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A COORDINATION OF LATTICES
BY ONE-SIDED BAER ASSEMBLIES .

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A Coördinatisation of Lattices by One-sided Baer Assemblies. By
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SYNOPSIS

In an earlier publication [1] we introduced the notion of a *Baer assembly* and applied it to obtain a coördinatisation theory for semilattices. This was achieved by considering the semigroup of quasi-residuated (i.e., 0-preserving and isotone) mappings on a bounded semilattice. In the present paper we consider the semigroup of quasi-residuated $\cup$ -homomorphisms (or *semimorphisms*) on a bounded lattice and thus show how a particular type of one-sided Baer assembly can be used to provide a coördinatisation theory for lattices; and in particular for complemented, modular and distributive lattices.

1. Introduction.

We have shown in a previous publication [1] how the fundamental coördinatisation of bounded lattices by Baer semigroups can be extended to semilattices through the concept of a Baer assembly. We recall ([1], §3) that $\langle A; B, D; K \rangle$ is a Baer assembly if A is a semigroup, B and D are distinguished subsets of A and K is a two-sided ideal of A such that

$$\begin{cases} (\forall x \in D) (\exists e = e^2 \in B) & R_K(x) = eA; \\ (\forall y \in B) (\exists f = f^2 \in D) & L_K(y) = Af, \end{cases}$$

where $R_K(x) = \{z \in A; xz \in K\}$ and $L_K(y) = \{z \in A; zy \in K\}$. In the case where A has a zero element 0 and $K = \{0\}$ we shall agree to drop the suffix K . We shall assume throughout the present paper that the reader is familiar with the terminology and notation used in [1]. Our goal here will be to isolate a type of Baer assembly which will

coördinatise a bounded lattice and do so in such a way that the lattice operations are readily recaptured as semigroup conditions. This we then use to provide Baer assembly coördinatisations of complemented, modular and distributive lattices. The latter two results do not as yet have analogues in the theory of Baer semigroups.

As in [1], the results which follow depend heavily on the choice of the semigroup of mappings and the correct choice of idempotents. To establish these, we recall the following definitions and results from [1].

Let E be an ordered set with minimum element 0 and maximum element 1 and let Q denote the semigroup of quasi-residuated (i.e., 0 -preserving and isotone) mappings on E . If E is a $\vee$ -semilattice we denote by T the subsemigroup of quasi-residuated $\vee$ -homomorphisms (or *hemimorphisms*) on E . Finally, we denote by C the subset of Q consisting of those mappings with principal kernels. It is shown in [1] that the mappings θ_x defined by

$$(\forall x \in E) \quad \theta_x(y) = \begin{cases} y & \text{if } y \leq x; \\ x & \text{if } y \not\leq x, \end{cases}$$

are idempotent elements of T , that the mappings $\hat{\theta}_x$ defined by

$$(\forall x \in E) \quad \hat{\theta}_x(y) = \begin{cases} 0 & \text{if } y \leq x; \\ x \vee y & \text{if } y \not\leq x, \end{cases}$$

are idempotents in $T \cap C$ and that ([1], §5)

$$\begin{cases} (\forall f \in T \cap C) & R(f) = \theta_{f^+(0)} \circ T; \\ (\forall g \in T) & L(g) = T \circ \hat{\theta}_{g(1)}, \end{cases}$$

where $f^+(0) = \max\{x \in E; f(x) = 0\}$. Moreover, if E is a lattice then θ_x is an idempotent in the semigroup $\text{Res}(E)$ of residuated

mappings on E .

In this paper we shall restrict our attention to the semigroup T and the mappings θ_x and $\hat{\psi}_x$ will be the idempotents of primary interest in the coördination of lattices.

We recall ([1], §3) that a Baer assembly is said to be *normal* if $K \subseteq B \cap D$ and ([1], §4) that a *right* Baer assembly is a normal Baer assembly $\langle A; B, D; K \rangle$ in which

- (1) B is a subsemigroup of A ;
- (2) $B \subseteq D$;
- (3) in the semigroup $P(A)$, $B \subseteq D \cdot D$ [i.e., $DB \subseteq D$].

The notion of a *left* Baer assembly is defined similarly. A right [resp. left] Baer assembly $\langle A; B, D; K \rangle$ is said to *coördinate* an ordered set E if $E \cong \mathcal{R}(D) = \{R_K(x); x \in D\}$ [resp. $E \cong \mathcal{L}(B) = \{L_K(x); x \in B\}$ where $\cong$ denotes dual order isomorphism].

2. Coördination of lattices by one-sided Baer assemblies.

Let $\langle A; B, D; K \rangle$ be a right Baer assembly. We shall use the notation

$$[B] = \{e \in B; e = e^2 \text{ and } eA \in \mathcal{R}(D)\};$$

$$[D] = \{f \in D; f = f^2 \text{ and } Af \in \mathcal{L}(B)\}.$$

We shall say that $\langle A; B, D; K \rangle$ is a *balanced* right Baer assembly if and only if there exist subsets B_* of $[B]$ and D^* of $[D]$ such that

- (1) for each $x \in D$ there is a unique $x_* \in B_*$ such that $R_K(x) = x_* A$;
- (2) for each $x \in B$ there is a unique $x^* \in D^*$ such that $L_K(x) = A x^*$;
- (3) $D^* \subseteq B$ [the "balancing factor"];

(4) if $e, f \in B_*$ are such that $e^*f \notin K$ and $f^*e \notin K$ then

$$\begin{cases} L_K(e^*f) = L_K(f^*e) = L_K(e) \wedge L_K(f); \\ R_K(e^*f) = R_K(f^*e) = R_K(e^*) \wedge R_K(f^*). \end{cases}$$

