Division of Engineering
BROWN UNIVERSITY
PROVIDENCE, R. I.

THE EFFECT OF SHEAR ON THE PLASTIC BENDING OF BEAMS

BY

D. C. DRUCKER

U.S. Army Ordnance Corps
Office of Ordnance Research
Contract DA-19-020-ORD-3172
Project TB 2 0001(1086)
OOR-3172/7 Tech. Rept. 7 October, 1955

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(10) D. C. Drucker**

(11) Oct 55 (12) 31

Summary

The end-loaded cantilever beam of perfectly plastic material has been studied in considerable detail but many questions remain unanswered. As a first step in extension to plates, the concept is explored of an interaction curve relating limiting values of shearing force and bending moment for perfectly plastic beams. Simple illustrations demonstrate that, far more than in the elastic range, such interaction is not just a local matter but depends upon the geometry and loading of the entire beam. Useful interaction curves are obtained, nevertheless, with the aid of the upper and lower bound techniques of limit analysis, choosing the maximum shearing stress criterion of yielding for convenience.

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It is shown, in particular, that although a small amount of shear produces but a second order reduction in the limit moment of beams, a small moment reduces the limiting shear value by a first order term.

(18) ARO, NA

(19) OOR-3172/7 140/19-14

*The results presented in this paper were obtained in the course of research sponsored by the Office of Ordnance Research under Contract No. DA-19-020-ORD-3172, Project No. TB2-0001 (1086).

**Chairman, Division of Engineering, Brown University, Providence, R. I.

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Introduction

All beams to be considered are assumed to be made of an idealized material which is termed perfectly plastic. Perfectly plastic material is elastic up to the yield point and then flows under constant stress. The analysis of perfectly plastic beams in the plastic range is at present in a very satisfactory state. Bending usually predominates so that the concept of simple plastic hinges is sufficient in most cases. Should there be axial force in addition, the extending or contracting hinge described by Onat and Prager (1)* takes care of the situation. It might be expected that shear force could be included in a similar manner. If shearing force, V, and bending moment, M, alone are considered, it would seem a simple matter to determine whether or not the beam is fully plastic at the section. In those problems where shear is important, an interaction curve relating V and M for fully plastic action would be most desirable for beams of rectangular cross-section, for I-beams and for each shape in common use. Unfortunately such a curve does not really exist, even for any one shape, because the geometry and loading of the entire beam are important, not the properties of the section alone.

The rectangular beam only will be considered in what follows and an attempt will be made to clarify the reasons for the lack of a unique interaction curve. Studies of the cantilever beam under end load have been made by Horne (2), by Onat and Shield (3), by Green (4), and by Leth (5). Much of the information to be presented here is contained, therefore, in this previous work but the relevent parts of each have not yet been compared in principle and some of the peculiarities of the results have not previously been explained.

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*Numbers in parentheses refer to the Bibliography at the end of the paper.

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A start will be made by the analysis of cantilever and simple beams with constant shear force. The lower bound technique of limit analysis (6) will be employed first to find a safe relation between V and M and to provide reference values for the subsequent work. A local criterion will then be sought to relate limit values of V and M. The impossibility of complete success with such an approach will be discussed. The upper bound technique of limit analysis (6) will then be applied to the simple span and comparison made with the lower bound and the local criteria. The cantilever will also be studied and its peculiarities noted. The influence of the loading and the geometry away from the section should then become clearer. Finally by comparison of all the results, a useful but by no means unique or exact interaction curve will be proposed.

Lower Bounds for Beam of Rectangular Cross-Section

The lower bound theorem of limit analysis deals with states of stress which satisfy equilibrium and which do not violate the yield condition. For convenience the maximum shearing stress criterion will be assumed so that the maximum shear stress may not exceed $\sigma_0/2$ where σ_0 is the yield point in tension and in compression. Any such equilibrium states of stress correspond to loads which are safe or at most at the limit load.

Figs. 1 and 2 show problems which are almost but not quite equivalent, a cantilever beam under end load and a simple beam under central loading. The equations of equilibrium to be satisfied are, in the usual notation,

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0 \qquad [1]$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0 \qquad [2]$$

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If in Equation [2] σ_y is taken as identically zero, τ_{xy} is seen to be independent of x. Then from Equation [1], observing that σ_x is zero at $x = 0$

$$\sigma_x = - x \frac{d\tau_{xy}}{dy} \qquad [3]$$

The usual elastic solution with linearly varying σ_x and parabolic τ_{xy}, Fig. 3a, satisfies equilibrium and will not violate yield anywhere if the maximum bending stress does not exceed σ_0 and the shear stress at the neutral axis is no more than $\sigma_0/2$. Calling the maximum moment $M = PL$ and the shear $V = P$, the lower bound result is

$$\begin{aligned} M &\leq \sigma_0 bh^2/6 \\ V &\leq \sigma_0 bh/3 \end{aligned} \qquad [4]$$

To obtain an interaction plot, designate the known limit moment for moment alone as M_0 and the limit shear for shear alone as V_0

$$\begin{aligned} M_0 &= \sigma_0 bh^2/4 \\ V_0 &= \frac{\sigma_0}{2} bh \end{aligned} \qquad [5]$$

These values are obtained respectively by σ_0 in tension below the neutral axis and σ_0 in compression above and by a uniformly distributed shear stress $\sigma_0/2$.

The elastic solution then gives the lower bound interaction plot shown as a square on Fig. 4:

$$\frac{M}{M_0} \leq \frac{2}{3}, \quad \frac{V}{V_0} \leq \frac{2}{3}$$

Although most of the points are far too low (too close to the origin) the

2/3, 2/3 point alone could be quite useful. Two other points which are known are 0,1 and 1,0. As a yield or interaction curve must be convex (7) any line joining two lower bound points must be a lower bound. Therefore all points lying inside the two inclined dashed straight lines of Fig. 4 give permissible combinations of V and M.

The lower bound can be improved by taking a more elaborate distribution of normal and shearing stress than in Fig. 3a to satisfy the limiting maximum shear condition

$$\sigma_x^2 + 4\tau_{xy}^2 = \sigma_0^2 \qquad [6]$$

over the entire critical cross-section. Substituting the value of σ_x at $x = L$ from Equation [3] gives the differential equation:

$$\left(-L\,\frac{d\tau_{xy}}{dy}\right)^2 + 4\tau_{xy}^2 = \sigma_0^2 \qquad [7]$$

The solution for positive y is

$$\frac{2\tau_{xy}}{\sigma_0} = \sin\frac{h}{L}\left(1 - \frac{2y}{h}\right)$$

$$[8]$$

$$\frac{\sigma_x}{\sigma_0} = \cos\frac{h}{L}\left(1 - \frac{2y}{h}\right)$$

and is valid for $\frac{h}{L} \leq \frac{\pi}{2}$ as illustrated in Figs. 3b,c. For larger values of $\frac{h}{L}$ the normal stress distribution separates into two 1/4 cycle loops, as shown in Fig. 3d, and the shearing stress is constant at $\sigma_0/2$ in the central region D.

