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Large Elastic-Plastic Deformations of Built-In Circular Plates under Uniform Load,

Part I—Theoretical Analysis

by

H. G. Hopkins

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Large Elastic-Plastic Deformations of Built-In Circular Plates under Uniform Load[1]

Part I - Theoretical Analysis

by H. G. Hopkins[2]

Abstract

Considerable effort has been given to the study of the behavior of built-in circular plates under uniform load. This problem is not only of great practical interest but, because of its relative simplicity, it is also of great significance in structural theory. The information at present available in the plastic range is not extensive, and further information would be of considerable use in design. This fact, together with the opportunity afforded by digital computing machines for handling extensive numerical work, has stimulated the present study.

The purpose of this paper is to give a treatment of this problem which is more general than has hitherto been attempted. Briefly the essential task is to modify the analysis given by von Kármán for large elastic deformations to include plastic deformations. A flow theory for non-hardening Tresca material is used. The analysis leading to the fundamental equations is given here. Details of the method of computation, the numerical results, and comparison with experimental results will be given in Part II of this paper.

1. Introduction

The behavior of a built-in circular plate under uniform load is an important technical problem with several applications. It is also of very considerable significance in the theory of structures because it is one of the simplest problems, involving a two-dimensional structural element, that may be formulated. It is natural therefore that this problem should possess a long

1. The results presented in this paper were obtained in the course of research sponsored by Watertown Arsenal Laboratories, under Contract DA-19-020-ORD-2598.

2. Visiting Professor of Applied Mathematics, Brown University, Providence, R. I.

history, and great efforts have been made to analyse various of its aspects with a view to providing design data. The literature on the theory of plates is very extensive. In the following summary attention will be confined to the contributions of most significance in relation to the present study. Results applicable within the elastic range have been described admirably by Timoshenko [1]*.

A theoretical analysis of this problem leading to a simple solution is possible only if very restrictive assumptions are made. This is the situation in the Kirchhoff bending theory of thin plates which involves the following assumptions: linear elements normal to the undeformed middle surface remain normal to the deformed middle surface; the transverse displacements are sufficiently small for extension of the middle surface to be neglected and simplified curvature formulas to be applied; Hooke's law is obeyed; and the stresses normal to the middle surface are, on the average, much smaller than the stresses parallel to this surface. The solution of the problem subject to these assumptions is due to Poisson [2] (see also [3]). Now let the first of Kirchhoff's assumptions be replaced by the less restrictive one that linear elements normal to the undeformed middle surface merely remain linear. Thus account is now taken of transverse shearing deformations. A simple closed expression is still found for the transverse displacement: this expression is in fact the exact (three-dimensional) one first given by

* Numbers in square brackets refer to the bibliography at the end of the paper.

Love [3]. The results show that neglect of transverse shearing deformation introduces an error in the displacement of order h^2/a^2 where h and a are the plate thickness and radius, respectively. This error is often quite small in practical applications.

In general if the assumptions of the Kirchhoff theory are not made, then treatment of the problem is more complicated and numerical methods must be applied directly to the basic equations. This is the situation in the analyses due to Foeppl [4] and von Kármán [5] in which the displacements are not sufficiently small for the extension of the middle surface to be neglected although simplified curvature formulas may still be used. The theories both lead to two non-linear simultaneous partial differential equations, but differ in that the former does not, whereas the latter does, take into account bending action in addition to membrane action. In Foeppl's theory the only boundary conditions that may be imposed are those directly related to edge displacement. There are no bending stresses, and accordingly edge rotation must be permitted. Further, Foeppl's theory may be expected on general grounds to approximate closely von Kármán's theory when the displacements are relatively large. Away from the edge membrane action will predominate over bending action. In the immediate neighborhood of the built-in edge the reverse situation applies, and there must be a narrow boundary layer region across which the transition occurs. The first detailed numerical treatment of the Foeppl equations is due to Hencky [6], and that of the von Kármán equations to Way [7]. It should be noted particularly that the

Kirchhoff theory is linear whereas the von Kármán (and of course the Foeppl) theory is non-linear. For example the latter theory predicts that the displacements are ultimately proportional to $p^{1/3}$ where p is the uniform intensity of applied load. It is also known that the neglect of membrane action in the Kirchhoff theory introduces errors of order w^2/h^2 where w is (say) the central displacement, and accordingly this theory is certainly inadequate if it predicts $w/h > \frac{1}{2}$.

