	.1063594
.58	.2809573	.2020203	1.18	.1190001	.1046391
.59	.2775953	.2005361	1.19	.1170232	.1033288

TABLE I

STANDARD SAMPLING ERRORS
E NORMAL CURVE

P = 1 - Q



T	Q	PQ
.60	.2742531	.1990383
.81	.2709309	.1975273
.82	.2876289	.1960037
.83	.2843473	.1944878
.64	.2810863	.1929202
.65	.2578481	.1913615
.86	.2548289	.1897920
.87	.2514289	.1882124
.88	.2482522	.1866230
.89	.2450971	.1850245
.70	.2419637	.1834173
.71	.2388521	.1818018
.72	.2357825	.1801785
.73	.2326951	.1785481
.74	.2296500	.1769109
.75	.2266274	.1752674
.78	.2236273	.1736181
.77	.2206499	.1719635
.78	.2175954	.1703041
.79	.2147639	.1686404
.80	.2118554	.1669727
.81	.2089701	.1653013
.82	.2061081	.1636276
.83	.2032694	.1619510
.84	.2004542	.1602723
.85	.1976625	.1585920
.86	.1948945	.1569106
.87	.1921502	.1552285
.88	.1894297	.1535461
.89	.1867329	.1518637
.90	.1840601	.1501820
.91	.1814113	.1485012
.92	.1787864	.1468218
.93	.1761855	.1451442
.94	.1736088	.1434688
.95	.1710561	.1417959
.96	.1685276	.1401260
.97	.1660203	.1384595
.98	.1635431	.1367968
.99	.1610871	.1351380
1.00	.1586553	.1334838
1.01	.1562476	.1318343
1.02	.1538642	.1301900
1.03	.1515050	.1285512
1.04	.1491700	.1269183
1.05	.1468591	.1252915
1.06	.1445723	.1236712
1.07	.1423097	.1220576
1.08	.1400711	.1204512
1.09	.1378566	.1188522
1.10	.1356661	.1172608
1.11	.1334995	.1156774
1.12	.1313569	.1141023
1.13	.1292381	.1125356
1.14	.1271432	.1109778
1.15	.1250719	.1094289
1.16	.1230244	.1078894
1.17	.1210005	.1063594
1.18	.1190001	.1048391
1.19	.1170232	.1033288

TABLE I - Continued

T	Q	PQ
1.20	.1150897	.1018287
1.21	.1131394	.1003389
1.22	.1112324	.0988598
1.23	.1093486	.0973915
1.24	.1074877	.0959341
1.25	.1056498	.0944879
1.26	.1038347	.0930531
1.27	.1020423	.0916297
1.28	.1002726	.0902180
1.29	.0985253	.0888181
1.30	.0968005	.0874302
1.31	.0950979	.0860543
1.32	.0934175	.0846907
1.33	.0917591	.0833394
1.34	.0901227	.0820006
1.35	.0885080	.0806743
1.36	.0889150	.0793608
1.37	.0853435	.0780600
1.38	.0837933	.0767720
1.39	.0822644	.0754970
1.40	.0807567	.0742351
1.41	.0792898	.0729861
1.42	.0778038	.0717504
1.43	.0763585	.0705279
1.44	.0749337	.0693186
1.45	.0735293	.0681227
1.48	.0721450	.0669401
1.47	.0707809	.0657710
1.48	.0694366	.0646152
1.49	.0681121	.0634728
1.50	.0668072	.0623440
1.51	.0655217	.0612286
1.52	.0642555	.0601267
1.53	.0630084	.0590383
1.54	.0617802	.0579634
1.55	.0605708	.0569020
1.56	.0593799	.0558539
1.57	.0582076	.0548195
1.58	.0570534	.0537983
1.59	.0559174	.0527906
1.60	.0547993	.0517963
1.61	.0536989	.0508153
1.62	.0526161	.0498476
1.63	.0515507	.0488932
1.64	.0505026	.0479521
1.65	.0494715	.0470241
1.66	.0484572	.0461091
1.67	.0474597	.0452073
1.68	.0464787	.0443184
1.69	.0455140	.0434425
1.70	.0445655	.0425794
1.71	.0436329	.0417291
1.72	.0427162	.0408915
1.73	.0418151	.0400666
1.74	.0409295	.0392543
1.75	.0400592	.0384545
1.76	.0392039	.0376670
1.77	.0383636	.0368918
1.78	.0375380	.0361289
1.79	.0367270	.0353781
1.80	.0359303	.0346393
1.81	.0351479	.0339125
1.82	.0343795	.0331975
1.83	.0336250	.0324944
1.84	.0328841	.0318027
1.85	.0321568	.0311227
1.86	.0314428	.0304542
1.87	.0307419	.0297968
1.88	.0300540	.0291508
1.89	.0293790	.0285159
1.90	.0287166	.0278920
1.91	.0280666	.0272789
1.92	.0274289	.0266766
1.93	.0268034	.0260850
1.94	.0261898	.0255039
1.95	.0255881	.0249333
1.96	.0249979	.0243730
1.97	.0244192	.0238229
1.98	.0238518	.0232829
1.99	.0232955	.0227528
2.00	.0227501	.0222325
2.01	.0222156	.0217221
2.02	.0216917	.0212212
2.03	.0211783	.0207298
2.04	.0206752	.0202477
2.05	.0201822	.0197749
2.06	.0196993	.0193112
2.07	.0192262	.0188566
2.08	.0187628	.0184108
2.09	.0183089	.0179737
2.10	.0178644	.0175453
2.11	.0174292	.0171254
2.12	.0170030	.0167139
2.13	.0165858	.0163107
2.14	.0161774	.0159157
2.15	.0157776	.0155287
2.16	.0153863	.0151496
2.17	.0150034	.0147783
2.18	.0146287	.0144147
2.19	.0142621	.0140587
2.20	.0139034	.0137101
2.21	.0135526	.0133689
2.22	.0132094	.0130349
2.23	.0128737	.0127080
2.24	.0125455	.0123881
2.25	.0122245	.0120751
2.26	.0119106	.0117687
2.27	.0116038	.0114692
2.28	.0113038	.0111760
2.29	.0110107	.0108895
2.30	.0107241	.0106091
2.31	.0104441	.0103350
2.32	.0101704	.0100670
2.33	.0099031	.0098050
2.34	.0096419	.0095489
2.35	.0093867	.0092986
2.36	.0091375	.0090540
2.37	.0088940	.0088149
2.38	.0086563	.0085814
2.39	.0084242	.0083532
2.40	.0081975	.0081303
2.41	.0079763	.0079127
2.42	.0077603	.0077001
2.43	.0075494	.0074924
2.44	.0073436	.0072897
2.45	.0071428	.0070918
2.46	.0069469	.0068986
2.47	.0067557	.0067101
2.48	.0065691	.0065259
2.49	.0063872	.0063464
2.50	.0062097	.0061711
2.51	.0060366	.0060002

TABLE I - Continued

T	Q	PQ	T	Q	PQ
2.52	.0058677	.0058333	3.18	.0007364	.0007359
2.53	.0057031	.0056706	3.19	.0007114	.0007109
2.54	.0055426	.0055119	3.20	.0006871	.0006866
2.55	.0053861	.0053571	3.21	.0006637	.0006633
2.56	.0052336	.0052062	3.22	.0006410	.0006406
2.57	.0050849	.0050590	3.23	.0006190	.0006186
2.58	.0049400	.0049156	3.24	.0005976	.0005972
2.59	.0047988	.0047758	3.25	.0005770	.0005767
2.60	.0046612	.0046395	3.26	.0005571	.0005568
2.61	.0045271	.0045066	3.27	.0005377	.0005374
2.62	.0043965	.0043772	3.28	.0005190	.0005187
2.63	.0042692	.0042510	3.29	.0005009	.0005006
2.64	.0041453	.0041281	3.30	.0004834	.0004832
2.65	.0040246	.0040084	3.31	.0004665	.0004663
2.66	.0039070	.0038917	3.32	.0004501	.0004499
2.67	.0037926	.0037782	3.33	.0004342	.0004340
2.68	.0036811	.0036675	3.34	.0004189	.0004187
2.69	.0035726	.0035598	3.35	.0004041	.0004039
2.70	.0034670	.0034550	3.36	.0003897	.0003895
2.71	.0033642	.0033529	3.37	.0003758	.0003757
2.72	.0032641	.0032534	3.38	.0003624	.0003623
2.73	.0031667	.0031567	3.39	.0003495	.0003494
2.74	.0030720	.0030626	3.40	.0003369	.0003368
2.75	.0029708	.0029709	3.41	.0003248	.0003247
2.76	.0028901	.0028817	3.42	.0003131	.0003130
2.77	.0028028	.0027949	3.43	.0003018	.0003017
2.78	.0027179	.0027105	3.44	.0002909	.0002908
2.79	.0026354	.0026285	3.45	.0002803	.0002802
2.80	.0025551	.0025486	3.46	.0002701	.0002700
2.81	.0024771	.0024710	3.47	.0002602	.0002601
2.82	.0024012	.0023954	3.48	.0002507	.0002506
2.83	.0023274	.0023220	3.49	.0002415	.0002414
2.84	.0022557	.0022506	3.50	.0002326	.0002325
2.85	.0021860	.0021812	3.51	.0002241	.0002240
2.86	.0021182	.0021137	3.52	.0002158	.0002158
2.87	.0020524	.0020482	3.53	.0002078	.0002078
2.88	.0019884	.0019844	3.54	.0002001	.0002001
2.89	.0019262	.0019225	3.55	.0001926	.0001926
2.90	.0018658	.0018623	3.56	.0001854	.0001854
2.91	.0018071	.0018038	3.57	.0001785	.0001785
2.92	.0017502	.0017471	3.58	.0001718	.0001718
2.93	.0016948	.0016919	3.59	.0001653	.0001653
2.94	.0016411	.0016384	3.60	.0001591	.0001591
2.95	.0015889	.0015864	3.61	.0001531	.0001531
2.96	.0015382	.0015358	3.62	.0001473	.0001473
2.97	.0014890	.0014868	3.63	.0001417	.0001417
2.98	.0014412	.0014391	3.64	.0001363	.0001363
2.99	.0013949	.0013930	3.65	.0001311	.0001311
3.00	.0013449	.0013481	3.66	.0001261	.0001261
3.01	.0013062	.0013045	3.67	.0001213	.0001213
3.02	.0012639	.0012623	3.68	.0001166	.0001166
3.03	.0012228	.0012213	3.69	.0001121	.0001121
3.04	.0011829	.0011815	3.70	.0001078	.0001078
3.05	.0011442	.0011429	3.71	.0001036	.0001036
3.06	.0011067	.0011055	3.72	.0000996	.0000996
3.07	.0010703	.0010692	3.73	.0000957	.0000957
3.08	.0010350	.0010339	3.74	.0000920	.0000920
3.09	.0010008	.0009998	3.75	.0000884	.0000884
3.10	.0009676	.0009667	3.76	.0000850	.0000850
3.11	.0009354	.0009345	3.77	.0000816	.0000816
3.12	.0009043	.0009035	3.78	.0000784	.0000784
3.13	.0008740	.0008732	3.79	.0000753	.0000753
3.14	.0008447	.0008440	3.80	.0000723	.0000723
3.15	.0008164	.0008157	3.81	.0000695	.0000695
3.16	.0007888	.0007882	3.82	.0000667	.0000667
3.17	.0007622	.0007616	3.83	.0000641	.0000641

T	Q
3.84	.0000615
3.85	.0000591
3.86	.0000567
3.87	.0000544
3.88	.0000522
3.89	.0000501
3.90	.0000481
3.91	.0000461
3.92	.0000443
3.93	.0000425
3.94	.0000407
3.95	.0000391
3.96	.0000375
3.97	.0000359
3.98	.0000345
3.99	.0000330
4.00	.0000317
4.01	.0000304
4.02	.0000291
4.03	.0000279
4.04	.0000267
4.05	.0000256
4.06	.0000245
4.07	.0000235
4.08	.0000225
4.09	.0000216
4.10	.0000207
4.11	.0000198
4.12	.0000189
4.13	.0000181
4.14	.0000174
4.15	.0000166
4.16	.0000159
4.17	.0000152
4.18	.0000146
4.19	.0000139
4.20	.0000133
4.21	.0000128
4.22	.0000122
4.23	.0000117
4.24	.0000112
4.25	.0000107

Continued

T	Q	PQ
3.18	.0007364	.0007359
3.19	.0007114	.0007109
3.20	.0006871	.0006866
3.21	.0006637	.0006633
3.22	.0006410	.0006406
3.23	.0006190	.0006186
3.24	.0005976	.0005972
3.25	.0005770	.0005767
3.26	.0005571	.0005568
3.27	.0005377	.0005374
3.28	.0005190	.0005187
3.29	.0005009	.0005006
3.30	.0004834	.0004832
3.31	.0004665	.0004663
3.32	.0004501	.0004499
3.33	.0004342	.0004340
3.34	.0004189	.0004187
3.35	.0004041	.0004039
3.36	.0003897	.0003895
3.37	.0003758	.0003757
3.38	.0003624	.0003623
3.39	.0003495	.0003494
3.40	.0003369	.0003368
3.41	.0003248	.0003247
3.42	.0003131	.0003130
3.43	.0003018	.0003017
3.44	.0002909	.0002908
3.45	.0002803	.0002802
3.46	.0002701	.0002700
3.47	.0002602	.0002601
3.48	.0002507	.0002506
3.49	.0002415	.0002414
3.50	.0002326	.0002325
3.51	.0002241	.0002240
3.52	.0002158	.0002158
3.53	.0002078	.0002078
3.54	.0002001	.0002001
3.55	.0001926	.0001926
3.56	.0001854	.0001854
3.57	.0001785	.0001785
3.58	.0001718	.0001718
3.59	.0001653	.0001653
3.60	.0001591	.0001591
3.61	.0001531	.0001531
3.62	.0001473	.0001473
3.63	.0001417	.0001417
3.64	.0001363	.0001363
3.65	.0001311	.0001311
3.66	.0001261	.0001261
3.67	.0001213	.0001213
3.68	.0001166	.0001166
3.69	.0001121	.0001121
3.70	.0001078	.0001078
3.71	.0001036	.0001036
3.72	.0000996	.0000996
3.73	.0000957	.0000957
3.74	.0000920	.0000920
3.75	.0000884	.0000884
3.76	.0000850	.0000850
3.77	.0000816	.0000816
3.78	.0000784	.0000784
3.79	.0000753	.0000753
3.80	.0000723	.0000723
3.81	.0000695	.0000695
3.82	.0000667	.0000667
3.83	.0000641	.0000641

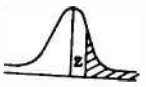
TABLE I - Continued

T	Q	PQ	T	Q	PQ
3.84	.0000615	.0000615	4.26	.0000102	.0000102
3.85	.0000591	.0000591	4.27	.0000098	.0000098
3.86	.0000567	.0000567	4.28	.0000093	.0000093
3.87	.0000544	.0000544	4.29	.0000089	.0000089
3.88	.0000522	.0000522	4.30	.0000085	.0000085
3.89	.0000501	.0000501	4.31	.0000082	.0000082
3.90	.0000481	.0000481	4.32	.0000078	.0000078
3.91	.0000461	.0000461	4.33	.0000075	.0000075
3.92	.0000443	.0000443	4.34	.0000071	.0000071
3.93	.0000425	.0000425	4.35	.0000068	.0000068
3.94	.0000407	.0000407	4.36	.0000065	.0000065
3.95	.0000391	.0000391	4.37	.0000062	.0000062
3.96	.0000375	.0000375	4.38	.0000059	.0000059
3.97	.0000359	.0000359	4.39	.0000057	.0000057
3.98	.0000345	.0000345	4.40	.0000054	.0000054
3.99	.0000330	.0000330	4.41	.0000052	.0000052
4.00	.0000317	.0000317	4.42	.0000049	.0000049
4.01	.0000304	.0000304	4.43	.0000047	.0000047
4.02	.0000291	.0000291	4.44	.0000045	.0000045
4.03	.0000279	.0000279	4.45	.0000043	.0000043
4.04	.0000267	.0000267	4.46	.0000041	.0000041
4.05	.0000256	.0000256	4.47	.0000039	.0000039
4.06	.0000245	.0000245	4.48	.0000037	.0000037
4.07	.0000235	.0000235	4.49	.0000036	.0000036
4.08	.0000225	.0000225	4.50	.0000034	.0000034
4.09	.0000216	.0000216	4.51	.0000032	.0000032
4.10	.0000207	.0000207	4.52	.0000031	.0000031
4.11	.0000198	.0000198	4.53	.0000030	.0000030
4.12	.0000189	.0000189	4.54	.0000028	.0000028
4.13	.0000181	.0000181	4.55	.0000027	.0000027
4.14	.0000174	.0000174	4.56	.0000026	.0000026
4.15	.0000166	.0000166	4.57	.0000024	.0000024
4.16	.0000159	.0000159	4.58	.0000023	.0000023
4.17	.0000152	.0000152	4.59	.0000022	.0000022
4.18	.0000146	.0000146	4.60	.0000021	.0000021
4.19	.0000139	.0000139	4.61	.0000020	.0000020
4.20	.0000133	.0000133	4.62	.0000019	.0000019
4.21	.0000128	.0000128	4.63	.0000018	.0000018
4.22	.0000122	.0000122	4.64	.0000017	.0000017
4.23	.0000117	.0000117	4.65	.0000017	.0000017
4.24	.0000112	.0000112	4.66	.0000016	.0000016
4.25	.0000107	.0000107	4.67	.0000015	.0000015

TABLE II

ORDINATES, PROBITS, AND PROBIT WEIGHTS ASSOCIATED WITH THE NORMAL CURVE OF ERROR.

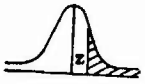
$$z = \frac{1}{\sqrt{2\pi}} Q^{-T^2/2}, \quad W = \frac{z}{PQ}$$



T	Z	W	T	Z	W
.00	.3989423	.63662	.59	.3352132	.56034
.01	.3989223	.63660	.60	.3332246	.55788
.02	.3988625	.63653	.61	.3312147	.55538
.03	.3987628	.63641	.62	.3291840	.55286
.04	.3986233	.63625	.63	.3271330	.55030
.05	.3984439	.63604	.64	.3250623	.54772
.06	.3982248	.63579	.65	.3229724	.54510
.07	.3979661	.63549	.66	.3208638	.54246
.08	.3976677	.63514	.67	.3187371	.53978
.09	.3973298	.63475	.68	.3165929	.53708
.10	.3969525	.63431	.69	.3144317	.53435
.11	.3965360	.63383	.70	.3122539	.53159
.12	.3960802	.63330	.71	.3100603	.52880
.13	.3955854	.63272	.72	.3078513	.52599
.14	.3950517	.63210	.73	.3056274	.52315
.15	.3944793	.63143	.74	.3033693	.52029
.16	.3938664	.63072	.75	.3011374	.51740
.17	.3932190	.62996	.76	.2988724	.51449
.18	.3925315	.62916	.77	.2965948	.51153
.19	.3918060	.62832	.78	.2943050	.50859
.20	.3910427	.62742	.79	.2920036	.50561
.21	.3902419	.62649	.80	.2896916	.50260
.22	.3894038	.62551	.81	.2873689	.49958
.23	.3885266	.62446	.82	.2850364	.49653
.24	.3876166	.62341	.83	.2826945	.49346
.25	.3866661	.62230	.84	.2803438	.49037
.26	.3856834	.62115	.85	.2779849	.48726
.27	.3846627	.6199-	.86	.2756182	.48413
.28	.3836063	.61870	.87	.2732444	.48098
.29	.3825146	.61742	.88	.2708640	.47782
.30	.3813876	.61609	.89	.2684774	.47464
.31	.3802264	.61472	.90	.2660852	.47144
.32	.3790305	.61331	.91	.2636880	.46822
.33	.3776007	.61165	.92	.2612863	.46499
.34	.3765372	.61035	.93	.2586805	.46174
.35	.3752403	.60862	.94	.2564713	.45848
.36	.3739106	.60724	.95	.2540591	.45520
.37	.3725483	.60562	.96	.2516443	.45191
.38	.3711539	.60396	.97	.2492277	.44861
.39	.3697277	.60226	.98	.2466095	.44530
.40	.3662701	.60052	.99	.2443904	.44197
.41	.3667617	.59874	1.00	.2419707	.43863
.42	.3652627	.59692	1.01	.2395511	.43528
.43	.3637136	.59506	1.02	.2371320	.43192
.44	.3621349	.59316	1.03	.2347138	.42855
.45	.3605270	.59123	1.04	.2322970	.42517
.46	.3586903	.58925	1.05	.2298821	.42178
.47	.3572253	.58724	1.06	.2274696	.41839
.48	.3555325	.58520	1.07	.2250599	.41498
.49	.3536124	.58311	1.08	.2226535	.41157
.50	.3520653	.58099	1.09	.2202506	.40816
.51	.3502919	.57884	1.10	.2178522	.40474
.52	.3484925	.57664	1.11	.2154582	.40131
.53	.3466677	.57442	1.12	.2140691	.39788
.54	.3448160	.57215	1.13	.2106856	.39444
.55	.3429439	.56966	1.14	.2083076	.39100
.56	.3410458	.56753	1.15	.2059363	.38756
.57	.3391243	.56516	1.16	.2035714	.38411
.58	.3371799	.56277	1.17	.2012135	.38066

T
1.18
1.19
1.20
1.21
1.22
1.23
1.24
1.25
1.26
1.27
1.28
1.29
1.30
1.31
1.32
1.33
1.34
1.35
1.36
1.37
1.38
1.39
1.40
1.41
1.42
1.43
1.44
1.45
1.46
1.47
1.48
1.49
1.50
1.51
1.52
1.53
1.54
1.55
1.56
1.57
1.58
1.59
1.60
1.61
1.62
1.63
1.64
1.65
1.66
1.67
1.68
1.69
1.70
1.71
1.72
1.73
1.74
1.75
1.76
1.77
1.78
1.79
1.80
1.81
1.82
1.83

T WEIGHTS ASSOCIATED
VE OF ERROR.



T	Z	W
	.3352132	.56034
	.3332246	.55788
	.3312147	.55538
	.3291840	.55286
	.3271330	.55030
	.3250623	.54772
	.3229724	.54510
	.3208638	.54246
	.3187371	.53978
	.3165929	.53708
	.3144317	.53435
	.3122539	.53159
	.3100603	.52880
	.3078513	.52599
	.3056274	.52315
	.3033893	.52029
	.3011374	.51740
	.2988724	.51449
	.2965948	.51153
	.2943050	.50859
	.2920038	.50561
	.2896916	.50260
	.2873689	.49958
	.2850364	.49653
	.2826945	.49346
	.2803438	.49037
	.2779849	.48726
	.2756182	.48413
	.2732444	.48098
	.2708640	.47782
	.2684774	.47464
	.2660852	.47144
	.2636880	.46822
	.2612863	.46499
	.2588805	46174
	.2564713	.45848
	.2540591	.45520
	.2516443	.45191
	.2492277	.44861
	.2468095	.44530
	.2443904	.44197
	.2419707	.43863
	.2395511	.43528
	.2371320	.43192
	.2347138	.42855
	.2322970	.42517
	.2298821	.42176
	.2274696	.41839
	.2250599	.41498
	.2226535	.41157
	.2202506	.40816
	.2178522	.40474
	.2154582	.40131
	.2140691	.39788
	.2106856	.39444
	.2083078	.39100
	.2059363	.38756
	.2035714	.38411
	.2012135	.38066

TABLES

TABLE II - Continued

T	Z	W	T	Z	W
1.18	.1966831	.37721	1.84	.0734068	.18944
1.19	.1965205	.37376	1.85	.0720649	.16687
1.20	.1941861	.37031	1.68	.0707404	.16432
1.21	.1918602	.36686	1.67	.0894333	.16180
1.22	.1695432	.36341	1.88	.0681438	.15929
1.23	.1872354	.35998	1.89	.0668711	.15861
1.24	.1649373	.35651	1.90	.0656158	.15438
1.25	.1626491	.35307	1.91	.0643777	.15193
1.28	.1803712	.34963	1.92	.0631566	.14952
1.27	.1781038	.34619	1.93	.0819524	.14714
1.28	.1756474	.34275	1.94	.0807852	.14478
1.29	.1736022	.33932	1.95	.0595947	.14244
1.30	.1713686	.33589	1.98	.0584409	.14013
1.31	.1691488	.33247	1.97	.0573038	.13784
1.32	.1889370	.32906	1.98	.0581831	.13557
1.33	.1847397	.32565	1.99	.0550789	.13333
1.34	.1625551	.32224	2.00	.0539910	.13112
1.35	.1603833	.31865	2.01	.0529192	.12892
1.38	.1582248	.31546	2.02	.0518638	.12875
1.37	.1580797	.31208	2.03	.0508239	.12461
1.38	.1539463	.30871	2.04	.0498001	.12249
1.39	.1518308	.30534	2.05	.0487920	.12039
1.40	.1497275	.30199	2.06	.0477996	.11832
1.41	.1476365	.29885	2.07	.0486226	.11828
1.42	.1455641	.29531	2.06	.0458811	.11424
1.43	.1435046	.29199	2.09	.0449148	.11224
1.44	.1414800	.28888	2.10	.0439836	.11026
1.45	.1394308	.26538	2.11	.0430674	.10831
1.48	.1374185	.26209	2.12	.0421681	.10638
1.47	.1354161	.27882	2.13	.0412795	.10447
1.48	.1334353	.27555	2.14	.0404076	.10259
1.49	.1314684	.27231	2.15	.0395500	.10073
1.50	.1295176	.28907	2.16	.0387089	.09690
1.51	.1275830	.26585	2.17	.0378779	.09708
1.52	.1256648	.26264	2.18	.0370629	.09530
1.53	.1237828	.25945	2.19	.0382619	.09353
1.54	.1218775	.25827	2.20	.0354746	.09179
1.55	.1200090	.25310	2.21	.0347009	.09007
1.56	.1161573	.24996	2.22	.0339408	.08838
1.57	.1163225	.24663	2.23	.0331939	.08870
1.58	.1145048	.24371	2.24	.0324603	.08506
1.59	.1127042	.24062	2.25	.0317397	.08343
1.60	.1109208	.23754	2.26	.0310319	.08182
1.61	.1091548	.23447	2.27	.0303370	.08021
1.62	.1074061	.23143	2.28	.0296546	.07889
1.63	.1056748	.22840	2.29	.0289847	.07715
1.64	.1039611	.22539	2.30	.0283270	.07564
1.65	.1022649	.22240	2.31	.0278816	.07414
1.68	.1005864	.21943	2.32	.0270481	.07267
1.67	.0989255	.21648	2.33	.0264265	.07122
1.68	.0972823	.21354	2.34	.0258166	.06980
1.69	.0956568	.21063	2.35	.0252182	.06839
1.70	.0940491	.20774	2.36	.0246313	.06701
1.71	.0924591	.20486	2.37	.0240558	.08585
1.72	.0908870	.20201	2.38	.0234910	.06430
1.73	.0893326	.19918	2.39	.0229374	.06298
1.74	.0877961	.19638	2.40	.0223945	.06169
1.75	.0862773	.19357	2.41	.0218624	.06040
1.76	.0847764	.19080	2.42	.0213407	.05915
1.77	.0832932	.18806	2.43	.0208294	.05791
1.78	.0818278	.18533	2.44	.0203284	.05689
1.79	.0803801	.16263	2.45	.0198374	.05549
1.80	.0789502	.17994	2.48	.0193563	.05431
1.81	.0775379	.17726	2.47	.0188850	.05315
1.82	.0761433	.17465	2.48	.0184233	.05201
1.83	.0747663	.17203	2.49	.0179711	.05089

TABLE II - Continued

T	Z	W	T	Z	W
2.50	.0175283	.04979	3.16	.0027075	.00930
2.51	.0170947	.04870	3.17	.0026231	.00903
2.52	.0166701	.04764	3.18	.0025412	.00878
2.53	.0162545	.04659	3.19	.0024615	.00852
2.54	.0158476	.04556	3.20	.0023841	.00828
2.55	.0154493	.04455	3.21	.0023089	.00804
2.56	.0150596	.04356	3.22	.0022358	.00780
2.57	.0146782	.04259	3.23	.0021649	.00758
2.58	.0143051	.04163	3.24	.0020960	.00736
2.59	.0139401	.04069	3.25	.0020290	.00714
2.60	.0135830	.03977	3.26	.0019641	.00693
2.61	.0132337	.03886	3.27	.0019010	.00672
2.62	.0128921	.03797	3.28	.0018397	.00653
2.63	.0125581	.03710	3.29	.0017803	.00633
2.64	.0122315	.03624	3.30	.0017228	.00614
2.65	.0119122	.03540	3.31	.0018888	.00596
2.66	.0116001	.03458	3.32	.0016122	.00578
2.67	.0112951	.03377	3.33	.0015595	.00560
2.68	.0109969	.03297	3.34	.0015084	.00543
2.69	.0107056	.03220	3.35	.0014587	.00527
2.70	.0104209	.03143	3.38	.0014106	.00511
2.71	.0101428	.03068	3.37	.0013639	.00495
2.72	.0098712	.02995	3.38	.0013187	.00480
2.73	.0096058	.02923	3.39	.0012748	.00465
2.74	.0093466	.02852	3.40	.0012322	.00451
2.75	.0090936	.02783	3.41	.0011910	.00437
2.76	.0088465	.02718	3.42	.0011510	.00423
2.77	.0086052	.02649	3.43	.0011122	.00410
2.78	.0083697	.02584	3.44	.0010747	.00397
2.79	.0081398	.02521	3.45	.0010383	.00385
2.80	.0079155	.02458	3.46	.0010030	.00373
2.81	.0076965	.02397	3.47	.0009689	.00361
2.82	.0074829	.02338	3.48	.0009358	.00349
2.83	.0072744	.02279	3.49	.0009037	.00338
2.84	.0070711	.02222	3.50	.0008727	.00328
2.85	.0068728	.02166	3.51	.0008426	.00317
2.86	.0066793	.02111	3.52	.0008135	.00307
2.87	.0064907	.02057	3.53	.0007853	.00297
2.88	.0063067	.02004	3.54	.0007581	.00287
2.89	.0061274	.01953	3.55	.0007317	.00278
2.90	.0059525	.01903	3.56	.0007061	.00269
2.91	.0057821	.01853	3.57	.0006814	.00260
2.92	.0056160	.01805	3.58	.0006575	.00252
2.93	.0054541	.01758	3.59	.0006343	.00243
2.94	.0052963	.01712	3.60	.0006119	.00235
2.95	.0051426	.01667	3.61	.0005902	.00228
2.96	.0049929	.01623	3.62	.0005693	.00220
2.97	.0048470	.01580	3.63	.0005490	.00212
2.98	.0047050	.01538	3.64	.0005294	.00206
2.99	.0045666	.01497	3.65	.0005105	.00199
3.00	.0044318	.01457	3.66	.0004921	.00192
3.01	.0043007	.01418	3.67	.0004744	.00186
3.02	.0041729	.01380	3.68	.0004573	.00179
3.03	.0040486	.01342	3.69	.0004408	.00173
3.04	.0039276	.01306	3.70	.0004248	.00167
3.05	.0038098	.01270	3.71	.0004093	.00162
3.06	.0036951	.01235	3.72	.0003944	.00156
3.07	.0035836	.01201	3.73	.0003800	.00151
3.08	.0034751	.01168	3.74	.0003661	.00146
3.09	.0033695	.01136	3.75	.0003526	.00141
3.10	.0032668	.01104	3.76	.0003396	.00136
3.11	.0031669	.01073	3.77	.0003271	.00131
3.12	.0030698	.01043	3.78	.0003149	.00126
3.13	.0029754	.01014	3.79	.0003032	.00122
3.14	.0028835	.00985	3.80	.0002919	.00118
3.15	.0027943	.00957	3.81	.0002810	.00114

Continued

T	Z	W
3.16	.0027075	.00930
3.17	.0026231	.00903
3.18	.0025412	.00878
3.19	.0024615	.00852
3.20	.0023841	.00828
3.21	.0023089	.00804
3.22	.0022358	.00780
3.23	.0021649	.00758
3.24	.0020960	.00736
3.25	.0020290	.00714
3.26	.0019641	.00693
3.27	.0019010	.00672
3.28	.0018397	.00653
3.29	.0017803	.00633
3.30	.0017226	.00614
3.31	.0018886	.00596
3.32	.0016122	.00578
3.33	.0015595	.00560
3.34	.0015084	.00543
3.35	.0014587	.00527
3.36	.0014108	.00511
3.37	.0013639	.00495
3.38	.0013187	.00480
3.39	.0012748	.00465
3.40	.0012322	.00451
3.41	.0011910	.00437
3.42	.0011510	.00423
3.43	.0011122	.00410
3.44	.0010747	.00397
3.45	.0010383	.00385
3.46	.0010030	.00373
3.47	.0009689	.00361
3.48	.0009358	.00349
3.49	.0009037	.00338
3.50	.0008727	.00328
3.51	.0008426	.00317
3.52	.0008135	.00307
3.53	.0007853	.00297
3.54	.0007581	.00287
3.55	.0007317	.00278
3.56	.0007081	.00269
3.57	.0006814	.00260
3.58	.0006575	.00252
3.59	.0006343	.00243
3.60	.0006119	.00235
3.61	.0005902	.00228
3.62	.0005693	.00220
3.63	.0005490	.00212
3.64	.0005294	.00206
3.65	.0005105	.00199
3.66	.0004921	.00192
3.67	.0004744	.00186
3.68	.0004573	.00179
3.69	.0004408	.00173
3.70	.0004248	.00167
3.71	.0004093	.00162
3.72	.0003944	.00156
3.73	.0003800	.00151
3.74	.0003661	.00146
3.75	.0003526	.00141
3.76	.0003396	.00136
3.77	.0003271	.00131
3.78	.0003149	.00126
3.79	.0003032	.00122
3.80	.0002919	.00118
3.81	.0002810	.00114