Remark. As immediate consequences of this definition we have that $L_K(e^*f), L_K(f^*e) \in \mathcal{L}(B)$ and $R_K(e^*f), R_K(f^*e) \in \mathcal{R}(D)$; this follows from the fact that $e^*f, f^*e \in D^*B_* \subseteq BB_* \subseteq B \subseteq D$. Moreover, we have $x \in B_* \iff x = (x^*)_*$ and $x \in D^* \iff x = (x_*)^*$; for example, if $x \in B_*$ then by Theorem 8 of [1] we have $xA = \hat{R}_K[L_K(x)] = \hat{R}_K(Ax^*) = R_K(x^*) = (x^*)_*A$ whence $x = (x^*)_*$ by the uniqueness in (1). Finally, we note at this juncture that condition (4), which expresses conditions on the intersection of right and left annihilators of products of idempotents, can also be expressed as conditions on the elements of B_* and D^* . It can be shown (though we shall not do so here) that, if we define an element $e_* \in B_*$ to be *r-decreasing* whenever

$$(\forall f^* \in D^*) (e_* \notin R_K(f^*)) \quad R_K(f^*e_*) \subseteq R_K(f^*)$$

and an element $f^* \in D^*$ to be *l-decreasing* whenever

$$(\forall e_* \in B_*) (f^* \notin L_K(e_*)) \quad L_K(f^*e_*) \subseteq L_K(e_*),$$

then the conditions $L_K(f^*e) = L_K(e) \wedge L_K(f)$ and $R_K(f^*e) = R_K(e^*) \wedge R_K(f^*)$ are satisfied if and only if each $e_* \in B_*$ is *r-decreasing* and each $f^* \in D^*$ is *l-decreasing*. However, we shall use the former conditions in order to facilitate the presentation.

Our basic model of a balanced right Baer assembly is given in the following result.

THEOREM 1 *If E is a bounded lattice then $\langle T; \text{Res}(E), T \cap C; \{0\} \rangle$ is a balanced right Baer assembly which coordinatises E .*

Proof. That $\langle T; \text{Res}(E), T \wedge C; \{0\} \rangle$ is a right Baer assembly which coordinatises E is established by observing that the results of §2 of [1] carry over *in toto* to the semigroup T with ψ_y replaced by $\hat{\psi}_y$ so that Theorem 11 of [1] with Q replaced by T , C replaced by $T \wedge C$ and ψ_y replaced by $\hat{\psi}_y$ carries through in exactly the same fashion. To show that this right Baer assembly is balanced, write $\text{Res}(E) = S$ and consider the subsets S_* of $[S]$ and $(T \wedge C)^*$ of $[T \wedge C]$ given by

$$S_* = \{\theta_x; x \in E\} \quad \text{and} \quad (T \wedge C)^* = \{\hat{\psi}_x; x \in E\}.$$

(That each θ_x belongs to $[S]$ follows from Theorem 9 of [1]). Now for each $f \in T \wedge C$ we have $R(f) = \theta_{f^+(0)} \cdot T$ where $\theta_{f^+(0)} \in S_*$; and if $\theta_y \in S_*$ is also such that $R(f) = \theta_y \cdot T$ then the T -analogue of Theorem 3 of [1] shows that we must have $\theta_y(\pi) = f^+(0)$. But $\theta_y(\pi) = y$ and so $y = f^+(0)$ whence $\theta_y = \theta_{f^+(0)}$. It follows that $\theta_{f^+(0)}$ is the unique element f_* of (1) above. Similarly, for each $f \in S$ we have $L(f) = T \cdot \hat{\psi}_{f(\pi)}$ where $\hat{\psi}_{f(\pi)} \in (T \wedge C)^*$; and if $\hat{\psi}_y \in (T \wedge C)^*$ is such that $L(f) = T \cdot \hat{\psi}_y$ then the T -analogue of Theorem 4 of [1] gives $y = \hat{\psi}_y^+(0) = f(\pi)$. Thus $\hat{\psi}_{f(\pi)}$ is the unique element f^* of (2) above. To establish (3), we note that, by §5 of [1], each $\hat{\psi}_y \in S$ and so $(T \wedge C)^* \subseteq S$. As for (4), we have, for each $x \in E$,

$$\begin{cases} L(\theta_x) = T \cdot \hat{\psi}_{\theta_x(\pi)} = T \cdot \hat{\psi}_x; \\ R(\hat{\psi}_x) = \theta_{\hat{\psi}_x^+(0)} \cdot T = \theta_x \cdot T, \end{cases}$$

and so $(\theta_x)^* = \hat{\psi}_x$ and $(\hat{\psi}_x)_* = \theta_x$. Now given $\theta_x, \theta_y \in S_*$ we have

$$[(\theta_x)^* \cdot \theta_y](z) = (\hat{\psi}_x \cdot \theta_y)(z) = \begin{cases} \hat{\psi}_x(z) & \text{if } z \leq y; \\ \hat{\psi}_x(y) & \text{if } z \not\leq y, \end{cases}$$

$$= \begin{cases} 0 & \text{if } z \leq y \text{ and } z \leq x; \\ x \vee z & \text{if } z \leq y \text{ and } z \not\leq x; \\ 0 & \text{if } z \not\leq y \text{ and } y \leq x; \\ x \vee y & \text{if } z \not\leq y \text{ and } y \not\leq x. \end{cases}$$

It follows from this that $(\theta_x)^* \cdot \theta_y \neq 0$ implies $y \not\leq x$ and so if $(\theta_x)^* \cdot \theta_y \neq 0$ and $(\theta_y)^* \cdot \theta_x \neq 0$ we have $y \not\leq x$ and $x \not\leq y$, which we express in the usual way by writing $x \parallel y$. In this case we have

$$(x \parallel y) \quad (\hat{\psi}_x \cdot \theta_y)(z) = \begin{cases} 0 & \text{if } z \leq x \wedge y; \\ x \vee z & \text{if } z \not\leq x \text{ and } z \leq y; \\ x \vee y & \text{if } z \not\leq y, \end{cases}$$

with a similar formula for $(\theta_y)^* \cdot \theta_x$. It follows that, in the semi-group T ,

$$\begin{aligned} \alpha \in L(\hat{\psi}_x \cdot \theta_y) &\iff \alpha(x \vee y) = 0 \iff \alpha \in L(\hat{\psi}_y \cdot \theta_x) \\ &\iff \alpha(x) \vee \alpha(y) = 0 \\ &\iff \alpha(x) = 0 = \alpha(y) \\ &\iff \alpha \in L(\theta_x) \cap L(\theta_y), \end{aligned}$$

so that $L(\hat{\psi}_x \cdot \theta_y) = L(\hat{\psi}_y \cdot \theta_x) = L(\theta_x) \cap L(\theta_y)$. Similarly,

$$\begin{aligned} \alpha \in R(\hat{\psi}_x \cdot \theta_x) &\iff \text{Im } \alpha \subseteq \{+, x \wedge y\} \iff \alpha \in R(\hat{\psi}_y \cdot \theta_x) \\ &\iff \text{Im } \alpha \subseteq \{+, x\} \cap \{+, y\} \\ &\iff \alpha \in R(\hat{\psi}_x) \cap R(\hat{\psi}_y), \end{aligned}$$

so that $R(\hat{\psi}_x \cdot \theta_y) = R(\hat{\psi}_y \cdot \theta_x) = R(\hat{\psi}_x) \cap R(\hat{\psi}_y)$. This then shows that the right Baer assembly $\langle T, \text{Res}(E), T \cap C, \{0\} \rangle$ is balanced and completes the proof.