The above summary shows that analysis of the problem within the elastic range is highly satisfactory. It is true of course that the von Kármán theory has not yet been extended to take account of transverse shearing deformations, but this modification is not of great practical importance.

In contrast the analysis of the problem within the plastic range is very much less satisfactory. So far as the present writer is aware previously published work is confined either to bending or membrane action, and even here is by no means complete. Trifan [8] has given an elastic-plastic bending theory based upon an empirical approximation to an experimental stress-strain curve; the predictions of plastic flow and deformation theories were compared. More recently analyses have been made in which the elastic deformations are completely neglected. Thus Hopkins and Prager [9] have determined the transverse load required to initiate the bending deformation of a plastic-rigid plate whose material obeys Tresca's yield condition and associated flow rule. This analysis has been extended to other yield conditions by Hopkins and Wang [10]. In contrast Hill [11] has treated the membrane deformation of a plastic-rigid plate whose material obeys von Mises yield condition and associated flow rule; linear strain-hardening is included and no restriction is made on the magnitude of the displacements. This problem has also been discussed by Ross and Prager [12] for the case of the Tresca yield condition.

The information at present available in the plastic range is not therefore very extensive, and further information would be of considerable use in design. This fact, together with the opportunity provided by digital computing machines for handling extensive numerical work, has stimulated the present study.

The purpose of this paper is to give an analysis of this problem more general than has hitherto been attempted, and thereby to provide design data. The material, initially both homogeneous and isotropic, is taken to be non-hardening. The theory, which is of the flow (or incremental) type, involves these assumptions:

a) linear elements normal to the undeformed middle surface remain normal to the deformed middle surface;
b) the displacements are not so large that approximate curvature formulas cannot be used or that strains cannot be measured with respect to undeformed elements;
c) elastic strain-rates obey Hooke's law, and plastic strain-rates are given by the von Mises condition;
d) the yield function is that of Tresca, and also coincides with the plastic potential; and
e) the stresses normal to the middle surface are, on the

average, much smaller than the stresses parallel to this surface.

It is difficult to justify all of these assumptions, and the applicability of the numerical results based upon the present theory must perforce be judged directly through their comparison with experimental results. Experimental work in progress at Brown University will provide a basis for such a comparison. Here only brief comment can be made on the assumptions. Experimental work due to Shanley [13] suggests that assumption a) introduces little error if h^2/a^2 is sufficiently small. The reason for the choice of a flow theory is well-known, and, as already mentioned, basing this theory on the Tresca yield condition is a compromise. Assumptions b) and e) are conventionally made in thin plate theory.

The analysis leading to the fundamental equations is given here. Details of the method of computation, the numerical results, and comparison with experimental results will be given in Part II of this paper.

2. Notation

The notation used is now described, and for convenience the plate is taken to be horizontal. Let:

a = plate radius;

h = plate thickness;

p = uniform load intensity;

z = distance (positive below and) normal to middle surface;

r = distance from axis of symmetry;

w = displacement (positive vertically downwards) of points in middle surface;

u = outwards radial displacement of points in middle surface;

E = Young's modulus;

ν = Poisson's ratio;

D = $Eh^3/12\,(1 - \nu^2)$ = flexural rigidity;

σ_o = tensile yield stress of material;

$\sigma_r, \sigma_t, \sigma_z$ = radial, circumferential and transverse stresses;

$\varepsilon_r', \varepsilon_r'^e, \varepsilon_r'^p$; $\varepsilon_t', \varepsilon_t'^e, \varepsilon_t'^p$; $\varepsilon_z', \varepsilon_z'^e, \varepsilon_z'^p$ = total, elastic and plastic strain-rates in the radial, circumferential and transverse directions;

M_r, M_t = radial and circumferential bending moments;

N_r, N_t = radial and circumferential membrane forces; and

Q_r = radial transverse shear force.

The basic notation is that used by Timoshenko [1]; see also Fig. 1. In the plastic range flow theory necessitates the use of rates-of-change of certain quantities denoted here by primes, e.g. ε_r' etc. The material does not exhibit viscosity effects and accordingly any monotonic increasing quantity associated with the progressive deformation of the plate, e.g. p, is suitable for use as a "time" t. Additional notation is defined later when first introduced.