TABLE II - Continued

T	Z	W	T	Z	W
3.82	.0002705	.00110	4.25	.0000477	.00021
3.83	.0002604	.00106	4.26	.0000457	.00020
3.84	.0002506	.00102	4.27	.0000438	.00020
3.85	.0002411	.00098	4.28	.0000420	.00019
3.86	.0002320	.00095	4.29	.0000402	.00018
3.87	.0002232	.00092	4.30	.0000385	.00017
3.88	.0002147	.00088	4.31	.0000369	.00017
3.89	.0002065	.00085	4.32	.0000354	.00016
3.90	.0001987	.00082	4.33	.0000339	.00015
3.91	.0001910	.00079	4.34	.0000324	.00015
3.92	.0001837	.00076	4.35	.0000310	.00014
3.93	.0001766	.00073	4.36	.0000297	.00014
3.94	.0001698	.00071	4.37	.0000284	.00013
3.95	.0001633	.00068	4.38	.0000272	.00012
3.96	.0001569	.00066	4.39	.0000261	.00012
3.97	.0001508	.00063	4.40	.0000249	.00012
3.98	.0001449	.00061	4.41	.0000239	.00011
3.99	.0001393	.00059	4.42	.0000228	.00011
4.00	.0001338	.00056	4.43	.0000218	.00010
4.01	.0001288	.00054	4.44	.0000209	.00010
4.02	.0001235	.00052	4.45	.0000200	.00009
4.03	.0001186	.00050	4.46	.0000191	.00009
4.04	.0001140	.00049	4.47	.0000183	.00009
4.05	.0001094	.00047	4.48	.0000175	.00008
4.06	.0001051	.00045	4.49	.0000167	.00008
4.07	.0001009	.00043	4.50	.0000160	.00008
4.08	.0000969	.00042	4.51	.0000153	.00007
4.09	.0000930	.00040	4.52	.0000146	.00007
4.10	.0000893	.00038	4.53	.0000140	.00006
4.11	.0000857	.00037	4.54	.0000133	.00006
4.12	.0000822	.00036	4.55	.0000127	.00006
4.13	.0000789	.00034	4.56	.0000122	.00008
4.14	.0000757	.00033	4.57	.0000116	.00006
4.15	.0000726	.00032	4.58	.0000111	.00005
4.16	.0000697	.00031	4.59	.0000106	.00005
4.17	.0000668	.00029	4.60	.0000101	.00005
4.18	.0000641	.00028	4.61	.0000097	.00005
4.19	.0000615	.00027	4.62	.0000092	.00004
4.20	.0000589	.00026	4.63	.0000088	.00004
4.21	.0000565	.00025	4.84	.0000084	.00004
4.22	.0000542	.00024	4.85	.0000080	.00004
4.23	.0000519	.00023	4.66	.0000077	.00004
4.24	.0000498	.00022	4.67	.0000073	.00004

Table III*

TABLES OF PERCENTAGE POINTS OF THE INVERTED BETA (F) DISTRIBUTION

F DISTRIBUTION: 50 PER CENT POINTS

ν_2 \ ν_1	1	2	3	4	5	6	7	8	9
1	1·0000	1·5000	1·7092	1·8227	1·8937	1·9422	1·9774	2·0041	2·0250
2	0·66667	1·0000	1·1349	1·2071	1·2519	1·2824	1·3045	1·3213	1·3344
3	·58506	0·88110	1·0000	1·0632	1·1024	1·1289	1·1482	1·1627	1·1741
4	·54863	·82843	0·94054	1·0000	1·0367	1·0617	1·0797	1·0933	1·1040
5	0·52807	0·79877	0·90715	0·96456	1·0000	1·0240	1·0414	1·0545	1·0648
6	·51489	·77976	·88578	·94191	0·97654	1·0000	1·0169	1·0298	1·0398
7	·50572	·76655	·87095	·92619	·96026	0·98334	1·0000	1·0126	1·0224
8	·49898	·75683	·86004	·91464	·94831	·97111	0·98757	1·0000	1·0097
9	·49382	·74938	·85168	·90580	·93916	·96175	·97805	0·99037	1·0000
10	0·48973	0·74349	0·84508	0·89882	0·93193	0·95436	0·97054	0·98276	0·99232
11	·48644	·73872	·83973	·89316	·92608	·94837	·96445	·97661	·98610
12	·48369	·73477	·83530	·88848	·92124	·94342	·95943	·97152	·98097
13	·48141	·73145	·83159	·88454	·91718	·93926	·95520	·96724	·97665
14	·47944	·72862	·82842	·88119	·91371	·93573	·95161	·96360	·97298
15	0·47775	0·72619	0·82569	0·87830	0·91073	0·93267	0·94850	0·96046	0·96981
16	·47628	·72406	·82330	·87578	·90812	·93001	·94580	·95773	·96705
17	·47499	·72219	·82121	·87357	·90584	·92767	·94342	·95532	·96462
18	·47385	·72053	·81936	·87161	·90381	·92560	·94132	·95319	·96247
19	·47284	·71906	·81771	·86987	·90200	·92375	·93944	·95129	·96056
20	0·47192	0·71773	0·81621	0·86830	0·90038	0·92210	0·93776	0·94959	0·95884
21	·47108	·71653	·81487	·86688	·89891	·92060	·93624	·94805	·95728
22	·47033	·71545	·81365	·86559	·89759	·91924	·93486	·94665	·95588
23	·46965	·71446	·81255	·86442	·89638	·91800	·93360	·94538	·95459
24	·46902	·71356	·81153	·86335	·89527	·91687	·93245	·94422	·95342
25	0·46844	0·71272	0·81061	0·86236	0·89425	0·91583	0·93140	0·94315	0·95234
26	·46793	·71195	·80975	·86145	·89331	·91487	·93042	·94217	·95135
27	·46744	·71124	·80894	·86061	·89244	·91399	·92952	·94126	·95044
28	·46697	·71059	·80820	·85983	·89164	·91317	·92869	·94041	·94958
29	·46654	·70999	80753	·85911	·89089	·91241	·92791	·93963	·94879
30	0·46616	0·70941	0·80689	0·85844	0·89019	0·91169	0·92719	0·93889	0·94805
40	·46330	·70531	·80228	·85357	·88516	·90654	·92197	·93361	·94272
60	·46053	·70122	79770	·84873	·88017	·90144	·91679	·92838	·93743
120	·45774	·69717	·79314	·84392	·87521	·89637	·91164	·92318	·93218
∞	·45494	·69315	·78866	·83918	·87029	·89135	·90654	·91802	·92698

This table gives the values of F for which $I_F(\nu_1, \nu_2) = 0{\cdot}50$.

*Reproduced by special permission of the editors of Biometrika, from vol. 33, pp. 78-87, "Tables of Percentage Points of the Inverted Beta (F) Distribution," by M. Merrington and C. M. Thompson.

able III*

E POINTS OF THE INVERTED DISTRIBUTION

: 50 PER CENT POINTS

	5	6	7	8	9
	1·8937	1·9422	1·9774	2·0041	2·0250
	1·2519	1·2824	1·3045	1·3213	1·3344
	1·1024	1·1289	1·1482	1·1527	1·1741
	1·0367	1·0617	1·0797	1·0933	1·1040
	1·0000	1·0240	1·0414	1·0545	1·0648
	0·97654	1·0000	1·0169	1·0298	1·0393
	·96026	0·98334	1·0000	1·0126	1·0224
	·94831	·97111	0·98757	1·0000	1·0097
	·93916	·96175	·97805	0·99037	1·0000
	0·93193	0·95436	0·97054	0·98276	0·99232
	·92608	·94837	·96445	·97661	·98610
	·92124	·94342	·95943	·97152	·98097
	·91718	·93926	·95520	·96724	·97665
	·91371	·93573	·95161	·96360	·97298
	0·91073	0·93267	0·94850	0·96046	0·96981
	·90812	·93001	·94580	·95773	·96705
	·90584	·92767	·94342	·95532	·96462
	·90381	·92560	·94132	·95319	·96247
	·90200	·92375	·93944	·95129	·96056
	0·90038	0·92210	0·93776	0·94959	0·95884
	·89891	·92060	·93624	·94805	·95728
	·89759	·91924	·93486	·94665	·95588
	·89638	·91800	·93360	·94538	·95459
	·89527	·91687	·93245	·94422	·95342
	0·89425	0·91583	0·93140	0·94315	0·95234
	·89331	·91487	·93042	·94217	·95135
	·89244	·91399	·92952	·94126	·95044
	·89164	·91317	·92869	·94041	·94958
	·89089	·91241	·92791	·93963	·94879
	0·89019	0·91169	0·92719	0·93889	0·94805
	·88518	·90654	·92197	·93361	·94272
	·88017	·90144	·91679	·92838	·93743
	·87521	·89637	·91164	·92318	·93218
	·87029	·89135	·90654	·91802	·92698

of F for which $I_F(\nu_1, \nu_2) = 0{\cdot}50$.

ssion of the editors of Biometrika, bles of Percentage Points of the a," by M. Merrington and C. M.

Table III – Continued

F DISTRIBUTION: 50 PER CENT POINTS

ν_2 \ ν_1	10	12	15	20	24	30	40	60	120	∞
1	2·0419	2·0674	2·0931	2·1190	2·1321	2·1452	2·1584	2·1716	2·1848	2·1981
2	1·3450	1·3610	1·3771	1·3933	1·4014	1·4096	1·4178	1·4261	1·4344	1·4427
3	1·1833	1·1972	1·2111	1·2252	1·2322	1·2393	1·2464	1·2536	1·2608	1·2680
4	1·1126	1·1255	1·1386	1·1517	1·1583	1·1649	1·1716	1·1782	1·1849	1·1916
5	1·0730	1·0855	1·0980	1·1106	1·1170	1·1234	1·1297	1·1361	1·1426	1·1490
6	1·0478	1·0600	1·0722	1·0845	1·0907	1·0969	1·1031	1·1093	1·1156	1·1219
7	1·0304	1·0423	1·0543	1·0664	1·0724	1·0785	1·0846	1·0908	1·0969	1·1031
8	1·0175	1·0293	1·0412	1·0531	1·0591	1·0651	1·0711	1·0771	1·0832	1·0893
9	1·0077	1·0194	1·0311	1·0429	1·0489	1·0548	1·0608	1·0667	1·0727	1·0788
10	1·0000	1·0116	1·0232	1·0349	1·0408	1·0467	1·0526	1·0585	1·0645	1·0705
11	0·99373	1·0052	1·0168	1·0284	1·0343	1·0401	1·0460	1·0519	1·0578	1·0637
12	·98856	1·0000	1·0115	1·0231	1·0289	1·0347	1·0405	1·0464	1·0523	1·0582
13	·98421	0·99580	1·0071	1·0186	1·0243	1·0301	1·0360	1·0418	1·0476	1·0535
14	·98051	·99186	1·0033	1·0147	1·0205	1·0263	1·0321	1·0379	1·0437	1·0495
15	0·97732	0·98863	1·0000	1·0114	1·0172	1·0229	1·0287	1·0345	1·0403	1·0461
16	·97454	·98582	0·99716	1·0086	1·0143	1·0200	1·0258	1·0315	1·0373	1·0431
17	·97209	·98334	·99468	1·0060	1·0117	1·0174	1·0232	1·0289	1·0347	1·0405
18	·96993	·98116	·99245	1·0038	1·0095	1·0152	1·0209	1·0267	1·0324	1·0382
19	·96800	·97920	·99047	1·0018	1·0075	1·0132	1·0189	1·0246	1·0304	1·0361
20	0·96628	0·97746	0·98870	1·0000	1·0057	1·0114	1·0171	1·0228	1·0285	1·0343
21	·96470	·97587	·98710	0·99838	1·0040	1·0097	1·0154	1·0211	1·0268	1·0326
22	·96328	·97444	·98565	·99692	1·0026	1·0082	1·0139	1·0196	1·0253	1·0311
23	·96199	·97313	·98433	·99558	1·0012	1·0069	1·0126	1·0183	1·0240	1·0297
24	·96081	·97194	·98312	·99436	1·0000	1·0057	1·0113	1·0170	1·0227	1·0284
25	0·95972	0·97084	0·98201	0·99324	0·99887	1·0045	1·0102	1·0159	1·0215	1·0273
26	·95872	·98983	·98099	·99220	·99783	1·0035	1·0091	1·0148	1·0205	1·0262
27	·95779	·96889	·98004	·99125	·99687	1·0025	1·0082	1·0138	1·0195	1·0252
28	95694	·98802	·97917	·99036	·99598	1·0016	1·0073	1·0129	1·0186	1·0243
29	95614	·96722	·97835	·98954	·99515	1·0008	1·0064	1·0121	1·0177	1·0234
30	0·95540	0·96647	0·97759	0·98877	0·99438	1·0000	1·0056	1·0113	1·0170	1·0226
40	·95003	·96104	·97211	·98323	·98880	0·99440	1·0000	1·0056	1·0113	1·0169
60	·94471	·95568	·96667	·97773	·98328	·98884	0·99441	1·0000	1·0056	1·0112
120	93943	·95032	·96128	·97228	·97780	·98339	·98887	0·99443	1·0000	1·0056
∞	·93418	·94503	·95593	·96687	97236	·97787	·98339	·98891	0·99445	1·0000

$$F = \frac{s_1^2}{s_2^2} = \frac{\nu_2 N_1}{\nu_1 N_2}$$

Table III – Continued

F DISTRIBUTION: 25 PER CENT POINTS

ν_2 \ ν_1	1	2	3	4	5	6	7	8	9
1	5·8285	7·5000	8·1999	8·5810	8·8198	8·9833	9·1021	9·1922	9·2631
2	2·5714	3·0000	3·1534	3·2320	3·2799	3·3121	3·3352	3·3526	3·3661
3	2·0239	2·2798	2·3555	2·3901	2·4095	2·4218	2·4302	2·4364	2·4410
4	1·8074	2·0000	2·0487	2·0642	2·0723	2·0766	2·0790	2·0805	2·0814
5	1·6925	1·8528	1·8843	1·8927	1·8947	1·8945	1·8935	1·8923	1·8911
6	1·6214	1·7622	1·7844	1·7872	1·7852	1·7821	1·7789	1·7760	1·7733
7	1·5732	1·7010	1·7169	1·7157	1·7111	1·7059	1·7011	1·6969	1·6931
8	1·5384	1·6569	1·6683	1·6642	1·6575	1·6508	1·6448	1·6396	1·6350
9	1·5121	1·6236	1·6315	1·6253	1·6170	1·6091	1·6022	1·5961	1·5909
10	1·4915	1·5975	1·6028	1·5949	1·5853	1·5765	1·5688	1·5621	1·5563
11	1·4749	1·5767	1·5798	1·5704	1·5598	1·5502	1·5418	1·5346	1·5284
12	1·4613	1·5595	1·5609	1·5503	1·5389	1·5286	1·5197	1·5120	1·5054
13	1·4500	1·5452	1·5451	1·5336	1·5214	1·5105	1·5011	1·4931	1·4861
14	1·4403	1·5331	1·5317	1·5194	1·5068	1·4952	1·4854	1·4770	1·4697
15	1·4321	1·5227	1·5202	1·5071	1·4938	1·4820	1·4718	1·4631	1·4556
16	1·4249	1·5137	1·5103	1·4965	1·4827	1·4705	1·4601	1·4511	1·4433
17	1·4186	1·5057	1·5015	1·4873	1·4730	1·4606	1·4497	1·4405	1·4325
18	1·4130	1·4988	1·4938	1·4790	1·4644	1·4516	1·4406	1·4312	1·4230
19	1·4081	1·4925	1·4870	1·4717	1·4568	1·4437	1·4325	1·4228	1·4145
20	1·4037	1·4870	1·4808	1·4652	1·4500	1·4366	1·4252	1·4153	1·4069
21	1·3997	1·4820	1·4753	1·4593	1·4438	1·4302	1·4186	1·4086	1·4000
22	1·3961	1·4774	1·4703	[illegible]	1·4382	1·4244	1·4126	1·4025	1·3937
23	1·3928	1·4733	1·4658	1·4491	1·4331	1·4191	1·4072	1·3969	1·3880
24	1·3898	1·4695	1·4618	1·4447	1·4285	1·4143	1·4022	1·3918	1·3828
25	1·3870	1·4661	1·4577	1·4406	1·4242	1·4099	1·3976	1·3871	1·3780
26	1·3845	1·4629	1·4542	1·4368	1·4203	1·4058	1·3935	1·3828	1·3737
27	1·3822	1·4600	1·4510	1·4334	1·4166	1·4021	1·3896	1·3788	1·3696
28	1·3800	1·4572	1·4480	1·4302	1·4133	1·3986	1·3860	1·3752	1·3658
29	1·3780	1·4547	1·4452	1·4272	1·4102	1·3953	1·3826	1·3717	1·3623
30	1·3761	1·4524	1·4426	1·4244	1·4073	1·3923	1·3795	1·3685	1·3590
40	1·3626	1·4355	1·4239	1·4045	1·3863	1·3706	1·3571	1·3455	1·3354
60	1·3493	1·4188	1·4055	1·3848	1·3657	1·3491	1·3349	1·3226	1·3119
120	1·3362	1·4024	1·3873	1·3654	1·3453	1·3278	1·3128	1·2999	1·2886
∞	1·3233	1·3863	1·3694	1·3463	1·3251	1·3068	1·2910	1·2774	1·2654

This table gives the values of F for which $I_F(\nu_1, \nu_2) = 0{\cdot}25$.

ν_2 \ ν_1	10	12	
1	9·3202	9·4064	
2	3·3770	3·3934	
3	2·4447	2·4500	
4	2·0820	2·0826	
5	1·8899	1·8877	
6	1·7708	1·7668	
7	1·6898	1·6843	
8	1·6310	1·6244	
9	1·5863	1·5788	
10	1·5513	1·5430	
11	1·5230	1·5140	
12	1·4996	1·4902	
13	1·4801	1·4701	
14	1·4634	1·4530	
15	1·4491	1·4383	
16	1·4366	1·4255	
17	1·4256	1·4142	
18	1·4159	1·4042	
19	1·4073	1·3953	
20	1·3995	1·3873	
21	1·3925	1·3801	
22	1·3861	1·3735	
23	1·3803	1·3675	
24	1·3750	1·3621	
25	1·3701	1·3570	
26	1·3656	1·3524	
27	1·3615	1·3481	
28	1·3576	1·3441	
29	1·3541	1·3404	
30	1·3507	1·3369	
40	1·3266	1·3119	
60	1·3026	1·2870	
120	1·2787	1·2621	
∞	1·2549	1·2371	

Table III – Continued

F DISTRIBUTION: 25 PER CENT POINTS

ν_2 \ ν_1	10	12	15	20	24	30	40	60	120	∞
1	9·3202	9·4064	9·4934	9·5813	9·6255	9·6696	9·7144	9·7591	9·8041	9·8492
2	3·3770	3·3934	3·4098	3·4263	3·4345	3·4426	3·4511	3·4594	3·4677	3·4761
3	2·4447	2·4580	2·4552	2·4602	2·4626	2·4650	2·4674	2·4697	2·4720	2·4742
4	2·0820	2·0826	2·0829	2·0828	2·0827	2·0825	2·0821	2·0617	2·0612	2·0806
5	1·8899	1·8877	1·8851	1·8820	1·8802	1·8784	1·8763	1·8742	1·8719	1·8694
6	1·7708	1·7668	1·7621	1·7569	1·7540	1·7510	1·7477	1·7443	1·7407	1·7368
7	1·6898	1·6843	1·6781	1·6712	1·6675	1·6635	1·6593	1·6548	1·6502	1·6452
8	1·6310	1·6244	1·6170	1·6086	1·6043	1·5996	1·5945	1·5892	1·5836	1·5777
9	1·5863	1·5788	1·5705	1·5611	1·5560	1·5506	1·5450	1·5389	1·5325	1·5257
10	1·5513	1·5430	1·5338	1·5235	1·5179	1·5119	1·5056	1·4990	1·4919	1·4843
11	1·5230	1·5140	1·5041	1·4930	1·4869	1·4805	1·4737	1·4664	1·4587	1·4504
12	1·4996	1·4902	1·4796	1·4678	1·4613	1·4544	1·4471	1·4393	1·4310	1·4221
13	1·4801	1·4701	1·4590	1·4465	1·4397	1·4324	1·4247	1·4164	1·4075	1·3980
14	1·4634	1·4530	1·4414	1·4284	1·4212	1·4136	1·4055	1·3967	1·3874	1·3772
15	1·4491	1·4383	1·4263	1·4127	1·4052	1·3973	1·3888	1·3796	1·3698	1·3591
16	1·4366	1·4255	1·4130	1·3990	1·3913	1·3830	1·3742	1·3646	1·3543	1·3432
17	1·4256	1·4142	1·4014	1·3869	1·3790	1·3704	1·3613	1·3514	1·3406	1·3290
18	1·4159	1·4042	1·3911	1·3762	1·3680	1·3592	1·3497	1·3395	1·3284	1·3162
19	1·4073	1·3953	1·3819	1·3666	1·3582	1·3492	1·3394	1·3289	1·3174	1·3048
20	1·3995	1·3873	1·3736	1·3580	1·3494	1·3401	1·3301	1·3193	1·3074	1·2943
21	1·3925	1·3801	1·3661	1·3502	1·3414	1·3319	1·3217	1·3105	1·2983	1·2848
22	1·3861	1·3735	1·3593	1·3431	1·3341	1·3245	1·3140	1·3025	1·2900	1·2761
23	1·3803	1·3675	1·3531	1·3366	1·3275	1·3176	1·3069	1·2952	1·2824	1·2681
24	1·3750	1·3621	1·3474	1·3307	1·3214	1·3113	1·3004	1·2885	1·2754	1·2607
25	1·3701	1·3570	1·3422	1·3252	1·3158	1·3056	1·2945	1·2823	1·2689	1·2538
26	1·3656	1·3524	1·3374	1·3202	1·3106	1·3002	1·2889	1·2765	1·2628	1·2474
27	1·3615	1·3481	1·3329	1·3155	1·3058	1·2953	1·2838	1·2712	1·2572	1·2414
28	1·3578	1·3441	1·3288	1·3112	1·3013	1·2906	1·2790	1·2662	1·2519	1·2358
29	1·3541	1·3404	1·3249	1·3071	1·2971	1·2863	1·2745	1·2615	1·2470	1·2306
30	1·3507	1·3369	1·3213	1·3033	1·2933	1·2823	1·2703	1·2571	1·2424	1·2256
40	1·3266	1·3119	1·2952	1·2758	1·2649	1·2529	1·2397	1·2249	1·2080	1·1883
60	1·3026	1·2870	1·2691	1·2481	1·2361	1·2229	1·2081	1·1912	1·1715	1·1474
120	1·2787	1·2621	1·2428	1·2200	1·2068	1·1921	1·1752	1·1555	1·1314	1·0987
∞	1·2549	1·2371	1·2163	1·1914	1·1767	1·1600	1·1404	1·1164	1·0838	1·0880

$$F = \frac{s_1^2/\sigma_1^2 - \nu_2 N_1}{s_2^2/\sigma_2^2 - \nu_1 N_2}$$

Table III – Continued

F DISTRIBUTION: 10 PER CENT POINTS

ν_2 \ ν_1	1	2	3	4	5	6	7	8	9
1	39·864	49·500	53·593	55·833	57·241	58·204	58·906	59·439	59·858
2	8·5263	9·0000	9·1618	9·2434	9·2926	9·3255	9·3491	9·3668	9·3805
3	5·5383	5·4624	5·3908	5·3427	5·3092	5·2847	5·2662	5·2517	5·2400
4	4·5448	4·3246	4·1908	4·1073	4·0506	4·0098	3·9790	3·9549	3·9357
5	4·0604	3·7797	3·6195	3·5202	3·4530	3·4045	3·3679	3·3393	3·3163
6	3·7760	3·4633	3·2888	3·1808	3·1075	3·0546	3·0145	2·9830	2·9577
7	3·5894	3·2574	3·0741	2·9605	2·8833	2·8274	2·7849	2·7516	2·7247
8	3·4579	3·1131	2·9238	2·8064	2·7265	2·6683	2·6241	2·5893	2·5612
9	3·3603	3·0065	2·8129	2·6927	2·6106	2·5509	2·5053	2·4694	2·4403
10	3·2850	2·9245	2·7277	2·6053	2·5216	2·4606	2·4140	2·3772	2·3473
11	3·2252	2·8595	2·6602	2·5382	2·4512	2·3891	2·3416	2·3040	2·2735
12	3·1765	2·8068	2·6055	2·4801	2·3940	2·3310	2·2828	2·2446	2·2135
13	3·1362	2·7632	2·5603	2·4337	2·3467	2·2830	2·2341	2·1953	2·1638
14	3·1022	2·7265	2·5222	2·3947	2·3069	2·2426	2·1931	2·1539	2·1220
15	3·0732	2·6952	2·4898	2·3614	2·2730	2·2081	2·1582	2·1185	2·0862
16	3·0481	2·6682	2·4618	2·3327	2·2438	2·1783	2·1280	2·0880	2·0553
17	3·0262	2·6446	2·4374	2·3077	2·2183	2·1524	2·1017	2·0613	2·0284
18	3·0070	2·6239	2·4160	2·2858	2·1958	2·1296	2·0785	2·0379	2·0047
19	2·9899	2·6056	2·3970	2·2663	2·1760	2·1094	2·0580	2·0171	1·9836
20	2·9747	2·5893	2·3801	2·2489	2·1582	2·0913	2·0397	1·9985	1·9649
21	2·9609	2·5746	2·3649	2·2333	2·1423	2·0751	2·0232	1·9819	1·9480
22	2·9486	2·5613	2·3512	2·2193	2·1279	2·0605	2·0084	1·9668	1·9327
23	2·9374	2·5493	2·3387	2·2065	2·1149	2·0472	1·9949	1·9531	1·9189
24	2·9271	2·5383	2·3274	2·1949	2·1030	2·0351	1·9826	1·9407	1·9063
25	2·9177	2·5283	2·3170	2·1843	2·0922	2·0241	1·9714	1·9292	1·8947
26	2·9091	2·5191	2·3075	2·1745	2·0822	2·0139	1·9610	1·9188	1·8841
27	2·9012	2·5106	2·2987	2·1655	2·0730	2·0045	1·9515	1·9091	1·8743
28	2·8939	2·5028	2·2906	2·1571	2·0645	1·9959	1·9427	1·9001	1·8652
29	2·8871	2·4955	2·2831	2·1494	2·0566	1·9878	1·9345	1·8918	1·8568
30	2·8807	2·4887	2·2761	2·1422	2·0492	1·9803	1·9269	1·8841	1·8490
40	2·8354	2·4404	2·2261	2·0909	1·9968	1·9269	1·8725	1·8289	1·7929
60	2·7914	2·3932	2·1774	2·0410	1·9457	1·8747	1·8194	1·7748	1·7380
120	2·7478	2·3473	2·1300	1·9923	1·8959	1·8238	1·7675	1·7220	1·6843
∞	2·7055	2·3026	2·0838	1·9449	1·8473	1·7741	1·7167	1·6702	1·6315

This table gives the values of F for which $I_F(\nu_1, \nu_2) = 0{\cdot}10$.

Continued

PER CENT POINTS

5	6	7	8	9
241	58·204	58·906	59·439	59·858
9226	9·3255	9·3491	9·3668	9·3805
3092	5·2847	5·2662	5·2517	5·2400
3506	4·0098	3·9790	3·9549	3·9357
4530	3·4045	3·3679	3·3393	3·3163
3075	3·0546	3·0145	2·9830	2·9577
2833	2·8274	2·7849	2·7516	2·7247
2665	2·6683	2·6241	2·5893	2·5612
2106	2·5509	2·5053	2·4694	2·4403
1816	2·4606	2·4140	2·3772	2·3473
1612	2·3891	2·3416	2·3040	2·2735
1440	2·3310	2·2828	2·2446	2·2135
1467	2·2830	2·2341	2·1953	2·1638
1069	2·2426	2·1931	2·1539	2·1220
0730	2·2081	2·1582	2·1185	2·0862
0438	2·1783	2·1280	2·0880	2·0553
0183	2·1524	2·1017	2·0613	2·0284
0958	2·1296	2·0785	2·0379	2·0047
0760	2·1094	2·0580	2·0171	1·9836
0582	2·0913	2·0397	1·9985	1·9649
0423	2·0751	2·0232	1·9819	1·9480
0279	2·0605	2·0084	1·9668	1·9327
0149	2·0472	1·9949	1·9531	1·9189
0030	2·0351	1·9826	1·9407	1·9063
0922	2·0241	1·9714	1·9292	1·8947
0822	2·0139	1·9610	1·9188	1·8841
0730	2·0045	1·9515	1·9091	1·8743
0645	1·9959	1·9427	1·9001	1·8652
0566	1·9878	1·9345	1·8918	1·8568
0492	1·9803	1·9269	1·8841	1·8490
0426	1·9269	1·8725	1·8289	1·7929
0367	1·8745	1·8194	1·7748	1·7380
0313	1·8238	1·7675	1·7220	1·6843
0265	1·7741	1·7167	1·6702	1·6315

for which $I_F(\nu_1, \nu_2) = 0{\cdot}10$.