Integration of [8] leads to

$$\frac{V}{V_0} = \frac{L}{h}\left(1 - \cos\frac{h}{L}\right)$$

$$\frac{M}{M_0} = 2\left(\frac{L}{h}\right)^2\left(1 - \cos\frac{h}{L}\right) \qquad [9]$$

for $\frac{h}{L} \leq \frac{\pi}{2}$ or $\frac{V}{V_0} \leq \frac{2}{\pi}$.

For $\frac{V}{V_0} \geq \frac{2}{\pi}$

$$\frac{V}{V_0} = \frac{2}{\pi}\left[1 + \frac{D}{h}\left(\frac{\pi}{2} - 1\right)\right]$$

$$\frac{M}{M_0} = \frac{4}{\pi}\left[1 - \frac{D}{h}\right]\frac{V}{V_0} \qquad [10]$$

The composite result is plotted on Fig. 4 and should be a very good lower bound indeed because equilibrium and yield are satisfied in a reasonable manner. A modification of the linear distribution of bending stress and parabolic distribution of shear along similar lines to Figs. 3c and 3d would give a fairly good lower bound.

An implicit assumption has been made, however, that the distribution of shearing stress on the cross-section $x = 0$ can be whatever is called for by the lower bound solution. In a sense, therefore, the lower bounds for Fig. 3a and for Figs. 3b, c, d, do not apply to exactly the same problem. There is no simple way of resolving this difficulty. St. Venant's principle cannot be appealed to for short beams whether elastic or plastic and does not generally have as much meaning in the plastic range.

A Local Criterion of Limit Loading

It is customary in the derivation of the elastic moment-curvature relation for beams in bending to analyze a very short length of beam between two neighboring cross-sections. Shear, if included at all, is added by supposing constancy along the length of the beam. Free end or support conditions are satisfied in the same nominal manner as in Fig. 3. In effect, therefore, the assumption is made for general loading that each element of the beam behaves independently and exerts no restraint upon its neighbor.

The same assumption of independent action in the plastic range has much less justification as will be seen. It will, however, lead to interesting interaction relations between shear and moment. Suppose, two neighboring cross-sections are rotated and transversely displaced with respect to each other as in Fig. 5 or in some more complicated pattern as in Fig. 8. Transverse strain increments ε_y, ε_z accompanying the longitudinal strain increment ε_x are assumed unimpeded. Quite a bit of information can then be deduced about the state of stress and of strain increment at each point in the plastically deforming body.

It has been established within the framework of small displacement theory that, at the limit load, the stresses are constant and the deformation is purely plastic (6). Consider a small element of the beam which is stretched plastically with strain increment ε_x and sheared plastically with increment γ_{xy}, Fig. 6a. As in the previous section, the normal stress σ_y will be taken as zero or negligible. The Mohr's circle for stress is as shown in Fig. 6b. Assumption of the maximum shearing stress criterion of yielding then requires that all shearing be in the xy plane, $\varepsilon_z = 0$. As a consequence of the incompressibility in the plastic range of a material obeying the maximum shear rule the plastic and, therefore, total strain increments must satisfy

$$\varepsilon_x + \varepsilon_y = 0 \quad \text{or} \quad \varepsilon_y = - \varepsilon_x \qquad [11]$$

The Mohr's circle for strain increment is thus centered at the origin, Fig. 6c. As the principal directions of stress and of strain increment coincide,

$$\tan 2\theta = \frac{\gamma_{xy}}{2\varepsilon_x} = \frac{2\tau_{xy}}{\sigma_x} \qquad [12]$$

Substitution in the yield criterion [6] gives

$$\sigma_x = \frac{\sigma_0}{\sqrt{1 + \left(\frac{\gamma_{xy}}{2\varepsilon_x}\right)^2}}$$

$$\tau_{xy} = \frac{\sigma_0/2}{\sqrt{\left(\frac{2\varepsilon_x}{\gamma_{xy}}\right)^2 + 1}} \qquad [13]$$

at any point in the plastically deforming section.

The assumed deformation of Fig. 5 in analytical form is $\gamma_{xy} = \gamma$, a constant over the depth of the beam, and $\varepsilon_x = \frac{2y}{h}\varepsilon = \eta\varepsilon$ where ε is the maximum strain increment at the extreme fiber $y = h/2$ or $\eta = 1$. Integrating expressions [13] to obtain moment and shear leads to

$$M = M_0 \int_{-1}^{+1} \frac{\eta^2 d\eta}{\sqrt{\eta^2 + \left(\frac{\gamma}{2\varepsilon}\right)^2}} \quad \text{or} \quad \frac{M}{M_0} = \sqrt{1 + \left(\frac{\gamma}{2\varepsilon}\right)^2} - \left(\frac{\gamma}{2\varepsilon}\right)^2 \sinh^{-1}\frac{2\varepsilon}{\gamma}$$

$$V = V_0 \int_{-1}^{+1} \frac{d\eta/2}{\sqrt{\left(\frac{2\varepsilon}{\gamma}\right)^2 \eta^2 + 1}} \quad \text{or} \quad \frac{V}{V_0} = \frac{\gamma}{2\varepsilon}\sinh^{-1}\frac{2\varepsilon}{\gamma} \qquad [14]$$

Fig. 7 gives the interaction curve of M/M_0 vs. V/V_0.

The question which arises immediately is whether the curve represents actual limiting values, upper bounds or lower bounds. If the length Δx of the beam were indeed free at its end cross-sections to carry V and M as it liked, the result would be an upper bound. A deformation pattern was assumed and the answer is the same as would be found by following the upper bound procedure of equating the work done by M and V to the energy dissipation (6)(8). Stress or equilibrium conditions are not satisfied because τ_{xy} is not zero at $y = \pm h/2$, except for $\gamma = 0$.

A plastic deformation pattern of a quite different type can be taken as in Fig. 8 in which the outer regions of the beam stretch or contract without shearing and the inner portion D shears and changes length. An interaction curve can then be obtained for each value of D/h, Fig. 7. The lowest values will be found as γ/ε becomes indefinitely large. At this stage the inner region D is effectively under plastic shear alone and the shear stress will be $\sigma_0/2$.

$$V = \frac{\sigma_0}{2} bD$$

$$M = \sigma_0 \frac{bh^2}{4} - \sigma_0 \frac{bD^2}{4}$$

or

$$\frac{M}{M_0} = 1 - \left(\frac{V}{V_0}\right)^2 \qquad [15]$$

which is appreciably below the local criterion corresponding to Fig. 5 (see Fig. 7).

Stresses corresponding to this deformation pattern are admissible, the surfaces $y = \pm h/2$ are free of stress. Does this mean that one of the individual interaction curves or their lower limit is the true interaction curve? The answer must be no because [15] is in fact below the lower bounds, Figs. 3,4. The confusion arises because of the attempt to find a local criterion. Neither the deformation pattern of Fig. 5 nor of Fig. 8 will ordinarily be permissible because of the remaining portions of the beam. Transverse strains required by Fig. 5 will be restrained by neighboring elastic (or rigid) regions as will even more the peculiar distortions of Fig. 8. If the upper bound procedure of limit analysis is followed, it is necessary to include the energy dissipated by the mis-matching of the length Δx and the undeforming remainder of the beam, Fig. 9. Such dissipation terms are finite and independent of Δx. They predominate, therefore, as Δx approaches zero.