3. Analysis

The problem is one of rotational symmetry, and accordingly all quantities are functions at most of r, z and t. The essential task is to modify the analysis of von Kármán [5] through the introduction of stress-rate v. strain-rate equations that are valid for plastic deformations.

1) Equilibrium equations

The present theory is a two-dimensional one in that the overall equations of equilibrium are expressed in terms of stress resultants. The form of these equations (see [1]) is of course independent of the mechanical behavior of the plate:

$$\frac{\partial}{\partial r}(rN_r) - N_t = 0, \tag{1}$$

$$\frac{\partial}{\partial r}\left(rQ_r + r\,\frac{\partial w}{\partial r}\,N_r\right) + pr = 0, \tag{2}$$

$$\frac{\partial}{\partial r}(rM_r) - M_t - rQ_r = 0. \tag{3}$$

It is straightforward to eliminate Q_r between Eqs. (2) and (3):

$$\frac{\partial}{\partial r}(rM_r) - M_t + r\,\frac{\partial w}{\partial r}\,N_r + \frac{1}{2}\,pr^2 = 0. \tag{4}$$

Note that in the general case when p is not constant the last term in Eq. (4) is $\int_0^r p(r)r dr$. Equations (1) and (4) are the fundamental statical equations. These equations may be differentiated with respect to t to give

$$\frac{\partial}{\partial r}(rN_r') - N_t' = 0, \tag{5}$$

$$\frac{\partial}{\partial r}(rM_r') - M_t' + r\left(\frac{\partial w'}{\partial r}\,N_r + \frac{\partial w}{\partial r}\,N_r'\right) + \frac{1}{2}\,p'r^2 = 0. \tag{6}$$

The membrane forces and bending moments are defined by the equations

$$(N_r, N_t) = \int_{-\frac{1}{2}h}^{\frac{1}{2}h} (\sigma_r, \sigma_t) dz, \quad (M_r, M_t) = \int_{-\frac{1}{2}h}^{\frac{1}{2}h} z(\sigma_r, \sigma_t) dz. \quad (7)$$

These equations may also be differentiated with respect to t to give

$$(N_r', N_t') = \int_{-\frac{1}{2}h}^{\frac{1}{2}h} (\sigma_r', \sigma_t') dz, \quad (M_r', M_t') = \int_{-\frac{1}{2}h}^{\frac{1}{2}h} z(\sigma_r', \sigma_t') dz. \quad (8)$$

2) <u>Stress-rate v. strain-rate relationships</u>

Assumptions a) and b) of our theory lead to the kinematical relations (see [1])

$$\varepsilon_r = \frac{\partial u}{\partial r} + \frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2 - z\frac{\partial^2 w}{\partial r^2}, \quad \varepsilon_t = \frac{u}{r} - \frac{z}{r}\frac{\partial w}{\partial r}, \quad (9)$$

and hence

$$\varepsilon_r' = \frac{\partial u'}{\partial r} + \frac{\partial w}{\partial r}\frac{\partial w'}{\partial r} - z\frac{\partial^2 w'}{\partial r^2}, \quad \varepsilon_t' = \frac{u'}{r} - \frac{z}{r}\frac{\partial w'}{\partial r}. \quad (10)$$

Assumptions c) and e) lead to

$$E\varepsilon_r'^e = \sigma_r' - \nu\sigma_t', \quad E\varepsilon_t'^e = \sigma_t' - \nu\sigma_r'. \quad (11)$$

The plastic strain-rates $\varepsilon_r'^p$, $\varepsilon_t'^p$ are defined by the equations

$$\varepsilon_r'^p = \varepsilon_r' - \varepsilon_r'^e, \quad \varepsilon_t'^p = \varepsilon_t' - \varepsilon_t'^e. \quad (12)$$