TABLES

Table III – Continued

F DISTRIBUTION: 10 PER CENT POINTS

ν_2 \ ν_1	10	12	15	20	24	30	40	60	120	∞
1	60·195	60·705	61·220	61·740	62·002	62·265	62·529	62·794	63·061	63·328
2	9·3916	9·4081	9·4247	9·4413	9·4496	9·4579	9·4663	9·4746	9·4829	9·4913
3	5·2304	5·2156	5·2003	5·1845	5·1764	5·1681	5·1597	5·1512	5·1425	5·1337
4	3·9199	3·8955	3·8689	3·8443	3·8310	3·8174	3·8036	3·7896	3·7753	3·7607
5	3·2974	3·2682	3·2380	3·2067	3·1905	3·1741	3·1573	3·1402	3·1228	3·1050
6	2·9369	2·9047	2·8712	2·8363	2·8183	2·8000	2·7812	2·7620	2·7423	2·7222
7	2·7025	2·6681	2·6322	2·5947	2·5753	2·5555	2·5351	2·5142	2·4928	2·4708
8	2·5380	2·5020	2·4642	2·4246	2·4041	2·3830	2·3614	2·3391	2·3162	2·2926
9	2·4163	2·3789	2·3396	2·2983	2·2768	2·2547	2·2320	2·2085	2·1843	2·1592
10	2·3226	2·2841	2·2435	2·2007	2·1784	2·1554	2·1317	2·1072	2·0818	2·0554
11	2·2482	2·2087	2·1671	2·1230	2·1000	2·0762	2·0516	2·0261	1·9997	1·9721
12	2·1878	2·1474	2·1049	2·0597	2·0360	2·0115	1·9861	1·9597	1·9323	1·9036
13	2·1376	2·0966	2·0532	2·0070	1·9827	1·9576	1·9315	1·9043	1·8759	1·8462
14	2·0954	2·0537	2·0095	1·9625	1·9377	1·9119	1·8852	1·8572	1·8280	1·7973
15	2·0593	2·0171	1·9722	1·9243	1·8990	1·8728	1·8454	1·8168	1·7867	1·7551
16	2·0281	1·9854	1·9399	1·8913	1·8656	1·8388	1·8108	1·7816	1·7507	1·7182
17	2·0009	1·9577	1·9117	1·8624	1·8362	1·8090	1·7805	1·7506	1·7191	1·6856
18	1·9770	1·9333	1·8868	1·8368	1·8103	1·7827	1·7537	1·7232	1·6910	1·6567
19	1·9557	1·9117	1·8647	1·8142	1·7873	1·7592	1·7298	1·6988	1·6659	1·6308
20	1·9367	1·8924	1·8449	1·7938	1·7667	1·7382	1·7083	1·6768	1·6433	1·6074
21	1·9197	1·8750	1·8272	1·7756	1·7481	1·7193	1·6890	1·6569	1·6228	1·5862
22	1·9043	1·8593	1·8111	1·7590	1·7312	1·7021	1·6714	1·6389	1·6042	1·5668
23	1·8903	1·8450	1·7964	1·7439	1·7159	1·6864	1·6554	1·6224	1·5871	1·5490
24	1·8775	1·8319	1·7831	1·7302	1·7019	1·6721	1·6407	1·6073	1·5715	1·5327
25	1·8658	1·8200	1·7708	1·7175	1·6890	1·6589	1·6272	1·5934	1·5570	1·5176
26	1·8550	1·8090	1·7596	1·7059	1·6771	1·6468	1·6147	1·5805	1·5437	1·5036
27	1·8451	1·7989	1·7492	1·6951	1·6662	1·6356	1·6032	1·5686	1·5313	1·4906
28	1·8359	1·7895	1·7395	1·6852	1·6560	1·6252	1·5925	1·5575	1·5198	1·4784
29	1·8274	1·7808	1·7306	1·6759	1·6465	1·6155	1·5825	1·5472	1·5090	1·4670
30	1·8195	1·7727	1·7223	1·6673	1·6377	1·6065	1·5732	1·5376	1·4989	1·4564
40	1·7627	1·7146	1·6624	1·6052	1·5741	1·5411	1·5056	1·4672	1·4248	1·3769
60	1·7070	1·6574	1·6034	1·5435	1·5107	1·4755	1·4373	1·3952	1·3476	1·2915
120	1·6524	1·6012	1·5450	1·4821	1·4472	1·4094	1·3676	1·3203	1·2646	1·1925
∞	1·5987	1·5458	1·4871	1·4206	1·3832	1·3419	1·2951	1·2400	1·1686	1·0900

$$F = \frac{s_1^2}{s_2^2} = \frac{\nu_2 S_1}{\nu_1 S_2}.$$

Table III – Continued

F DISTRIBUTION: 5 PER CENT POINTS

ν_2 \ ν_1	1	2	3	4	5	6	7	8	9
1	161·45	199·50	215·71	224·58	230·16	233·99	236·77	238·88	240·54
2	18·513	19·000	19·164	19·247	19·296	19·330	19·353	19·371	19·385
3	10·128	9·5521	9·2766	9·1172	9·0135	8·9406	8·8868	8·8452	8·8123
4	7·7086	6·9443	6·5914	6·3883	6·2560	6·1631	6·0942	6·0410	5·9988
5	6·6079	5·7861	5·4095	5·1922	5·0503	4·9503	4·8759	4·8183	4·7725
6	5·9874	5·1433	4·7571	4·5337	4·3874	4·2839	4·2066	4·1468	4·0990
7	5·5914	4·7374	4·3468	4·1203	3·9715	3·8660	3·7870	3·7257	3·6767
8	5·3177	4·4590	4·0662	3·8378	3·6875	3·5806	3·5005	3·4381	3·3881
9	5·1174	4·2565	3·8626	3·6331	3·4817	3·3738	3·2927	3·2296	3·1789
10	4·9646	4·1028	3·7083	3·4780	3·3258	3·2172	3·1355	3·0717	3·0204
11	4·8443	3·9823	3·5874	3·3567	3·2039	3·0946	3·0123	2·9480	2·8962
12	4·7472	3·8853	3·4903	3·2592	3·1059	2·9961	2·9134	2·8486	2·7964
13	4·6672	3·8056	3·4105	3·1791	3·0254	2·9153	2·8321	2·7669	2·7144
14	4·6001	3·7389	3·3439	3·1122	2·9582	2·8477	2·7642	2·6987	2·6458
15	4·5431	3·6823	3·2874	3·0556	2·9013	2·7905	2·7066	2·6408	2·5876
16	4·4940	3·6337	3·2389	3·0069	2·8524	2·7413	2·6572	2·5911	2·5377
17	4·4513	3·5915	3·1968	2·9647	2·8100	2·6987	2·6143	2·5480	2·4943
18	4·4139	3·5546	3·1599	2·9277	2·7729	2·6613	2·5767	2·5102	2·4563
19	4·3808	3·5219	3·1274	2·8951	2·7401	2·6283	2·5435	2·4768	2·4227
20	4·3513	3·4928	3·0984	2·8661	2·7109	2·5990	2·5140	2·4471	2·3928
21	4·3248	3·4668	3·0725	2·8401	2·6848	2·5727	2·4876	2·4205	2·3661
22	4·3009	3·4434	3·0491	2·8167	2·6613	2·5491	2·4638	2·3965	2·3419
23	4·2793	3·4221	3·0280	2·7955	2·6400	2·5277	2·4422	2·3748	2·3201
24	4·2597	3·4028	3·0088	2·7763	2·6207	2·5082	2·4226	2·3551	2·3002
25	4·2417	3·3852	2·9912	2·7587	2·6030	2·4904	2·4047	2·3371	2·2821
26	4·2252	3·3690	2·9751	2·7426	2·5868	2·4741	2·3883	2·3205	2·2655
27	4·2100	3·3541	2·9604	2·7278	2·5719	2·4591	2·3732	2·3053	2·2501
28	4·1960	3·3404	2·9467	2·7141	2·5581	2·4453	2·3593	2·2913	2·2360
29	4·1830	3·3277	2·9340	2·7014	2·5454	2·4324	2·3463	2·2782	2·2229
30	4·1709	3·3158	2·9223	2·6896	2·5336	2·4205	2·3343	2·2662	2·2107
40	4·0848	3·2317	2·8387	2·6060	2·4495	2·3359	2·2490	2·1802	2·1240
60	4·0012	3·1504	2·7581	2·5252	2·3683	2·2540	2·1665	2·0970	2·0401
120	3·9201	3·0718	2·6802	2·4472	2·2900	2·1780	2·0867	2·0164	1·9588
∞	3·8415	2·9957	2·6049	2·3719	2·2141	2·0986	2·0096	1·9384	1·8799

This table gives the values of F for which $I_F(\nu_1, \nu_2) = 0{\cdot}05$.

Continued

CENT POINTS

	6	7	8	9
	233·99	236·77	238·88	240·54
	19·330	19·353	19·371	19·385
	8·9406	8·8868	8·8452	8·8123
	6·1631	6·0942	6·0410	5·9988
	4·9503	4·8759	4·8183	4·7725
	4·2839	4·2066	4·1468	4·0990
	3·8660	3·7870	3·7257	3·6767
	3·5806	3·5005	3·4381	3·3881
	3·3738	3·2927	3·2296	3·1789
	3·2172	3·1355	3·0717	3·0204
	3·0946	3·0123	2·9480	2·8962
	2·9961	2·9134	2·8486	2·7964
	2·9153	2·8321	2·7669	2·7144
	2·8477	2·7642	2·6987	2·6458
	2·7905	2·7066	2·6408	2·5876
	2·7413	2·6572	2·5911	2·5377
	2·6987	2·6143	2·5480	2·4943
	2·6613	2·5767	2·5102	2·4563
	2·6283	2·5435	2·4768	2·4227
	2·5990	2·5140	2·4471	2·3928
	2·5727	2·4876	2·4205	2·3661
	2·5491	2·4638	2·3965	2·3419
	2·5277	2·4422	2·3748	2·3201
	2·5082	2·4226	2·3551	2·3002
	2·4904	2·4047	2·3371	2·2821
	2·4741	2·3883	2·3205	2·2655
	2·4591	2·3732	2·3053	2·2501
	2·4453	2·3593	2·2913	2·2360
	2·4324	2·3463	2·2782	2·2229
	2·4205	2·3343	2·2662	2·2107
	2·2359	2·2490	2·1802	2·1240
	2·1540	2·1665	2·0970	2·0401
	2·1750	2·0867	2·0154	1·9588
	2·0986	2·0096	1·9384	1·8799

which $I_p(\nu_1, \nu_2) = 0{\cdot}05$.

Table III – Continued

F DISTRIBUTION: 5 PER CENT POINTS

ν_2 \ ν_1	10	12	15	20	24	30	40	60	120	∞
1	241·88	243·91	245·95	248·01	249·05	250·09	251·14	252·20	253·25	254·32
2	19·396	19·413	19·429	19·446	19·454	19·462	19·471	19·479	19·487	19·496
3	8·7855	8·7446	8·7029	8·6602	8·6385	8·6166	8·5944	8·5720	8·5494	8·5265
4	5·9644	5·9117	5·8578	5·8025	5·7744	5·7459	5·7170	5·6878	5·6581	5·6281
5	4·7351	4·6777	4·6188	4·5581	4·5272	4·4957	4·4638	4·4314	4·3984	4·3650
6	4·0600	3·9999	3·9381	3·8742	3·8415	3·8082	3·7743	3·7398	3·7047	3·6688
7	3·6365	3·5747	3·5108	3·4445	3·4105	3·3758	3·3404	3·3043	3·2674	3·2298
8	3·3472	3·2840	3·2184	3·1503	3·1152	3·0794	3·0428	3·0053	2·9669	2·9276
9	3·1373	3·0729	3·0061	2·9365	2·9005	2·8637	2·8259	2·7872	2·7475	2·7067
10	2·9782	2·9130	2·8450	2·7740	2·7372	2·6998	2·6609	2·6211	2·5801	2·5379
11	2·8536	2·7876	2·7186	2·6464	2·6090	2·5705	2·5309	2·4901	2·4480	2·4045
12	2·7534	2·6866	2·6169	2·5436	2·5055	2·4663	2·4259	2·3842	2·3410	2·2962
13	2·6710	2·6037	2·5331	2·4589	2·4202	2·3803	2·3392	2·2966	2·2524	2·2064
14	2·6021	2·5342	2·4630	2·3879	2·3487	2·3082	2·2664	2·2230	2·1778	2·1307
15	2·5437	2·4753	2·4035	2·3275	2·2878	2·2468	2·2043	2·1601	2·1141	2·0658
16	2·4935	2·4247	2·3522	2·2756	2·2354	2·1938	2·1507	2·1058	2·0589	2·0096
17	2·4499	2·3807	2·3077	2·2304	2·1898	2·1477	2·1040	2·0584	2·0107	1·9604
18	2·4117	2·3421	2·2686	2·1906	2·1497	2·1071	2·0629	2·0166	1·9681	1·9168
19	2·3779	2·3080	2·2341	2·1555	2·1141	2·0712	2·0264	1·9796	1·9302	1·8780
20	2·3479	2·2776	2·2033	2·1242	2·0825	2·0391	1·9938	1·9464	1·8963	1·8432
21	2·3210	2·2504	2·1757	2·0960	2·0540	2·0102	1·9645	1·9165	1·8657	1·8117
22	2·2967	2·2258	2·1508	2·0707	2·0283	1·9842	1·9380	1·8895	1·8380	1·7831
23	2·2747	2·2036	2·1282	2·0476	2·0050	1·9605	1·9139	1·8649	1·8128	1·7570
24	2·2547	2·1834	2·1077	2·0267	1·9838	1·9390	1·8920	1·8424	1·7897	1·7331
25	2·2365	2·1649	2·0889	2·0075	1·9643	1·9192	1·8718	1·8217	1·7684	1·7110
26	2·2197	2·1479	2·0716	1·9898	1·9464	1·9010	1·8533	1·8027	1·7488	1·6906
27	2·2043	2·1323	2·0558	1·9736	1·9299	1·8842	1·8361	1·7851	1·7307	1·6717
28	2·1900	2·1179	2·0411	1·9586	1·9147	1·8687	1·8203	1·7689	1·7138	1·6541
29	2·1768	2·1045	2·0275	1·9446	1·9005	1·8543	1·8055	1·7537	1·6981	1·6377
30	2·1646	2·0921	2·0148	1·9317	1·8874	1·8409	1·7918	1·7396	1·6835	1·6223
40	2·0772	2·0035	1·9245	1·8389	1·7929	1·7444	1·6928	1·6373	1·5766	1·5089
60	1·9926	1·9174	1·8364	1·7480	1·7001	1·6491	1·5943	1·5343	1·4673	1·3893
120	1·9105	1·8337	1·7505	1·6587	1·6084	1·5543	1·4952	1·4290	1·3519	1·2539
∞	1·8307	1·7522	1·6664	1·5705	1·5173	1·4591	1·3940	1·3180	1·2214	1·0000

$$F = \frac{s_1^2}{s_2^2} = \frac{\nu_2 S_1}{\nu_1 S_2}$$

Table III - Continued

F DISTRIBUTION: 2·5 PER CENT POINTS

ν_2 \ ν_1	1	2	3	4	5	6	7	8	9
1	647·79	799·50	864·16	899·58	921·85	937·11	948·22	956·66	963·28
2	38·506	39·000	39·165	39·248	39·298	39·331	39·355	39·373	39·387
3	17·443	16·044	15·439	15·101	14·885	14·735	14·624	14·540	14·473
4	12·218	10·649	9·9792	9·6045	9·3645	9·1973	9·0741	8·9796	8·9047
5	10·007	8·4336	7·7636	7·3879	7·1464	6·9777	6·8531	6·7572	6·6810
6	8·8131	7·2598	6·5988	6·2272	5·9876	5·8197	5·6955	5·5996	5·5234
7	8·0727	6·5415	5·8898	5·5226	5·2852	5·1186	4·9949	4·8994	4·8232
8	7·5709	6·0595	5·4160	5·0526	4·8173	4·6517	4·5286	4·4332	4·3572
9	7·2093	5·7147	5·0781	4·7181	4·4844	4·3197	4·1971	4·1020	4·0260
10	6·9367	5·4564	4·8256	4·4683	4·2361	4·0721	3·9498	3·8549	3·7790
11	6·7241	5·2559	4·6300	4·2751	4·0440	3·8807	3·7586	3·6638	3·5879
12	6·5538	5·0959	4·4742	4·1212	3·8911	3·7283	3·6065	3·5118	3·4358
13	6·4143	4·9653	4·3472	3·9959	3·7667	3·6043	3·4827	3·3880	3·3120
14	6·2979	4·8567	4·2417	3·8919	3·6634	3·5014	3·3799	3·2853	3·2093
15	6·1995	4·7650	4·1528	3·8043	3·5764	3·4147	3·2934	3·1987	3·1227
16	6·1151	4·6867	4·0768	3·7294	3·5021	3·3406	3·2194	3·1248	3·0488
17	6·0420	4·6189	4·0112	3·6648	3·4379	3·2767	3·1556	3·0610	2·9849
18	5·9781	4·5597	3·9539	3·6083	3·3820	3·2209	3·0999	3·0053	2·9291
19	5·9216	4·5075	3·9034	3·5587	3·3327	3·1718	3·0509	2·9563	2·8800
20	5·8715	4·4613	3·8587	3·5147	3·2891	3·1283	3·0074	2·9128	2·8365
21	5·8266	4·4199	3·8188	3·4754	3·2501	3·0895	2·9686	2·8740	2·7977
22	5·7863	4·3828	3·7829	3·4401	3·2151	3·0546	2·9338	2·8392	2·7628
23	5·7498	4·3492	3·7505	3·4083	3·1838	3·0232	2·9024	2·8077	2·7313
24	5·7167	4·3187	3·7211	3·3794	3·1548	2·9946	2·8738	2·7791	2·7027
25	5·6864	4·2909	3·6943	3·3530	3·1287	2·9685	2·8478	2·7531	2·6766
26	5·6586	4·2655	3·6697	3·3289	3·1048	2·9447	2·8240	2·7293	2·6528
27	5·6331	4·2421	3·6472	3·3067	3·0828	2·9228	2·8021	2·7074	2·6309
28	5·6096	4·2205	3·6264	3·2863	3·0625	2·9027	2·7820	2·6872	2·6106
29	5·5878	4·2006	3·6072	3·2674	3·0438	2·8840	2·7633	2·6686	2·5919
30	5·5675	4·1821	3·5894	3·2499	3·0265	2·8667	2·7460	2·6513	2·5746
40	5·4239	4·0510	3·4633	3·1261	2·9037	2·7444	2·6238	2·5289	2·4519
60	5·2857	3·9253	3·3425	3·0077	2·7863	2·6274	2·5068	2·4117	2·3344
120	5·1524	3·8046	3·2270	2·8943	2·6740	2·5154	2·3948	2·2994	2·2217
∞	5·0239	3·6889	3·1161	2·7858	2·5665	2·4082	2·2875	2·1918	2·1136

This table gives the values of F for which $I_F(\nu_1, \nu_2) = 0{\cdot}025$.

– Continued

5 PER CENT POINTS

5	6	7	8	9
1·95	937·11	948·22	956·66	963·28
9·298	39·331	39·355	39·373	39·387
4·885	14·735	14·624	14·540	14·473
9·3645	9·1973	9·0741	8·9796	8·9047
7·1464	6·9777	6·8531	6·7572	6·6810
5·9876	5·8197	5·6955	5·5996	5·5234
5·2652	5·1186	4·9949	4·8994	4·8232
4·6173	4·6517	4·5286	4·4332	4·3572
4·4844	4·3197	4·1971	4·1020	4·0260
4·2361	4·0721	3·9498	3·8549	3·7790
4·0440	3·8807	3·7586	3·6638	3·5879
3·8911	3·7283	3·6065	3·5118	3·4358
3·7667	3·6043	3·4927	3·3880	3·3120
3·6634	3·5014	3·3799	3·2853	3·2093
3·5764	3·4147	3·2934	3·1987	3·1227
3·5021	3·3406	3·2194	3·1248	3·0488
3·4379	3·2767	3·1556	3·0610	2·9849
3·3820	3·2209	3·0999	3·0053	2·9291
3·3327	3·1718	3·0509	2·9563	2·8800
3·2891	3·1283	3·0074	2·9128	2·8365
3·2501	3·0895	2·9686	2·8740	2·7977
3·2151	3·0546	2·9338	2·8392	2·7628
3·1835	3·0222	2·9024	2·8077	2·7313
3·1548	2·9946	2·8728	2·7791	2·7027
3·1287	2·9685	2·8478	2·7531	2·6766
3·1048	2·9447	2·8240	2·7293	2·6528
3·0828	2·9228	2·8021	2·7074	2·6309
3·0625	2·9027	2·7820	2·6872	2·6106
3·0438	2·8840	2·7633	2·6686	2·5919
3·0265	2·8667	2·7460	2·6513	2·5746
3·0057	2·7444	2·6238	2·5289	2·4519
2·7863	2·6274	2·5068	2·4117	2·3344
2·6740	2·5154	2·3948	2·2994	2·2217
2·5665	2·4082	2·2875	2·1918	2·1136

for which $I_p(\nu_1, \nu_2) = 0{\cdot}025$.

Table III – Continued

F DISTRIBUTION: 2·5 PER CENT POINTS

ν_2 \ ν_1	10	12	15	20	24	30	40	60	120	∞
1	968·63	976·71	984·87	993·10	997·25	1001·4	1005·6	1009·8	1014·0	1018·3
2	39·398	39·415	39·431	39·448	39·456	39·465	39·473	39·481	39·490	29·498
3	14·419	14·337	14·253	14·167	14·124	14·081	14·037	13·992	13·947	13·902
4	8·8439	8·7512	8·6565	8·5599	8·5109	8·4613	8·411[illegible]	8·3604	8·3092	8·2573
5	6·6192	6·5246	6·4277	6·3285	6·2780	6·2259	6·1751	6·1225	6·0693	6·0152
6	5·4613	5·3662	5·2687	5·1684	5·1172	5·0652	5·0125	4·9589	4·9045	4·8491
7	4·7611	4·6658	4·5678	4·4667	4·4150	4·3624	4·3089	4·2544	4·1989	4·1423
8	4·2951	4·1997	4·1012	3·9995	3·9472	3·8940	3·8398	2·7844	3·7279	3·6702
9	3·9639	3·8682	3·7694	2·6669	3·6142	3·5604	3·5055	2·4493	3·3918	2·2229
10	3·7168	3·6209	3·5217	3·4186	3·3654	2·3110	2·2554	3·1984	3·1299	2·0798
11	3·5257	3·4296	3·3299	3·2261	3·1725	3·1176	3·0613	3·0035	2·9441	2·8628
12	3·3736	3·2773	3·1772	3·0728	3·0187	2·9633	2·9063	2·8478	2·7874	2·7249
13	3·2497	3·1532	3·0527	2·9477	2·8932	2·8373	2·7797	2·7204	2·6590	2·5955
14	3·1469	3·0501	2·9493	2·8437	2·7888	2·7224	2·6742	2·6142	2·5519	2·4872
15	3·0602	2·9633	2·8621	2·7559	2·7006	2·6437	2·5850	2·5242	2·4611	2·3953
16	2·9862	2·8890	2·7875	2·6808	2·6252	2·5678	2·5085	2·4471	2·3831	2·3163
17	2·9222	2·8249	2·7230	2·6158	2·5598	2·5021	2·4422	2·3801	2·2153	2·2474
18	2·8664	2·7689	2·6667	2·5590	2·5027	2·4445	2·3842	2·3214	2·2568	2·1869
19	2·8173	2·7196	2·6171	2·5089	2·4523	2·2937	2·3329	2·2695	2·2022	2·1333
20	2·7737	2·6758	2·5731	2·4645	2·4076	2·3486	2·2872	2·2234	2·1562	2·0853
21	2·7348	2·6368	2·5338	2·4247	2·3675	2·3082	2·2465	2·1819	2·1141	2·0422
22	2·6998	2·6017	2·4984	2·3890	2·3315	2·2718	2·2097	2·1446	2·0760	2·0032
23	2·6682	2·5699	2·4665	2·3567	2·2989	2·2389	2·1763	2·1107	2·0415	1·9677
24	2·6396	2·5412	2·4374	2·3273	2·2393	2·2090	2·1460	2·0799	2·0099	1·9353
25	2·6135	2·5149	2·4110	2·3005	2·2422	2·1816	2·1182	2·0617	1·9811	1·9055
26	2·5895	2·4909	2·3867	2·2759	2·2174	2·1565	2·0928	2·0257	1·9545	1·8781
27	2·5676	2·4688	2·3644	2·2523	2·1946	2·1234	2·0693	2·0018	1·9299	1·8527
28	2·5473	2·4484	2·3438	2·2324	2·1735	2·1121	2·0477	1·9796	1·9072	1·8291
29	2·5286	2·4295	2·3248	2·2131	2·1540	2·0922	2·0276	1·9591	1·8861	1·8072
30	2·5112	2·4120	2·3072	2·1952	2·1359	2·0739	2·0089	1·9406	1·8664	1·7867
40	2·3882	2·2882	2·1819	2·0677	2·0068	1·9429	1·8752	1·8028	1·7242	1·6371
60	2·2702	2·1692	2·0613	1·9445	1·8817	1·8152	1·7449	1·6668	1·5810	1·4822
120	2·1570	2·0548	1·9450	1·8249	1·7597	1·6899	1·6141	1·5299	1·4327	1·3194
∞	2·0483	1·9447	1·8326	1·7085	1·6402	1·5660	1·4835	1·3883	1·2684	1·0000

$$F = \frac{s_1^2}{s_2^2} = \frac{\nu_2 S_1}{\nu_1 S_2}.$$

Table III – Continued

F DISTRIBUTION: 1 PER CENT POINTS

ν_2 \ ν_1	1	2	3	4	5	6	7	8	9
1	4052·2	4999·5	5403·3	5624·6	5763·7	5859·0	5928·3	5981·6	6022·5
2	98·503	99·000	99·166	99·249	99·299	99·332	99·356	99·374	99·388
3	34·116	30·817	29·457	28·710	28·237	27·911	27·672	27·489	27·345
4	21·198	18·000	16·694	15·977	15·522	15·207	14·976	14·799	14·659
5	16·258	13·274	12·060	11·392	10·967	10·672	10·456	10·289	10·158
6	13·745	10·925	9·7795	9·1483	8·7459	8·4661	8·2600	8·1016	7·9761
7	12·246	9·5466	8·4513	7·8467	7·4604	7·1914	C·9928	6·8401	6·7188
8	11·259	8·6491	7·5910	7·0060	6·6318	6·3707	6·1776	6·0289	5·9106
9	10·561	8·0215	6·9919	6·4221	6·0569	5·8018	5·6129	5·4671	5·3511
10	10·044	7·5594	6·5523	5·9943	5·6363	5·3858	5·2001	5·0567	4·9424
11	9·6460	7·2057	6·2167	5·6683	5·3160	5·0692	4·8861	4·7445	4·6315
12	9·3302	6·9266	5·9526	5·4119	5·0643	4·8206	4·6395	4·4994	4·3875
13	9·0738	6·7010	5·7394	5·2053	4·8616	4·6204	4·4410	4·3021	4·1911
14	8·8616	6·5149	5·5639	5·0354	4·6950	4·4558	4·2779	4·1399	4·0297
15	8·6831	6·3589	5·4170	4·8932	4·5556	4·3183	4·1415	4·0045	3·8948
16	8·5310	6·2262	5·2922	4·7726	4·4374	4·2016	4·0259	3·8896	3·7804
17	8·3997	6·1121	5·1850	4·6690	4·3359	4·1015	3·9267	3·7910	3·6822
18	8·2854	6·0129	5·0919	4·5790	4·2479	4·0146	3·8403	3·7054	3·5971
19	8·1850	5·9259	5·0103	4·5003	4·1708	3·9386	3·7653	3·6305	3·5225
20	8·0960	5·8489	4·9382	4·4307	4·1027	3·8714	3·6987	3·5644	3·4567
21	8·0166	5·7804	4·8740	4·3688	4·0421	3·8117	3·6396	3·5056	3·3981
22	7·9454	5·7190	4·8166	4·3134	3·9880	3·7583	3·5867	3·4530	3·3458
23	7·8811	5·6637	4·7649	4·2635	3·9392	3·7102	3·5390	3·4057	3·2986
24	7·8229	5·6136	4·7181	4·2184	3·8951	3·6667	3·4959	3·3629	3·2560
25	7·7698	5·5680	4·6755	4·1774	3·8550	3·6272	3·4568	3·3239	3·2172
26	7·7213	5·5263	4·6366	4·1400	3·8183	3·5911	3·4210	3·2884	3·1816
27	7·6767	5·4881	4·6009	4·1056	3·7848	3·5580	3·3882	3·2558	3·1494
28	7·6356	5·4529	4·5681	4·0740	3·7539	3·5276	3·3581	3·2259	3·1195
29	7·5976	5·4205	4·5378	4·0449	3·7254	3·4995	3·3302	3·1982	3·0920
30	7·5625	5·3904	4·5097	4·0179	3·6990	3·4735	3·3045	3·1726	3·0665
40	7·3141	5·1785	4·3126	3·8283	3·5138	3·2910	3·1238	2·9930	2·8876
60	7·0771	4·9774	4·1259	3·6491	3·3389	3·1187	2·9530	2·8323	2·7185
120	6·8510	4·7865	3·9493	3·4796	3·1735	2·9559	2·7918	2·6629	2·5586
∞	6·6349	4·6052	3·7816	3·3192	3·0173	2·8020	2·6393	2·5113	2·4073

This table gives the values of F for which $I_F(\nu_1, \nu_2) = 0{\cdot}01$.

Table III – Continued

F DISTRIBUTION: 1 PER CENT POINTS

ν_2 \ ν_1	10	12	15	20	24	30	40	60	120	∞
1	6055·8	6106·3	6157·3	6208·7	6234·6	6260·7	6286·8	6313·0	6339·4	6366·0
2	99·399	99·416	99·432	99·449	99·458	99·466	99·474	99·483	99·491	99·501
3	27·229	27·052	26·872	26·690	26·598	26·505	26·411	26·316	26·221	26·125
4	14·546	14·374	14·198	14·020	13·929	13·838	13·745	13·652	13·558	13·463
5	10·051	9·8882	9·7222	9·5527	9·4665	9·3793	9·2912	9·2020	9·1118	9·0204
6	7·8741	7·7183	7·5590	7·3958	7·3127	7·2285	7·1432	7·0568	6·9690	6·8801
7	6·6201	6·4691	6·3143	6·1554	6·0743	5·9921	5·9084	5·8236	5·7372	5·6495
8	5·8143	5·6668	5·5151	5·3591	5·2793	5·1981	5·1156	5·0316	4·9460	4·8588
9	5·2565	5·1114	4·9621	4·8080	4·7290	4·6486	4·5667	4·4831	4·3978	4·3105
10	4·8492	4·7059	4·5582	4·4054	4·3269	4·2469	4·1653	4·0819	3·9965	3·9090
11	4·5393	4·3974	4·2509	4·0990	4·0209	3·9411	3·8596	3·7761	3·6904	3·6025
12	4·2961	4·1553	4·0096	3·8584	3·7805	3·7008	3·6192	3·5355	3·4494	3·3608
13	4·1003	3·9603	3·8154	3·6646	3·5868	3·5070	3·4253	3·3413	3·2548	3·1654
14	3·9394	3·8001	3·6557	3·5052	3·4274	3·3476	3·2656	3·1813	3·0942	3·0040
15	3·8049	3·6662	3·5222	3·3719	3·2940	3·2141	3·1319	3·0471	2·9595	2·8684
16	3·6909	3·5527	3·5089	3·2588	3·1808	3·1007	3·0182	2·9330	2·8447	2·7528
17	3·5931	3·4552	3·3117	3·1615	3·0835	3·0032	2·9205	2·8348	2·7459	2·6530
18	3·5082	3·3706	3·2273	3·0771	2·9990	2·9185	2·8354	2·7493	2·6597	2·5660
19	3·4338	3·2965	3·1533	3·0031	2·9249	2·8442	2·7608	2·6742	2·5839	2·4893
20	3·3682	3·2311	3·0880	2·9377	2·8594	2·7785	2·6947	2·6077	2·5168	2·4212
21	3·3098	3·1729	3·0299	2·8796	2·8011	2·7200	2·6359	2·5484	2·4568	2·3603
22	3·2576	3·1209	2·9780	2·8274	2·7488	2·6675	2·5831	2·4951	2·4029	2·3055
23	3·2106	3·0740	2·9311	2·7805	2·7017	2·6202	2·5355	2·4471	2·3542	2·2559
24	3·1681	3·0316	2·8887	2·7380	2·6591	2·5773	2·4923	2·4035	2·3099	2·2107
25	3·1294	2·9931	2·8502	2·6995	2·6203	2·5383	2·4530	2·3637	2·2695	2·1694
26	3·0941	2·9578	2·8150	2·6640	2·5848	2·5026	2·4170	2·3273	2·2325	2·1315
27	3·0618	2·9256	2·7827	2·6316	2·5522	2·4699	2·3840	2·2938	2·1984	2·0965
28	3·0320	2·8959	2·7530	2·6017	2·5223	2·4397	2·3535	2·2629	2·1670	2·0642
29	3·0045	2·8685	2·7256	2·5742	2·4946	2·4118	2·3253	2·2344	2·1378	2·0342
30	2·9791	2·8431	2·7002	2·5487	2·4689	2·3860	2·2992	2·2079	2·1107	2·0062
40	2·8005	2·6648	2·5216	2·3689	2·2880	2·2034	2·1142	2·0194	1·9172	1·8047
60	2·6318	2·4961	2·3523	2·1978	2·1154	2·0285	1·9360	1·8363	1·7263	1·6006
120	2·4721	2·3363	2·1915	2·0346	1·9500	1·8600	1·7628	1·6557	1·5330	1·3805
∞	2·3209	2·1848	2·0385	1·8743	1·7908	1·6964	1·5923	1·4730	1·3246	1·0000

$$F = \frac{s_1^2}{s_2^2} = \frac{\nu_2 S_1}{\nu_1 S_2}.$$

Table III – Continued

F Distribution: 0·5 per cent Points

ν_2 \ ν_1	1	2	3	4	5	6	7	8	9
1	16211	20000	21615	22500	23056	23437	23715	23925	24091
2	198·50	199·00	199·17	199·25	199·30	199·33	199·36	199·37	199·39
3	55·552	49·799	47·467	46·195	45·392	44·838	44·434	44·126	43·882
4	31·333	26·284	24·259	23·155	22·456	21·975	21·622	21·352	21·139
5	22·785	18·314	16·530	15·556	14·940	14·513	14·200	13·961	13·772
6	18·635	14·544	12·917	12·028	11·464	11·073	10·786	10·566	10·391
7	16·238	12·404	10·882	10·050	9·5221	9·1554	8·8854	8·6781	8·5138
8	14·688	11·042	9·5965	8·8051	8·3018	7·9520	7·6942	7·4960	7·3386
9	13·614	10·107	8·7171	7·9559	7·4711	7·1338	6·8849	6·6933	6·5411
10	12·826	9·4270	8·0807	7·3428	6·8723	6·5446	6·3025	6·1159	5·9676
11	12·226	8·9122	7·6004	6·8809	6·4217	6·1015	5·8648	5·6821	5·5368
12	11·754	8·5096	7·2258	6·5211	6·0711	5·7570	5·5245	5·3451	5·2021
13	11·374	8·1865	6·9257	6·2335	5·7910	5·4819	5·2529	5·0761	4·9351
14	11·060	7·9217	6·6803	5·9984	5·5623	5·2574	5·0313	4·8566	4·7173
15	10·798	7·7008	6·4760	5·8029	5·3721	5·0708	4·8473	4·6743	4·5364
16	10·575	7·5138	6·3034	5·6378	5·2117	4·9134	4·6920	4·5207	4·3838
17	10·384	7·3536	6·1556	5·4967	5·0746	4·7789	4·5594	4·3893	4·2535
18	10·218	7·2148	6·0277	5·3748	4·9560	4·6627	4·4448	4·2759	4·1410
19	10·073	7·0935	5·9161	5·2681	4·8526	4·5614	4·3448	4·1770	4·0428
20	9·9439	6·9865	5·8177	5·1743	4·7818	4·4721	4·2569	4·0900	3·9564
21	9·8295	6·8914	5·7304	5·0911	4·6808	4·3931	4·1789	4·0128	3·8799
22	9·7271	6·8064	5·6524	5·0168	4·6088	4·3225	4·1094	3·9440	3·8116
23	9·6348	6·7300	5·5823	4·9500	4·5441	4·2591	4·0469	3·8822	3·7502
24	9·5513	6·6610	5·5190	4·8898	4·4857	4·2019	3·9905	3·8264	3·6949
25	9·4753	6·5982	5·4615	4·8351	4·4327	4·1500	3·9394	3·7758	3·6447
26	9·4059	6·5409	5·4091	4·7852	4·3844	4·1027	3·8928	3·7297	3·5989
27	9·3423	6·4885	5·3611	4·7396	4·3402	4·0594	3·8501	3·6875	3·5571
28	9·2838	6·4405	5·3170	4·6977	4·2996	4·0197	3·8110	3·6487	3·5186
29	9·2297	6·3958	5·2764	4·6591	4·2622	3·9830	3·7749	3·6130	3·4832
30	9·1797	6·3547	5·2388	4·8233	4·2278	3·9492	3·7416	3·5801	3·4505
40	8·8278	6·0664	4·9759	4·3738	3·9860	3·7129	3·5088	3·3498	3·2220
60	8·4946	5·7950	4·7290	4·1399	3·7600	3·4918	3·2911	3·1344	3·0083
120	8·1790	5·5393	4·4973	3·9207	3·5482	3·2849	3·0874	2·9330	2·8083
∞	7·8794	5·2983	4·2794	3·7151	3·3499	3·0913	2·8968	2·7444	2·6210

This table gives the values of F for which $I_F(\nu_1, \nu_2) = 0{\cdot}005$.