The fundamental coördination of bounded lattices by one-sided Baer assemblies may now be stated as follows:

THEOREM 2 *If E is an ordered set then the following conditions are equivalent :*

- (1) E is a bounded lattice;
- (2) there exists a balanced right Baer assembly $\langle A; B, D; K \rangle$ which coordinatises E .

Proof. (1) $\Rightarrow$ (2): This is immediate from Theorem 1.

(2) $\Rightarrow$ (1): Suppose that $\langle A; B, D; K \rangle$ is a balanced right Baer assembly which is such that $E \cong \mathcal{R}(D)$. For each $x \in D$ we have, by Theorem 8 of [1],

$$R_X(x) = (\hat{R}_X \circ \hat{L}_X)[R_X(x)] = (\hat{R}_X \circ \hat{L}_X)(x, A) = \hat{R}_X[L_X(x)] = \hat{R}_X[A(x_*)^*] = R_X[(x_*)^*]$$

and, by the uniqueness in (1) and (2) of the definition,

$$R_X(x) = R_X(y) \iff x_* = y_* \iff (x_*)^* = (y_*)^*.$$

It follows that $\mathcal{R}(D) = \{R_X(x); x \in D\} = \{R_X(x); x \in D^*\}$. Moreover,

$e \in B_*$ implies $e^* \in D^*$, so $\{R_X(e^*); e \in B_*\} \subseteq \{R_X(x); x \in D^*\}$; and

$x \in D^* \iff x = (x_*)^* = e^*$ for some $e \in B_*$, so $\{R_X(x); x \in D^*\} \subseteq$

$\{R_X(e^*); e \in B_*\}$. It follows that $\mathcal{R}(D) = \{R_X(x); x \in D^*\} = \{R_X(e^*); e \in B_*\}$.

To show that, when ordered by set inclusion, $\mathcal{R}(D)$ is an $\wedge$ -semilattice,

it therefore suffices to show that, for $e, f \in B_*$, $R_X(e^*) \cap R_X(f^*)$ is

in $\mathcal{R}(D)$ whenever $R_X(e^*) \parallel R_X(f^*)$. Note that this is the same as the

condition $e\lambda \parallel f\lambda$ since $R_X(e^*) = (e^*)_*\lambda = e\lambda$. Suppose then that

$e, f \in B_*$ are such that $e\lambda \parallel f\lambda$. Then necessarily $e^*f \notin K$ and $f^*e \notin K$;

for if, for example, $e^*f \in K$ then $f \in R_X(e^*) = e\lambda$ and so $f\lambda \subseteq e\lambda$,

a contradiction. Applying part (4) of the definition, we then have

$$(e\lambda \parallel f\lambda) \quad e\lambda \cap f\lambda = R_X(e^*) \cap R_X(f^*) = R_X(e^*f)$$

and since $R_X(e^*f) = R_X[(e^*f)_*]^* \in \mathcal{R}(D)$ we see that $e\lambda \cap f\lambda \in \mathcal{R}(D)$.

This shows that $\mathcal{R}(D)$ forms an $\wedge$ -semilattice. To show that $\mathcal{R}(D)$

also forms a $\bigcup$ semilattice, suppose that $e, f \in B_*$ with $eA \parallel fA$. Then again $e*f \notin K$ and $f*e \notin K$ so, by (4), $L_K(e*f) = L_K(e) \cap L_K(f)$ from which we have $L_K(e*f) \subseteq L_K(e)$ and $L_K(e*f) \subseteq L_K(f)$. It follows by Theorem 8 of [1] that $[(e*f)^*]_* A = \hat{R}_K[L_K(e*f)] \supseteq \hat{R}_K[L_K(e)] = R_K(e^*)$ and similarly $\hat{R}_K[L_K(e*f)] \supseteq R_K(f^*)$. If now $h \in B_*$ is such that both $hA \supseteq eA = R_K(e^*)$ and $hA \supseteq fA = R_K(f^*)$ then $L_K(h) \subseteq \hat{L}_K[R_K(e^*)] = Ae^*$ and similarly $L_K(h) \subseteq Af^*$. It follows that $L_K(h) \subseteq Ae^* \cap Af^* = L_K(e*f)$ whence $hA = \hat{R}_K[L_K(h)] \supseteq \hat{R}_K[L_K(e*f)]$. This shows that

$$(eA \parallel fA) \quad eA \vee fA = \hat{R}_K[L_K(e*f)] = [(e*f)^*]_* A = R_K[(e*f)^*]$$

so that $\mathcal{R}(D)$ is a union semilattice in which unions are given by $\vee$ (which is in general different from set-theoretic union). We have thus shown that $\mathcal{R}(D)$ forms a bounded lattice, whence so also is E .

Remarks. (1) Although we shall not develop the details here, we point out that a parallel coöordinatisation of bounded lattices can be obtained by using balanced left Baer assemblies $\langle A; B, D; K \rangle$. In such an assembly we have $D \subseteq B$ and $DB \subseteq B$ and the "balancing" is obtained by changing part (3) of the previous definition to (3') $B_* \subseteq D$. In this case the left Baer assembly used as a model is the left Baer assembly $\langle T; T, \text{Res}(E); \{0\} \rangle$ of §5 in [1].