According to assumptions c) and d) the ratios and signs, but not the absolute magnitudes, of the plastic strain-rates are given from the following condition. The plastic strain-rate vector corresponding to any plastic stress state is directed in the sense of the outwards-drawn normal to the yield locus at this stress state. This condition was proposed originally by von Mises [14] and was later extended by Prager [15] (see also Refs. [16] and [17]). The present approach is due to Prager. The foregoing statement pre-supposes that there is a well-defined normal, and otherwise some degree of arbitrariness within the obvious extremes is permissible for the direction of this vector (see [18] and [19]). In general the yield locus is a surface in a three-dimensional space in which σ_r, σ_t and σ_z are taken as rectangular cartesian co-ordinates, and the plastic strain-rate vector has components $\varepsilon_r'^p$, $\varepsilon_t'^p$, $\varepsilon_z'^p$. Now according to assumption e) $\sigma_z \ll \sigma_r, \sigma_t$, and hence the relevant part of the yield locus lies in the immediate neighborhood of its intersection with the plane $\sigma_z = 0$. Here Tresca's yield condition is assumed, and hence yielding is possible only if the absolute value of the shear stress acting parallel to an arbitrarily oriented plane is equal to the yield stress in shear ($\frac{1}{2}\sigma_o$). In the present application the yield locus is therefore adequately represented by the plane curve given by

$$\max. \left| \tfrac{1}{2}\sigma_r, \tfrac{1}{2}\sigma_t, \tfrac{1}{2}(\sigma_r - \sigma_t) \right| = \tfrac{1}{2}\sigma_o. \qquad (13)$$

This yield criterion is represented by the hexagon ABCDEF drawn in the σ_r,σ_t-plane shown in Fig. 2. The full yield criterion is a cylinder based on this hexagon with its axis equally inclined to the co-ordinate axes. The plastic strain-rate vector $\varepsilon_r'^p$, $\varepsilon_t'^p$, $\varepsilon_z'^p$ does not in general lie in the plane $\sigma_z = 0$. However the normality of the plastic strain-rate vector $\varepsilon_r'^p$, $\varepsilon_t'^p$, $\varepsilon_z'^p$ to the yield surface implies the normality of the vector

$\varepsilon_r^{\prime p}$, $\varepsilon_t^{\prime p}$, 0 to the yield curve. Thus the ratio $\varepsilon_r^{\prime p} : \varepsilon_t^{\prime p}$ is easily found. The strain-rate $\varepsilon_z^{\prime}$ is not essential to our discussion but may be found as follows. First from Hooke's law

$$E\varepsilon_z^{\prime e} = -\nu(\sigma_r^{\prime} + \sigma_t^{\prime}); \tag{14}$$

and second from the condition of plastic strain-rate incompressibility

$$\varepsilon_z^{\prime p} = -(\varepsilon_r^{\prime p} + \varepsilon_t^{\prime p}), \tag{15}$$

and hence all the ratios $\varepsilon_r^{\prime p} : \varepsilon_t^{\prime p} : \varepsilon_z^{\prime p}$ may be found. The simplicity of the relations governing the plastic strain-rates is at once evident. Let

$$f = f(\sigma_r, \sigma_t; \sigma_o) \tag{16}$$

denote the (simplified) yield function, the sign of f being so chosen that $f < 0$ inside the hexagon and $f > 0$ outside the hexagon. Note that only the region $f \leq 0$ is of physical significance. The yield function f is always linear in σ_r and σ_t, i.e. $f = \alpha\sigma_r + \beta\sigma_t + \text{const.}$, and it follows that

$$\varepsilon_r^{\prime p} = \alpha\lambda^{\prime}, \quad \varepsilon_t^{\prime p} = \beta\lambda^{\prime} \quad \text{if} \quad f = f^{\prime} = 0 \text{ where } \lambda^{\prime} \geq 0. \tag{17}$$

The quantities α and β for the non-singular regimes corresponding to the sides of the hexagon are exhibited in the following table, Relations(18):

Regime	AB	BC	CD	DE	EF	FA
f	$+\sigma_t-\sigma_o$	$+\sigma_t-\sigma_r-\sigma_o$	$-\sigma_r-\sigma_o$	$-\sigma_t-\sigma_o$	$-\sigma_t+\sigma_r-\sigma_o$	$+\sigma_r-\sigma_o$
α	0	-1	-1	0	+1	+1
β	+1	+1	0	-1	-1	0

Permissible values of α and β for the singular regimes corresponding to the vertices of the hexagon are easily written down, e.g.

$$\text{Regime A: } \sigma_r = \sigma_t = \sigma_o; \ \alpha = 1 - \Theta, \ \beta = \Theta \quad \text{where} \quad 0 < \Theta < 1 \tag{19}$$

so that Θ increasing from 0 to 1 corresponds to a continuous transition between the common terminal point of the regimes FA and AB, etc. The quantity $\lambda' = \lambda'(r,z;t)$ determines the magnitude of the plastic strain-rates and its determination is discussed later.