III – Continued

F: 0·5 PER CENT POINTS

	5	6	7	8	9
	23056	23437	23715	23925	24091
	199·30	199·33	199·36	199·37	199·39
	45·392	44·838	44·434	44·126	43·882
	22·456	21·975	21·622	21·352	21·139
	14·940	14·513	14·200	13·961	13·772
	11·464	11·073	10·786	10·566	10·391
	9·5221	9·1554	8·8854	8·6781	8·5138
	8·3018	7·9520	7·6942	7·4960	7·3386
	7·4711	7·1338	6·8849	6·6933	6·5411
	6·8723	6·5446	6·3025	6·1159	5·9676
	6·4217	6·1015	5·8648	5·6821	5·5368
	6·0711	5·7570	5·5245	5·3431	5·2021
	5·7910	5·4819	5·2529	5·0761	4·9351
	5·5623	5·2574	5·0313	4·8566	4·7173
	5·3721	5·0708	4·8473	4·6743	4·5364
	5·2117	4·9134	4·6920	4·5207	4·3838
	5·0746	4·7789	4·5594	4·3893	4·2535
	4·9560	4·6627	4·4448	4·2759	4·1410
	4·8526	4·5614	4·3448	4·1770	4·0428
	4·7616	4·4721	4·2569	4·0900	3·9564
	4·6808	4·3931	4·1789	4·0128	3·8799
	4·6088	4·3225	4·1094	3·9440	3·8118
	4·5441	4·2591	4·0469	3·8822	3·7502
	4·4857	4·2019	3·9905	3·8264	3·6949
	4·4327	4·1500	3·9394	3·7758	3·6447
	4·3844	4·1027	3·8928	3·7297	3·5989
	4·3402	4·0594	3·8501	3·6875	3·5571
	4·2996	4·0197	3·8110	3·6487	3·5186
	4·262*	3·9830	3·7749	3·6130	3·4832
	4·2276	3·9492	3·7416	3·5801	3·4505
	3·9860	3·7129	3·5088	3·3498	3·2220
	3·7600	3·4916	3·2911	3·1344	3·0063
	3·5482	3·2849	3·0874	2·9330	2·8063
	3·3499	3·0913	2·8968	2·7444	2·6210

* F for which $I_F(\nu_1, \nu_2) = 0{\cdot}005$.

Table III – Continued

F DISTRIBUTION: 0·5 PER CENT POINTS

ν_2 \ ν_1	10	12	15	20	24	30	40	60	120	∞
1	24224	24426	24630	24836	24940	25044	25148	25253	25359	25465
2	199·40	199·42	199·43	199·45	199·46	199·47	199·47	199·48	199·49	199·51
3	43·686	43·387	43·085	42·776	42·622	42·466	42·308	42·149	41·989	41·829
4	20·967	20·705	20·436	20·167	20·030	19·892	19·752	19·611	19·466	19·325
5	13·618	13·384	13·146	12·903	12·780	12·656	12·530	12·402	12·274	12·144
6	10·250	10·034	9·8140	9·5888	9·4741	9·3583	9·2406	9·1219	9·0015	6·8793
7	8·3803	8·1764	7·9678	7·7540	7·6450	7·5345	7·4225	7·3088	7·1933	7·0760
8	7·2107	7·0149	6·8143	6·6082	6·5029	6·3961	6·2875	6·1772	6·0649	5·9505
9	6·4171	6·2274	6·0325	5·8318	5·7292	5·6248	5·5186	5·4104	5·3001	5·1875
10	5·8467	5·6613	5·4707	5·2740	5·1732	5·0705	4·9659	4·8592	4·7501	4·6385
11	5·4182	5·2363	5·0489	4·8552	4·7557	4·6543	4·5508	4·4450	4·3367	4·2256
12	5·0855	4·9063	4·7214	4·5299	4·4315	4·3309	4·2282	4·1229	4·0149	3·9039
13	4·8199	4·6429	4·4600	4·2703	4·1726	4·0727	3·9704	3·8655	3·7577	3·6465
14	4·6034	4·4281	4·2468	4·0585	3·9614	3·8619	3·7600	3·6553	3·5473	3·4359
15	4·4236	4·2498	4·0698	3·8826	3·7859	3·6867	3·5850	3·4803	3·3722	3·2602
16	4·2719	4·0994	3·9205	3·7342	3·6378	3·5388	3·4372	3·3324	3·2240	3·1115
17	4·1423	3·9709	3·7929	3·6073	3·5112	3·4124	3·3107	3·2058	3·0971	2·9839
18	4·0305	3·8599	3·6827	3·4977	3·4017	3·3030	3·2014	3·0962	2·9871	2·8735
19	3·9329	3·7631	3·5866	3·4020	3·3062	3·2075	3·1058	3·0004	2·8908	2·7762
20	3·8470	3·6779	3·5020	3·3178	3·2220	3·1234	3·0215	2·9159	2·8058	2·6904
21	3·7709	3·6024	3·4270	3·2431	3·1474	3·0488	2·9467	2·8408	2·7302	2·6140
22	3·7030	3·5350	3·3600	3·1764	3·0807	2·9821	2·8799	2·7736	2·6625	2·5455
23	3·6420	3·4745	3·2999	3·1165	3·0208	2·9221	2·8198	2·7132	2·6015	2·4837
24	3·5870	3·4199	3·2456	3·0624	2·9667	2·8679	2·7654	2·6585	2·5463	2·4276
25	3·5370	3·3704	3·1963	3·0133	2·9176	2·8187	2·7160	2·6088	2·4960	2·3765
26	3·4916	3·3252	3·1515	2·9685	2·8728	2·7738	2·6709	2·5633	2·4501	2·3297
27	3·4499	3·2839	3·1104	2·9275	2·8318	2·7327	2·6296	2·5217	2·4078	2·2867
28	3·4117	3·2460	3·0727	2·8899	2·7941	2·6949	2·5916	2·4834	2·3689	2·2469
29	3·3765	3·2111	3·0379	2·8551	2·7594	2·6601	2·5565	2·4479	2·3330	2·2102
30	3·3440	3·1787	3·0057	2·8230	2·7272	2·6278	2·5241	2·4151	2·2997	2·1760
40	3·1167	2·9531	2·7811	2·5984	2·5020	2·4015	2·2958	2·1838	2·0635	1·9318
60	2·9042	2·7418	2·5705	2·3872	2·2898	2·1874	2·0789	1·9722	1·8341	1·6885
120	2·7052	2·5439	2·3727	2·1881	2·0890	1·9839	1·8709	1·7469	1·6055	1·4311
∞	2·5188	2·3583	2·1868	1·9998	1·8983	1·7891	1·6691	1·5325	1·3637	1·0000

$$F = \frac{s_1^2}{s_2^2} = \frac{\nu_2 S_1}{\nu_1 S_2}.$$

Table IV*

TABLE OF PERCENTAGE POINTS OF THE *t*-DISTRIBUTION

COMPUTED BY MAXINE MERRINGTON

THE following table has been derived from Miss Thompson's Tables of Percentage Points of the Incomplete Beta Function (*Biometrika*, **32**, 168–81), by taking

$$t = \sqrt{(\nu_2(1-x)/\nu_1 x)}$$

for the case $\nu_1 = 1$. t is the usual "Student" ratio based on an estimate of variance having $\nu = \nu_2$ degrees of freedom. If t_P is the quantity tabled, P/100 is the chance that $|t| \geq t_P$, i.e. represents the area in the two tails of the t-distribution. The table includes certain levels for t not previously available and should be accurate to the five significant figures shown.

ν \ P	50	25	10	5	2·5	1	0·5
1	1·00000	2·4142	6·3138	12·706	25·452	63·657	127·32
2	0·81650	1·6036	2·9200	4·3027	6·2053	9·9248	14·089
3	0·76489	1·4226	2·3534	3·1825	4·1765	5·8409	7·4533
4	0·74070	1·3444	2·1318	2·7764	3·4954	4·6041	5·5976
5	0·72669	1·3009	2·0150	2·5706	3·1634	4·0321	4·7733
6	0·71756	1·2733	1·9432	2·4469	2·9687	3·7074	4·3168
7	0·71114	1·2543	1·8946	2·3646	2·8412	3·4995	4·0293
8	0·70639	1·2403	1·8595	2·3060	2·7515	3·3554	3·8325
9	0·70272	1·2297	1·8331	2·2622	2·6850	3·2498	3·6897
10	0·69981	1·2213	1·8125	2·2281	2·6338	3·1693	3·5814
11	0·69745	1·2145	1·7959	2·2010	2·5931	3·1058	3·4966
12	0·69548	1·2089	1·7823	2·1788	2·5600	3·0545	3·4284
13	0·69384	1·2041	1·7709	2·1604	2·5326	3·0123	3·3725
14	0·69242	1·2001	1·7613	2·1448	2·5096	2·9768	3·3257
15	0·69120	1·1967	1·7530	2·1315	2·4899	2·9467	3·2860
16	0·69013	1·1937	1·7459	2·1199	2·4729	2·9208	3·2520
17	0·68919	1·1910	1·7396	2·1098	2·4581	2·8982	3·2225
18	0·68837	1·1887	1·7341	2·1009	2·4450	2·8784	3·1966
19	0·68763	1·1866	1·7291	2·0930	2·4334	2·8609	3·1737
20	0·68696	1·1848	1·7247	2·0860	2·4231	2·8453	3·1534
21	0·68635	1·1830	1·7207	2·0796	2·4138	2·8314	3·1352
22	0·68580	1·1816	1·7171	2·0739	2·4055	2·8188	3·1188
23	0·68531	1·1802	1·7139	2·0687	2·3979	2·8073	3·1040
24	0·68485	1·1789	1·7109	2·0639	2·3910	2·7969	3·0905
25	0·68443	1·1777	1·7081	2·0595	2·3846	2·7874	3·0782
26	0·68405	1·1766	1·7056	2·0555	2·3788	2·7787	3·0669
27	0·68370	1·1757	1·7033	2·0518	2·3734	2·7707	3·0565
28	0·68335	1·1748	1·7011	2·0484	2·3685	2·7633	3·0469
29	0·68304	1·1739	1·6991	2·0452	2·3638	2·7564	3·0380
30	0·68276	1·1731	1·6973	2·0423	2·3596	2·7500	3·0298
40	0·68066	1·1673	1·6839	2·0213	2·3289	2·7045	2·9712
60	0·67862	1·1616	1·6707	2·0003	2·2991	2·6603	2·9146
120	0·67656	1·1559	1·6577	1·9799	2·2699	2·6174	2·8599
∞	0·67449	1·1503	1·6449	1·9600	2·2414	2·5758	2·8070

*Reproduced by special permission of the editors of Biometrika from vol. 32, p. 300, "Table of Percentage Points of the t-Distribution" by M. Merrington.

Table IV*

NTAGE POINTS OF THE TRIBUTION

AXINE MERRINGTON

Miss Thompson's Tables of Percentage Points of the 166–81), by taking

$(\nu_2(1-x), \nu_1 x)$

ratio based on an estimate of variance having $\nu = \nu_2$ ed, P_t 100 is the chance that $|t| \geqslant t_P$, i.e. represents The table includes certain levels for t not previously significant figures shown.

5	2·5	1	0·5
12·706	25·452	63·657	127·32
4·3027	6·2053	9·9248	14·089
3·1825	4·1765	5·8409	7·4533
2·7764	3·4954	4·6041	5·5076
2·5706	3·1634	4·0321	4·7733
2·4469	2·9687	3·7074	4·3168
2·3646	2·8412	3·4995	4·0293
2·3060	2·7515	3·3554	3·8325
2·2622	2·6850	3·2498	3·6897
2·2281	2·6338	3·1693	3·5814
2·2010	2·5931	3·1058	3·4966
2·1788	2·5600	3·0545	3·4284
2·1604	2·5326	3·0123	3·3725
2·1448	2·5096	2·9768	3·3257
2·1315	2·4899	2·9467	3·2860
2·1199	2·4729	2·9208	3·2520
2·1098	2·4581	2·8982	3·2225
2·1009	2·4450	2·8784	3·1966
2·0930	2·4334	2·8609	3·1737
2·0860	2·4231	2·8453	3·1534
2·0796	2·4138	2·8314	3·1352
2·0739	2·4055	2·8188	3·1188
2·0687	2·3979	2·8073	3·1040
2·0639	2·3910	2·7969	3·0905
2·0595	2·3846	2·7874	3·0782
2·0555	2·3788	2·7787	3·0669
2·0518	2·3734	2·7707	3·0565
2·0484	2·3685	2·7633	3·0469
2·0452	2·3638	2·7564	3·0380
2·0423	2·3596	2·7500	3·0298
2·0211	2·3289	2·7045	2·9712
2·0003	2·2991	2·6603	2·9146
1·9799	2·2699	2·6174	2·8599
1·9600	2·2414	2·5758	2·8070

on of the editors of Biometrika
Percentage Points of the
on.

TABLE V*

x^2

Degrees of freedom	$P = 0.99$	0.98	0.95	0.90	0.80	0.70	0.50	0.30	0.20	0.10	0.05	0.02	0.01
1	0.0001	0.0006	0.0039	0.0158	0.0642	0.148	0.455	1.074	1.642	2.706	3.841	5.412	6.635
2	0.0201	0.0404	0.103	0.211	0.446	0.713	1.386	2.408	3.219	4.605	5.991	7.824	9.210
3	0.115	0.185	0.352	0.584	1.005	1.424	2.366	3.665	4.642	6.251	7.815	9.837	11.341
4	0.297	0.429	0.711	1.064	1.649	2.195	3.357	4.878	5.989	7.779	9.488	11.668	13.277
5	0.554	0.752	1.145	1.610	2.343	3.000	4.351	6.064	7.289	9.236	11.070	13.388	15.086
6	0.872	1.134	1.635	2.204	3.070	3.828	5.348	7.231	8.558	10.645	12.592	15.033	16.812
7	1.239	1.564	2.167	2.833	3.822	4.671	6.346	8.383	9.803	12.017	14.067	16.622	18.475
8	1.646	2.032	2.733	3.490	4.594	5.527	7.344	9.524	11.030	13.362	15.507	18.168	20.090
9	2.088	2.532	3.325	4.168	5.380	6.393	8.343	10.656	12.242	14.684	16.919	19.679	21.666
10	2.558	3.059	3.940	4.865	6.179	7.267	9.342	11.781	13.442	15.987	18.307	21.161	23.209
11	3.053	3.609	4.575	5.578	6.989	8.148	10.341	12.899	14.631	17.275	19.675	22.618	24.725
12	3.571	4.178	5.226	6.304	7.807	9.034	11.340	14.011	15.812	18.549	21.026	24.054	26.217
13	4.107	4.765	5.892	7.042	8.634	9.926	12.340	15.119	16.985	19.812	22.362	25.472	27.688
14	4.660	5.368	6.571	7.790	9.467	10.821	13.339	16.222	18.151	21.064	23.685	26.873	29.141
15	5.229	5.985	7.261	8.547	10.307	11.721	14.339	17.322	19.311	22.307	24.996	28.259	30.578
16	5.812	6.614	7.962	9.312	11.152	12.624	15.338	18.418	20.465	23.542	26.296	29.633	32.000
17	6.408	7.255	8.672	10.085	12.002	13.531	16.338	19.511	21.615	24.769	27.587	30.995	33.409
18	7.015	7.906	9.390	10.865	12.857	14.440	17.338	20.601	22.760	25.989	28.869	32.346	34.805
19	7.633	8.567	10.117	11.651	13.716	15.352	18.338	21.689	23.900	27.204	30.144	33.687	36.191
20	8.260	9.237	10.851	12.443	14.578	16.266	19.337	22.775	25.038	28.412	31.410	35.020	37.566
21	8.897	9.915	11.591	13.240	15.445	17.182	20.337	23.858	26.171	29.615	32.671	36.343	38.932
22	9.542	10.600	12.338	14.041	16.314	18.101	21.337	24.939	27.301	30.813	33.924	37.659	40.289
23	10.196	11.293	13.091	14.848	17.187	19.021	22.337	26.018	28.429	32.007	35.172	38.968	41.638
24	10.856	11.992	13.848	15.659	18.062	19.943	23.337	27.096	29.553	33.196	36.415	40.270	42.980
25	11.524	12.697	14.611	16.473	18.940	20.867	24.337	28.172	30.675	34.382	37.652	41.566	44.314
26	12.198	13.409	15.379	17.292	19.820	21.792	25.336	29.246	31.795	35.563	38.885	42.856	45.642
27	12.879	14.125	16.151	18.114	20.703	22.719	26.336	30.319	32.912	36.741	40.113	44.140	46.963
28	13.565	14.847	16.928	18.939	21.588	23.647	27.336	31.391	34.027	37.916	41.337	45.419	48.278
29	14.256	15.574	17.708	19.768	22.475	24.577	28.336	32.461	35.139	39.087	42.557	46.693	49.588
30	14.953	16.306	18.493	20.599	23.364	25.508	29.336	33.530	36.250	40.256	43.773	47.962	50.892

For degrees of freedom greater than 30, the expression $\sqrt{2\chi^2} - \sqrt{2n'-1}$ may be used as a normal deviate with unit variance, where n' is the number of degrees freedom.

*Reproduced from Table III of R. A. Fisher's Statistical Methods for Research Workers, Oliver & Boyd, Ltd., of Edinburgh, by kind permission of the author and publishers.

Table VI*

L_1

FIVE PER CENT VALUES

Values of k (number of samples)	Values of n (size of samples)									
	2	3	4	5	10	15	20	30	40	50
2	0.0723	0.3107	0.4782	0.5842	0.7985	0.8673	0.9014	0.9349	0.9512	0.9612
3	0.0704	0.3040	0.4696	0.5755	0.7925	0.8632	0.8980	0.9325	0.9495	0.9598
4	0.0753	0.3152	0.4800	0.5849	0.7970	0.8662	0.9005	0.9341	0.9506	0.9608
5	0.0825	0.3278	0.4915	0.5950	0.8025	0.8699	0.9032	0.9358	0.9519	0.9618
10	0.1135	0.3738	0.5341	0.6318	0.8228	0.8813	0.9135	0.9427	0.9573	0.9659
20	0.1472	0.4191	0.5697	0.6658	0.8417	0.8961	0.9227	0.9498	0.9619	0.9694
25	0.1578	0.4320	0.5841	0.6747	0.8453	0.8989	0.9250	0.9508	0.9630	0.9697
50	0.1878	0.4723	0.6278	0.7137	0.8688	0.9150	0.9365	0.9584	0.9672	0.9730

ONE PER CENT VALUES

Values of k (number of samples)										
2	0.0126	0.1361	0.2818	0.3855	0.6782	0.7821	0.8359	0.8902	0.9171	0.9340
3	0.0169	0.1615	0.3138	0.4291	0.6992	0.7976	0.8476	0.8981	0.9234	0.9388
4	0.0233	0.1876	0.3414	0.4594	0.7194	0.8118	0.8586	0.9056	0.9291	0.9434
5	0.0285	0.2101	0.3703	0.4838	0.7350	0.8228	0.8645	0.9113	0.9334	0.9470
10	0.0624	0.2838	0.4483	0.5556	0.7791	0.8537	0.8908	0.9273	0.9458	0.9565
20	0.0997	0.3524	0.5138	0.6142	0.8131	0.8768	0.9082	0.9404	0.9546	0.9633
25	0.1129	0.3718	0.5304	0.6287	0.8201	0.8817	0.9121	0.9415	0.9565	0.9640
50	0.1517	0.4328	0.5915	0.6672	0.8576	0.9077	0.9307	0.9548	0.9643	0.9697

*Reproduced by special permission of the copyright owner from the 1942 edition of H. A. Freeman's "Industrial Statistics" published by John Wiley and Sons of New York City.

Table VI – Continued

L_2

FIVE PER CENT VALUES

					0.7137	0.8688	0.9150	0.9365	0.9584	0.9672	0.9730

One Per Cent Values

Values of k (number of samples)											
	2	0 0126	0.1361	0.2818	0.3855	0.6782	0.7821	0.8359	0.8902	0.9171	0.9340
	3	0 0169	0.1615	0.3138	0.4291	0.6992	0.7976	0.8476	0.8981	0.9234	0.9388
	4	0 0233	0.1876	0.3414	0.4594	0.7194	0.8118	0.8586	0.9056	0.9291	0.9434
	5	0 0285	0 2101	0.3703	0 4838	0.7350	0.8228	0.8645	0.9113	0.9334	0.9470
	10	0.0624	0 2838	0.4483	0 5556	0.7791	0.8537	0.8908	0.9273	0.9458	0.9565
	20	0.0997	0 3524	0.5138	0 6142	0.8131	0.8768	0.9082	0.9404	0.9546	0.9633
	25	0.1129	0 3718	0.5304	0 6287	0.8201	0.8817	0.9121	0.9415	0.9565	0.9640
	50	0 1517	0.4328	0.5915	0.6672	0.8576	0.9077	0.9307	0.9548	0.9643	0.9697

*Reproduced by special permission of the copyright owner from the 1942 edition of H. A. Freeman's "Industrial Statistics" published by John Wiley and Sons of New York City.

Table VI - Continued

L_0

Five Per Cent Values

Values of k (number of samples)	Values of n (size of samples)									
	2	3	4	5	10	15	20	30	40	50
2	0.0190	0.1719	0.3277	0.4424	0.7071	0.8028	0.8515	0.9007	0.9254	0.9403
3	0.0217	0.1745	0.3205	0.4290	0.6977	0.7950	0.8450	0.8960	0.9216	0.9372
4	0.0258	0.1846	0.3328	0.4416	0.7008	0.7968	0.8463	0.8967	0.9222	0.9376
5	0.0301	0.1946	0.3421	0.4511	0.7057	0.8000	0.8488	0.8983	0.9234	0.9384
10	0.0470	0.2296	0.3792	0.4857	0.7262	0.8142	0.8595	0.9057	0.9294	0.9432
20	0.0656	0.2640	0.4149	0.5193	0.7549	0.8358	0.8766	0.9183	0.9383	0.9511
25	0.0721	0.2744	0.4204	0.5338	0.7635	0.8422	0.8816	0.9210	0.9408	0.9519
50	0.0878	0.3036	0.4644	0.5731	0.7865	0.8550	0.8891	0.9275	0.9449	0.9554

One Per Cent Values

Values of k (number of samples)										
	2	0.0023	0.0590	0.1804	0.2856	0.5870	0.7135	0.7811	0.8516	0.8875	0.9097
	3	0.0024	0.0856	0.2053	0.3112	0.6043	0.7253	0.7900	0.8575	0.8922	0.9133
	4	0.0065	0.1031	0.2293	0.3353	0.6219	0.7382	0.8001	0.8644	0.8975	0.9176
	5	0.0092	0.1180	0.2478	0.3558	0.6367	0.7490	0.8086	0.8702	0.9018	0.9210
	10	0.0229	0.1679	0.3093	0.4171	0.6804	0.7809	0.8335	0.8877	0.9154	0.9321
	20	0.0422	0.2166	0.3656	0.4736	0.7286	0.8180	0.8632	0.9098	0.9316	0.9443
	25	0.0495	0.2315	0.3814	0.4954	0.7428	0.8285	0.8710	0.9142	0.9356	0.9473
	50	0.0688	0.2729	0.4389	0.5464	0.7727	0.8486	0.8833	0.9243	0.9422	0.9531

Table VII*

ORTHOGONAL POLYNOMIALS

(Values from $n = 32$ to $n = 51$ are due to V. Satakopan)

| 3 | | |

Table VII - Continued

ξ'_1	ξ'_2	19 ξ'_3	ξ'_4	ξ'_5	ξ'_1	ξ'_2	20 ξ'_3	ξ'_4	ξ'_5	ξ'_1	ξ'_2	21 ξ'_3	ξ'_4	ξ'_5
0	−30	0	+396	0	+1	−33	−99	+1188	+396	0	−110	0	+594	0
+1	−29	−44	+352	+44	+3	−31	−287	+948	+1076	+1	−107	−54	+540	+1404
+2	−26	−83	+227	+74	+5	−27	−445	+503	+1441	+2	−98	−103	+385	+1444
+3	−21	−112	+42	+79	+7	−21	−553	−77	+1351	+3	−83	−142	+150	+2819
+4	−14	−126	−168	+54	+9	−13	−591	−687	+771	+4	−62	−166	−130	+2354
+5	−5	−120	−354	+3	+11	−3	−539	−1187	−187	+5	−35	−170	−406	+1063
+6	+6	−89	−453	−58	+13	+9	−377	−1402	−1222	+6	−2	−149	−615	−788
+7	+19	−28	−388	−98	+15	+23	−85	−1222	−1802	+7	+37	−98	−680	−2618
+8	+34	+68	−68	−68	+17	+39	+357	−102	−1122	+8	+82	−12	−510	−3468
+9	+51	+204	+612	+102	+19	+57	+969	+1938	+1938	+9	+133	+114	0	−1938
										+10	+190	+285	+969	+3876
570	13,566	213,180	2,288,132	89,148	2,660	17,556	4,903,140	22,831,320	31,201,800	770	201,824	432,630	5,720,330	121,687,080
1	1	$\frac{1}{4}$	$\frac{1}{12}$	$\frac{1}{20}$	2	1	$\frac{1}{8}$	$\frac{11}{24}$	$\frac{1}{20}$	1	3	$\frac{1}{8}$	$\frac{1}{12}$	$\frac{11}{20}$

ξ'_1	ξ'_2	22 ξ'_3	ξ'_4	ξ'_5	ξ'_1	ξ'_2	23 ξ'_3	ξ'_4	ξ'_5	ξ'_1	ξ'_2	24 ξ'_3	ξ'_4	ξ'_5
+1	−20	−12	+702	+390	0	−44	0	+858	0	+1	−143	−143	+143	+715
+3	−19	−35	+585	+1079	+1	−45	−13	+793	+65	+3	−137	−419	+123	+2005
+5	−17	−55	+365	+1509	+2	−40	−25	+605	+116	+5	−125	−665	+85	+2893
+7	−14	−70	+70	+1554	+3	−35	−35	+325	+141	+7	−107	−861	+33	+3171
+9	−10	−78	−258	+1158	+4	−28	−42	−42	+132	+9	−83	−987	−27	+8721
+11	−5	−77	−563	+363	+5	−19	−45	−417	+87	+11	−53	−1023	−87	+2551
+13	+1	−65	−775	−663	+6	−8	−43	−747	+12	+13	−17	−949	−137	−169
+15	+8	−40	−810	−2598	+7	+5	−35	−955	−77	+15	+25	−745	−165	−2071
+17	+16	0	−570	−1938	+8	+20	−20	−950	−152	+17	+73	−391	−157	−3553
+19	+25	+57	+57	−969	+9	+37	+3	−627	−171	+19	+127	+133	−97	−3743
+21	+35	+133	+1197	+2261	+10	+56	+35	+133	−76	+21	+187	+647	+33	−1463
					+11	+77	+77	+1463	+209	+23	+253	+1771	+253	+4807
3,542	7,084	96,140	8,768,740	40,562,340	1,012	35,420	32,890	13,123,110	340,260	4,600	394,680	17,760,600	394,680	177,808,980
2	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{12}$	$\frac{1}{20}$	1	1	$\frac{1}{8}$	$\frac{1}{12}$	$\frac{1}{20}$	2	3	$\frac{11}{24}$	$\frac{1}{12}$	$\frac{1}{20}$

ξ'_1	ξ'_2	25 ξ'_3	ξ'_4	ξ'_5	ξ'_1	ξ'_2	26 ξ'_3	ξ'_4	ξ'_5	ξ'_1	ξ'_2	27 ξ'_3	ξ'_4	ξ'_5
0	−52	0	+858	0	+1	−28	−84	+1386	+330	0	−182	0	+1638	0
+1	−51	−77	+803	+275	+3	−27	−247	+1221	+935	+1	−179	−18	+1548	+3960
+2	−48	−149	+643	+500	+5	−25	−395	+905	+1381	+2	−170	−35	+1285	+7304
+3	−43	−211	+393	+631	+7	−22	−518	+466	+1582	+3	−155	−50	+870	+9479
+4	−36	−258	+78	+636	+9	−18	−606	−54	+1482	+4	−134	−62	+338	+10058
+5	−27	−285	−267	+501	+11	−13	−649	−599	+1067	+5	−107	−70	−262	+8802
+6	−16	−287	−597	+236	+13	−7	−637	−1099	+377	+6	−74	−73	−867	+5728
+7	−3	−259	−857	−119	+15	0	−560	−1470	−482	+7	−35	−70	−1400	+1262
+8	+12	−196	−982	−488	+17	+8	−408	−1614	−1326	+8	+10	−60	−1770	−4188
+9	+29	−93	−897	−753	+19	+17	−171	−1419	−1881	+9	+61	−42	−1872	−9174
+10	+48	+55	−517	−748	+21	+27	+161	−759	−1771	+10	+118	−15	−1587	−12144
+11	+69	+253	+253	−253	+23	+38	+598	+506	−506	+11	+181	+22	−782	−10879
+12	+92	+506	+1518	+1012	+25	+50	+1150	+2530	+2530	+12	+250	+70	+690	−2530
										+13	+325	+130	+2990	+16445
1,300	53,820	1,480,050	14,307,150	7,803,900	5,850	16,380	7,803,900	40,060,020	48,384,180	1,638	712,530	101,790	56,448,210	2,032,135,560
1	1	$\frac{1}{4}$	$\frac{1}{12}$	$\frac{1}{20}$	2	$\frac{1}{2}$	$\frac{1}{8}$	$\frac{1}{12}$	$\frac{1}{20}$	1	3	$\frac{1}{8}$	$\frac{1}{12}$	$\frac{11}{20}$

Table VII – Continued

28

ξ'_1	ξ'_2	ξ'_3	ξ'_4	ξ'_5
+1	−65	−39	+936	+1560
+3	−63	−115	+840	+4456
+5	−59	−185	+655	+6701
+7	−53	−245	+395	+7931
+9	−45	−291	+81	+7887
+11	−35	−319	−259	+6457
+13	−23	−325	−590	+3718
+15	−9	−305	−870	−22
+17	+7	−255	−1050	−4182
+19	+25	−171	−1074	−7866
+21	+45	−49	−879	−9821
+23	+67	+115	−395	−8395
+25	+91	+325	+455	−1495
+27	+117	+585	+1755	+13455

7,308 — 2,103,660 — 1,354,757,040
95,004 — 19,634,160
2 | 1 | $\frac{1}{2}$ | $\frac{1}{4}$ | $\frac{1}{6}$

29

ξ'_1	ξ'_2	ξ'_3	ξ'_4	ξ'_5
0	−70	0	+2184	0
+1	−69	−104	+2080	+1768
+2	−66	−203	+1775	+3298
+3	−61	−292	+1290	+4373
+4	−54	−366	+660	+4818
+5	−45	−420	−66	+4521
+6	−34	−449	−825	+3454
+7	−21	−448	−1540	+1694
+8	−6	−412	−2120	−556
+9	+11	−336	−2460	−2946
+10	+30	−215	−2441	−4958
+11	+51	−44	−1930	−5885
+12	+74	+182	−780	−4810
+13	+99	+468	+1170	−585
+14	+126	+819	+4095	+8190

2,030 — 4,207,320 — 500,071,080
113,274 — 107,087,880
1 | 1 | $\frac{1}{2}$ | $\frac{1}{12}$ | $\frac{1}{10}$