In the elastic regime, such mismatch or its equivalent is of second order with V constant. The curvature varies linearly along the beam and there is a gradual transition from the section of maximum moment to the section of zero moment. For elastic theory to have any validity, the length of the beam must be several times the depth so that shear strains γ_{xy} resulting from the variation of ε_y are not significant. At the limit load, on the other hand, the deformation is entirely plastic and is strongly localized. The transition between deforming and undeforming material is abrupt. When bending predominates, a small length only is plastic. If in Fig. 9a the curvature of the plastic region is assumed instead to vary smoothly along Δx from zero to a maximum and back to zero there will be no mis-match at the ends of the deforming portion. For Δx small compared with h, however, the γ_{xy} which is secondary in the elastic beam becomes primary and has large energy dissipation associated with it. The mis-match trouble for pure bending is avoided by the plastic hinge, Fig. 10, which spreads out over a distance equal to the depth of the beam so that the criterion is in reality no longer local.

In general, for both elastic and plastic members it is clear that a local criterion cannot apply in regions of rapidly changing cross-section. Roots of notches, abrupt changes in depth, and fixed ends all require more elaborate theory for accurate analysis. The influence of the complete geometry and load distribution on the limit load appears in more detail in the further analysis of cantilever and simple beams which follows.

Cantilever and Simple Rectangular Beams — Upper Bounds

If a relation had been established between V/V_0 and M/M_0, the cantilever problem of Fig. 1 would be solved. The limit load P would be determined by the value of shear P and moment PL.

$$\frac{V}{V_0} = \frac{P}{\frac{\sigma_0}{2}bh}, \qquad \frac{M}{M_0} = \frac{PL}{\sigma_0 \frac{bh^2}{4}}$$

therefore

$$\frac{V}{V_0} = \frac{h}{2L}\frac{M}{M_0} \qquad [16]$$

and the ratio $h/2L$ would give the proper point on the interaction curve. Conversely, a solution of the cantilever under end load will help to clarify the interaction relation.

Several solutions are available (2)-(5) including a fairly comprehensive treatment of upper bounds by A. P. Green (4). When the beam is very long, an ordinary plastic hinge may be assumed. Some of the confusion in the detailed analysis of results is apparent from Fig. 11a. Although the hinge is of the standard type, its center is $h/2$ from the fixed end. The support is assumed capable of applying stresses which keep an elastic triangular core adjacent to the fixed end and so strengthen the beam. When the beam is very short, Green (4) and Onat and Shield (3) propose the circular arc of sliding as in Fig. 11b. The kinematic picture proposed by Leth (5) for an I-beam is appropriate for the short rectangular beam, Fig. 11c. Again, as evidenced by the different points of view expressed in (3) (4) and (5) the length of the beam becomes questionable because of the restraint by the fixed end.

A discussion of the simple beam centrally loaded, Fig. 2, does not resolve the inherent and important problem associated with a fixed end or any abrupt change in section. It does, however, simplify the analysis of the interaction problem. In particular, if the central hinge kinematic picture is assumed, Fig. 12a, equating work done by the forces P to the energy dissipated in plastic deformation gives the upper bound result (6)

$$PL \leq M_0 \qquad \text{or} \qquad \frac{M}{M_0} \leq 1 \tag{17}$$

This answer cannot be said to be unexpected but it is not obtained for the cantilever, Fig. 11a.

Assuming a circular slip surface for very short beams, Fig. 12b, equating the work done by the external load to the internal dissipation gives as an upper bound:

$$P\Delta = \frac{\sigma_0}{2} b \frac{h}{2 \sin \psi} 2\psi \frac{\Delta}{L + \frac{h}{2} \operatorname{ctn} \psi} \frac{h}{2 \sin \psi} \tag{18}$$

The angle ψ should be chosen to minimize P because the least upper bound is desired. The result of Onat and Shield (3) is then found ($V=P$):

$$\frac{V}{V_0} = 2\psi \operatorname{ctn} \psi - 1 \tag{19}$$

For very short beams and consequently small M/M_0, series expansion leads to the upper bound on the interaction relation ($M = PL$) plotted on Fig. 7.

$$\frac{V}{V_0} = 1 - \frac{3}{8}\left(\frac{M}{M_0}\right)^2 \qquad [20]$$

The discontinuous shear upper bound picture, Fig. 12c, gives

$$P\Delta = \frac{\sigma_0}{2}\frac{\Delta}{L}\, bDL + \sigma_0 b\,\frac{(h - D)^2}{4}\frac{\Delta}{L}$$

or

$$\frac{PL}{\sigma_0 \frac{bh^2}{4}} = \frac{M}{M_0} = \frac{2L}{h}\frac{D}{h} + \left(1 - \frac{D}{h}\right)^2 \qquad [21]$$

Minimizing the upper bound by taking the derivative with respect to D/h and equating to zero

$$1 - \frac{D}{h} = \frac{L}{h} \qquad \text{or} \qquad D + L = h \qquad [22]$$

Substitution of [22] and [16] in [21] results in the upper bound interaction relation

$$\frac{M}{M_0} = 4\ \frac{V}{V_0}\left(1 - \frac{V}{V_0}\right) \qquad [23]$$

valid for $L/h \leq 1$ or $M/M_0 \leq 2(V/V_0)$ or $V/V_0 \geq 1/2$. Note that in Equation [23] first order changes in V/V_0 at $V/V_0 = 1$ correspond to first order changes in M/M_0. As shown in Fig. 7, the upper bound [23] is a much better answer than [20] for small M/M_0.

Extension of the simple beam to the left and to the right of the lines of action of the forces P as in Fig. 13 provides an excellent illustration of the non-local character of the interaction relation calculation. There will be no change in the upper bound computed from a hinge picture like Fig. 12a or a circular arc of sliding as Fig. 12b. However, comparing Fig. 12c and Fig. 13, it can easily be seen that

Equation [21] does not contain a complete expression for the dissipated energy for Fig. 13. The $(1 - D/h)^2$ term must be doubled because bending occurs at the loads P as well as at the center of the beam. Here again is the mis-match trouble discussed earlier, Fig. 9b, and the reason Fig. 8 leads to a result below the lower bounds of Fig. 3.

Returning then to a modified Equation [21] and minimizing the energy dissipated,

$$D + \frac{L}{2} = h \qquad [24]$$

and for $\frac{L}{2h} \leq 1$ or $\frac{M}{M_0} \leq 4 \frac{V}{V_0}$

$$\frac{M}{M_0} = 8 \frac{V}{V_0} \left(1 - \frac{V}{V_0}\right) \qquad [25]$$

is an upper bound on the interaction relation, Fig. 7.