The relevant stress-rate v. strain-rate relations may now be expressed in the form

$$E(\varepsilon_r' - \alpha\lambda') = \sigma_r' - \nu\sigma_t', \quad E(\varepsilon_t' - \beta\lambda') = \sigma_t' - \nu\sigma_r', \tag{20}$$

where α and β are now further defined to be zero whenever an element undergoes a purely elastic change in strain, i.e.

$$\alpha = \beta = 0 \text{ if either (i) } f < 0, \ f' \gtrless 0 \text{ or (ii) } f = 0, \ f' < 0. \tag{21}$$

Equations (20) are equivalent to

$$\left.\begin{aligned} \sigma_r' &= \frac{E}{(1-\nu^2)}\left\{\varepsilon_r' + \nu\varepsilon_t' - (\alpha + \beta\nu)\lambda'\right\}, \\ \sigma_t' &= \frac{E}{(1-\nu^2)}\left\{\varepsilon_t' + \nu\varepsilon_r' - (\beta + \alpha\nu)\lambda'\right\}. \end{aligned}\right\} \tag{22}$$

Next from Eqs. (10) and (22),

$$
\left.
\begin{aligned}
\sigma_r' &= \frac{E}{(1-\nu^2)}\left\{\frac{\partial u'}{\partial r} + \frac{\partial w}{\partial r}\frac{\partial w'}{\partial r} - z\frac{\partial^2 w'}{\partial r^2} + \nu\left(u' - z\frac{\partial w'}{\partial r}\right)/r - (\alpha+\beta\nu)\lambda'\right\}, \\
\sigma_t' &= \frac{E}{(1-\nu^2)}\left\{\left(u' - z\frac{\partial w'}{\partial r}\right)/r + \nu\left(\frac{\partial u'}{\partial r} + \frac{\partial w}{\partial r}\frac{\partial w'}{\partial r} - z\frac{\partial^2 w'}{\partial r^2}\right) - (\beta+\alpha\nu)\lambda'\right\}.
\end{aligned}
\right\} \tag{23}
$$

Note that α and β are generally functions of r, z and t. It now follows from Eqs. (7), (8) and (23) that

$$
\left.
\begin{aligned}
N_r' &= \frac{Eh}{(1-\nu^2)}\left\{\frac{\partial u'}{\partial r} + \frac{\partial w}{\partial r}\frac{\partial w'}{\partial r} + \nu\frac{u'}{r} - \frac{1}{h}\int_{-\frac{1}{2}h}^{\frac{1}{2}h}(\alpha+\beta\nu)\lambda' dz\right\}, \\
N_t' &= \frac{Eh}{(1-\nu^2)}\left\{\frac{u'}{r} + \nu\left(\frac{\partial u'}{\partial r} + \frac{\partial w}{\partial r}\frac{\partial w'}{\partial r}\right) - \frac{1}{h}\int_{-\frac{1}{2}h}^{\frac{1}{2}h}(\beta+\alpha\nu)\lambda' dz\right\},
\end{aligned}
\right\} \tag{24}
$$

and

$$
\left.
\begin{aligned}
M_r' &= -D\left\{\frac{\partial^2 w'}{\partial r^2} + \frac{\nu}{r}\frac{\partial w'}{\partial r} + \frac{12}{h^3}\int_{-\frac{1}{2}h}^{\frac{1}{2}h}(\alpha+\beta\nu)\lambda' z dz\right\}, \\
M_t' &= -D\left\{\frac{1}{r}\frac{\partial w'}{\partial r} + \nu\frac{\partial^2 w'}{\partial r^2} + \frac{12}{h^3}\int_{-\frac{1}{2}h}^{\frac{1}{2}h}(\beta+\alpha\nu)\lambda' z dz\right\}.
\end{aligned}
\right\} \tag{25}
$$