30

ξ'_1	ξ'_2	ξ'_3	ξ'_4	ξ'_5
+1	−112	−112	+12376	+1768
+3	−109	−331	+11271	+5083
+5	−103	−535	+9131	+7753
+7	−94	−714	+6096	+9408
+9	−82	−858	+2376	+9768
+11	−67	−957	−1749	+8679
+13	−49	−1001	−5989	+5149
+15	−28	−980	−9744	+1384
+17	−4	−884	−12704	−2176
+19	+23	−703	−14849	−6811
+21	+53	−427	−13749	−10535
+23	+86	−46	−10504	−11960
+25	+122	+450	−3744	−9360
+27	+161	+1071	+7371	−585
+29	+203	+1827	+23751	+16965

8,990 — 11,360,240 — 2,145,733,200
308,064 — 3,071,587,920
2 | $\frac{1}{2}$ | $\frac{1}{8}$ | $\frac{1}{12}$ | $\frac{1}{16}$

31

ξ'_1	ξ'_2	ξ'_3	ξ'_4	ξ'_5
+0	−80	0	+408	0
+1	−79	−119	+391	+221
+2	−76	−233	+341	+416
+3	−71	−337	+261	+561
+4	−64	−426	+156	+630
+5	−55	−495	+33	+627
+6	−44	−539	−99	+528
+7	−31	−553	−229	+343
+8	−16	−532	−344	+88
+9	+1	−471	−429	−207
+10	+20	−385	−467	−496
+11	+41	−299	−439	−715
+12	+64	−212	−324	−780
+13	+89	−273	−99	−585
+14	+116	−609	+261	0
+15	+145	+1015	+783	+1131

2,480 — 6,714,520 — 9,530,592
58,224 — 4,034,712
1 | 1 | $\frac{1}{2}$ | $\frac{1}{12}$ | $\frac{1}{10}$

32

ξ'_1	ξ'_2	ξ'_3	ξ'_4	ξ'_5
+1	85	−51	+459	+255
+3	83	−151	+423	+737
+5	79	−245	+353	+1137
+7	73	−329	+253	+1407
+9	65	−399	+129	+1509
+11	55	−451	−11	+1419
+13	43	−481	−157	+1131
+15	29	−485	−297	+661
+17	13	−459	−417	+51
+19	+5	−399	−501	−627
+21	+25	−301	−531	−1207
+23	+47	−161	−487	−1725
+25	+71	+25	−347	−1815
+27	+97	+261	−87	−1305
+29	+125	+551	+319	+87
+31	+155	+899	+899	+2697

10,912 — 5,379,616 — 54,285,210
135,504 — 5,379,616
2 | 1 | $\frac{1}{2}$ | $\frac{1}{12}$ | $\frac{1}{12}$

33

ξ'_1	ξ'_2	ξ'_3	ξ'_4	ξ'_5
0	−272	0	+3672	0
+1	−269	−27	+3537	+2565
+2	−260	−53	+3139	+4864
+3	−245	−77	+2499	+6649
+4	−224	−98	+1652	+7708
+5	−197	−115	+647	+7883
+6	−164	−127	−453	+7088
+7	−125	−133	−1571	+5327
+8	−80	−132	−2616	+2712
+9	−29	−123	−3483	−519
+10	+28	−105	−4053	−3084
+11	+91	−77	−4193	−7139
+12	+160	−38	−3756	−9260
+13	+235	+13	−2581	−9425
+14	+316	+77	−493	−6496
+15	+403	+155	+2647	+899
+16	+496	+248	+7192	+14384

2,982 — 417,384 — 1,567,188,096
1,947,792 — 348,330,120
1 | 3 | $\frac{1}{8}$ | $\frac{1}{16}$ | $\frac{1}{16}$

Continued

ξ'_4	ξ'_5	ξ'_1	ξ'_2	ξ'_3	ξ'_4	ξ'_5
2184	0	+1	−112	−112	+12376	+1768
2080	+1768	+3	−109	−331	+11271	+5083
1775	+3298	+5	−103	−535	+9131	+7753
1290	+4373	+7	−94	−714	+6096	+9008
660	+4818	+9	−82	−858	+2376	+9768
−66	+4521	+11	−67	−957	−1749	+8679
−825	+3454	+13	−49	−1001	−5929	+6149
1540	+1694	+15	−28	−980	−9744	+2384
2220	−556	+17	−4	−884	−12704	−2176
2460	−2946	+19	+23	−703	−14249	−6821
2441	−4958	+21	+53	−427	−13749	−10535
1932	−5885	+23	+86	−46	−10504	−11960
−780	−4810	+25	+122	+450	−3744	−9360
2870	−585	+27	+161	+1071	+7371	−585
4095	+8190	+29	+203	+1827	+23751	+16965
	900,671,080	8,990		21,360,240		2,145,733,200
	387,880		308,364		3,671,587,920	
	1/8	1/10	2	3	4	11

ξ'_4	ξ'_5	ξ'_1	ξ'_2	ξ'_3	ξ'_4	ξ'_5
459	+255	0	−272	0	+3672	0
423	+737	+1	−269	−27	+3537	+2565
353	+1137	+2	−260	−53	+3135	+4864
253	+1407	+3	−245	−77	+2499	+6749
129	+1509	+4	−224	−98	+1652	+7708
−21	+1419	+5	−197	−115	+647	+7883
−157	+1131	+6	−164	−127	−453	+7088
−297	+661	+7	−125	−133	−1571	+5327
−417	+51	+8	−80	−132	−2616	+2712
−501	−627	+9	−29	−123	−3483	−519
−531	−1267	+10	+28	−105	−4053	−3984
−487	−1725	+11	+91	−77	−4193	−7139
−347	−1815	+12	+160	−38	−3756	−9260
−87	−1305	+13	+235	+13	−2581	−9415
319	+8	+14	+316	+77	−493	−6496
899	+2697	+15	+403	+155	+2697	+899
		+16	+496	+248	+7192	+14384
	56,285,216	2,992		417,384		1,547,128,656
	2646		1,947,792		348,330,136	
	1/8	1/10	1	3	4	11

Table VII - Continued

ξ'_1	ξ'_2	ξ'_3	ξ'_4	ξ'_5
1	−48	−144	+4104	+6840
3	−47	−427	+3819	+19855
5	−45	−695	+3263	+30917
7	−42	−938	+2464	+38864
9	−38	−1146	+1464	+42744
11	−33	−1309	+319	+41899
13	−27	−1417	−901	+36049
15	−20	−1460	−2112	+25376
17	−12	−1428	−3216	+10608
19	−3	−1311	−4101	−6897
21	+7	−1099	−4641	−25067
23	+18	−782	−4696	−41032
25	+30	−350	−4112	−51040
27	+43	+207	−2721	−50373
29	+57	+899	−341	−33263
31	+72	+1736	+3224	+7192
33	+88	+2728	+8184	+79112
		13,090	51,477,360	46,909,569,232
		62,832		456,438,592
		2	1/2	4	1/8	1/6

ξ'_1	ξ'_2	ξ'_3	ξ'_4	ξ'_5
0	−102	0	+23256	0
1	−101	−152	+22496	+3800
2	−98	−299	+20251	+7250
3	−93	−436	+16626	+10021
4	−86	−558	+11796	+11826
5	−77	−660	+6006	+12441
6	−66	−737	−429	+11726
7	−53	−784	−7124	+9646
8	−38	−796	−13624	+6292
9	−21	−768	−19404	+1902
10	−2	−695	−23869	−3118
11	+19	−572	−26354	−8173
12	+42	−394	−26124	−12458
13	+67	−156	−22374	−14937
14	+94	+147	−14229	−14322
15	+123	+520	−744	−9052
16	+154	+968	+19096	+2728
17	+187	+1496	+46376	+23188
		3,570	15,775,320	4,045,652,520
			290,598	14,834,059,240
		1	1	4	11	1/6

ξ'_1	ξ'_2	ξ'_3	ξ'_4	ξ'_5
1	−323	−323	+2584	+12920
3	−317	−959	+2424	+37640
5	−305	−1565	+2111	+59063
7	−287	−2121	+1659	+75201
9	−263	−2607	+1089	+84381
11	−233	−3003	+429	+85371
13	−197	−3289	−286	+77506
15	−155	−3445	−1014	+60814
17	−107	−3451	−1706	+36142
19	−53	−3287	−2306	+5282
21	+7	−2933	−2751	−28903
23	+73	−2369	−2971	−62353
25	+145	−1575	−2889	−89685
27	+223	−531	−2421	−104067
29	+307	+783	−1476	−97092
31	+397	+2387	+44	−58652
33	+493	+4301	+2244	+23188
35	+595	+6545	+5236	+162316
		15,540	307,618,740	199,046,103,984
			3,011,652	191,407,216
		2	3	1/8	1/12	1/10

ξ'_1	ξ'_2	ξ'_3	ξ'_4	ξ'_5
0	−114	0	+5814	0
1	−113	−34	+5644	+680
2	−110	−67	+5141	+1304
3	−105	−98	+4326	+1819
4	−98	−126	+3234	+2178
5	−89	−150	+1914	+2343
6	−78	−169	+429	+2288
7	−65	−182	−1144	+2000
8	−50	−188	−2714	+1492
9	−33	−186	−4176	+786
10	−14	−175	−5411	−64
11	+7	−154	−6286	−979
12	+30	−122	−6654	−1890
13	+55	−78	−6354	−2535
14	+82	−21	−5211	−2898
15	+111	+50	−3036	−2986
16	+142	+136	+374	−1496
17	+175	+238	+5236	+748
18	+210	+357	+11781	+4488
		4,218	930,170	152,877,192
			383,838	980,681,982
		1	1	4	1/8	1/6

ξ'_1	ξ'_2	ξ'_3	ξ'_4	ξ'_5
1	−60	−36	+918	+1530
3	−59	−107	+867	+4471
5	−57	−175	+767	+7061
7	−54	−238	+622	+9086
9	−50	−294	+438	+10362
11	−45	−341	+223	+10747
13	−39	−377	−13	+10153
15	−32	−400	−258	+8538
17	−24	−408	−498	+6018
19	−15	−399	−717	+2670
21	−5	−371	−897	−1811
23	+6	−322	−1018	−5890
25	+18	−250	−1058	−9070
27	+31	−153	−993	−11985
29	+45	−29	−797	−13079
31	+60	+124	−442	−11594
33	+76	+308	+102	−6358
35	+93	+525	+867	+3927
37	+111	+777	+1887	+20757
		4,950	103,668	6,404,388	25,470,532	3,388,890,600
						1/10
		2	1/2	4	1/8	1/6

ξ'_1	ξ'_2	ξ'_3	ξ'_4	ξ'_5
0	−380	0	+1026	0
1	−377	−189	+999	+5049
2	−368	−373	+919	+9724
3	−353	−547	+789	+13669
4	−332	−706	+614	+16504
5	−305	−845	+401	+18143
6	−272	−959	+159	+18212
7	−233	−1043	−101	+16667
8	−188	−1092	−366	+13512
9	−137	−1101	−621	+8877
10	−80	−1065	−849	+3036
11	−17	−979	−1031	−3575
12	+52	−838	−1146	−10340
13	+123	−637	−1171	−16445
14	+208	−371	−1081	−20860
15	+295	−35	−849	−22321
16	+388	+376	−446	−19312
17	+487	+867	+159	−10047
18	+592	+1443	+999	+7548
19	+703	+2109	+2109	+35853
		9,880	6,004,388	33,722,010	30,204,114	9,880,578,884
						1/10
		1	3	4	1/8	1/6

Table VII – Continued

40

ξ'_1	ξ'_2	ξ'_3	ξ'_4	ξ'_5
1	133	399	+39501	+627
3	−131	−1187	+37521	+1837
5	−127	−1945	+33631	+2917
7	−121	−2653	+27971	+3787
9	−113	−3291	+20751	+4377
11	−103	−3839	+12251	+4631
13	−91	−4277	+2821	+4511
15	77	−4585	−7119	+4001
17	−61	−4743	−17079	+3111
19	−43	−4731	−26499	+1881
21	−23	−4529	−34749	+385
23	−1	−4117	−41129	−1265
25	+23	−3475	−44869	−2915
27	+49	−2583	−45129	−4365
29	+77	−1421	−40999	−5365
31	+107	+31	−31499	−5611
33	+139	+1793	−15579	−4741
35	+173	+3895	+7881	−2331
37	+209	+6327	+40071	+2109
39	+247	+9139	+82251	+9139

$21,320 \quad 644,482,280 \quad 644,482,280 \quad 5,740 \quad 47,900,210 \quad 14,370,164,700$

$507,112 \quad 49,025,135,560$

$2 \quad 1 \quad \frac{1}{2} \quad \frac{1}{12} \quad \frac{1}{30}$

41

ξ'_1	ξ'_2	ξ'_3	ξ'_4	ξ'_5
0	−140	0	+8778	0
1	−139	−209	+8569	+4807
2	136	−413	+7949	+9292
3	−131	−607	+6939	+13147
4	−124	−786	+5574	+16092
5	−115	−945	+3903	+17889
6	−104	−1079	+1989	+18356
7	−91	−1183	−91	+17381
8	−76	−1252	−2246	+14936
9	−59	−1281	−4371	+11091
10	−40	−1265	−6347	+6028
11	−19	−1199	−8041	+55
12	+4	−1078	−9306	−6380
13	+29	−897	−9981	−12975
14	+56	−651	−9891	−18660
15	+85	−335	−8847	−21583
16	+116	+56	−6646	−22096
17	+149	+527	−3071	−18241
18	+184	+1083	+2109	−8436
19	+221	+1729	+9139	+9139
20	+260	+2470	+18278	+36556

$641,732 \quad 2,481,350,278$

$2 \quad 1 \quad \frac{1}{2} \quad \frac{1}{12} \quad \frac{1}{30}$

42

ξ'_1	ξ'_2	ξ'_3	ξ'_4	ξ'_5
1	−220	−44	+9614	+48070
3	−217	−131	+9177	+141151
5	−211	−215	+8317	+225181
7	−202	−294	+7062	+294546
9	−190	−366	+5454	+344262
11	−175	−429	+3549	+370227
13	−157	−481	+1417	+369473
15	−136	−520	−858	+340418
17	−112	−544	−3178	+283118
19	−85	−551	−5431	+199519
21	−55	−539	−7491	+93709
23	−22	−506	−9218	−27830
25	+14	−450	−10458	−155970
27	+53	−369	−11043	−278685
29	+95	−261	−10791	−380799
31	+140	−124	−9506	−443734
33	+188	+44	−6978	−445258
35	+239	+245	−2983	−359733
37	+293	+481	+2717	−155363
39	+350	+754	+10374	+201058
41	+410	+1066	+20754	+749398

$1,020,012 \quad 3,084,805,724$

$2 \quad \frac{1}{2} \quad \frac{1}{3} \quad \frac{1}{12} \quad \frac{1}{16}$

43

ξ'_1	ξ'_2	ξ'_3	ξ'_4	ξ'_5
0	155	0	+10626	0
1	153	−46	+10396	+8740
2	150	91	+9713	+16948
3	145	134	+8598	+24113
4	138	174	+7086	+29766
5	129	210	+5226	+33501
6	118	243	+3081	+34996
7	−105	−269	+728	+34034
8	90	284	1742	+30524
9	73	294	4224	+24523
10	54	295	6500	+16252
11	33	286	8434	+6127
12	10	266	10082	5230
13	+15	234	11682	16965
14	+42	189	12150	27972
15	+71	131	11724	36852
16	+102	60	10174	41902
17	+135	+14	7292	41344
18	+170	+141	2847	32604
19	+207	+266	+3400	11091
20	+246	+410	+11726	+20254
21	+287	+574	+22380	+70889

$6,822 \quad 1,796,234 \quad 30,561,000,044 \quad 28,353$

$814,500 \quad 3,813,417,000$

$1 \quad 1 \quad \frac{1}{2} \quad \frac{1}{4} \quad \frac{1}{10}$

44

ξ'_1	ξ'_2	ξ'_3	ξ'_4	ξ'_5
1	−101	483	+5790	+1380
3	159	1439	+5550	+4060
5	1	2305	+5083	+6503
7	149	3241	+4301	+8661
9	143	4047	+3501	+10101
11	131	4763	+2441	+11011
13	119	5369	+1246	+11206
15	105	5845	−42	+10653
17	89	6171	1374	+9282
19	71	6327	2694	+7185
21	51	6291	3939	+4417
23	30	6049	5039	+1127
25	8	5575	5917	2485
27	+21	4851	6489	6147
29	+46	3857	6664	9512
31	+70	2573	6344	12152
33	+111	979	5474	13552
35	+145	+985	3792	13104
37	+181	+3219	1329	10101
39	+219	+5863	+2091	3731
41	+259	+8897	+6601	+6929
43	+301	+12341	+12341	+22919

$1,257,029,980 \quad 4,102,273,750$

$913,850 \quad 1,173,974,848$

$1 \quad \frac{1}{2} \quad \frac{1}{4} \quad \frac{1}{10}$

45

ξ'_1	ξ'_2	ξ'_3	ξ'_4	ξ'_5
0	506	0	+9108	0
1	503	252	+8928	+4500
2	494	499	+8393	+8750
3	479	736	+7518	+12509
4	458	958	+6328	+15554
5	431	1160	+4858	+17689
6	398	1337	+3153	+18754
7	359	1484	+1268	+18634
8	314	1596	−732	+17268
9	263	1668	−2772	+14658
10	206	1695	−4767	+10878
11	143	1672	−6622	+6083
12	74	1594	−8232	+518
13	+1	1456	−9482	−5473
14	+82	1253	−10247	−11438
15	+169	−980	−10392	−16808
16	+262	632	−9772	−20888
17	+361	204	−8232	−22848
18	+466	+309	−5607	−21774
19	+577	+912	−1722	−16359
20	+694	+1610	+3608	−5494
21	+817	+2408	+10578	+12341
22	+946	+3311	+19393	+38586

$92,036,340 \quad 11,036,558,900$

$9,203,634 \quad 2,934,930,620$

$1 \quad 3 \quad \frac{1}{4} \quad \frac{1}{12} \quad \frac{1}{16}$

- Continued

ξ'_4	ξ'_6
8778	0
9569	+4807
7949	+9292
9139	+13147
8374	+16092
9903	+17889
9689	+18356
7991	+17381
6746	+14936
3972	+11091
7987	+6028
8211	+55
8256	-6380
8112	-12575
7918	-18060
8287	-21583
8115	-22096
7912	-18242
7909	-8430
7904	+9139
7900	+36556

ξ'_1	ξ'_2	ξ'_3	ξ'_4	ξ'_6
		42		
1	-220	-44	+9614	+48070
3	-217	-132	+9177	+141151
5	-211	-215	+8317	+225181
7	-202	-294	+7062	+294546
9	-190	-366	+5454	+344262
11	-175	-429	+3549	+370227
13	-157	-481	+1417	+369473
15	-136	-520	-858	+340418
17	-112	-544	-3178	+283118
19	-85	-551	-5431	+199519
21	-55	-539	-7491	+93709
23	-22	-506	-9218	-27830
25	+14	-450	-10458	-155970
27	+53	-369	-11043	-278685
29	+95	-261	-10791	-380799
31	+140	-124	-9506	-443734
33	+188	+44	-6978	-445258
35	+239	+245	-2983	-359233
37	+293	+481	+2717	-155363
39	+350	+754	+10374	+201058
41	+410	+2066	+20254	+.49308

ξ'_1	ξ'_2	ξ'_3	ξ'_4	ξ'_6
		45		
+1380	0	-506	0	+9108	0
+4060	1	503	-252	+8928	+4500
+6503	2	494	499	+8393	+8750
+8561	3	-479	-736	+7518	+12509
+20101	4	-458	-958	+6328	+15554
22011	5	-431	-1160	+4858	+17689
21206	6	398	-1337	+3153	+18754
20634	7	-359	-1484	+1268	+18634
+9282	8	-314	1596	-732	+17268
+7182	9	-263	1668	-2772	+14658
4417	10	-206	-1695	-4767	+10878
8127	11	-143	1672	-6622	+6083
0685	12	-74	1594	8232	+518
6147	13	+1	-1456	9482	5471
5512	14	+82	-1253	-10247	11438
2153	15	+169	-980	-10392	16808
2852	16	+262	-632	-9772	20888
2004	17	+361	-204	-8232	22848
2001	18	+466	+309	-5607	21774
2231	19	+577	+913	-1723	16359
2099	20	+694	+1610	+3608	5494
2099	21	+817	+2508	+10578	+12341
	22	+946	+3511	+19393	+35786

ξ'_1	ξ'_2	ξ'_3	ξ'_4	ξ'_5
		46		
1	-88	-264	+1980	+3300
3	-87	-787	+1905	+9725
5	-85	-1295	+1757	+15631
7	-82	-1778	+1540	+20692
9	-78	-2226	+1260	+24612
11	-73	-2629	+925	+27137
13	-67	-2977	+545	+28067
15	-60	-3260	+132	+27268
17	-52	-3468	-300	+24684
19	-43	-3591	-735	+20349
21	-33	-3619	-1155	+14399
23	-22	-3542	-1540	+7084
25	-10	-3350	-1868	-1220
27	+3	-3033	-2115	-9999
29	+17	-2581	-2255	-18589
31	+32	-1984	-2260	-26164
33	+48	-1232	-2100	-31724
35	+65	-315	-1743	-34083
37	+83	+777	-1155	-31857
39	+102	+2054	-300	-23452
41	+122	+3526	+860	-7052
43	+143	+5203	+2365	+19393
45	+165	+7095	+4257	+58179

ξ'_1	ξ'_2	ξ'_3	ξ'_4	ξ'_5
		47		
0	-184	0	+15180	0
1	-183	-55	+24905	+3575
2	-180	-109	+14087	+6968
3	-175	-161	+12747	+10003
4	-268	-210	+.0920	+12516
5	-159	-255	+8655	+14362
6	-148	-295	+6015	+15426
7	-135	-329	+3077	+15589
8	-120	-356	-68	+14824
9	-103	-375	-3315	+13107
20	-84	-385	-6545	+10472
11	-63	-385	-9625	+7007
12	-40	-374	-12408	+2860
13	-15	-351	-14733	-1755
14	+12	-315	-16425	-6552
15	+41	-265	-17295	-11167
16	+72	-200	-17140	-15152
17	+105	-119	-15743	-17969
18	+140	-21	-12873	-18984
19	+177	+95	-8285	-17401
20	+216	+230	-1720	-12556
21	+257	+385	+7095	-3311
22	+300	+561	+18447	+11352
23	+345	+759	+.3637	+32637

ξ'_2	ξ'_3	ξ'_4	ξ'_6
	48		
2	-575	-125	+16445	+82225
3	-569	-343	+15873	+242671
5	-557	-565	+14743	+391232
7	-539	-777	+13083	+520402
9	-515	-975	+20935	+623307
11	-485	-1155	+8355	+693957
13	-449	-1313	+5413	+727493
15	-407	-1445	+2193	+720443
17	-359	-1547	-1207	+670973
19	-305	-1615	-4075	+579139
21	-245	-1645	-8085	+447139
23	-179	-1633	-11297	+279565
25	-107	-1575	-14157	+83655
27	-29	-1467	-16497	-130455
29	+55	-1305	-18135	-349479
31	+145	-1085	-18875	-556729
33	+241	-803	-18507	731863
35	+343	-455	-16807	-850633
37	+451	-37	-13537	-884633
39	+565	+455	-8445	-801047
41	+685	+1025	-1265	-562397
43	+811	+1677	+8283	-126292
45	+943	+2415	+20493	+554829
47	+1081	+3243	+35073	+1533939

ξ'_1	ξ'_2	ξ'_3	ξ'_4	ξ'_5
		49		
0	-200	0	+17940	0
1	-199	-299	+17641	+9867
2	-196	593	+16751	+19272
3	-191	-877	+15291	+27767
4	184	1146	+13296	+34932
5	175	-1395	+10815	+40389
6	164	1614	+7911	+43816
7	151	-1813	+4661	+44961
8	136	1972	+1156	+43656
9	119	2091	-2499	+39832
10	100	2165	-6185	+33518
11	79	2139	-9769	+24925
12	56	2158	-13104	+14300
13	31	2067	-16029	+2145
14	4	-1911	-18369	-10920

ξ'_1	ξ'_2	ξ'_3	ξ'_4	ξ'_5
		49—*continued*		
15	+25	-1685	-19935	-24083
16	+56	-1384	-20524	-36330
17	+89	-1003	-19019	-40461
18	+124	537	-17889	-53016
19	+161	+19	-14189	-54321
20	+200	+670	-8560	-48444
21	+241	+1421	-729	-33187
22	+283	+2277	+9501	-6072
23	+329	+3243	+22701	+35673
24	+376	+4324	+38916	+95128

Table VII – Continued

50

ξ_1	ξ_2	ξ_3	ξ_4	ξ_5
1	−104	−313	+96870	+10704
3	−103	−931	+93771	+31809
5	101	−1535	+87031	+51419
7	98	−2114	+78596	+68684
9	94	−2658	+66876	+82764
11	89	−3157	+52751	+92917
13	83	−3601	+36571	+98527
15	76	−3980	+18756	+99132
17	68	−4284	204	+94452
19	59	−4503	−19749	+84417
21	49	−4627	−39249	+69195
23	38	−4646	−58004	+49220
25	26	−4550	−75244	+25220
27	13	−4329	−90239	−1755
29	+1	−3973	−101749	−30305
31	+16	−3472	−109124	−58652
33	+32	−2816	−111200	−84612
35	+49	−1995	−106869	−105567
37	+67	−999	−94029	−118437
39	+86	+182	−74124	−119652
41	+106	+1558	−43124	−105124
43	+127	+3139	−529	−70219
45	+149	+4935	+55131	−9729
47	+172	+6956	+125306	+82156
49	+196	+9212	+211876	+211876

41,650 770,715,900 372,455,535,200
433,180 372,455,535,200
2 4 8 $\frac{11}{2}$ $\frac{11}{4}$

51

ξ_1	ξ_2	ξ_3	ξ_4	ξ_5
0	−650	0	+21060	0
1	−647	−314	+20736	+7452
2	−638	−643	+19771	+14582
3	−623	−952	+18186	+21077
4	−602	−1246	+16016	+26643
5	−575	−1520	+13310	+31009
6	−542	−1769	+10131	+33946
7	−503	−1988	+6556	+35266
8	−458	−2173	+2676	+34836
9	−407	−2316	−1404	+32586
10	−350	−2415	−5565	+28518
11	−287	−2464	−9674	+21715
12	−218	−2458	−13584	+15350
13	−143	−2392	−17134	+6695
14	−62	−2261	−20149	−2870
15	+25	−2060	−22440	−12848
16	+118	−1784	−23804	−22616
17	+217	−1428	−24024	−31416
18	+322	−987	−22869	−38345
19	+433	−456	−20094	−42351
20	+550	+170	−15440	−42214
21	+673	+896	−8634	−36547
22	+802	+1727	+611	−23782
23	+937	+2668	+12596	−2162
24	+1078	+3724	+27636	+30268
25	+1225	+4900	+46060	+75670

11,050 321,375,700 47,801,426,360
17,215,110 17,803,525,740
1 3 8 $\frac{11}{2}$ $\frac{11}{4}$

52

ξ_1	ξ_2	ξ_3	ξ_4	ξ_5
2	−225	−135	+1620	+370
3	−223	−493	+1572	+7988
5	−219	−665	+1477	+13943
7	−213	−917	+1337	+17353
9	−205	−1155	+1155	+21021
11	−195	−1375	+935	+23771
13	−183	−1573	+682	+25454
15	−169	−1745	+402	+25954
17	−153	−1887	+102	+25194
19	−135	−1995	−210	+23142
21	−115	−2065	−515	+19817
23	−93	−2093	−833	+15295
25	−69	−2075	−1153	+9715
27	−43	−2007	−1383	+3285
29	−15	−1885	−1700	−3712
31	+15	−1705	−1760	−10912
33	+47	−1463	−1848	−17864
35	+81	−1155	−1848	−24024
37	+117	−777	−1743	−28749
39	+155	−325	−1515	−31291
41	+195	+205	−1145	−30791
43	+237	+817	−613	−26273
45	+281	+1515	+102	−16638
47	+327	+2303	+1022	−658
49	+375	+3185	+2170	+23030
51	+425	+4105	+3570	+55930

66,852 102,342,180 26,355,466,880
2,108,340 108,228,120
2 1 $\frac{11}{2}$ $\frac{11}{4}$ $\frac{11}{4}$

Table VIII*

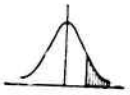
SIGNIFICANCE LEVELS AND STANDARD ERRORS OF δ^2/s^2

P / n	.001	.01	.05	.95	.99	.999
4	.5898	.6258	7805	3.2105	3.3744	3.4102
5	.4181	.5379	.8204	3.1798	3.4621	3.5839
6	.3834	.5815	.8902	3.1098	3.4385	3.6368
7	.3895	.8140	.9359	3.0641	3.3860	3.6305
8	.4038	.8828	.9825	3.0175	3.3372	3.5984
9	.4420	.7088	1.0244	2.9758	3.2911	3.5580
10	.4818	.7518	1.0823	2.9378	3.2482	3.5184
11	.5197	.7915	1.0985	2.9035	3.2085	3.4803
12	.5557	.8280	1.1278	2.8724	3.1720	3.4443
13	.5898	.8818	1.1558	2.8442	3.1381	3.4102
14	.8223	.8931	1.1818	2.8184	3.1068	3.3777
15	.8532	.9221	1.2053	2.7947	3.0778	3.3467
18	.8828	.9491	1.2272	2.7728	3.0509	3.3174
17	.7104	9743	1.2473	2.7527	3.0257	3.2896
18	.7368	.9979	1.2660	2.7340	3.0021	3.2632
19	.7617	1.0199	1.2834	2.7168	2.9800	3.2383
20	.7852	1.0406	1.2998	2.7004	2.9593	3.2148
21	.8073	1.0801	1.3148	2.6852	2.9399	3.1927
22	.8283	1.0785	1.3290	2.8710	2.9216	3.1718
23	.8481	1.0958	1.3425	2.8578	2.9042	3 1520
24	.8668	1.1122	1.3552	2.6449	2.8877	3.1333
25	.8848	1.1278	1.3871	2.6329	2.8722	3.1154
28	.9017	1.1426	1.3785	2.6215	2.8575	3.0983
27	.9182	1.1587	1.3892	2.8108	2.8434	3.0818
28	.9341	1.1702	1.3994	2.6006	2.8300	3.0659
29	.9496	1.1830	1.4091	2.5909	2.8171	3.0505
30	.9645	1.1951	1.4183	2.5817	2.8048	3.0355
31	.9789	1.2067	1.4270	2.5729	2.7933	3.0201
32	.9925	1.2177	1.4354	2.5648	2.7823	3.0076
33	1.0055	1.2283	1.4434	2.5586	2.7717	2.9946
34	1.0180	1.2386	1.4511	2.5490	2.7814	2.9821
35	1.0300	1.2485	1.4585	2.5416	2.7515	2.9701
36	1.0416	1.2581	1.4656	2.5344	2.7419	2.9584
37	1.0529	1.2873	1.4726	2.5274	2.7325	2.9469
38	1.0639	1.2763	1.4793	2.5208	2.7237	2.9360
39	1.0746	1.2850	1.4858	2.5142	2.7151	2.9254
40	1.0850	1.2934	1.4921	2.5079	2.7066	2.9151
41	1.0950	1.3017	1.4982	2.5018	2.8983	2.9050
42	1.1048	1.3096	1.5041	2.4958	2.6904	2.8952
43	1.1142	1.3172	1.5098	2.4901	2.8827	2.8858
44	1.1233	1.3246	1.5154	2.4846	3.6754	2.8767
45	1.1320	1.3317	1.5206	2.4794	2.6683	2.8680
46	1.1404	1.3387	1.5257	2.4743	2.8614	2.8596
47	1.1484	1.3453	1.5305	2.4695	2.8548	2.8516
48	1.1561	1.3515	1.5351	2.4649	2.6486	2.8439
49	1.1635	1.3573	1.5395	2.4604	2.6426	2.8385
50	1.1705	1.3629	1.5437	2.4563	2.6370	2.8295
51	1.1774	1.3683	1.5477	2.4522	2.6316	2.8225
52	1.1843	1.3738	1.5518	2.4483	2.6262	2.8157
53	1.1910	1.3792	1.5557	2.4444	2.6208	2.8090
54	1.1976	1.3846	1.5596	2.4405	2.6154	2.8024
55	1.2041	1.3899	1.5834	2.4368	2.6102	2.7959
56	1.2104	1.3949	1.5870	2.4330	2.6050	2.7896
57	1.2166	1.3999	1.5707	2.4294	2.6001	2.7834
58	1.2227	1.4048	1.5743	2.4258	2.5952	2.7773
59	1.2288	1.4096	1.5779	2.4222	2.5903	2.7712
60	1.2349	1.4144	1.5814	2.4186	2.5856	2.7651

*Reproduced from "Methods of Detecting Non-Randomness in a given Series of Observations," by A. H. J. Baines in Technical Report-Series 'R', No. Q.C./R/12, distributed by the British Ministry of Supply.