(2) Although the definition of a balanced right Baer assembly is somewhat complicated, nevertheless the lattice operations are more readily represented in the coöordinatising right Baer assembly than for instance was the case with the semilattice operation in Theorem 11 of [1]. Note that in the lattice case the mappings $\hat{\psi}_y$ belong to $\text{Res}(E)$ whereas in the semilattice case in [1] the mappings ψ_y do not.

3. Coördinatisation of complemented lattices.

If $\langle A; B, D; K \rangle$ is a normal Baer assembly then, by Theorem 7 of [1], A has an identity element 1 with $1 \in B \cap D$. Also, K is generated by a central idempotent k^0 with $k^0 \in B \cap D$ and k^0 is the identity element of K . If now $\langle A; B, D; K \rangle$ is a balanced right Baer assembly then from $1 \in B \cap D$ we have $K = R_K(1) = 1_* A$ and $K = L_K(1) = A 1^*$. It follows that 1_* is a left identity for K and 1^* is a right identity for K , whence necessarily $1_* = 1^* = k^0$ and consequently $k^0 \in B_* \cap D^*$. Similarly, from $k^0 \in B \cap D$ we deduce that $A = R_K(k^0) = (k^0)_* A$ and $A = L_K(k^0) = A (k^0)^*$. These equalities show that $(k^0)_*$ is a left identity for A and $(k^0)^*$ is a right identity for A , whence we have $(k^0)_* = (k^0)^* = 1$ and consequently $1 \in B_* \cap D^*$. Note also that $K \cap B_* = \{k^0\} = K \cap D^*$; for example, if $k \in K \cap B_*$ then $A = L_K(k) = A k^*$ so $1 = 1 k^* = k^*$ and it follows that $k = (k^*)_* = 1_* = k^0$.

We can now isolate a balanced right Baer assembly which will coördinatisate a complemented lattice.

THEOREM 3 *If E is an ordered set then the following conditions are equivalent :*

- (1) E is a complemented lattice;
- (2) E can be coördinatised by a balanced right Baer assembly $\langle A; B, D; K \rangle$ which satisfies the property

$$(C) \quad \begin{cases} (\forall e \in B_* \setminus \{k^0, 1\}) (\exists f \in B_* \setminus \{k^0, 1\}) \\ R_K(e^* f) = R_K(f^* e) = L_K(e^* f) = L_K(f^* e) = K. \end{cases}$$

Proof. (1) $\Rightarrow$ (2): Let E be a complemented lattice and consider the balanced right Baer assembly $\langle T; \text{Res}(E), T \cap C; \{0\} \rangle$. Here we have

$k^* = 0$ and $1 = id_E$. If $0, \pi$ denote respectively the minimum and maximum elements of E then, as is readily seen, $\theta_x = 0 \iff x = 0$, $\theta_x = id_E \iff x = \pi$, $\hat{\psi}_x = 0 \iff x = \pi$ and $\hat{\psi}_x = id_E \iff x = 0$. It follows that, with $S = Res(E)$,

$$\begin{cases} S \setminus \{0, id_E\} = \{\theta_x; x \notin \{0, \pi\}\}; \\ (T \cap C) \setminus \{0, id_E\} = \{\hat{\psi}_x; x \notin \{0, \pi\}\}. \end{cases}$$

Suppose then that $x \notin \{0, \pi\}$ and let x' denote any complement of x .

We have

$$\begin{aligned} (\hat{\psi}_x \circ \theta_{x'}) (y) &= \begin{cases} \hat{\psi}_x(y) & \text{if } y \leq x'; \\ \hat{\psi}_x(x') & \text{if } y \not\leq x'; \end{cases} \\ &= \begin{cases} 0 & \text{if } y = 0; \\ x \cup y & \text{if } y \leq x' \text{ and } y \not\leq x; \\ \pi & \text{otherwise,} \end{cases} \end{aligned}$$

from which it readily follows that $R(\hat{\psi}_x \circ \theta_{x'}) = \{0\} = L(\hat{\psi}_x \circ \theta_{x'})$. Interchanging x, x' gives similarly $R(\hat{\psi}_{x'} \circ \theta_x) = \{0\} = L(\hat{\psi}_{x'} \circ \theta_x)$ and the result follows.

(2) $\Rightarrow$ (1): Suppose now that (2) holds and, given $e \in B \setminus \{k^*, 1\}$, let $f \in B \setminus \{k^*, 1\}$ be such that $R_K(e*f) = R_K(f*e) = L_K(e*f) = L_K(f*e) = K$. Then clearly $e*f \notin K$ and $f*e \notin K$ (for otherwise we would be reduced to the trivial case $K = A$) and hence $eA \parallel fA$. Applying the formulae of Theorem 2, we obtain

$$(eA \parallel fA) \quad \begin{cases} eA \cap fA = R_K(e*f) = K; \\ eA \vee fA = \hat{R}_K[L_K(e*f)] = R_K(K) = A, \end{cases}$$

which shows that fA is a complement of eA in $\mathcal{R}(D)$. Observing that when $e = k^*$ we have $eA = k^*A = K$, which admits A as a complement in $\mathcal{R}(D)$, and that when $e = 1$ we have $eA = A$, which admits K as a

complement in $\mathcal{R}(D)$, it follows that $\mathcal{R}(D)$ is a complemented lattice, whence so also is E .

Remark. For a coöordinatisation of uniquely complemented lattices, it clearly suffices to postulate the uniqueness of f in part (2) of the above.

4. Coöordinatisation of modular lattices.

In order to obtain a coöordinatisation of bounded modular lattices, we require the assistance of other idempotents. By way of introducing these, we prove the following characterisation of modular lattices.

THEOREM 4 *Let E be a lattice and let U denote the semigroup of $\vee$ -homomorphisms on E . Then the following conditions are equivalent:*

(1) E is modular;

(2) for all $a, b \in E$ with $a \parallel b$ the mapping $\theta_{a,b} : E \rightarrow E$ given by

$$\theta_{a,b}(x) = \begin{cases} a \wedge (b \vee x) & \text{if } x \not\leq a \text{ and } x \leq a \vee b; \\ \theta_a(x) & \text{otherwise,} \end{cases}$$

is an element of U .