Let us now return to the determination of λ'. Throughout the plastic region $f' = 0$, i.e.

$$\alpha\sigma_r' + \beta\sigma_t' = 0 \tag{26}$$

and hence from Eqs. (23) it follows that λ' is given from

$$(\alpha^2 + \beta^2 + 2\alpha\beta\nu)\lambda' = (\alpha + \beta\nu)\left(\frac{\partial u'}{\partial r} + \frac{\partial w}{\partial r}\frac{\partial w'}{\partial r} - z\,\frac{\partial^2 w'}{\partial r^2}\right) + (\beta + \alpha\nu)\left(u' - z\,\frac{\partial w'}{\partial r}\right)/r. \tag{27}$$

The indefinite integrals $\int^z (1,z)\lambda' dz$ are easily written down:

$$\left.\begin{aligned}(\alpha^2 + \beta^2 + 2\alpha\beta\nu)\int^z \lambda' dz &= (\alpha + \beta\nu)\left\{\left(\frac{\partial u'}{\partial r} + \frac{\partial w}{\partial r}\frac{\partial w'}{\partial r}\right)z - \frac{\partial^2 w'}{\partial r^2}\frac{z^2}{2}\right\} + (\beta + \alpha\nu)\left(u'z - \frac{\partial w'}{\partial r}\frac{z^2}{2}\right)/r,\\ (\alpha^2 + \beta^2 + 2\alpha\beta\nu)\int^z \lambda' z dz &= (\alpha + \beta\nu)\left\{\left(\frac{\partial u'}{\partial r} + \frac{\partial w}{\partial r}\frac{\partial w'}{\partial r}\right)\frac{z^2}{2} - \frac{\partial^2 w'}{\partial r^2}\frac{z^3}{3}\right\} + (\beta + \alpha\nu)\left(u'\frac{z^2}{2} - \frac{\partial w'}{\partial r}\frac{z^3}{3}\right)/r.\end{aligned}\right\} \tag{28}$$

If it is assumed that α and β are, for fixed r and t, piecewise constant in z then the integrals occurring in Eqs. (24) and (25) are formally given through use of Eqs. (28). These integrals involve the position of the elastic-plastic boundary $z = z_{p\ell}(r;t)$ as an unknown, and this boundary must be found from the condition that the elastic material immediately adjacent to the plastic-elastic boundary is just on the point of yielding. Thus Eq. (26) also applies at the boundary of the elastic region. In addition the same plastic regime, for fixed r and t, may not apply throughout a plastic region; in this case the position of the boundary between adjacent plastic regimes occurs as an additional unknown but its determination is probably straightforward. The foregoing discussion rules out the possibility of the occurrence of singular plastic regimes. In this case λ' is not piecewise constant (see Eq. (19)), and the integrals in

Eqs. (24) and (25) cannot be formally evaluated as before. These remarks indicate the difficulties due to the presence of moving elastic-plastic and plastic-plastic interfaces, and their precise extent can only be known through the details of the numerical treatment of the problem.

Note that, in general, it is not true that σ_r' and σ_t' are continuous across elastic-plastic and plastic-plastic interfaces.

3). The fundamental equations

The fundamental equations for the problem are now immediately deduced through the elimination of the rates of change of the stress resultants and stress couples between the statical equations (5) and (6) and Eqs. (24) and (25):