Table IX*

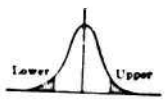
PROBABILITY POINTS OF $\sqrt{b_1}$



Size of sample n	Probability points		Size of sample n	Probability points		Size of sample n	Probability points	
	5%	1%		5%	1%		5%	1%
25	0.711	1.061	200	0.280	0.403	1000	0.127	0.180
30	0.661	0.982	250	0.251	0.360	1200	0.116	0.165
35	0.621	0.921	300	0.230	0.329	1400	0.107	0.152
40	0.587	0.869	350	0.213	0.303	1600	0.100	0.142
45	0.558	0.825	400	0.200	0.285	1800	0.095	0.134
50	0.533	0.787	450	0.188	0.269	2000	0.090	0.127
			500	0.179	0.255			
60	0.492	0.724	550	0.171	0.243	2500	0.080	0.114
70	0.459	0.673	600	0.163	0.233	3000	0.073	0.104
80	0.432	0.631	650	0.157	0.224	3500	0.068	0.096
90	0.409	0.595	700	0.151	0.215	4000	0.064	0.090
100	0.389	0.562	750	0.146	0.208	4500	0.060	0.085
			800	0.142	0.202	5000	0.057	0.081
125	0.350	0.508	850	0.138	0.196			
150	0.321	0.464	900	0.134	0.190			
175	0.298	0.430	950	0.130	0.185			
200	0.280	0.403	1000	0.127	0.180			

Note. As the sampling distribution of $\sqrt{b_1}$ is symmetrical about zero, the same values, with negative sign, correspond to the lower limits.

PROBABILITY POINTS OF b_2



Size of sample n	Probability points				Size of sample n	Probability points			
	Upper 1%	Upper 5%	Lower 5%	Lower 1%		Upper 1%	Upper 5%	Lower 5%	Lower 1%
200	3.98	3.57	2.51	2.37	1000	3.41	3.26	2.76	2.68
250	3.87	3.52	2.55	2.42	1200	3.37	3.24	2.78	2.71
300	3.79	3.47	2.59	2.46	1400	3.34	3.22	2.80	2.72
350	3.72	3.44	2.62	2.50	1600	3.32	3.21	2.81	2.74
400	3.67	3.41	2.64	2.52	1800	3.30	3.20	2.82	2.76
450	3.63	3.39	2.66	2.55	2000	3.29	3.18	2.83	2.77
500	3.60	3.37	2.67	2.57					
550	3.57	3.35	2.69	2.58	2500	3.25	3.16	2.85	2.79
600	3.54	3.34	2.70	2.60	3000	3.22	3.15	2.86	2.81
650	3.52	3.33	2.71	2.61	3500	3.21	3.14	2.87	2.82
700	3.50	3.31	2.72	2.62	4000	3.19	3.13	2.88	2.83
750	3.48	3.30	2.73	2.64	4500	3.18	3.12	2.88	2.84
800	3.46	3.29	2.74	2.65	5000	3.17	3.12	2.89	2.85
850	3.45	3.28	2.74	2.66					
900	3.43	3.28	2.75	2.66					
950	3.42	3.27	2.76	2.67					
1000	3.41	3.26	2.76	2.68					

*Reproduced by special permission of the copyright owner from the 1942 edition of H. A. Freeman's "Industrial Statistics" published by John Wiley and Sons of New York City.

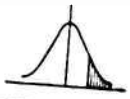
Table X*

TABLES FOR TESTING RANDOMNESS OF GROUPING IN A SEQUENCE OF ALTERNATIVES

$P(u \leq u')$

When $m = n$, the largest possible value of u' is $2m$; when $m < n$, the largest possible value of u' is $2m + 1$.

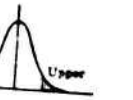
Table IX*



ENTS OF $\sqrt{b_1}$

Size of sample n	Probability points 5%	1%	Size of sample n	Probability points 5%	1%
200	0 280	0 403	1000	0 127	0 180
220	0 251	0 360	1200	0 116	0 165
240	0 230	0 329	1400	0 107	0 152
260	0 213	0 305	1600	0 100	0 142
280	0 200	0 285	1800	0 095	0 134
300	0 188	0 269	2000	0 090	0 127
320	0 179	0 255			
340	0 171	0 243	2500	0 080	0 114
360	0 163	0 233	3000	0 073	0 104
380	0 157	0 224	3500	0 068	0 096
400	0 151	0 215	4000	0 064	0 090
420	0 146	0 208	4500	0 060	0 085
440	0 142	0 202	5000	0 057	0 081
460	0 138	0 196			
480	0 134	0 190			
500	0 130	0 185			
520	0 127	0 180			

$\sqrt{b_1}$ is symmetrical about zero, the same values, with nega-



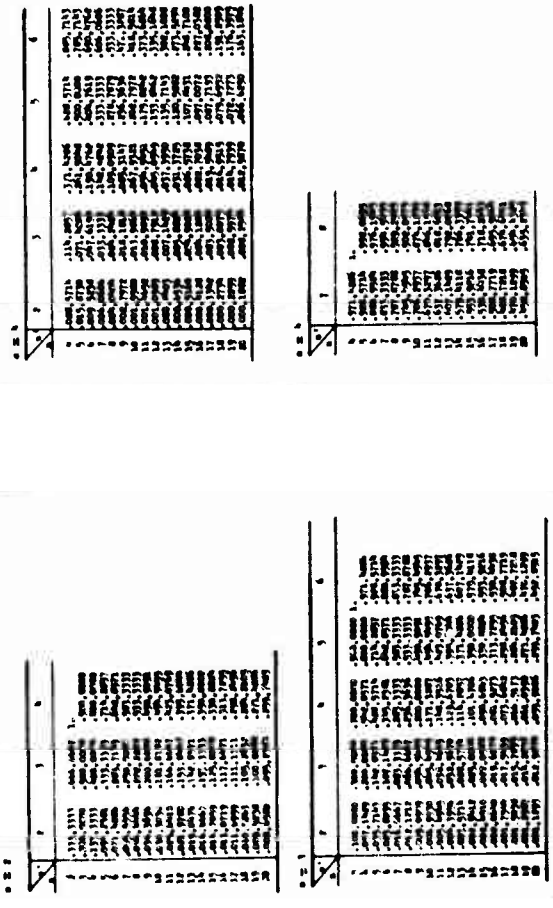
OF b_1

Lower 1%	Size of sample n	Probability points Upper 1%	Upper 5%	Lower 5%	Lower 1%
3.37	1000	3 41	3 26	2 76	2 58
	1200	3 37	3 24	2 78	2 71
	1400	3 34	3 22	2 80	2 72
	1600	3 32	3 21	2 81	2 74
	1800	3 30	3 20	2 82	2 76
	2000	3 28	3 18	2 83	2 77
	2500	3 25	3 16	2 85	2 79
	3000	3 22	3 15	2 86	2 81
	3500	3 21	3 14	2 87	2 82
	4000	3 19	3 13	2 88	2 83
	4500	3 18	3 12	2 88	2 84
	5000	3 17	3 2	2 89	2 85

ission of the copyright owner from
eman's "Industrial Statistics"
ons of New York City.

Table X*

TABLES FOR TESTING RANDOMNESS OF GROUPING IN A SEQUENCE OF ALTERNATIVES



*Reproduced from a paper by F. S. Swed and C. Eisenhart, "Tables for Testing Randomness of grouping in a Sequence of Alternatives," Annals of Mathematical Statistics," vol. XIV, 1943, pp. 66-87, by kind permission of the authors and the editor.

Table X - Continued

n = 5

n \ r	2	3	4	5	6
5	.007,9365	.039,6825	.166,6667	.357,1429	.642,8571
6	.004,3290	.023,8095	.110,5886	.261,9048	.521,6450
7	.002,5253	.015,1515	.075,7576	.196,9697	.436,3636
8	.001,5360	.010,1010	.053,6131	.151,5152	.367,3193
9	.000,9980	.006,9930	.038,9610	.118,8811	.308,7133
10	.000,6660	.004,9950	.028,9710	.094,9051	.258,7612
11	.000,4579	.003,6630	.021,8180	.076,9231	.216,8531
12	.000,3252	.002,7972	.016,9643	.063,1868	.180,5696
13	.000,2334	.002,1008	.013,3053	.052,5210	.149,8849
14	.000,1720	.001,6340	.010,5779	.044,1176	.124,6130
15	.000,1290	.001,2900	.008,5139	.037,4097	.103,8831
16	.000,0983	.001,0320	.006,9971	.031,9917	.093,9112
17	.000,0759	.000,8348	.005,6960	.027,5689	.082,2511
18	.000,0594	.000,6839	.004,7292	.023,9238	.072,4261
19	.000,0471	.000,5647	.003,9426	.020,8922	.063,0881
20	.000,0376	.000,4705	.003,3314	.018,3412	.054,9735

n = 6

n \ r	2	3	4	5	6
6	.002,1645	.012,9870	.067,0996	.174,1247	.341,7749
7	.001,1515	.007,5758	.042,5908	.121,2121	.298,0373
8	.000,6660	.004,6620	.027,9720	.086,7471	.226,1072
9	.000,3996	.002,9970	.018,9810	.062,9371	.179,8252
10	.000,2498	.001,9980	.013,2368	.046,9530	.136,8631
11	.000,1616	.001,3736	.009,4538	.035,7143	.108,4357
12	.000,1077	.000,9696	.006,8951	.027,6341	.085,9886
13	.000,0737	.000,7003	.005,1231	.021,7087	.070,3597
14	.000,0516	.000,5160	.003,8700	.017,2859	.057,5335
15	.000,0369	.000,3870	.002,9670	.013,9319	.047,4716
16	.000,0268	.000,2949	.002,3052	.011,3519	.039,4973
17	.000,0198	.000,2278	.001,8128	.009,3415	.033,1164
18	.000,0149	.000,1783	.001,4414	.007,7565	.027,9652
19	.000,0113	.000,1412	.001,1575	.006,4935	.023,7719
20	.000,0087	.000,1129	.000,9382	.005,4771	.020,3318

n = 5

n \ r	7	8	9	10
5	.833,3333	.960,3175	.992,0639	1.
6	.738,0952	.911,2554	.976,1905	.997,6394
7	.651,5152	.853,5998	.954,5695	.992,4242
8	.575,7576	.793,3178	.929,8989	.983,6730
9	.510,4895	.734,7657	.902,0379	.972,0280
10	.453,9844	.678,3217	.874,1249	.958,0420
11	.406,9934	.626,3736	.846,1538	.942,3077
12	.365,3846	.578,7007	.818,6813	.925,3394
13	.329,8319	.535,2675	.792,0168	.907,6630
14	.299,0196	.495,7860	.766,3399	.889,3189
15	.272,1878	.460,0101	.741,7641	.870,8720
16	.248,7300	.427,5886	.718,2662	.852,4252
17	.228,2702	.398,1934	.695,9064	.834,1308
18	.209,8828	.371,5118	.674,6411	.816,1015
19	.193,6799	.347,2618	.654,4329	.798,4190
20	.179,2772	.325,1855	.635,2543	.781,1406

n = 6

n \ r	7	8	9	10	11	12
6	.606,2251	.826,6753	.932,9004	.967,0130	.997,8355	1.
7	.500,0000	.733,1002	.878,7879	.966,2005	.992,4242	.999,4172
8	.412,5874	.646,6876	.820,5128	.937,0629	.983,6830	.997,8690
9	.342,6573	.566,4336	.762,2378	.902,0979	.972,0280	.994,4056
10	.286,7133	.496,9035	.706,2937	.863,6364	.958,0420	.989,5105
11	.241,7582	.435,6820	.653,8462	.823,5294	.942,3077	.983,0317
12	.205,1975	.383,1810	.605,3652	.783,1286	.925,3394	.975,1131
13	.175,7703	.337,9404	.560,9244	.743,3658	.907,6630	.965,9443
14	.151,8648	.299,0196	.520,1818	.704,8504	.889,3189	.955,7276
15	.131,3209	.265,4799	.483,4481	.667,9567	.870,8720	.944,6594
16	.114,9511	.236,5137	.449,9435	.632,8924	.852,4252	.932,9205
17	.100,4785	.211,4278	.419,4577	.599,7501	.834,1308	.920,6712
18	.088,5910	.189,6342	.391,7208	.568,5[illegible]	.816,1015	.908,0508
19	.078,4867	.170,6381	.366,4596	.539,2834	.798,4190	.895,1779
20	.069,8475	.154,0242	.343,4218	.511,7752	.781,1406	.882,1586

Table X - Continued

n = 7

n \ r	2	3	4	5	6
7	.001,1655	.007,5758	.042,5908	.121,2121	.298,0373
8	.000,6660	.004,6620	.027,9720	.086,7471	.226,1072
9	.000,3996	.002,9970	.018,9810	.062,9371	.179,8252
10	.000,2498	.001,9980	.013,2368	.046,9530	.136,8631
11	.000,1616	.001,3736	.009,4538	.035,7143	.108,4357
12	.000,1077	.000,9696	.006,8951	.027,6341	.085,9886
13	.000,0737	.000,7003	.005,1231	.021,7087	.070,3597
14	.000,0516	.000,5160	.003,8700	.017,2859	.057,5335
15	.000,0369	.000,3870	.002,9670	.013,9319	.047,4716
16	.000,0268	.000,2949	.002,3052	.011,3519	.039,4973
17	.000,0198	.000,2278	.001,8128	.009,3415	.033,1164
18	.000,0149	.000,1783	.001,4414	.007,7565	.027,9652
19	.000,0113	.000,1412	.001,1575	.006,4935	.023,7719
20	.000,0087	.000,1129	.000,9382	.005,4771	.020,3318

n = 7

n \ r	12	13	14
7	.992,9807	.999,4172	1.
8	.997,8690	.999,8946	.999,9944
9	.994,4056	.999,6730	.999,9719
10	.989,5105	.999,2980	.999,9438
11	.983,0317	.998,8312	.999,9102
12	.975,1131	.998,2338	.999,8711
13	.965,9443	.997,4670	.999,8264
14	.955,7276	.996,4917	.999,7760
15	.944,6594	.995,2688	.999,7197
16	.932,9205	.993,7594	.999,6574
17	.920,6712	.991,9250	.999,5889
18	.908,0508	.989,7276	.999,5140
19	.895,1779	.987,1299	.999,4325
20	.882,1586	.984,1960	.999,3442

Table X - Continued

Table X - Continued

a = 8

n \ a	7	8	9	10	11
8	.714,6522	.804,8174	.899,1826	.785,5478	.899,7669
9	.157,3427	.318,5932	.900,0000	.701,5631	.842,6573
10	.117,0300	.251,4095	.419,3747	.620,9379	.782,1884
11	.088,2353	.199,3729	.352,1870	.546,6778	.721,7195
12	.067,3970	.159,0855	.296,6182	.479,9952	.663,3722
13	.052,1196	.127,7990	.250,7740	.421,0526	.608,3591
14	.040,7637	.103,3712	.212,9343	.369,8630	.557,2755
15	.032,2263	.084,1930	.181,6305	.324,5399	.510,3199
16	.025,7305	.069,0368	.155,8472	.285,0639	.467,4474
17	.020,7337	.056,9771	.133,9944	.251,7856	.428,4724
18	.016,8504	.047,5188	.115,8727	.222,5120	.393,1350
19	.011,3036	.039,5324	.100,6385	.197,1217	.361,1432
20	.011,3915	.033,2151	.087,7741	.175,0684	.332,1962

a = 9

n \ a	2	3	4	5	6
9	.000,0411	.000,3702	.003,0029	.012,7172	.044,4673
10	.000,0217	.000,2057	.001,7645	.007,6100	.029,4334
11	.000,0119	.000,1191	.001,0717	.004,8821	.019,6857
12	.000,0068	.000,0718	.000,6702	.003,2151	.013,6937
13	.000,0040	.000,0442	.000,4302	.002,1672	.009,5975
14	.000,0024	.000,0281	.000,2827	.001,4917	.006,8368
15	.000,0015	.000,0184	.000,1897	.001,0463	.004,9455
16	.000,0010	.000,0122	.000,1297	.000,7465	.003,6246
17	.000,0006	.000,0083	.000,0903	.000,5409	.002,6916
18	.000,0004	.000,0058	.000,0638	.000,3975	.002,0225
19	.000,0003	.000,0041	.000,0458	.000,2959	.001,5364
20	.000,0002	.000,0029	.000,0333	.000,2230	.001,1791

a = 8

n \ a	12	13	14	15	16
8	.968,2948	.991,1422	.998,7568	.999,8443	1.
9	.939,4076	.979,7203	.995,8453	.999,3007	.999,9589
10	.903,1263	.963,5952	.990,4703	.998,1489	.999,7943
11	.861,7528	.943,4389	.982,3371	.996,2293	.999,4046
12	.817,4089	.920,1000	.971,6456	.993,4508	.998,6902
13	.771,8266	.894,4272	.957,9979	.989,7833	.997,5673
14	.726,3154	.867,1827	.942,3117	.985,2425	.995,9752
15	.681,8100	.839,0093	.926,7543	.979,8762	.993,8754
16	.638,9375	.810,4276	.909,6999	.973,7515	.991,2505
17	.598,0917	.781,8459	.892,4022	.966,9368	.988,1007
18	.559,4924	.753,5760	.874,4810	.959,5423	.984,4394
19	.523,2350	.725,8498	.856,4161	.951,6206	.980,2899
20	.489,3882	.698,8345	.838,0466	.943,2567	.975,6823

a = 9

n \ a	7	8	9	10	11
9	.108,9675	.237,9679	.399,2184	.580,7816	.762,0321
10	.076,7174	.178,5598	.318,5932	.501,5477	.681,4068
11	.054,8940	.134,9131	.254,9417	.429,9833	.605,0250
12	.039,8905	.102,7626	.204,0297	.362,1100	.535,0083
13	.029,4118	.078,9478	.165,6347	.304,9536	.472,1352
14	.021,9814	.061,1792	.134,6749	.257,1679	.416,4057
15	.016,6363	.047,8163	.110,1763	.217,3577	.367,4115
16	.012,7388	.037,6828	.090,6888	.184,3289	.324,5330
17	.009,8606	.029,9339	.075,0988	.156,6466	.287,1229
18	.007,7099	.023,9597	.062,9530	.133,6459	.254,5038
19	.006,0849	.019,3169	.052,3969	.114,4218	.226,0667
20	.004,8444	.015,6810	.044,1269	.098,3096	.201,2567

Table X - Continued

n = 8

n \ n'	12	13	14	15	16
8	.968,2988	.991,1422	.998,7588	.999,8446	1.
9	.939,4076	.979,7203	.995,8453	.999,3007	.999,9589
10	.883,1283	.963,5952	.990,4703	.998,1489	.999,7943
11	.801,7528	.963,4389	.982,3371	.996,2293	.999,6046
12	.717,8089	.920,1000	.971,6456	.993,4508	.998,6902
13	.778,8246	.898,4272	.957,9979	.989,7833	.997,9674
14	.726,3158	.867,1827	.942,3117	.985,2425	.995,9752
15	.681,8100	.839,0093	.924,7543	.979,8762	.993,8754
16	.638,9375	.810,4276	.905,6999	.973,7518	.991,7505
17	.598,0917	.781,8459	.885,5022	.966,9464	.988,1007
18	.559,4924	.753,5760	.864,4810	.959,5423	.986,4394
19	.523,7350	.725,8498	.842,9161	.951,6206	.980,2899
20	.489,3882	.698,8345	.821,0466	.943,2547	.975,5423

n = 9

n \ n'	7	8	9	10	11
9	.108,9675	.237,9679	.399,2184	.600,7816	.762,0321
10	.076,7174	.178,5598	.318,5932	.509,5477	.681,4068
11	.054,8940	.134,9131	.254,9417	.429,9833	.605,0250
12	.039,8905	.102,7626	.204,0297	.362,1100	.535,0083
13	.029,4118	.078,9474	.165,6347	.304,9536	.472,1352
14	.021,9414	.061,1792	.134,6749	.257,1679	.416,4057
15	.016,6363	.047,8163	.110,1763	.217,3577	.367,4115
16	.012,7388	.037,6828	.090,6888	.184,2259	.324,5330
17	.009,8606	.029,9339	.075,0988	.156,6466	.287,1229
18	.007,7099	.023,9597	.062,9530	.133,6459	.254,5038
19	.006,0849	.019,3169	.052,3969	.114,4215	.226,0867
20	.004,8444	.015,6810	.044,1869	.098,1096	.201,2567

Table X - Continued

n = 9

n \ n'	12	13	14	15	16
9	.891,0525	.999,4327	.987,7826	.998,9971	.999,8298
10	.838,1705	.985,7826	.978,2038	.992,3900	.998,4252
11	.773,0650	.885,0417	.959,1088	.985,1154	.996,5468
12	.711,0802	.883,0817	.931,1026	.975,1131	.993,0766
13	.650,6668	.799,0712	.903,0480	.962,4387	.988,0141
14	.592,7885	.754,8452	.872,0829	.947,4780	.981,2761
15	.538,5611	.710,3917	.838,0093	.930,8796	.972,8746
16	.489,1695	.667,5198	.800,7123	.912,5090	.962,9025
17	.443,9986	.626,3918	.759,6856	.892,5062	.951,4674
18	.402,3771	.587,2186	.715,0919	.872,8027	.938,7998
19	.365,0895	.550,2503	.700,7842	.851,2780	.925,9991
20	.331,2958	.515,5164	.667,2279	.829,7760	.910,2760

n = 9

n \ n'	17	18
9	.999,9489	1.
10	.999,7353	.999,9898
11	.999,4086	.999,9405
12	.998,6802	.999,8139
13	.997,9674	.999,5977
14	.995,9752	.999,1758
15	.993,8754	.998,6688
16	.991,2505	.997,9501
17	.988,1007	.996,3387
18	.986,4394	.995,8131
19	.980,2899	.992,9807
20	.975,5423	.990,7760

n = 10

n \ n'	2	3	4	5	6
10	.000,0108	.000,1083	.000,9851	.004,4924	.016,5217
11	.000,0057	.000,0495	.000,4698	.002,7187	.011,9244
12	.000,0031	.000,0240	.000,2302	.001,7181	.007,8420
13	.000,0017	.000,0201	.000,2089	.001,1747	.005,2593
14	.000,0010	.000,0122	.000,1315	.000,7281	.003,4316
15	.000,0006	.000,0076	.000,0847	.000,4895	.002,4333
16	.000,0004	.000,0049	.000,0557	.000,3353	.001,7585
17	.000,0002	.000,0032	.000,0373	.000,2336	.0012578
18	.000,0002	.000,0021	.000,0255	.000,1644	.000,9115
19	.000,0001	.000,0014	.000,0176	.000,1187	.000,6687
20	.000,0001	.000,0010	.000,0124	.000,0864	.000,4862

n = 10

n \ n'	7	8	9	10	11
10	.051,2468	.127,6186	.282,2113	.414,0704	.585,9296
11	.034,8893	.092,0657	.188,9290	.334,9607	.500,0000
12	.024 1726	.067,0398	.142,0576	.270,6597	.424,9821
13	.017,9279	.049,3337	.109,8071	.218,9393	.360,6811
14	.012,7880	.036,6806	.085,6777	.177,4478	.306,1650
15	.008,8933	.027,5493	.067,3038	.144,4744	.260,2302
16	.006,9088	.020,8935	.053,2728	.118,0513	.221,6449
17	.004,8643	.015,9941	.042,4797	.096,8489	.189,2657
18	.003,6475	.012,3527	.034,1158	.079,8184	.162,0831
19	.002,7749	.009,4211	.027,5869	.066,0851	.139,2318
20	.002,1353	.007,5536	.022,4538	.054,9634	.119,9827

n = 10

n \ n'	12	13	14	15	16
10	.757,7887	.872,1414	.948,7532	.981,4743	.995,9075
11	.680,0429	.815,0750	.915,0989	.965,1107	.989,5064
12	.605,0290	.755,0607	.875,0693	.943,6771	.980,4805
13	.535,1526	.697,0664	.830,7309	.917,9567	.967,7999
14	.471,5395	.636,8999	.783,8875	.888,8814	.951,8778
15	.413,9713	.581,7761	.736,1152	.857,3832	.932,9789
16	.364,1136	.530,3271	.688,6257	.824,3101	.911,5359
17	.319,7430	.482,8375	.642,3047	.790,3830	.888,0248
18	.280,4098	.439,1648	.597,7810	.756,2166	.862,9182
19	.247,0868	.399,7365	.555,4805	.722,2661	.836,6609
20	.217,5119	.363,8048	.515,5164	.688,9010	.809,6510

Table X - Continued

n = 10

n' \ m	17	18	19	20
10	.999,0134	.999,9917	.999,9998	1.
11	.997,9613	.999,9977	.999,9995	.999,9998
12	.998,1993	.998,7989	.999,9129	.999,9830
13	.989,6448	.997,3984	.999,9577	.999,9825
14	.973,1760	.995,1878	.999,1258	.999,9582
15	.937,9014	.990,0588	.998,4688	.999,6938
16	.916,0580	.987,8586	.997,9981	.999,9366
17	.893,1487	.988,6087	.998,1367	.999,0908
18	.867,9843	.973,2889	.999,0131	.998,5188
19	.849,6066	.964,9298	.998,9807	.997,8148
20	.819,3073	.960,4889	.998,7760	.998,4293

n = 11

n' \ m	12	13	14	15	16
11	.590,0818	.740,0578	.865,0889	.936,5325	.977,3586
12	.567,1796	.666,0393	.806,9527	.900,8097	.959,3862
13	.433,4692	.593,2831	.746,7589	.899,8062	.935,9556
14	.369,0039	.536,6523	.688,3490	.819,3896	.907,7803
15	.313,6559	.466,0183	.629,2617	.769,1845	.875,7976
16	.266,4962	.411,6032	.572,8532	.722,5868	.841,0015
17	.226,5846	.363,2582	.519,0575	.678,4811	.804,3389
18	.192,8191	.320,5187	.470,7598	.631,7315	.766,6415
19	.164,3840	.282,9585	.425,6872	.588,8096	.728,8218
20	.140,1525	.250,0211	.384,6040	.548,0860	.690,0486

n = 11

n' \ m	2	3	4	5	6
11	.009,0000	.000,0918	.000,3147	.001,9905	.007,3317
12	.000,0009	.000,0170	.000,1797	.000,9586	.004,4136
13	.000,0002	.000,0096	.000,1098	.000,5865	.002,9662
14	.001,0000	.000,0058	.000,0699	.000,3708	.001,9951
15	.001,0003	.000,0034	.000,0346	.000,2349	.001,2990
16	.001,0002	.000,0025	.000,0278	.000,1574	.000,8822
17	.000,0001	.000,0013	.000,0162	.000,1058	.000,6085
18	.000,0001	.000,0008	.000,0107	.000,0721	.000,4259
19	.000,0000	.000,0005	.000,0071	.000,0508	.000,3060
20	.000,0000	.000,0004	.000,0049	.000,0358	.000,2169

n = 11

n' \ m	17	18	19	20	21
11	.992,6665	.998,4099	.999,6853	.999,9648	.999,9978
12	.985,0136	.995,9964	.999,0474	.999,8610	.999,9850
13	.976,0305	.991,8779	.997,4870	.999,4098	.999,9623
14	.959,7525	.985,7583	.999,8140	.999,0521	.999,8942
15	.942,6883	.977,4095	.992,9967	.998,1390	.999,6918
16	.922,4198	.966,8956	.989,0699	.996,7875	.999,4346
17	.900,2888	.953,1680	.984,3342	.994,7889	.999,0908
18	.876,2799	.939,4969	.978,1410	.992,1981	.998,5180
19	.850,5602	.923,0528	.971,1184	.988,9151	.997,8154
20	.824,7446	.905,0629	.963,1042	.984,9243	.996,9853

n = 11

n' \ m	7	8	9	10	11
11	.088,6616	.063,1675	.136,9831	.299,9988	.609,9788
12	.054,9046	.044,2746	.099,1983	.251,4998	.534,9807
13	.038,1058	.031,2978	.073,9831	.196,0515	.473,4998
14	.026,9573	.022,3318	.055,0985	.128,4698	.387,4883
15	.019,0716	.016,1928	.041,0803	.099,9998	.187,0998
16	.013,5838	.011,7586	.031,5884	.075,9585	.158,4106
17	.009,6586	.008,7113	.024,3988	.059,9982	.134,0879
18	.006,7838	.006,4958	.018,8827	.047,1788	.102,6697
19	.004,3133	.005,9998	.014,7942	.038,8889	.085,3743
20	.003,9749	.003,7288	.011,6173	.030,0936	.073,7188

n = 11

n' \ m	22
11	1.
12	.999,9993
13	.999,9992
14	.999,9885
15	.999,9949
16	.999,9993
17	.999,7966
18	.999,8453
19	.999,4176
20	.999,3879

Table X - Continued

n = 12

n' \ m	17	18	19	20	21

Table X – Continued

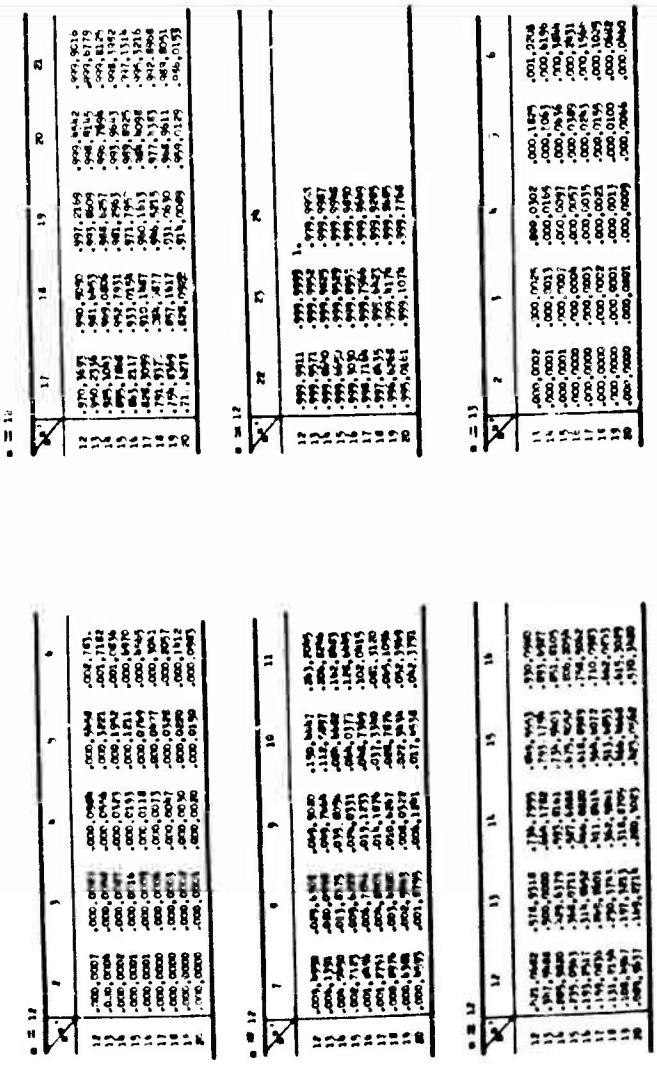


Table X - Continued

n = 13

m \ k	7	8	9	10	11
13	.003,8123	.013,1194	.038,0605	.081,1780	.156,5660
14	.002,4162	.008,0899	.023,5900	.054,8796	.110,8720
15	.001,4607	.005,4582	.016,5321	.042,9993	.090,6803
16	.001,0260	.003,9760	.011,7138	.031,6825	.069,4665
17	.000,6855	.002,7853	.009,0011	.025,4462	.053,3554
18	.000,4642	.001,9156	.006,0854	.021,5103	.041,5003
19	.000,1701	.001,3537	.004,4586	.013,1753	.032,3612
20	.000,2732	.000,9671	.003,2917	.009,9465	.025,3865

n = 13

m \ k	12	13	14	15	16
13	.277,1867	.417,9109	.582,0891	.722,8133	.843,8340
14	.220,5061	.347,4488	.505,6463	.652,4512	.787,9634
15	.175,3358	.288,2822	.436,4787	.588,6951	.729,6867
16	.139,5490	.238,8967	.375,1877	.521,1738	.671,3720
17	.111,3098	.197,9694	.321,5399	.469,7634	.618,0742
18	.089,0258	.164,2688	.275,1557	.409,8046	.599,1635
19	.071,4307	.136,4866	.235,3064	.362,2814	.507,3967
20	.057,2196	.113,7960	.201,2349	.319,9562	.499,2062

n = 13

m \ k	17	18	19	20	21
13	.918,8230	.966,9395	.986,8806	.996,1877	.998,9799
14	.881,1280	.944,6493	.976,8105	.992,0983	.997,5838
15	.838,7805	.918,1871	.962,2942	.985,8206	.995,2312
16	.795,4081	.857,2820	.944,6493	.977,0496	.991,7022
17	.746,4711	.852,6825	.923,7845	.965,8193	.986,8197
18	.699,1805	.815,8726	.900,1865	.942,6467	.980,4930
19	.652,5120	.777,2204	.874,2155	.915,8001	.972,7510
20	.607,1635	.737,7122	.846,4029	.887,4385	.963,5133

n = 13

m \ k	22	23	24	25	26
13	.999,8175	.999,9698	.999,9975	.999,9998	1.
14	.999,4659	.999,8937	.999,9870	.999,9987	.999,9999
15	.998,7601	.999,7226	.999,9559	.999,9948	.999,9996
16	.997,5432	.999,4017	.999,8845	.999,9850	.999,9985
17	.995,6611	.998,8707	.999,7861	.999,9649	.999,9953
18	.992,9766	.998,0683	.999,5084	.999,9285	.999,9885
19	.989,3787	.996,9368	.999,1355	.999,8585	.999,9753
20	.984,7879	.995,4258	.998,5900	.999,7768	.999,9527

n = 14

m \ k	2	3	4	5	6
14	.000,0000	.000,0007	.000,0095	.000,0597	.000,3630
15	.000,0000	.000,0004	.000,0051	.000,0344	.000,2174
16	.000,0000	.000,0002	.000,0029	.000,0203	.000,1330
17	.000,0000	.000,0001	.000,0017	.000,0123	.000,0829
18	.000,0000	.000,0001	.000,0010	.000,0076	.000,0526
19	.000,0000	.000,0000	.000,0006	.000,0048	.000,0333
20	.000,0000	.000,0000	.000,0004	.000,0030	.000,0222

n = 14

m \ k	7	8	9	10	11
14	.002,4751	.005,5531	.015,7478	.041,2348	.087,1113
15	.000,9398	.003,6036	.010,6505	.029,1065	.068,1731
16	.000,5835	.002,3732	.007,2948	.020,7174	.047,5626
17	.000,3770	.001,5849	.005,0577	.014,6721	.035,4823
18	.000,2476	.001,0726	.03,5478	.010,7670	.026,6493
19	.000,1651	.000,7351	.002,5165	.007,9606	.020,1520
20	.000,1116	.000,5098	.001,8039	.005,7858	.015,3423