Proof. (1) $\Rightarrow$ (2): Suppose that E is modular. Since clearly $\theta_a \in U$ for each $a \in E$ it suffices to show that $\theta_{a,b}(x \vee y) = \theta_{a,b}(x) \vee \theta_{a,b}(y)$ whenever $x \not\leq a$, $x \leq a \vee b$ and y is any element of E . For this, we consider the following three cases:

(i) $x \not\leq a$, $x \leq a \vee b$ and $y \leq a$:

In this case we have $\theta_{a,b}(x) = a \wedge (b \vee x)$ and $\theta_{a,b}(y) = \theta_a(y) = y$. Thus, since E is modular,

$$\theta_{a,b}(x) \vee \theta_{a,b}(y) = [a \wedge (b \vee x)] \vee y = a \wedge (b \vee x \vee y).$$

But from $x \not\leq a$ we have $x \vee y \not\leq a$; and since $y \leq a$ and $x \leq a \vee b$

we have $x \vee y \leq b \vee x \vee y \leq b \vee a \vee b \vee a = a \vee b$. Thus

$$\theta_{a,b}(x \vee y) = a \wedge (b \vee x \vee y) = \theta_{a,b}(x) \vee \theta_{a,b}(y).$$

(ii) $x \not\leq a$, $x \leq a \vee b$ and $y \not\leq a$, $y \leq a \vee b$:

In this case we have

$\theta_{a,b}(x) = a \wedge (b \vee x)$ and $\theta_{a,b}(y) = a \wedge (b \vee y)$. The modularity of E then gives

$$\begin{aligned} \theta_{a,b}(x) \vee \theta_{a,b}(y) &= [a \wedge (b \vee x)] \vee [a \wedge (b \vee y)] \\ &= \{[a \wedge (b \vee x)] \vee b \vee y\} \wedge a && \text{[since } a \wedge (b \vee x) \leq a] \\ &= \{[(b \vee x) \wedge (a \vee b)] \vee y\} \wedge a && \text{[since } b \leq b \vee x] \\ &= (b \vee x \vee y) \wedge a && \text{[since } b \vee x \leq a \vee b]. \end{aligned}$$

But from $x \not\leq a$ we have $x \vee y \not\leq a$; and since both $x, y \leq a \vee b$ we have $x \vee y \leq a \vee b$. Hence

$$\theta_{a,b}(x \vee y) = a \wedge (b \vee x \vee y) = \theta_{a,b}(x) \vee \theta_{a,b}(y).$$

(iii) $x \not\leq a$, $x \leq a \vee b$ and $y \not\leq a$, $y \not\leq a \vee b$:

In this case we have

$\theta_{a,b}(x) = a \wedge (b \vee x)$ and $\theta_{a,b}(y) = \theta_a(y) = a$ so that

$$\theta_{a,b}(x) \vee \theta_{a,b}(y) = [a \wedge (b \vee x)] \vee a = a.$$

But since also $x \vee y \not\leq a$ and $x \vee y \leq a \vee b$ we see that

$$\theta_{a,b}(x \vee y) = \theta_a(x \vee y) = a = \theta_{a,b}(x) \vee \theta_{a,b}(y).$$

Parts (i), (ii), (iii) now show that, whenever $a \parallel b$, we have

$\theta_{a,b} \in U$ as required.

(2) $\Rightarrow$ (1): By way of obtaining a contradiction, suppose that (2) holds and that E is not modular. As is well known, E then contains a sublattice of the form



Now in this sublattice we observe that

$$\begin{cases} \theta_{a,b}(b \vee c) = a \wedge (b \vee b \vee c) = a \wedge (a \vee b) = a; \\ \theta_{a,b}(b) \vee \theta_{a,b}(c) = [a \wedge (b \vee b)] \vee c = c, \end{cases}$$

so that $\theta_{a,b} \notin U$, a contradiction. This then establishes $(2) \Rightarrow (1)$.

Remark. Observe that if we remove the restriction that $a \parallel b$ in the definition of $\theta_{a,b}$ then we obtain $\theta_{a,b} = \theta_a$. For if $b \leq a$ then $x \leq a \vee b$ implies $x \leq a$, so $\{x \in E; \theta_{a,b}(x) = a \wedge (b \vee x)\} = \emptyset$; and if $b > a$ then $x \not\leq a$, $x \leq a \vee b = b$ imply $\theta_{a,b}(x) = a \wedge (b \vee x) = a \wedge b = a = \theta_a(x)$. Thus we need distinguish $\theta_{a,b}$ only in the case where $a \parallel b$.

The principal properties of the mappings $\theta_{a,b}$ when $a \parallel b$ in a bounded modular lattice are listed in the following result.

THEOREM 5 *Let E be a bounded modular lattice and let T be the semigroup of quasi-residuated $\vee$ -homomorphisms on E . If $a, b \in E$ are such that $a \parallel b$ then*

- (1) $\theta_{a,b}$ is an idempotent of T ;
- (2) $R(\hat{\psi}_a) = \theta_a \circ T = \theta_{a,b} \circ T$;
- (3) $L(\theta_{a,b}) = L(\theta_a) = T \circ \hat{\psi}_a$;
- (4) $L(\theta_{a,b} \circ \theta_b) = \hat{L}[R(\hat{\psi}_a \circ \theta_b)] = L(\theta_{a \wedge b})$;
- (5) $L(\hat{\psi}_a \circ \theta_{a,b}) = L(\hat{\psi}_b \circ \theta_a) = L(\theta_{a \vee b})$.

Proof. (1) Theorem 4 shows that $\theta_{a,b}$ is a $\vee$ -homomorphism and is consequently isotone. To show that $\theta_{a,b} \in T$ it then suffices to note

that $\theta_{a,b}(0) = \theta_a(0) = 0$. To show that $\theta_{a,b}$ is idempotent, it suffices, by virtue of the fact that θ_a is idempotent, to show that whenever $x \perp a$ and $x \leq a \vee b$ we have $\theta_{a,b}(x) = \theta_{a,b}[\theta_{a,b}(x)]$.

Now for such an element x we have

$$\theta_{a,b}[\theta_{a,b}(x)] = \theta_{a,b}[a \wedge (b \vee x)] = \theta_a[a \wedge (b \vee x)] = a \wedge (b \vee x) = \theta_{a,b}(x).$$

(2) In the proof of Theorem 1 we observed that $R(\hat{\psi}_a) = \theta_a \circ T$.