Comparisons and Comments

Fig. 14 compares several of the results obtained. It should be kept in mind that if a unique interaction curve existed it would be convex (7). Although the local criterion is not necessarily either an upper or a lower bound and the lower bound corresponds to shear stress distributions somewhat different from those of the upper bounds, it seems reasonable to take

$$\frac{M}{M_0} = 0.98 \left[1 - \left(\frac{V}{V_0}\right)^4\right] \qquad [26]$$

or an expression close to [26] as a working hypothesis. Such an approximation which nearly coincides with a lower bound and is not too far from possible upper bounds would seem satisfactory for practical and theoretical

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use. Polygonal or other approximations to the curve may well be more useful in particular problems (8)(9).

As V/V_0 will rarely exceed 1/2, in most practical problems the effect of shear may be ignored completely. As the purpose of this paper is to elucidate the nature of the shear-moment interaction and not to solve problems, none will be solved. One of the main points is that the interaction is not a local affair but depends upon the loading and geometry of the entire beam. Nevertheless when all possible loadings are considered so that appreciable lengths of beam are at or close to yield, the local criterion of Fig. 5 may be close to a useful limit loading for large moment. Complete end fixity as in the cantilever of Fig. 11 or reinforcement of the central region of the simple beam will, of course, raise the limit loads still further above those discussed here in detail. All things considered, it does appear that the concept of an interaction curve has enough value to warrant the selection of an approximation such as Equation [26] or for simplicity

$$\frac{M}{M_0} = 1 - \left(\frac{V}{V_0}\right)^4 . \qquad [27]$$

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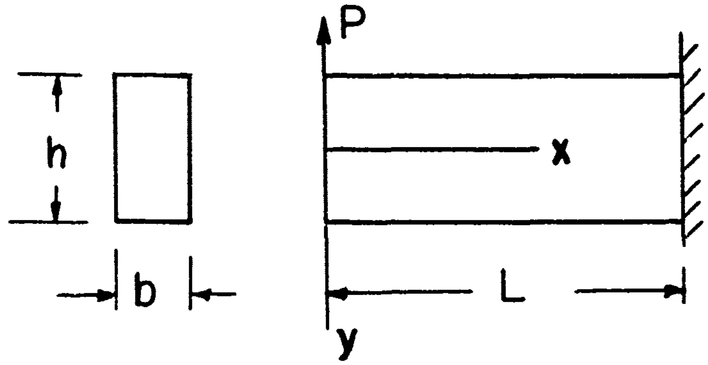