Table X - Continued

n'	16	17	18	19	20
16	.139,9999	.298,9987	.375,1977	.391,1798	.371,3780
17	.111,3996	.197,9698	.321,5999	.468,7634	.614,0742
18	.089,0954	.164,2688	.275,1957	.409,8046	.599,1630
19	.071,4507	.136,4866	.235,9046	.362,2814	.507,3967
20	.057,5196	.113,7960	.201,2744	.319,9962	.459,8042

n = 13

n'	17	18	19	20	21
13	.918,8720	.964,9395	.986,8806	.996,1877	.998,9799
14	.881,1288	.944,6493	.976,4107	.992,0983	.997,5836
15	.838,7805	.918,1871	.962,2942	.985,8206	.995,2312
16	.795,4081	.887,2820	.944,6443	.977,0496	.991,7029
17	.746,4711	.852,8525	.923,7885	.965,8193	.986,8347
18	.699,1864	.815,8728	.900,1452	.952,0067	.980,5350
19	.652,5120	.777,2204	.874,2159	.935,8003	.972,7510
20	.607,1635	.737,7122	.846,5029	.917,4185	.963,5133

n = 14

n'	7	8	9	10	11
14	.001,6751	.005,6631	.015,7478	.041,2948	.087,1113
15	.000,9191	.003,5036	.010,6505	.029,1045	.064,1731
16	.000,5835	.002,3732	.007,2948	.020,7174	.047,5626
17	.000,3770	.001,5849	.005,0577	.014,8721	.035,4823
18	.000,2476	.001,0726	.003,5478	.010,7670	.026,6493
19	.000,1651	.000,7351	.002,5165	.007,8606	.020,1520
20	.000,1116	.000,5098	.001,8039	.005,7858	.015,5423

Table X - Continued

n = 14

n'	12	13	14	15	16
14	.169,6891	.279,7927	.426,6976	.573,4024	.720,3073
15	.130,6189	.224,7809	.357,6246	.500,0000	.651,8671
16	.100,7161	.180,4463	.298,6452	.433,5548	.585,4252
17	.077,8803	.145,0107	.248,6504	.374,4997	.522,5555
18	.060,4351	.116,7853	.206,8412	.322,6788	.464,2882
19	.047,0863	.093,2213	.172,0314	.277,6389	.411,0197
20	.036,8849	.074,2652	.143,1507	.238,7258	.363,9605

n = 15

n'	2	3	4	5	6
15	.000,0000	.000,0008	.000,0027	.000,0131	.000,1259
16	.000,0000	.000,0001	.000,0015	.000,0109	.000,0745
17	.000,0000	.000,0001	.000,0004	.000,0044	.000,0450
18	.000,0000	.000,0000	.000,0005	.000,0038	.000,0277
19	.000,0000	.000,0000	.000,0003	.000,0023	.000,0173
20	.000,0000	.000,0000	.000,0002	.000,0014	.000,0110

n = 14

n'	17	18	19	20	21	22
14	.830,3109	.912,8867	.958,7652	.980,2522	.991,1869	.996,5248
15	.775,2591	.876,9719	.935,8259	.972,7391	.989,3695	.996,7320
16	.718,3080	.832,8093	.904,1428	.957,3990	.981,9671	.993,7730
17	.661,3588	.786,2817	.866,8959	.938,1962	.972,1239	.989,3972
18	.605,8175	.738,4681	.823,8030	.913,7858	.959,7858	.983,3824
19	.552,7642	.690,3074	.804,9334	.885,0453	.943,0180	.975,6062
20	.502,7244	.642,4884	.766,7231	.851,6246	.921,0956	.966,0162

n = 15

n'	7	8	9	10	11
15	.000,5530	.002,2611	.006,9592	.013,7786	.045,7179
16	.000,3335	.001,4416	.004,6103	.013,1030	.032,7977
17	.000,2123	.000,9329	.003,0948	.009,9355	.023,7050
18	.000,1351	.000,6124	.002,1040	.006,6480	.017,2643
19	.000,0873	.000,4074	.001,4477	.004,7484	.012,6703
20	.000,0573	.000,2745	.001,0076	.003,1967	.009,3695

n = 14

n'	23	24	25	26	27	28
14	.999,6370	.999,8403	.999,9509	.999,9893	.999,9959	1.
15	.999,0809	.999,8131	.999,9646	.999,9861	.999,9956	.999,99999
16	.998,0742	.999,5385	.999,9046	.999,9859	.999,9964	.999,9999
17	.998,4639	.999,0331	.999,7825	.999,9810	.999,9951	.999,9999
18	.998,1081	.998,2011	.999,5688	.999,9097	.999,9885	.999,9986
19	.998,8904	.996,9935	.999,2772	.999,8166	.999,9751	.999,9963
20	.986,7220	.995,1925	.998,7219	.999,6631	.999,9527	.999,9916

n = 15

n'	12	13	14	15	16
15	.097,3240	.178,9095	.291,1827	.424,0664	.575,9336
16	.072,8057	.136,1517	.236,1717	.367,6246	.504,5928
17	.054,6203	.106,1457	.191,1627	.300,4703	.432,2736
18	.041,1934	.082,9593	.154,6265	.251,8442	.380,6328
19	.031,1546	.066,0425	.125,1165	.210,9365	.328,6328
20	.023,7043	.041,1792	.101,3509	.178,6075	.283,0953

SCIENTIFIC AND TECHNICAL INFORMATION

Table X - Continued

n = 15

n \ u'	17	18	19	20	21
15	.706,8173	.825,0905	.902,6060	.954,2830	.980,1215
16	.642,3754	.770,9725	.863,8483	.930,5283	.967,2023
17	.578,0768	.714,7113	.820,9825	.901,9511	.950,5323
18	.517,1909	.658,1253	.775,4377	.869,2877	.930,2701
19	.461,0404	.602,6434	.728,5128	.833,4039	.906,8276
20	.409,5499	.549,3124	.681,3118	.795,1935	.880,6049

n = 15

n \ u'	22	23	24	25	26
15	.993,0898	.997,7387	.999,4470	.999,8741	.999,9809
16	.987,2063	.995,3897	.998,6962	.999,6625	.999,9741
17	.978,8713	.992,7526	.997,3736	.999,2872	.999,8327
18	.967,8301	.988,6081	.995,8870	.998,5846	.999,6305
19	.954,0286	.979,7786	.992,8575	.997,4587	.999,2792
20	.937,5447	.971,1926	.988,1337	.995,8984	.998,7215

n = 15

n \ u'	27	28	29	30
15	.999,9975	.999,9998	.999,9999	1.
16	.999,9891	.999,9988	.999,9999	1.
17	.999,9679	.999,9956	.999,9999	1.
18	.999,9288	.999,9871	.999,9986	.999,9999
19	.999,8583	.999,9686	.999,9963	.999,9996
20	.999,6979	.999,9332	.999,9916	.999,9988

n = 16

n \ u'	2	3	4	5	6
16	.000,0000	.000,0001	.000,0008	.000,0060	.000,0427
17	.000,0000	.000,0000	.000,0004	.000,0034	.000,0250
18	.000,0000	.000,0000	.000,0002	.000,0020	.000,0149
19	.000,0000	.000,0000	.000,0001	.000,0012	.000,0091
20	.000,0000	.000,0000	.000,0001	.000,0007	.000,0056

n = 16

n \ u'	7	8	9	10	11
16	.000,2017	.000,8905	.002,9571	.009,1566	.022,7957
17	.000,1222	.000,5590	.001,9238	.006,1821	.015,9762
18	.000,0754	.000,3562	.001,2687	.004,2167	.011,2921
19	.000,0473	.000,2302	.000,8475	.002,9051	.008,0492
20	.000,0302	.000,1509	.000,5732	.002,0211	.005,7858

n = 16

n \ u'	12	13	14	15	16
16	.052,8017	.102,5117	.186,1617	.293,3260	.431,1087
17	.038,4600	.077,8067	.146,5073	.239,7439	.365,9287
18	.028,1549	.059,0702	.115,2798	.195,5792	.309,1455
19	.020,7242	.045,0178	.090,7885	.159,4445	.260,3267
20	.015,3423	.034,4554	.071,6196	.130,0205	.218,7595

Table X - Continued

n = 16

n \ u'	17	18	19	20	21
16	.468,8913	.706,6740	.813,8181	.897,1683	.947,1983
17	.400,0000	.641,9579	.760,7961	.858,3998	.922,1933
18	.436,9076	.578,8665	.705,0503	.815,4620	.892,7401
19	.380,1245	.526,8576	.649,8644	.769,7200	.859,5266
20	.329,6033	.468,7919	.599,5008	.722,4357	.823,6640

n = 17

n \ u'	2	3	4	5	6
17	.000,0000	.000,0000	.000,0002	.000,0019	.000,0142
18	.000,0000	.000,0000	.000,0001	.000,0011	.000,0085
19	.000,0000	.000,0000	.000,0001	.000,0006	.000,0049
20	.000,0000	.000,0000	.000,0000	.000,0004	.000,0029

n	n'	2	28	29	30
15		.999,5973	.999,9998	.999,9999	1.
16		.999,9998	.999,9998	.999,9999	1.
17		.999,9679	.999,9996	.999,9999	1.
18		.999,9888	.999,9871	.999,9946	.999,9999
19		.999,8983	.999,9686	.999,9967	.999,9998
20		.999,6973	.999,9932	.999,9916	.999,9988

n	n'	17	18	19	20	21
16		.468,8913	.706,6740	.813,8383	.897,1883	.947,1983
17		.300,0000	.641,9579	.760,7563	.858,3998	.922,1933
18		.436,9070	.478,8669	.705,0503	.815,4620	.892,7401
19		.380,1248	.418,8376	.649,8448	.769,7200	.859,5265
20		.329,6833	.362,7919	.595,9008	.722,4357	.823,6640

n	n'	22	23	24	25	26
16		.977,2043	.990,8833	.997,0429	.999,1095	.999,7983
17		.963,4136	.984,0238	.994,2437	.998,0762	.999,4956
18		.945,7478	.974,6566	.989,9854	.996,3129	.998,9279
19		.926,3498	.962,6108	.984,0090	.993,8179	.997,9789
20		.895,5854	.947,8986	.976,1338	.990,2514	.996,5258

n	n'	27	28	29	30	31	32
16		.939,9473	.999,9940	.999,9992	.999,9999	1.	1.
17		.999,8778	.999,9786	.999,9966	.999,9997	1.	1.
18		.999,7140	.999,9408	.999,9894	.999,9986	.999,9999	1.
19		.999,4192	.999,8624	.999,9732	.999,9958	.999,9996	1.
20		.998,9391	.999,7188	.999,9415	.999,9893	.999,9968	.999,9999

n	n'	17	18	19	20
17		.000,1772	.000,9990	.001,9238	.006,1821
18		.000,0754	.000,3962	.001,2687	.004,2167
19		.000,0473	.000,2302	.000,8475	.002,9051
20		.000,0302	.000,1509	.000,5732	.002,0211

n	n'	12	13	14	15	16
16		.058,8017	.102,5117	.186,1617	.293,3260	.431,1087
17		.038,4400	.077,8067	.146,4073	.239,7439	.365,9287
18		.028,1549	.059,0702	.115,2798	.195,5792	.309,1455
19		.020,7242	.045,0178	.090,7885	.159,4445	.260,3267
20		.015,3423	.034,4554	.071,6196	.130,0205	.218,7595

n	n'	2	3	4	5	6
17		.000,0000	.000,0000	.000,0002	.000,0019	.000,0142
18		.000,0000	.000,0000	.000,0001	.000,0011	.000,0083
19		.000,0000	.000,0000	.000,0001	.000,0006	.000,0049
20		.000,0000	.000,0000	.000,0000	.000,0004	.000,0029

n	n'	7	8	9	10	11
17		.000,0718	.000,3406	.001,2141	.004,0529	.010,8662
18		.000,0430	.000,2109	.000,7773	.002,6855	.007,4596
19		.000,0262	.000,1325	.000,5046	.001,8001	.005,1885
20		.000,0163	.000,0845	.000,3318	.001,2129	.003,6139

n	n'	12	13	14	15	16
17		.027,2181	.057,1966	.112,1570	.190,6720	.302,8363
18		.019,3731	.042,2073	.085,8902	.151,4145	.249,4781
19		.013,8746	.031,8856	.065,8689	.120,2125	.204,9018
20		.010,0005	.023,3061	.050,6268	.095,5108	.167,9944

n	n'	17	18	19	20	21
17		.429,0211	.570,9789	.697,1637	.809,3280	.897,8430
18		.365,9287	.503,8306	.634,0713	.756,6508	.848,5855
19		.310,7679	.441,7747	.572,7816	.702,1711	.805,6826
20		.263,1290	.385,4450	.514,5463	.647,4428	.760,3963

Table X - Continued

Table X - Continued

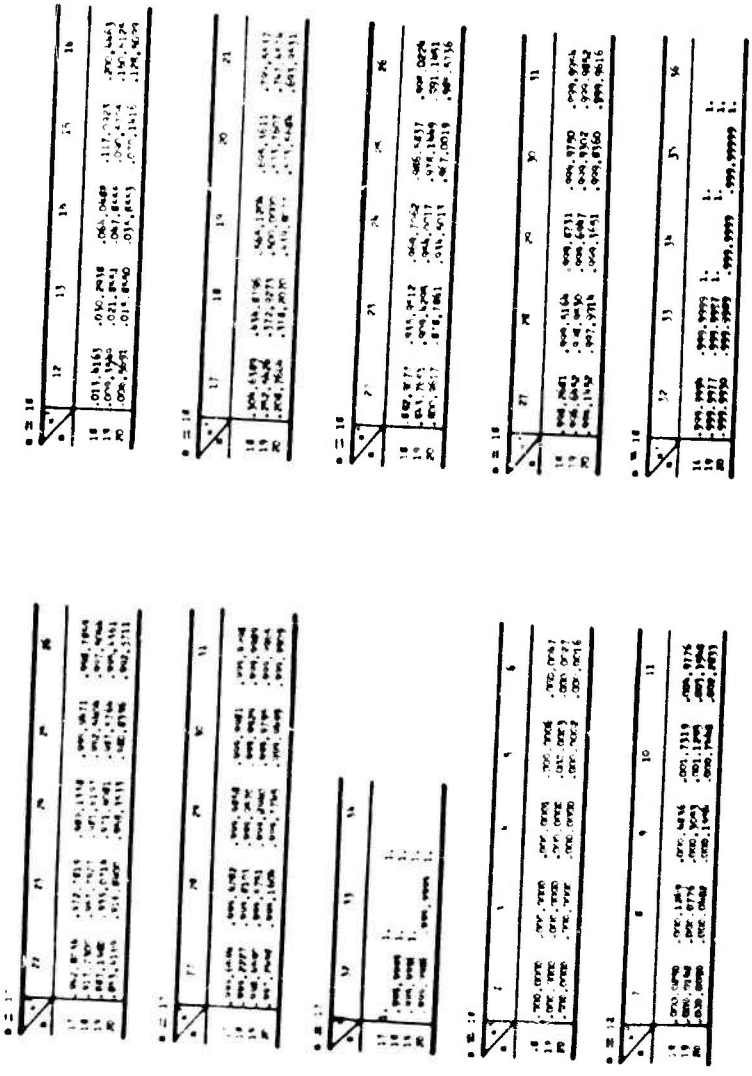
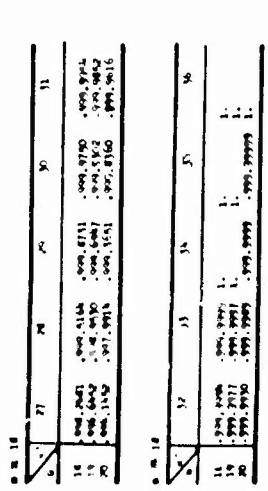
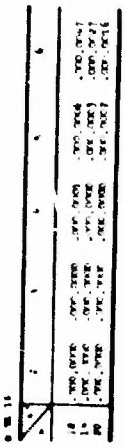
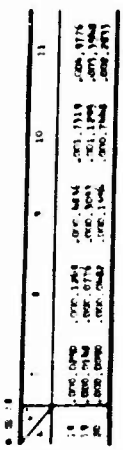
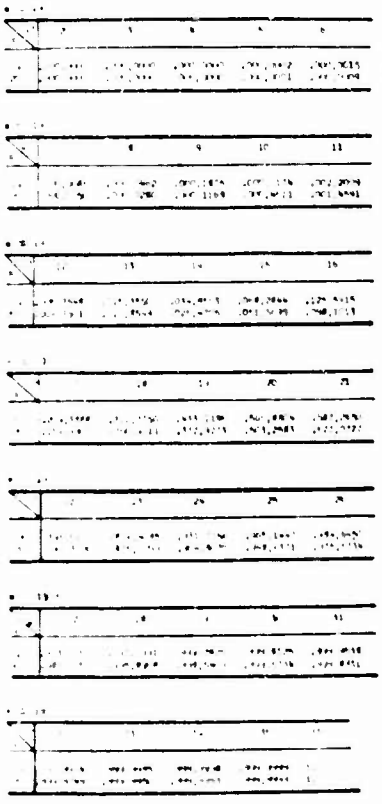


Table X - Continued









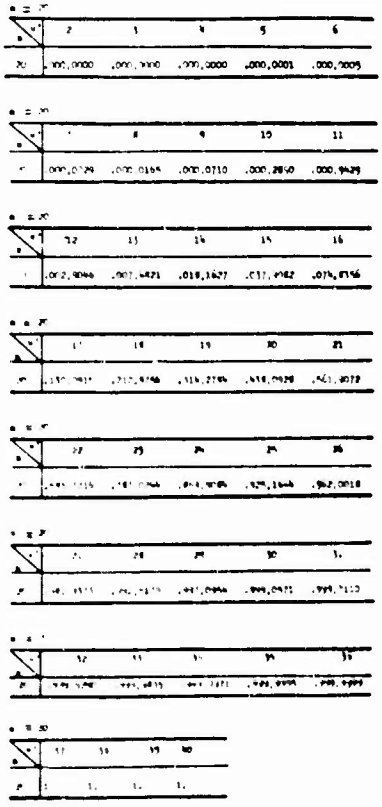


TABLE XI*

TEST OF SIGNIFICANCE OF MEAN USING RANGE

	Test of H_0: μ = a against H_1: $\mu \neq$ a.			Test of H_0: $\mu_1 = \mu_2$ against H_1: $\mu_1 \neq \mu_2$		
	Type I error			Type I error		
n \ α	0.05	0.01	0.001	0.05	0.01	0.001
2	6.353	31.828	318.31	3.427	7.916	25.23
3	1.304	3.008	9.58	1.272	2.093	4.18
4	0.717	1.316	2.85+	0.813	1.237	1.99
5	.507	.843	1.58	.613	.896	1.35
6	.399	.628	1.07	.499	.714	1.03
7	.333	.507	.82	.426	.600	.85
8	.288	.429	.67	.373	.521	.73
9	.255	.374	.57	.334	.464	.64
10	.230	.333	.50	.304	.419	.58
11	.210	.302	.44	.280	.384	.52
12	.194	.277	.40	.260	.355	.48
13	.181	.256	.37	.243	.331	.45-
14	.170	.239	.34	.228	.311	.42
15	.160	.224	.32	.216	.293	.39
16	.151	.212	.30	.205	.278	.37
17	.144	.201	.28	.195+	.264	.35
18	.137	.191	.26	.187	.252	.34
19	.131	.182	.25+	.179	.242	.32
20	.126	.175	.24	.172	.232	.31

*Table XI is reproduced from "The Use of Range in Place of Standard Deviation in the t-test," by E. Lord, Biometrika, Vol. 34, 1947, pp. 64-67, by kind permission of the editor and the author.

TABLE XII*

TABLE FOR TESTING HOMOGENEITY OF VARIANCES

Type I Error = 0.05

TABLE XII*

TABLE FOR TESTING HOMOGENEITY OF VARIANCES

Type I Error = 0.05

k \ C	0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0	6.0	7.0	8.0	9.0	10.0	12.0
3	5.99	6.47	6.89	7.20	7.38	7.39	7.22	9.37	9.18								
4	7.81	8.24	8.63	8.96	9.21	9.38	9.43	11.32	11.31	11.21	11.02						
5	9.49	9.86	10.24	10.57	10.86	11.08	11.24	13.01	13.11	13.14	13.10	12.78					
6	11.07	11.43	11.78	12.11	12.40	12.65	12.86	14.58	14.73	14.83	14.88	14.81	14.49				
7	12.59	12.94	13.27	13.59	13.88	14.15	14.38	16.06	16.25	16.40	16.51	16.60	16.49	16.16			
8	14.07	14.40	14.72	15.03	15.32	15.60	15.84	17.49	17.70	17.88	18.03	18.22	18.26	18.12	17.79		
9	15.51	15.83	16.14	16.44	16.73	17.01	17.26	18.89	19.11	19.31	19.48	19.75	19.89	19.89	19.73	19.40	
10	16.92	17.23	17.54	17.83	18.12	18.39	18.65	20.26	20.49	20.70	20.89	21.21	21.42	21.52	21.49	21.32	
11	18.31	18.61	18.91	19.20	19.48	19.76	20.02	21.61	21.84	22.06	22.27	22.62	22.88	23.06	23.12	23.07	22.56
12	19.68	19.97	20.26	20.55	20.83	21.10	21.36	22.94	23.18	23.40	23.62	23.99	24.30	24.53	24.66	24.70	24.44
13	21.03	21.32	21.60	21.89	22.16	22.43	22.69	24.26	24.50	24.73	24.95	25.34	25.68	25.95	26.14	26.25	26.17
14	22.36	22.65	22.93	23.21	23.48	23.75	24.01	25.56	25.80	26.04	26.26	26.67	27.03	27.33	27.56	27.73	27.80
15	23.68	23.97	24.24	24.52	24.79	25.05	25.31										

*Table XII is reproduced from "Tables for Testing the Homogeneity of a Set of Estimated Variances," by C. M. Thompson and M. Harrington, Biometrika, Vol. 33, 1946, pp. 295-304, by kind permission of the editor.

KI*

MEAN USING RANGE

Test of H_0: $\mu_1 = \mu_2$ against H_1: $\mu_1 \neq \mu_2$

Type I error		
.05	0.01	0.001
3.427	7.916	25.23
1.272	2.093	4.18
0.813	1.237	1.99
.613	.896	1.35
.499	.714	1.03
.426	.600	.85
.373	.521	.73
.334	.464	.64
.304	.419	.58
.280	.384	.52
.260	.355	.48
.243	.331	.45-
.228	.311	.42
.216	.293	.39
.205	.278	.37
.195	.264	.35
.187	.252	.34
.179	.242	.32
.172	.232	.31

Use of Range in Place of ," by E. Lord, Biometrika, permission of the editor

TABLE XII – Continued

TABLE FOR TESTING HOMOGENEITY OF VARIANCES

Type I Error = 0.01

k \ C_1	0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0	6.0	7.0	8.0	9.0	10.0	12.0
3	9.21	9.92	10.47	10.78	10.81	10.50	9.83										
4	11.34	11.95	12.46	12.86	13.11	13.18	13.03	12.65	12.03								
5	13.28	13.81	14.30	14.71	15.03	15.25	15.34	15.28	15.06	14.66	14.07						
6	15.09	15.58	16.03	16.41	16.79	17.07	17.27	17.37	17.37	17.24	16.98	16.03					
7	16.81	17.27	17.70	18.10	18.46	18.77	19.02	19.21	19.32	19.35	19.28	18.84	17.92				
8	18.48	18.91	19.32	19.71	20.07	20.39	20.67	20.90	21.08	21.20	21.25	21.13	20.64	19.76			
9	20.09	20.50	20.90	21.28	21.64	21.97	22.26	22.52	22.74	22.91	23.03	23.10	22.91	22.41	21.56		
10	21.67	22.06	22.45	22.82	23.17	23.50	23.80	24.08	24.32	24.52	24.69	24.90	24.90	24.61	24.15	23.33	
11	23.21	23.59	23.97	24.33	24.67	25.00	23.31	25.59	25.85	26.08	26.28	26.57	26.70	26.65	26.38	25.85	
12	24.72	25.10	25.46	25.81	26.15	26.48	26.79	27.08	27.35	27.59	27.81	28.16	28.30	28.46	28.37	28.07	26.79
13	26.22	26.56	26.93	27.28	27.62	27.94	28.25	28.54	28.81	29.07	29.30	29.70	29.99	30.16	30.19	30.06	29.22
14	27.69	28.04	28.39	28.73	29.06	29.38	29.69	29.98	30.26	30.52	30.77	31.19	31.53	31.77	31.89	31.88	31.39
15	29.14	29.49	29.83	30.16	30.49	30.80	30.11	31.40	31.68	31.95	32.20	32.66	33.03	33.32	33.51	33.59	33.37

TABL

n	
3	
4	
5	
6	
7	
8	
9	
10	
11	
12	
13	
14	
15	
16	
17	
18	
19	
20	
21	
22	
23	
24	
25	

S^2 =

S^2_n =

S_1^2 =

*Table XIII
Criteria fo
Frank E. C
vol. 21 (19
author and

TABLE XIII

TABLE OF PERCENTAGE POINTS FOR $\frac{S_n^2}{S^2}$ or $\frac{S_1^2}{S^2}$

	Percentage Points			
n	1%	2.5%	5%	10%
3	.0001	.0007	.0027	.0109
4	.0100	.0248	.0494	.0975
5	.0442	.0808	.1270	.1984
6	.0928	.1453	.2032	.2826
7	.1447	.2066	.2696	.3503
8	.1948	.2616	.3261	.4050
9	.2411	.3101	.3742	.4502
10	.2831	.3526	.4154	.4881
11	.3211	.3901	.4511	.5204
12	.3554	.4232	.4822	.5483
13	.3864	.4528	.5097	.5727
14	.4145	.4792	.5340	.5942
15	.4401	.5030	.5559	.6134
16	.4634	.5246	.5755	.6306
17	.4848	.5442	.5933	.6461
18	.5044	.5621	.6095	.6601
19	.5225	.5785	.6243	.6730
20	.5393	.5937	.6379	.6848
21	.5548	.6076	.6504	.6958
22	.5692	.6206	.6621	.7058
23	.5827	.6327	.6728	.7151
24	.5953	.6439	.6829	.7238
25	.6071	.6544	.6923	.7319

$$S^2 = \sum_{i=1}^{n} (x_i - \bar{x})^2 \text{ where } \bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$$

$$S^2_n = \sum_{i=1}^{n-1} (x_i - \bar{x}_n)^2 \text{ where } \bar{x}_n = \frac{1}{n-1} \sum_{i=1}^{n-1} x_i$$

$$S_1^2 = \sum_{i=2}^{n} (x_i - \bar{x}_1)^2 \text{ where } \bar{x}_1 = \frac{1}{n-1} \sum_{i=2}^{n} x_i$$

*Table XIII was reproduced from Table 1 of "Sample Criteria for Testing Outlying Observations," by Frank E. Grubbs, Annals of Mathematical Statistics, vol. 21 (1950), pp. 27-58, by kind permission of the author and the Institute of Mathematical Statistics.

TABLE XIV*

TABLE OF CRITICAL VALUES FOR THE SIGN TEST

n	Per Cent Level of Significance				n	Per Cent Level of Significance			
	1	5	10	25		1	5	10	25
1	-	-	-	-	51	15	18	19	20
2	-	-	-	-	52	16	18	19	21
3	-	-	-	0	53	16	18	20	21
4	-	-	-	0	54	17	19	20	22
5	-	-	0	0	55	17	19	20	22
6	-	0	0	1	56	17	20	21	23
7	-	0	0	1	57	18	20	21	23
8	0	0	1	1	58	18	21	22	24
9	0	1	1	2	59	19	21	22	24
10	0	1	1	2	60	19	21	23	25
11	0	1	2	3	61	20	22	23	25
12	1	2	2	3	62	20	22	24	25
13	1	2	3	3	63	20	23	24	26
14	1	2	3	4	64	21	23	24	26
15	2	3	3	4	65	21	24	25	27
16	2	3	4	5	66	22	24	25	27
17	2	4	4	5	67	22	25	26	28
18	3	4	5	6	68	22	25	26	28
19	3	4	5	6	69	23	25	27	29
20	3	5	5	6	70	23	26	27	29
21	4	5	6	7	71	24	26	28	30
22	4	5	6	7	72	24	27	28	30
23	4	6	7	8	73	25	27	28	31
24	5	6	7	8	74	25	28	29	31
25	5	7	7	9	75	25	28	29	32
26	6	7	8	9	76	26	28	30	32
27	6	7	3	10	77	26	29	30	32
28	6	8	9	10	78	27	29	31	33
29	7	8	9	10	79	27	30	31	33
30	7	9	10	11	80	28	30	32	34
31	7	9	10	11	81	28	31	32	34
32	8	9	10	12	82	28	31	33	35
33	8	10	11	12	83	29	32	33	35
34	9	10	11	13	84	29	32	33	36
35	9	11	12	13	85	30	32	34	36
36	9	11	12	14	86	30	33	34	37
37	10	12	13	14	87	31	33	35	37
38	10	12	13	14	88	31	34	35	38
39	11	12	13	15	89	31	34	36	38
40	11	13	14	15	90	32	35	36	39
41	11	13	14	16	91	32	35	37	39
42	12	14	15	16	92	33	36	37	39
43	12	14	15	17	93	33	36	38	40
44	13	15	16	17	94	34	37	38	40
45	13	15	16	18	95	34	37	38	41
46	13	15	16	18	96	34	37	39	41
47	14	16	17	19	97	35	38	39	42
48	14	16	17	19	98	35	38	40	42
49	15	17	18	19	99	36	39	40	43
50	15	17	18	20	100	36	39	41	43

For $n > 100$, approximate values of r may be found by taking the nearest integer less than $\frac{1}{2}(n - k\sqrt{n})$, where k = 1.3, 1, .82, .58 for the 1, 5, 10, 25 per cent values respectively. A closer approximation to the values of r is obtained from $\frac{1}{2}[(n - 1) - k\sqrt{n + 1}]$ and the more exact values of k, 1.2879, .9800, .8224, .5752.

*Reproduced from "The Statistical Sign Test," by W. J. Dixon and A. M. Mood, Journal of the American Statistical Association, vol. 41, 1946, pp. 557-566, by kind permission of the authors and the editors.

XIV*

UES FOR THE SIGN TEST

n	Per Cent Level of Significance			
	1	5	10	25
51	15	18	19	20
52	16	18	19	21
53	16	18	20	21
54	17	19	20	22
55	17	19	20	22
56	17	20	21	23
57	18	20	21	23
58	18	21	22	24
59	19	21	22	24
60	19	21	23	25
61	20	22	23	25
62	20	22	24	25
63	20	23	24	26
64	21	23	24	26
65	21	24	25	27
66	22	24	25	27
67	22	25	26	28
68	22	25	26	28
69	23	25	27	29
70	23	26	27	29
	24	26	28	30
	24	27	28	30
	25	27	28	31
	25	28	29	31
	25	28	29	32
	26	28	30	32
	26	29	30	32
	27	29	31	33
	27	30	31	33
	28	30	32	34
	28	31	32	34
	28	31	33	35
	29	32	33	35
	29	32	33	36
	30	32	34	36
	30	33	34	37
	31	33	35	37
	31	34	35	38
	31	34	36	38
	32	35	36	39
	32	35	37	39
	33	36	37	39
	33	36	38	40
	34	37	38	40
	34	37	38	41
	34	37	39	41
	35	38	39	42
	35	38	40	42
	36	39	40	43
	36	39	41	43

be found by taking the nearest
.8, 1, .82, .58 for the 1, 5,
ner approximation to the values
and the more exact values of

" by W. J. Dixon and A. M.
ssociation, vol. 41, 1946,
ors and the editors.