To establish the equality $\theta_{a,b} \circ T = \theta_a \circ T$ it is sufficient, by (1), to show that $\theta_{a,b} = \theta_a \circ \theta_{a,b}$ and $\theta_a = \theta_{a,b} \circ \theta_a$. Now $\theta_a(x) \leq a$ for all $x \in E$ and when $y \leq a$ we have $\theta_{a,b}(y) = \theta_a(y)$. It follows that $\theta_{a,b}[\theta_a(x)] = \theta_a[\theta_a(x)] = \theta_a(x)$ for each $x \in E$ and so $\theta_{a,b} \circ \theta_a = \theta_a$. Also, $\theta_{a,b}(x) \leq a$ for each $x \in E$ and so we have $\theta_a[\theta_{a,b}(x)] = \theta_{a,b}(x)$, which shows that $\theta_a \circ \theta_{a,b} = \theta_{a,b}$.

(3) This is immediate from (2) on taking left annihilators.

(4) As observed in the proof of Theorem 1, we have

$$\alpha \in R(\hat{\psi}_a \circ \theta_b) \iff \text{Im } \alpha \subseteq [\leftarrow, a \wedge b] \iff \alpha \in R(\hat{\psi}_{a \wedge b}).$$

It follows from this that $R(\hat{\psi}_a \circ \theta_b) = R(\hat{\psi}_{a \wedge b}) = \theta_{a \wedge b} \circ T$. Taking left annihilators we obtain $\hat{L}[R(\hat{\psi}_a \circ \theta_b)] = L(\theta_{a \wedge b})$. Now

$$(\theta_{a,b} \circ \theta_b)(x) = \begin{cases} \theta_{a,b}(x) \leq \theta_{a,b}(b) = a \wedge b & \text{if } x \leq b; \\ \theta_{a,b}(b) = a \wedge b & \text{if } x \perp b, \end{cases}$$

and so we deduce that

$$\alpha \in L(\theta_{a,b} \circ \theta_b) \iff \alpha(a \wedge b) = 0 \iff \alpha \in L(\theta_{a \wedge b}),$$

which completes the proof of (4).

(5) As observed in the proof of Theorem 1, we have

$$\alpha \in L(\hat{\psi}_b \circ \theta_a) \iff \alpha(a \vee b) = 0 \iff \alpha \in L(\theta_{a \vee b})$$

and so $L(\hat{\psi}_b \circ \theta_a) = L(\theta_{a \vee b})$. Now

$$(\hat{\psi}_b \circ \theta_{a,b})(x) = \begin{cases} \hat{\psi}_b[a \wedge (b \vee x)] \leq \hat{\psi}_b(a) = a \vee b & \text{if } x \not\leq a, x \leq a \\ \hat{\psi}_b[\theta_a(x)] = \hat{\psi}_b(x) = x \vee b \leq a \vee b & \text{otherwise,} \end{cases}$$

with $(\hat{\psi}_b \circ \theta_{a,b})(a) = a \vee b$. It follows that

$$\alpha \in L(\hat{\psi}_a \circ \theta_{a,b}) \iff \alpha(a \vee b) = 0 \iff \alpha \in L(\theta_{a \vee b}),$$

which completes the proof of (5).

We are now in a position to provide a coöordinatisation of bounded modular lattices.

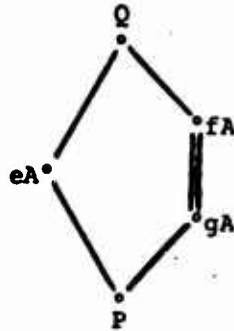
THEOREM 6 *If E is an ordered set then the following conditions are equivalent:*

- (1) E is a bounded modular lattice;
- (2) E can be coöordinatised by a balanced right Baer assembly $\langle A; B, D; K \rangle$ which satisfies the property

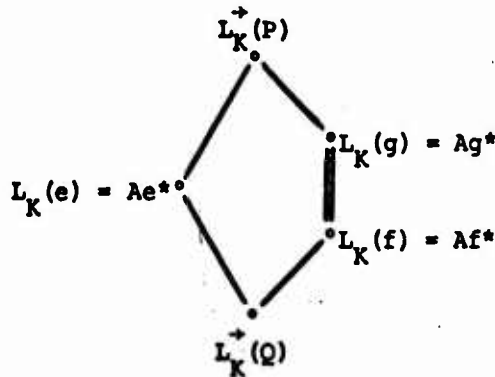
$$(M) \begin{cases} \text{if } e, f \in B_+ \text{ are such that } e^*f \notin K \text{ and } f^*e \notin K \\ \text{then there exists } \bar{e} = \bar{e}^2 \in A \text{ with } \bar{e}A = eA \text{ and} \\ R_K(e^*f) = \hat{R}_K[L_K(\bar{e}f)]. \end{cases}$$

Proof. (1) $\Rightarrow$ (2): Let E be a bounded modular lattice. To establish (2), it suffices to show that $\langle T; \text{Res}(E), T \cap C; \{0\} \rangle$ satisfies (M). Now given $\theta_a, \theta_b \in [\text{Res}(E)]_+$ such that $\hat{\psi}_a \circ \theta_b \neq 0$ and $\hat{\psi}_b \circ \theta_a \neq 0$ we have as before $a \parallel b$ and $\theta_{a,b} \in T$ with $\theta_{a,b}$ idempotent. Moreover, by Theorem 5, $\theta_{a,b} \circ T = \theta_a \circ T$. Taking right annihilators in Theorem 5(4), we obtain finally $\hat{R}[L(\theta_{a,b} \circ \theta_b)] = R(\hat{\psi}_a \circ \theta_b)$. This then shows that the condition (M) is satisfied.