FIG. 1 CANTILEVER

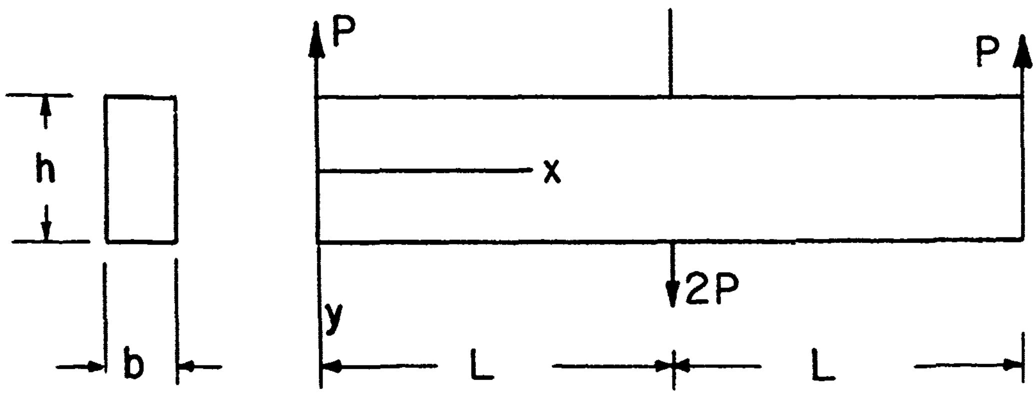

FIG. 2 SIMPLE SPAN

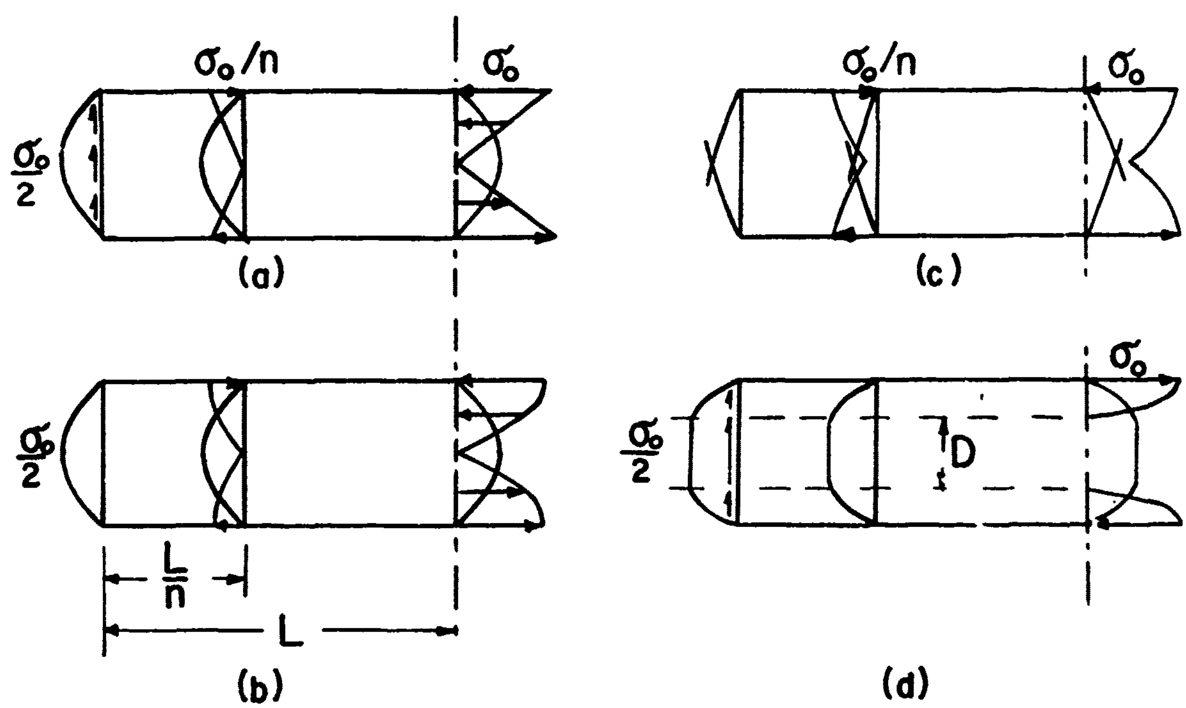

FIG. 3 EQUILIBRIUM DISTRIBUTIONS OF STRESS

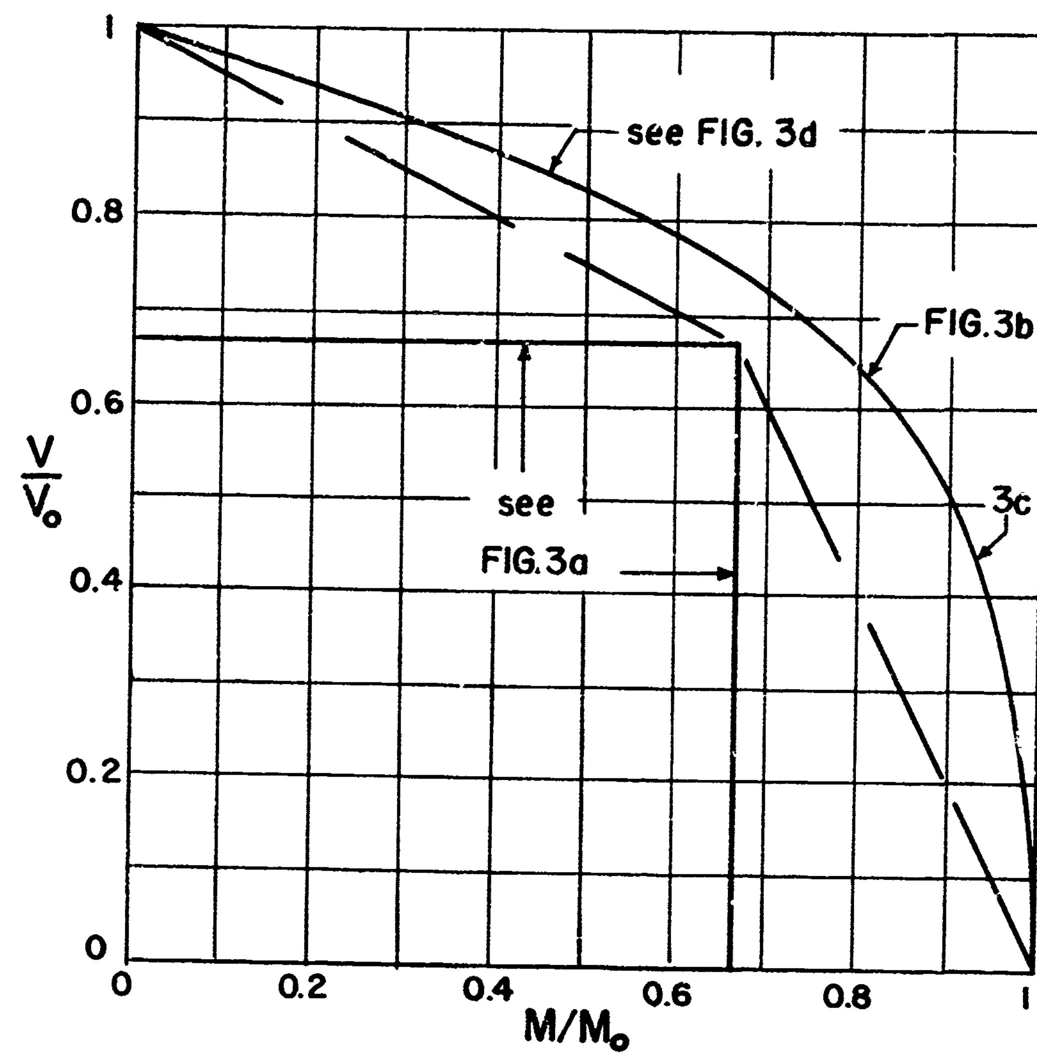

FIG. 4 LOWER BOUND INTERACTION PLOTS