TABLE XV*

MINIMUM VALUES OF n IN THE SIGN TEST NECESSARY TO FIND SIGNIFICANT DIFFERENCES 95% OF THE TIME FOR VARIOUS GIVEN PROPORTIONS

p	n			
	α = 1%	5%	10%	25%
.45 (.55)	1,777	1,297	1,080	780
.40 (.60)	442	327	267	193
.35 (.65)	193	143	118	86
.30 (.70)	106	79	67	47
.25 (.75)	66	49	42	32
.20 (.80)	44	35	28	21
.15 (.85)	32	23	18	14
.10 (.90)	24	17	13	11
.05 (.95)	15	12	11	6

The underlined values are approximate. The maximum error is about 5 for the value of n.

*Reproduced from "The Statistical Sign Test," by W. J. Dixon and A. M. Mood, Journal of the American Statistical Association, vol. 41, 1946, pp. 557-566, by kind permission of the authors and the editors.

TABLE XVI* †

SIGNIFICANCE LEVELS (ROUNDED OFF) FOR TESTS OF SIGNIFICANCE BY PAIRED OBSERVATIONS

Probability (P) of a chance occurrence of rank total of one sign, + or -, whichever is least, equal to or less than T. T is given in body of table, to nearest whole number. N is number of replicates.

A. Unpaired Replicates

N	P=0.05	P=0.02	P=0.01
5	18	16	15
6	27	24	23
7	37	34	32
8	49	46	44
9	63	59	56
10	79	74	71
11	97	91	87
12	116	110	105
13	137	130	125
14	160	152	147
15	185	176	170
16	212	202	196
17	241	230	223
18	271	259	252
19	303	291	282
20	338	324	315

B. Paired Replicates

N	P=0.05	P=0.02	P=0.01
6	0	-	-
7	2	0	-
8	4	2	0
9	6	3	2
10	8	5	3
11	11	7	5
12	14	10	7
13	17	13	10
14	21	16	13
15	25	20	16
16	30	24	20
17	35	28	23
18	40	33	28
19	46	38	32
20	52	43	38
21	59	49	43
22	66	56	49
23	73	62	55
24	81	69	61
25	89	77	68

*The values in Table XVI were obtained by rounding off values given by Tukey in Memorandum Report 17, "The Simplest Signed Rank Tests," Statistical Research Group, Princeton University, 1949.

†Reproduced from Table II of "Some Rapid Approximate Statistical Procedures," by Frank Wilcoxon, produced by the Insecticide and Fungicide Section, Stamford Research Laboratories, American Cyanamid Company, Stamford, Conn., by kind permission of the author and the American Cyanamid Co.

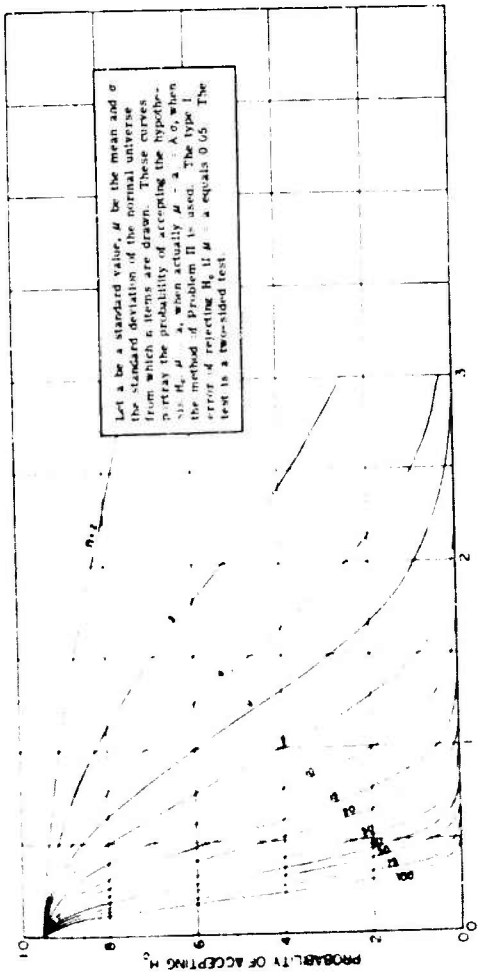


Fig. 1. Operating Characteristic Curve for the Method of Problem II, Two-Sided Test.*

*Figures 1-VII have been reproduced from "Operating Characteristics for the Common Statictical Tests of Significance," by C. D. Ferris, F. E. Grubbs, and C. L. Weaver, Annals of Mathematical Statistics, vol. XVII, 1946 pp. 178-192, by permission of the Institute of Mathematical Statistics.

VI* †

DED OFF) FOR TESTS
ED OBSERVATIONS

ccurrence of rank total
least, equal to or less
ble, to nearest whole
es.

B. Paired Replicates

P=0.05	P=0.02	P=0.01
0	-	-
2	0	-
4	2	0
6	3	2
8	5	3
11	7	5
14	10	7
17	13	10
21	16	13
25	20	16
30	24	20
35	28	23
40	33	28
46	38	32
52	43	38
59	49	43
66	56	49
73	62	55
81	69	61
89	77	68

tained by rounding off
andum Report 17, "The
tistical Research Group,

me Rapid Approximate
Wilcoxon, produced by
tion, Stamford Research
Company, Stamford,
author and the Ameri-

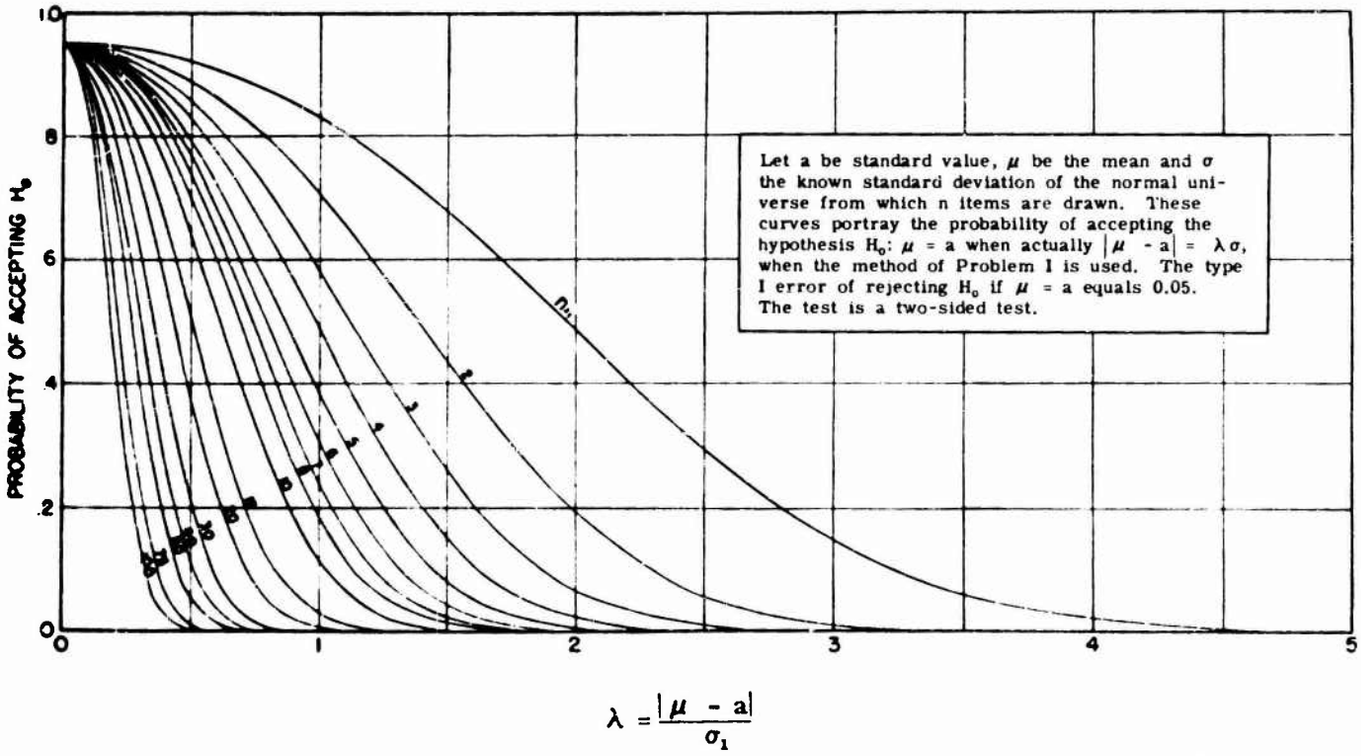
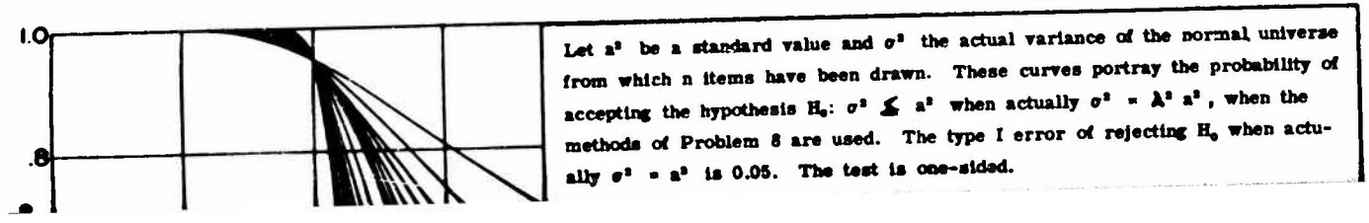
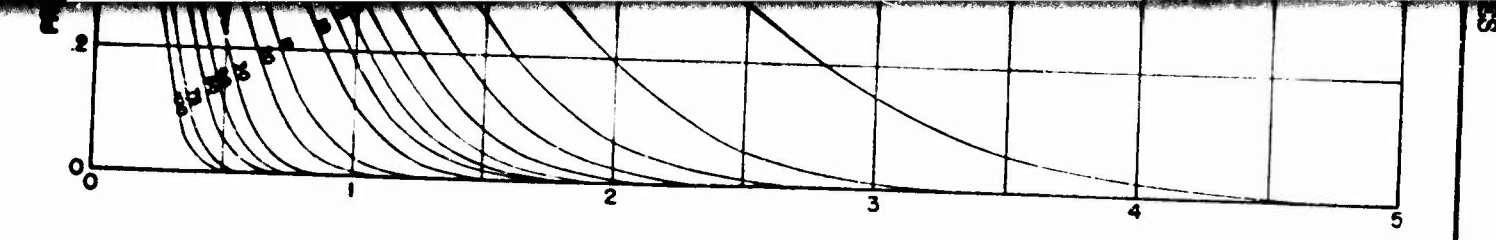


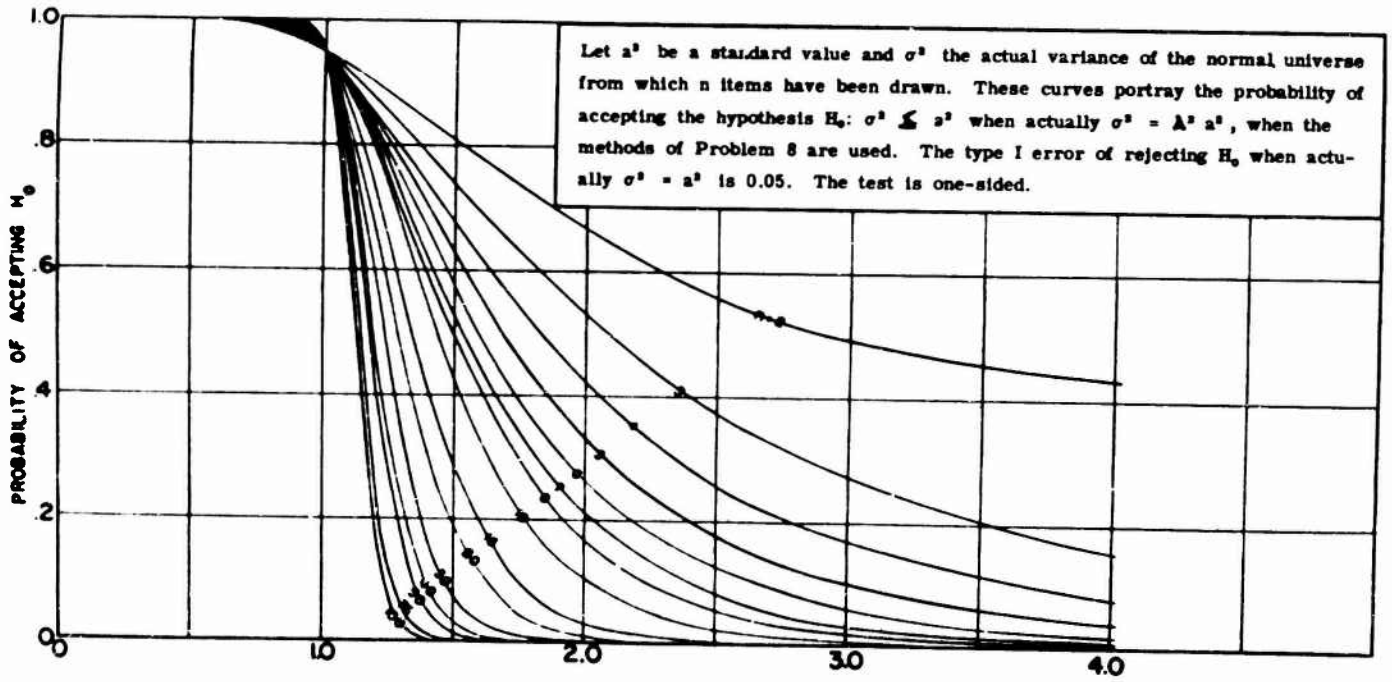
Fig. II. Operating Characteristic Curve for the Method of Problem I, Two-Sided Test.





$$\lambda = \frac{|\mu - a|}{\sigma_1}$$

Fig. II. Operating Characteristic Curve for the Method of Problem I, Two-Sided Test.



$$\lambda = \frac{\sigma_1}{\sigma}$$

Fig. III. Operating Characteristic Curve for the Method of Problem 8, One-Sided Test.

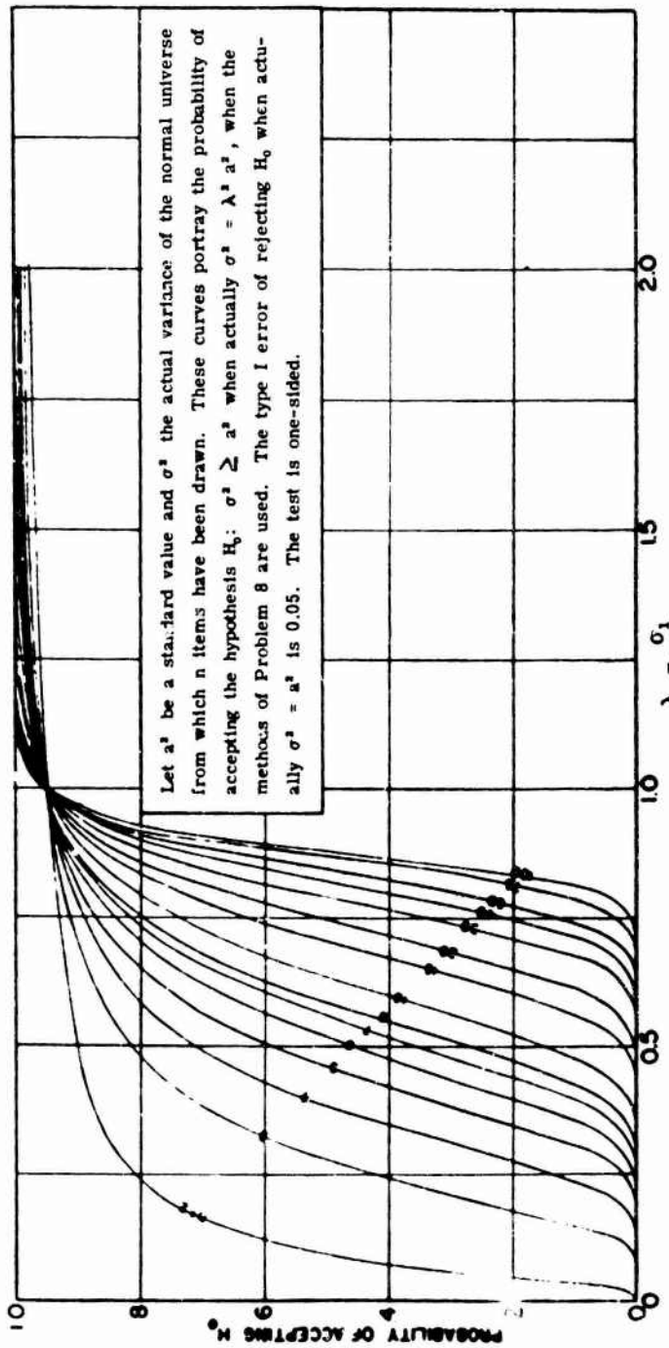


Fig. IV. Operating Characteristic Curve for the Method of Problem 8, One-Sided Test.

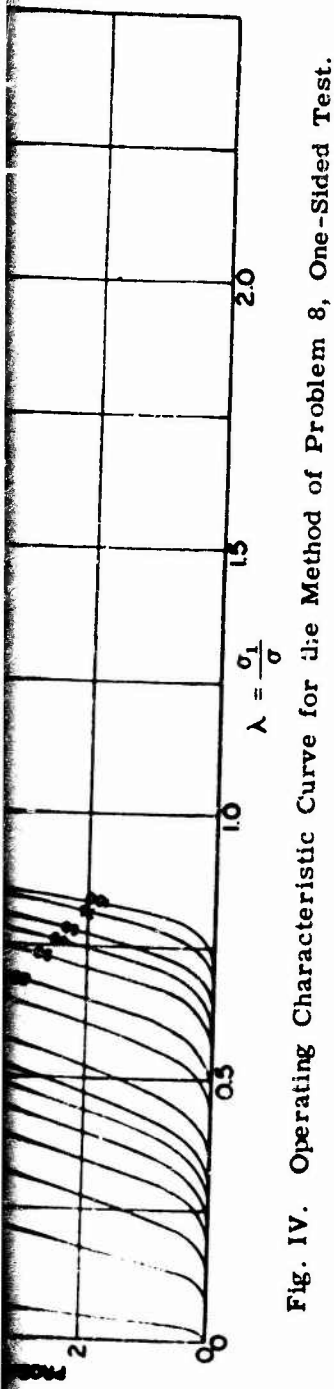


Fig. IV. Operating Characteristic Curve for the Method of Problem 8, One-Sided Test.

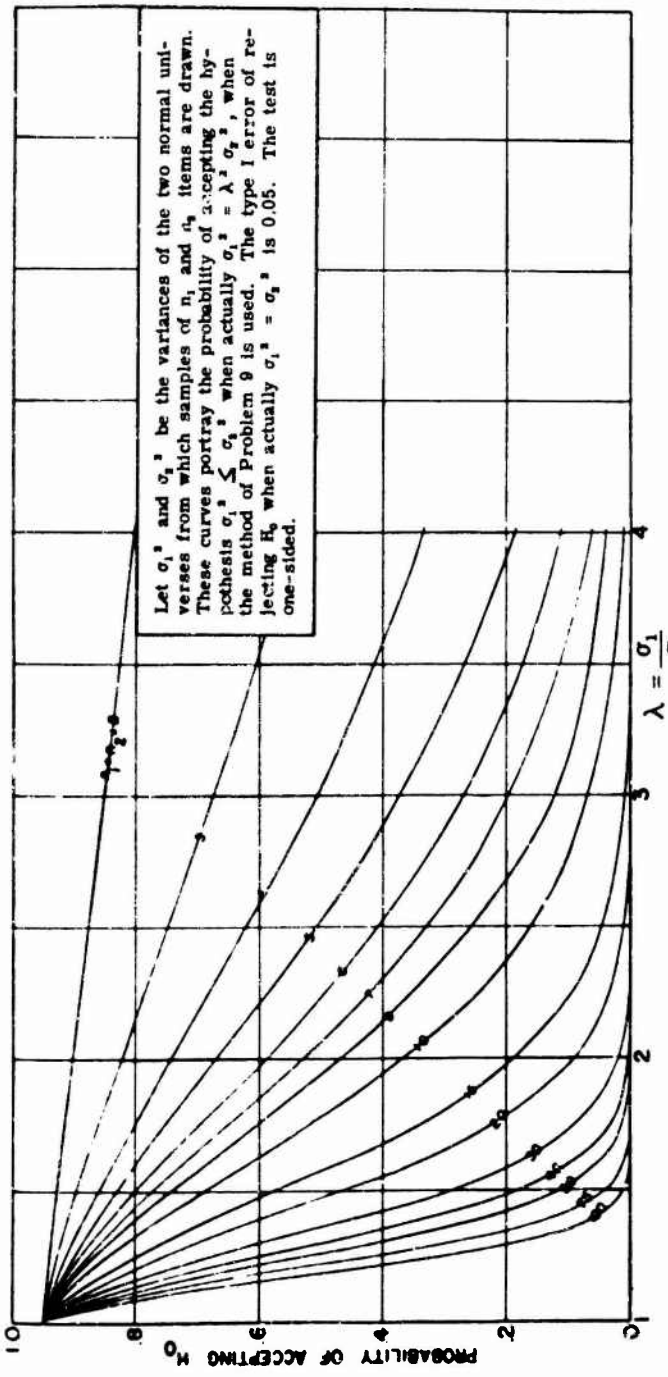


Fig. V. Operating Characteristic Curve for the Method of Problem 9, One-Sided Test.

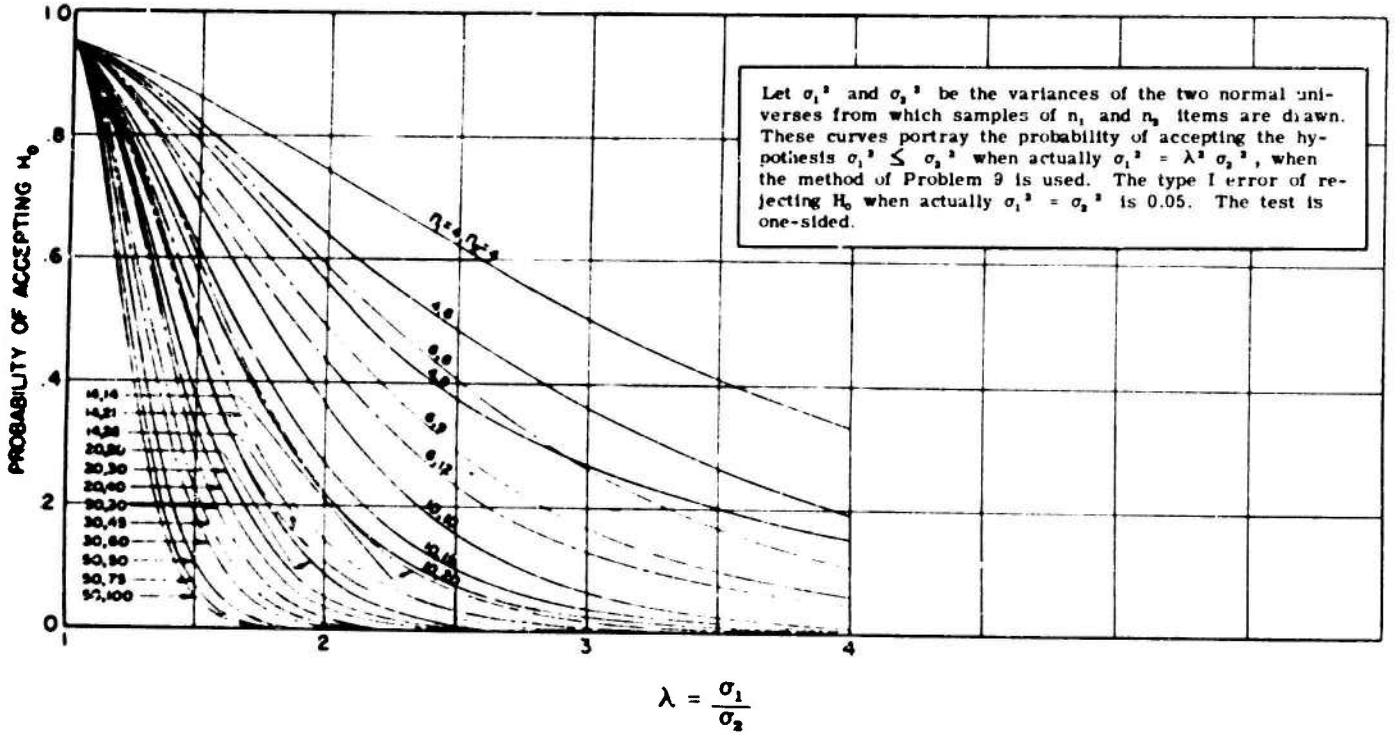


Fig. VI. Operating Characteristic Curve for the Method of Problem 9, One-Sided Test.

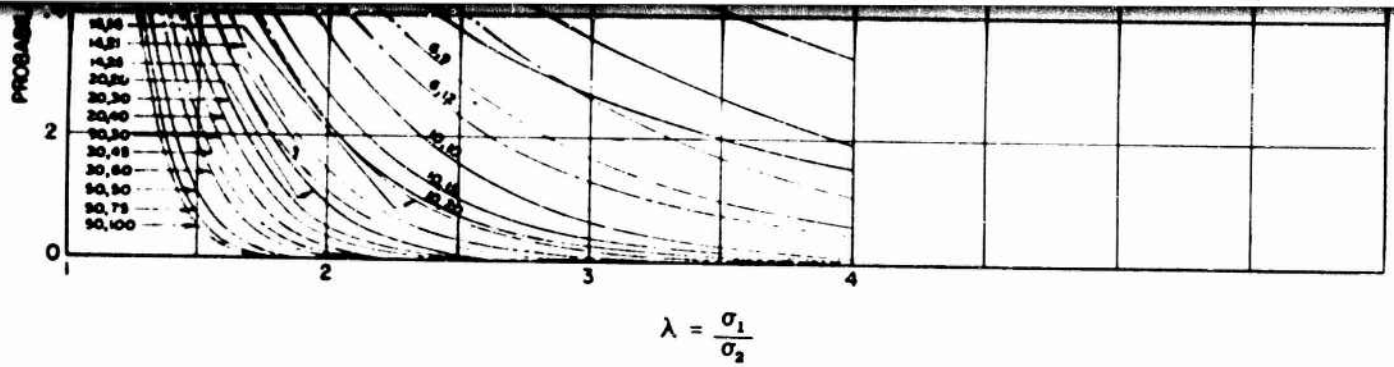


Fig. VI. Operating Characteristic Curve for the Method of Problem 9, One-Sided Test.

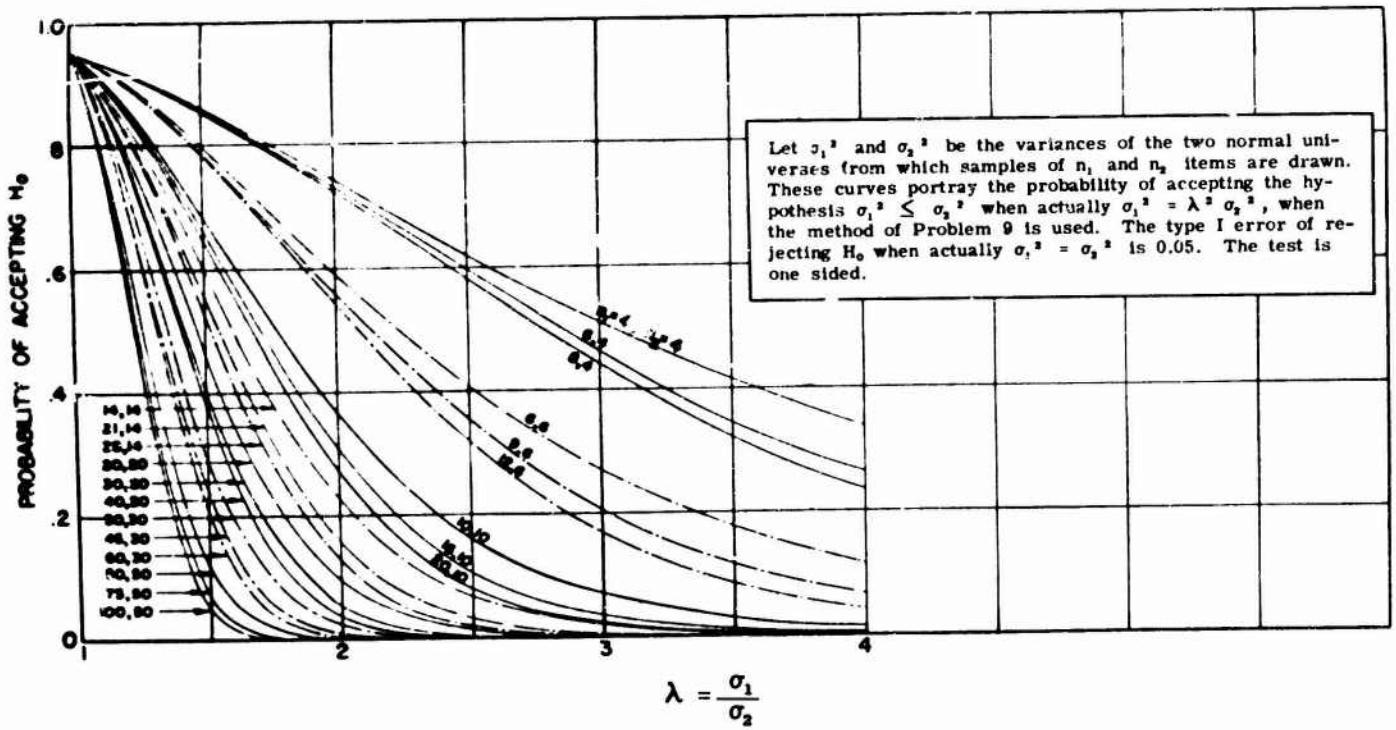


Fig. VII. Operating Characteristic Curve for the Method of Problem 9, One-Sided Test.

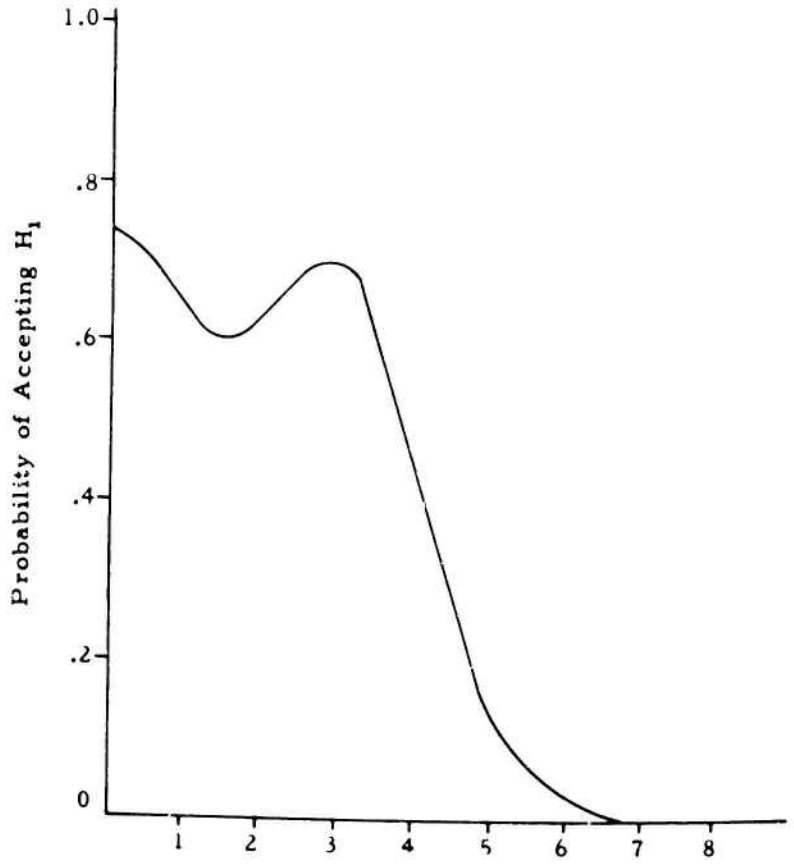


Fig. VIII. Operating Characteristics of Irregular Test for $\mu = a$ against $\mu \neq a$.

Let σ be the known standard deviation of a normal universe from which a sample of n items is drawn. This curve portrays the probability of accepting the hypothesis, H_1: $\mu = a$ when actually $(\mu - a) = \lambda \sigma_{\bar{x}}$ where $\sigma_{\bar{x}} = \sigma/n$. The basis of acceptance here is: accept if either

a) $(\bar{x} - a) \leq \sigma_{\bar{x}}$

or

b) $2\,\sigma_{\bar{x}} \leq (\bar{x} - a) \leq 4\,\sigma_{\bar{x}}$

otherwise reject.

ARMY-RARITAN ARSENAL(RAPO),METUCHEN,NJ-51-2235