(2) $\Rightarrow$ (1): Suppose now that (2) holds. To show that E is a modular lattice it suffices to show that, if $\mathcal{A}(D)$ has a sublattice of the form



in which increasing single lines denote $\subset$, the increasing double line denotes $\subseteq$, $Q = eA \vee fA = eA \vee gA$ and $P = eA \wedge fA = eA \wedge gA$, then necessarily $fA = gA$. Now since the mapping $L_K^+ : \mathcal{R}(D) \rightarrow \mathcal{L}(B)$ which is defined by $L_K^+(eA) = \hat{L}_K(eA) = L_K(e) = Ae^*$ is a dual isomorphism (see [1], Theorem 8), it is equivalent to show that if $\mathcal{L}(B)$ has a sublattice of the form



then necessarily $Af^* = Ag^*$. Using the formulae of Theorem 2, we see that $Q = eA \vee gA = \hat{R}_K[L_K(e^*g)]$ and, by the hypothesis (2), that $P = fA \wedge eA = R_K(f^*e) = \hat{R}_K[L_K(\bar{f}e)]$. It follows that $L_K^+(P) = L_K(\bar{f}e)$ and $L_K^+(Q) = L_K(e^*g)$. The problem therefore reduces to showing that

$$L_K(f) \supseteq L_K(g) = L_K(g) \cap L_K^+(P),$$

which is equivalent to

$$Af^* \supseteq Ag^* = Ag^* \cap L_K(\bar{f}e).$$

Now if $x \in Ag^* \cap L_K(\bar{f}e)$ then on the one hand $x = xg^*$ and on the other $x\bar{f}e \in K$. The second of these gives $x\bar{f} \in L_K(e) = Ae^*$ and so $x\bar{f} = x\bar{f}e^*$. But $gA \subseteq fA = \bar{f}A$ and so $g = \bar{f}g$ whence $xg = x\bar{f}g = x\bar{f}e^*g$. Since $x = xg^*$ we then have $xg = xg^*g \in K$ and hence $x\bar{f}e^*g \in K$. It follows that $x\bar{f} \in L_K(e^*g) = L_K^+(Q) \subseteq L_K(f)$ and so $x\bar{f}f \in K$. But since $\bar{f}A = fA$ we have $\bar{f}f = f$. We thus deduce that $xf \in K$ and so $x \in L_K(f) = Af^*$ as required. This then establishes that $\mathcal{R}(D)$, and hence E , is a bounded modular lattice.

Remark. Note that in the condition (M) the element $\bar{e}$ is simply an element of A and not necessarily in B or D . However, since L_K^+ is a dual isomorphism and $\hat{R}_K[L_K(\bar{e}f)] \in \mathcal{R}(D)$, we see that $L_K(\bar{e}f) = L_K^+ \hat{R}_K[L_K(\bar{e}f)] \in \mathcal{L}(B)$.

5. Coördinatisation of distributive lattices.

In the case of a distributive lattice, the idempotents which we require are of a much more obvious nature. These are given in the following result, the proof of which is clear.

THEOREM 7 Let E be a lattice and for each $x \in E$ let $t_x : E \rightarrow E$ be given by $t_x(y) = x \wedge y$. Then each t_x is idempotent and, when E is bounded, the following conditions are equivalent :

- (1) E is distributive;
- (2) $(\forall x \in E) \quad t_x \in T$.

The next result shows how the idempotents t_x enjoy, in a bounded distributive lattice, properties similar to those listed above for the mappings $\theta_{a,b}$ in a bounded modular lattice.

THEOREM 8 Let E be a bounded distributive lattice and let T be the semigroup of quasi-residuated $\vee$ -homomorphisms on E . Then given $a, b \in E$ we have

- (1) $L(t_a) = L(\theta_a)$; $t_a \circ T = \theta_a \circ T$;
- (2) $L(t_a \circ \theta_b) = L(\theta_{a \wedge b}) = L(t_a \circ t_b)$;
- (3) $L(\hat{\psi}_a \circ t_b) = L(\theta_{a \vee b})$.

Proof. (1) Since for each $\alpha \in T$ we have $(\alpha \circ t_a)(x) = \alpha(a \wedge x) \leq \alpha(a) = (\alpha \circ t_a)(a)$, we see that $\alpha \circ t_a = 0$ if and only if $\alpha(a) = 0$, so that $L(t_a) = L(\theta_a)$. That $t_a \circ T = \theta_a \circ T$ follows from the facts that both t_a and θ_a are idempotent,

$$t_a[\theta_a(x)] = a \wedge \theta_a(x) = \begin{cases} a \wedge x = x & \text{if } x \leq a; \\ a \wedge a = a & \text{if } x \not\leq a, \end{cases}$$

$$= \theta_a(x),$$

and $\theta_a[t_a(x)] = \theta_a(a \wedge x) = a \wedge x = t_a(x)$.

(2) Since

$$(t_a \circ \theta_b)(x) = \begin{cases} a \wedge x \leq a \wedge b & \text{if } x \leq b; \\ a \wedge b & \text{if } x \not\leq b, \end{cases}$$

we see that, in the semigroup T ,

$$\alpha \in L(t_a \circ \theta_b) \iff \alpha(a \wedge b) = 0 \iff \alpha \in L(\theta_{a \wedge b}).$$

Clearly, these are also equivalent to $\alpha \in L(t_a \circ t_b)$.

(3) Since

$$(\hat{\psi}_a \circ t_b)(x) = \hat{\psi}_a(b \wedge x) = \begin{cases} 0 & \text{if } a \wedge x \leq b; \\ (b \wedge x) \vee a \leq b \vee a & \text{if } a \wedge x \not\leq b, \end{cases}$$

with $(\hat{\psi}_a \circ t_a)(\pi) = a \vee b$, we see that

$$\alpha \in L(\hat{\psi}_a \circ t_b) \iff \alpha(a \vee b) = 0 \iff \alpha \in L(\theta_{a \vee b}).$$

We can now provide a coöordinatisation of bounded distributive lattices.