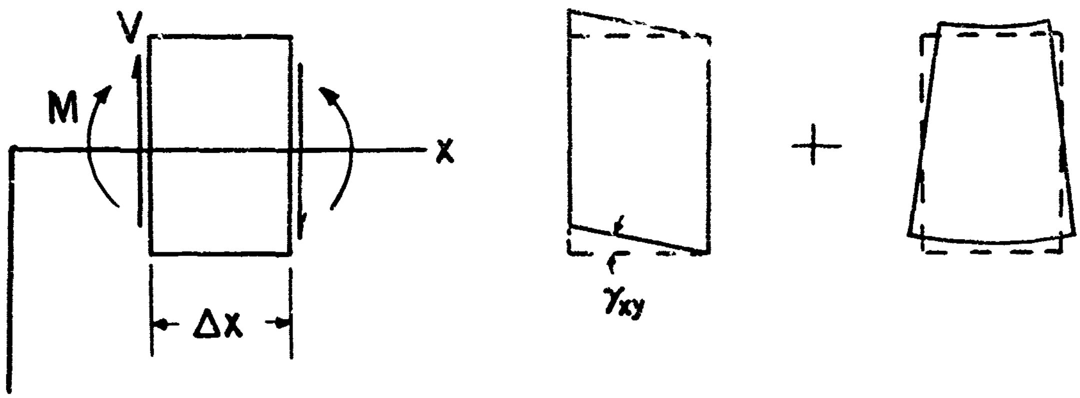

FIG. 5 SIMPLE SHEAR PLUS BENDING

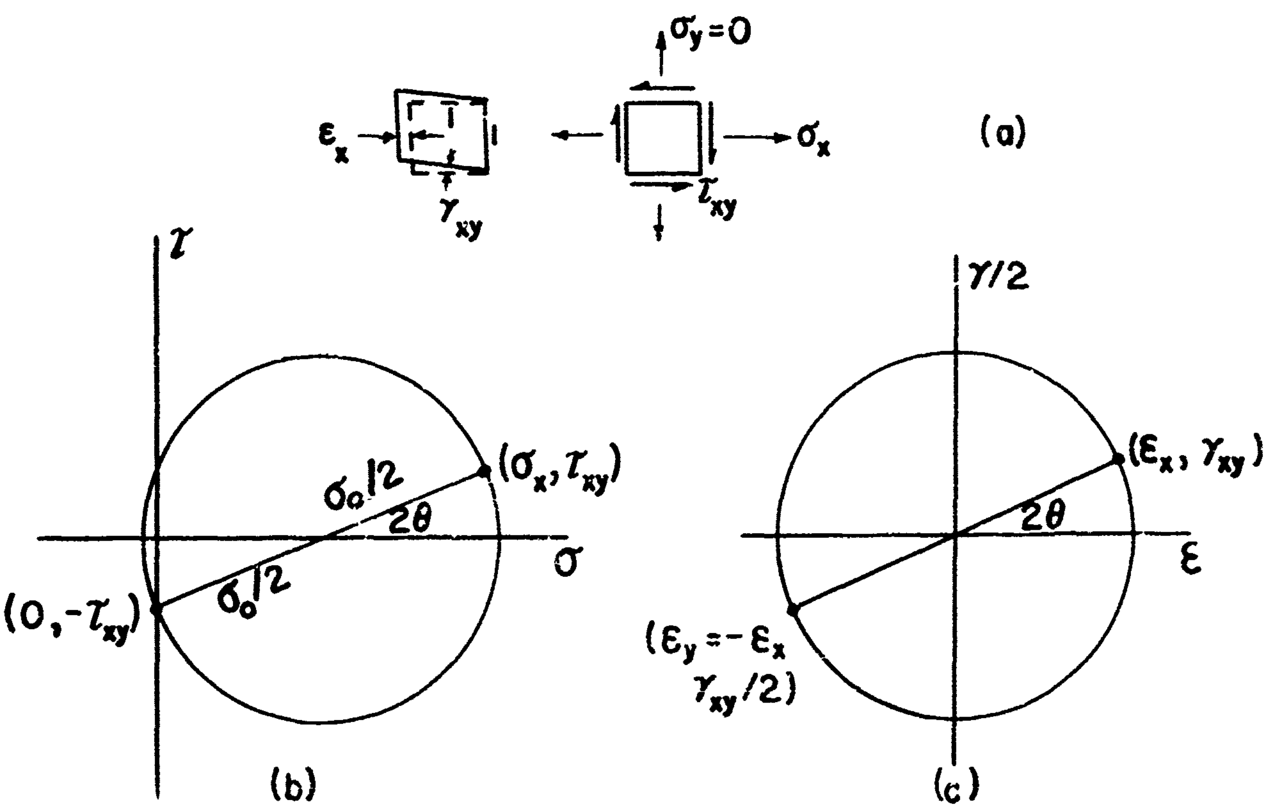

FIG. 6 STATE OF STRESS AND STRAIN INCREMENT AT EACH PLASTIC POINT IN A BENT AND SHEARED BEAM WITH $\sigma_y = 0$

(Maximum shearing stress criterion of yielding assumed)

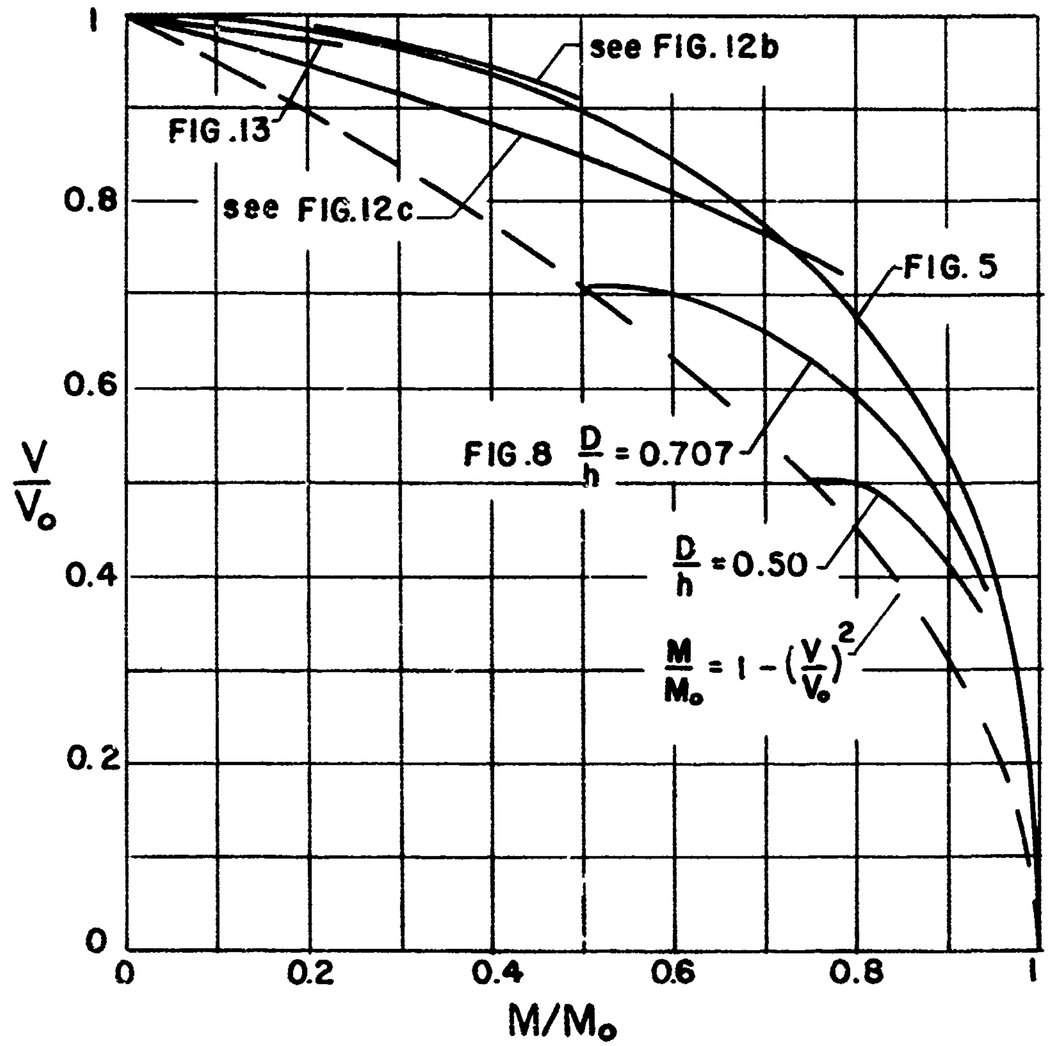

FIG. 7 INTERACTION CURVES

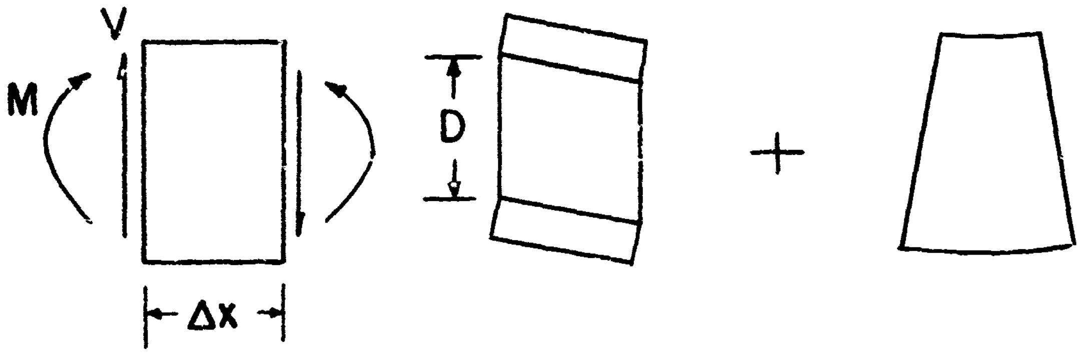

FIG. 8 DISCONTINUOUS SHEAR PLUS BENDING

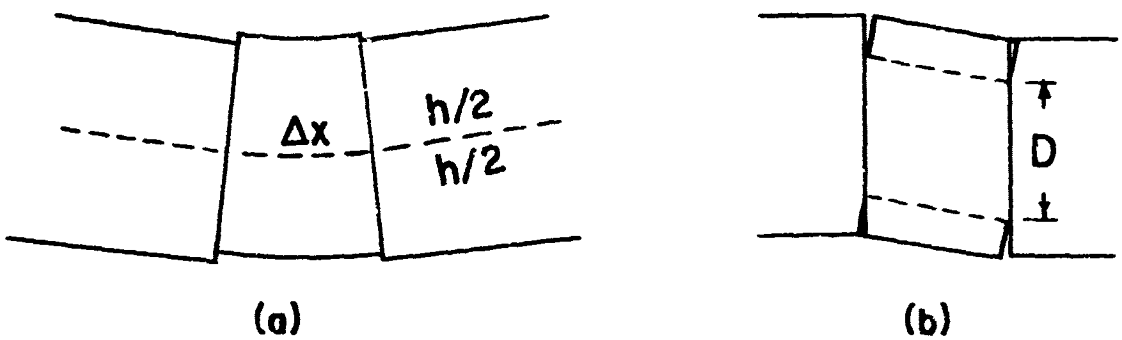

FIG. 9 MISMATCHING REQUIRES LARGE ENERGY DISSIPATION

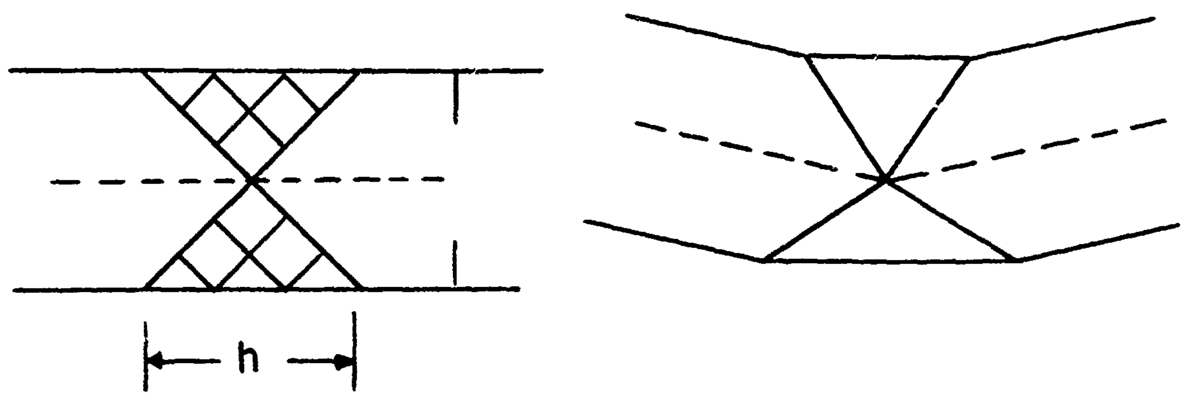

FIG. 10 PLASTIC HINGE

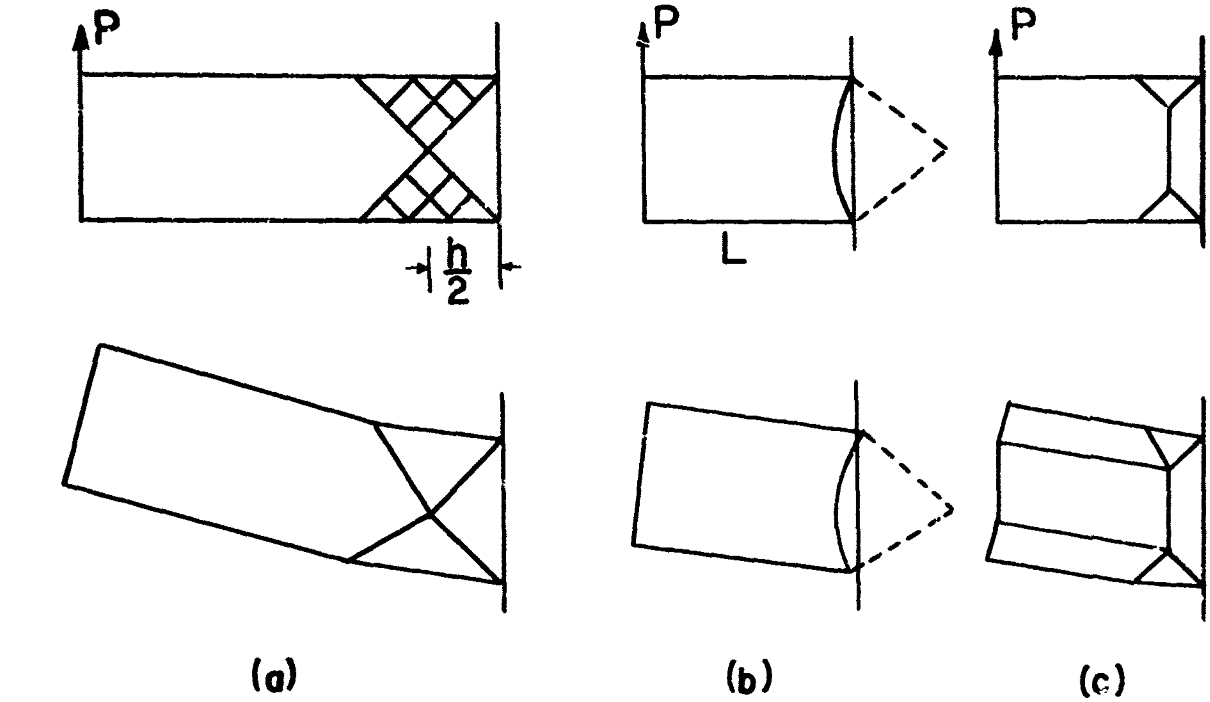