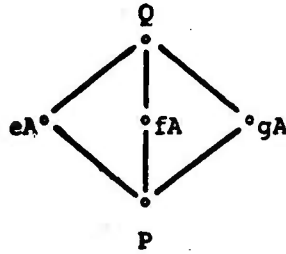
THEOREM 9 *If E is an ordered set then the following conditions are equivalent :*

- (1) E is a bounded distributive lattice;
- (2) E can be coöordinatised by a balanced right Baer assembly $\langle A; B, D; K \rangle$ which satisfies the property

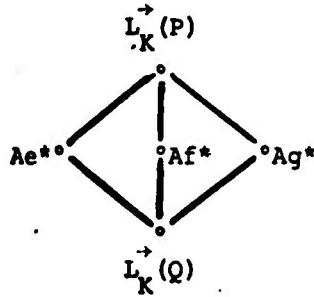
$$(D) \left\{ \begin{array}{l} \text{there exists an abelian idempotent subsemigroup } \hat{B}_A \\ \text{of } A \text{ such that} \\ \text{(i) for each } e \in B_A \text{ there is a unique } \bar{e} \in \hat{B}_A \\ \text{such that } \bar{e}A = eA; \\ \text{(ii) if } e, f \in B_A \text{ are such that } e^*f \notin K \text{ and } f^*e \notin K \\ \text{then } R_K(e^*f) = \hat{R}_K[L_K(\bar{e}f)] = \hat{R}_K[L_K(\bar{e}\bar{f})]. \end{array} \right.$$

Proof. (1) $\Rightarrow$ (2): Suppose that (1) holds. Then it suffices to show that $\langle T; \text{Res}(E), T \wedge C; \{0\} \rangle$ satisfies the condition (D). To this end we observe first that $\{t_a; a \in E\}$ clearly forms an abelian idempotent subsemigroup of T . Writing $S = \text{Res}(E)$, define $\hat{S}_* = \{t_a; a \in E\}$. We see that $\hat{S}_*$ satisfies (i) since, by Theorem 8(1), $\theta_a \cdot T = t_a \cdot T$ and if $t_a \cdot T = t_b \cdot T$ then clearly $t_a = t_b$. As for (ii), we note that if $\theta_a, \theta_b \in S_*$ are such that $\hat{\psi}_a \cdot \theta_b \neq 0$ and $\hat{\psi}_b \cdot \theta_a \neq 0$ then $a \parallel b$ and we see from Theorem 8(2) that $R(\hat{\psi}_a \cdot \theta_b) = \hat{R}[L(t_a \cdot \theta_b)] = \hat{R}[L(t_a \cdot t_b)]$.

(2) $\Rightarrow$ (1): Suppose that (2) holds. From the condition (D) we see that $\langle A; B, D; K \rangle$ satisfies the condition (M) of Theorem 6 and so $\mathcal{B}(D)$ is a bounded modular lattice. To show that it is distributive, it therefore suffices to show that it contains no sublattice of the form



As in the proof of Theorem 6, we can apply the dual isomorphism L_K^+ and show equivalently that $\mathcal{L}(B)$ contains no sublattice of the form



To fix ideas, we shall assume that Q, P, eA and fA are related as is shown above, so that $Q = eA \vee fA = \hat{R}_K[L_K(e^*f)]$ and, by (D), $P = eA \wedge fA = R_K(e^*f) = \hat{R}_K[L_K(\bar{e}\bar{f})]$ where $\bar{e}, \bar{f} \in \hat{B}_*$. Then $L_K^+(Q) = L_K(e^*f)$ and $L_K^+(P) = L_K(\bar{e}\bar{f})$. We shall now show that the conditions

$$L_K(e^*f) \subseteq Ag^* \subseteq L_K(\bar{e}\bar{f}), \quad Ag^* \wedge Af^* = L_K^+(Q) \quad \text{and} \quad Ag^* \parallel Af^*$$

together imply that $Ag^* \subseteq Ae^*$ from which it will follow that the above sublattice degenerates and $\mathcal{R}(D)$ is distributive. This we do as follows.

We note first that for $\bar{g} \in \hat{B}_*$ we have $R_K(g^*) = gA = \bar{g}A$ and so $g^*\bar{g} \in K$. Since $\bar{e}\bar{g} = \bar{g}\bar{e}$ we then have $g^*\bar{e}\bar{g} = g^*\bar{g}\bar{e} \in K$ and hence

$$g^*\bar{e} \in L_K(\bar{g}) = \hat{L}_K(\bar{g}A) = \hat{L}_K(gA) = Ag^*,$$

from which it follows that $g^*\bar{e} = g^*\bar{e}g^*$. Since $Ag^* \subseteq L_K^+(P) = L_K(\bar{e}\bar{f})$ we have $g^*\bar{e}\bar{f} \in K$. Using the previous equality, we deduce that $g^*\bar{e}g^*\bar{f} \in K$ so that $g^* \in L_K(\bar{e}g^*\bar{f})$ and consequently $Ag^* \subseteq L_K(\bar{e}g^*\bar{f})$.

Now using the readily established fact that

$$L_K(x) \subseteq L_K(y) \rightarrow (\forall z \in A) \quad L_K(zx) \subseteq L_K(zy)$$

we see from the fact that $L_K(f) = L_K(\bar{f})$ that

$$L_K(\bar{e}g\bar{f}) = L_K(\bar{e}g^*f).$$

Similarly, from $L_K(\bar{e}) \supseteq L_K^+(Q) = L_K(g^*f)$ we obtain

$$Ae^* = L_K(e) = L_K(\bar{e}) = L_K(\bar{e}\bar{e}) \supseteq L_K(\bar{e}g^*f).$$

Combining the above observations, we deduce that $Ae^* \supseteq Ag^*$ as required.

Remarks. (1) In the above proof it was not necessary to show that $L_K(\bar{e}g^*f) \subseteq \mathcal{L}(B)$. That this is so follows from the fact that a similar proof with e, g interchanged yields $Ae^* = Ag^* = L_K(\bar{e}g^*f) = L_K(\bar{g}e^*f)$.

(2) In [1] we showed that a bounded ordered set E forms an implicative (Glivenko-Brouwer) semilattice if and only if E can be coöordinatised by a right Baer assembly $\langle A; B, D; K \rangle$ in which there is an abelian idempotent subsemigroup $\hat{B}$ of B such that $\mathcal{R}(D) = \{eA; e \in \hat{B}\}$. Since a lattice E is implicative if and only if $t_a \in \text{Res}(E)$ for each $a \in E$, it is easy to see from the preceding result that a distributive lattice is implicative if and only if it can be coöordinatised by a balanced right Baer assembly which (using the notation introduced above) satisfies the additional condition that $B_* = \hat{B}_*$ or, equivalently, $\hat{B}_* \subseteq B$. This illustrates once again that the essential difference between the lattice and the semilattice coöordinatisations is the "balancing factor" of §2.

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