FIG. 11 CANTILEVER UNDER END LOAD – UPPER BOUND PICTURES

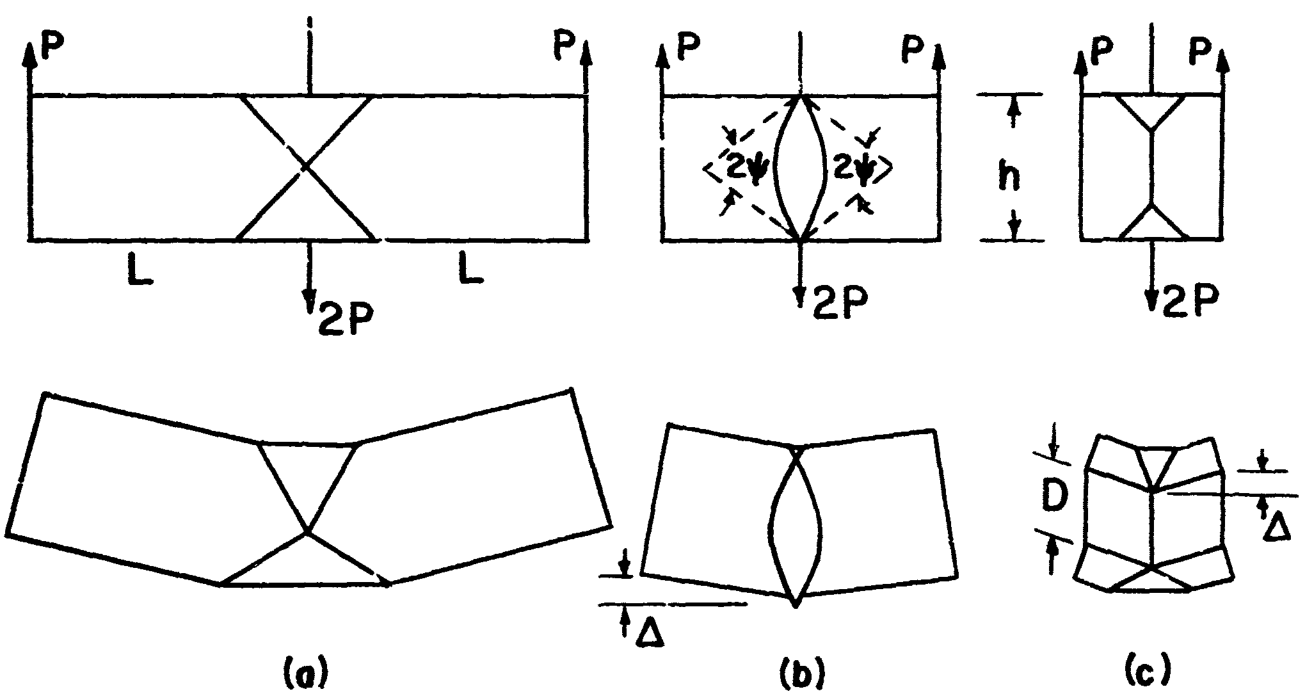

FIG. 12 SIMPLE SPAN UPPER BOUND PICTURES

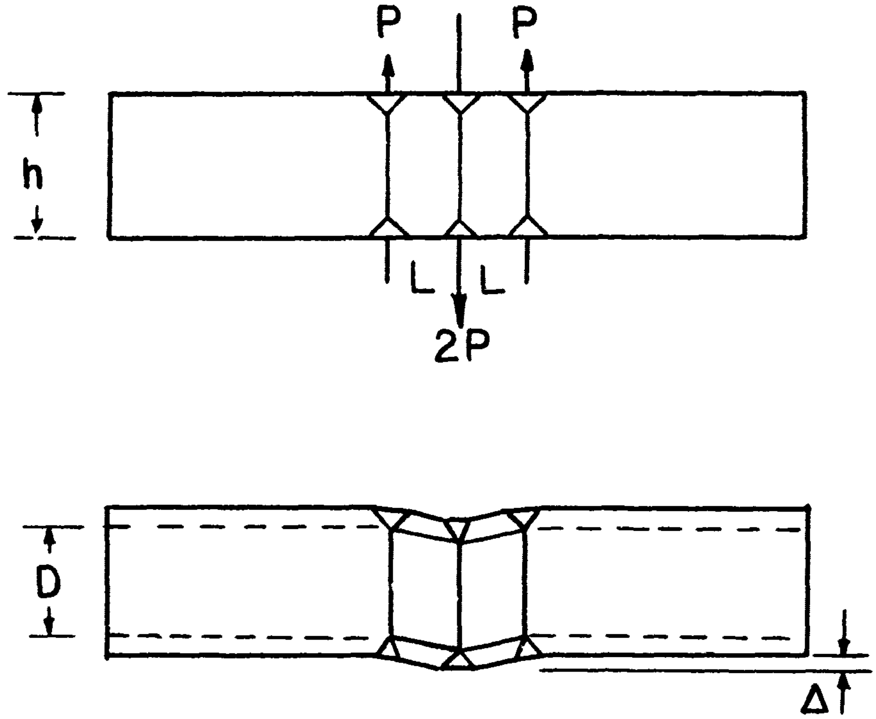

FIG. 13 FURTHER EVIDENCE OF NON-LOCAL EFFECTS

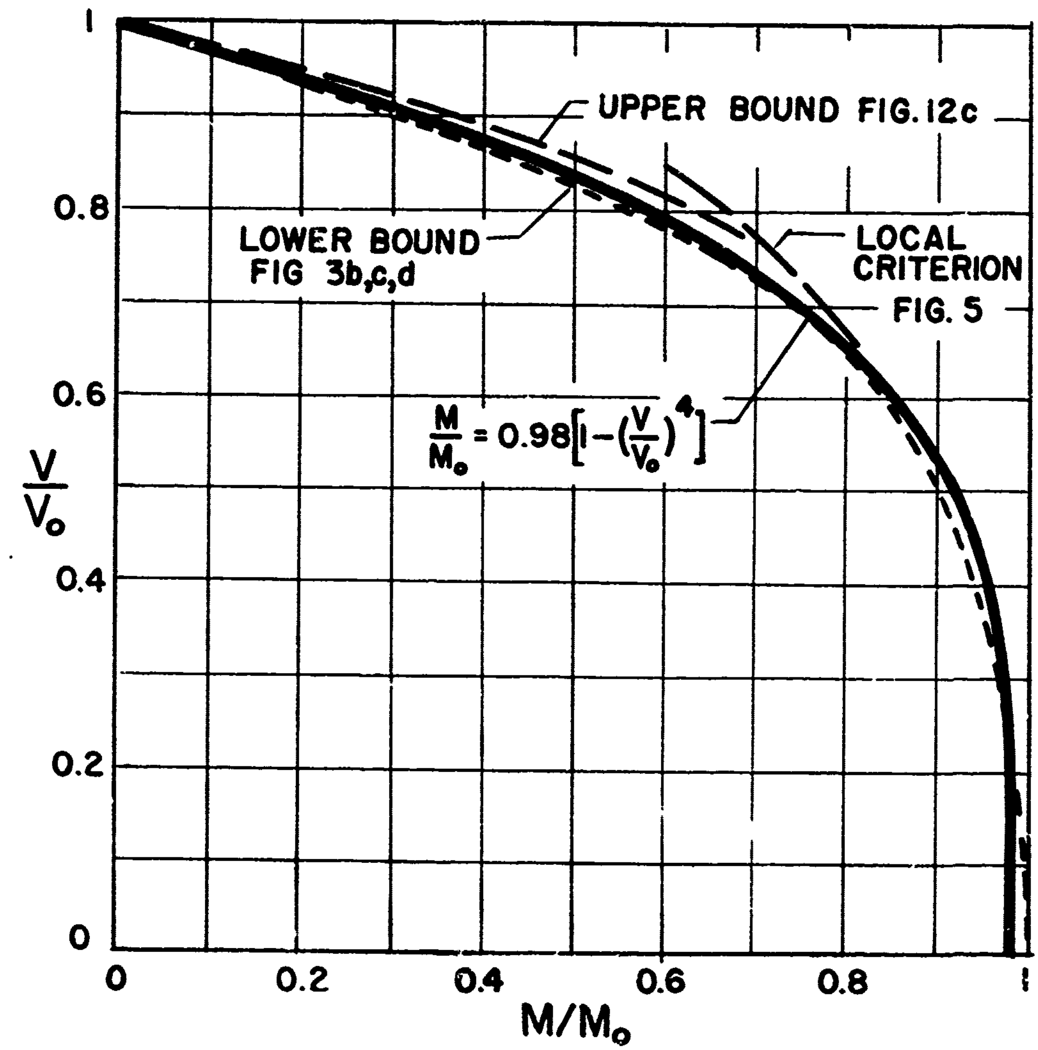

FIG. 14 PROPOSED INTERACTION CURVE AND COMPARISON OF RESULTS

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