$$\left.\begin{aligned}
&\frac{\partial}{\partial r}\left(r\frac{\partial u'}{\partial r}\right) - \frac{u'}{r} + \left\{(1-\nu) + r\frac{\partial}{\partial r}\right\}\frac{\partial w}{\partial r}\frac{\partial w'}{\partial r} + \varphi = 0,\\
&\varphi = -\frac{\partial}{\partial r}\left\{\frac{r}{h}\int_{-\frac{1}{2}h}^{\frac{1}{2}h}(\alpha+\beta\nu)\lambda' dz\right\} + \frac{1}{h}\int_{-\frac{1}{2}h}^{\frac{1}{2}h}(\beta+\alpha\nu)\lambda' dz,\\
&D\frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w'}{\partial r}\right)\right\} - N_r\frac{\partial w'}{\partial r} - \frac{Eh}{(1-\nu^2)}\frac{\partial w}{\partial r}\left\{\frac{\partial u'}{\partial r} + \frac{\partial w}{\partial r}\frac{\partial w'}{\partial r}\right.\\
&\qquad \left. + \nu\frac{u'}{r} - \frac{1}{h}\int_{-\frac{1}{2}h}^{\frac{1}{2}h}(\alpha+\beta\nu)\lambda' dz\right\} - \frac{1}{2}p'r + \frac{E}{(1-\nu^2)}\psi = 0,\\
&r\psi = \frac{\partial}{\partial r}\left\{r\int_{-\frac{1}{2}h}^{\frac{1}{2}h}(\alpha+\beta\nu)\lambda' z\,dz\right\} - \int_{-\frac{1}{2}h}^{\frac{1}{2}h}(\beta+\alpha\nu)\lambda' z\,dz.
\end{aligned}\right\} \quad (29)$$

Equations (29) are two non-linear simultaneous differential equations for the determination of the deformation of the plate. Integration of these equations must proceed in the following way. Let the plate be in a known state at time t. Then w, u, N_r etc. are all known as functions of r and t, and the position of the elastic-plastic boundary is known as a function of r, z and t. Now let t increase by a small amount Δt. Then the equations determine the resulting small changes Δw and Δu. A difficulty is that the resulting small change in position of the elastic-plastic boundary is not known beforehand. The movement of this boundary is to be found from the requirement that f is continuous across the boundary. The state of the plate at time $t + \Delta t$ is now known. This procedure is repeated step-by-step up to any level of applied load.

In the present problem the boundary conditions are

$$w' = \frac{\partial w'}{\partial r} = u' = 0 \quad \text{at} \quad r = a \quad \text{for} \quad t \geq 0. \tag{30}$$

So long as an element of the plate undergoes purely elastic changes in strain Eqs. (29) may be replaced by the von Kármán equations (see [1]),

$$\left.\begin{aligned} &\frac{d}{dr}\left(r\,\frac{du}{dr}\right) - \frac{u}{r} + \left\{(1-\nu) + r\,\frac{d}{dr}\right\}\frac{1}{2}\left(\frac{dw}{dr}\right)^2 = 0, \\ &D\,\frac{d}{dr}\left\{\frac{1}{r}\,\frac{d}{dr}\left(r\,\frac{dw}{dr}\right)\right\} - \frac{Eh}{(1-\nu^2)}\,\frac{dw}{dr}\left\{\frac{du}{dr} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2 + \nu\,\frac{u}{r}\right\} - \frac{1}{2}\,pr = 0. \end{aligned}\right\} \tag{31}$$

4). <u>The development of plasticity</u>

It is obvious that the major difficulty in the integration of Eqs. (26) is due to the presence of moving elastic-plastic boundaries. The computational task is eased by the fact that it is possible to make some qualitative remarks about the development of plasticity in the plate.

Attention is confined here to the case of a moderately thick plate for which the various types of action occur in this order: elastic bending, plastic bending, elastic membrane, and plastic membrane. This is not to say that all elements of the plate experience all types of action.

The known solution of the problem when there is elastic bending action shows that plastic deformation occurs first at the edge of the plate [1]. Since

$$\sigma_r = \sigma_t/\nu = \mp \frac{3}{4} p\, a^2/h^2 \qquad (r = a,\ z \pm \frac{1}{2} h), \tag{32}$$

the corresponding representative stress points move out along OG and OH (see Fig. 2), respectively, with increasing p. The yield limit is first reached at G and H when

$$p/\sigma_o = \frac{4}{3} h^2/a^2 = (p/\sigma_o)_{p\ell.}, \tag{33}$$

and then

$$w = \frac{a^2 h^2 \sigma_o}{48D} = (w)_{p\ell.} \qquad (r = 0). \tag{34}$$

As p/σ_o increases beyond $(p/\sigma_o)_{p\ell.}$, plastic regions must spread inwards from points $r = a$, $z = \pm \frac{1}{2} h$. The theory for elastic bending action shows that, for fixed z, the extreme values of σ_r and σ_t occur at the plate edge and center. At the center of the plate,

$$\sigma_r = \sigma_t = \pm \frac{1}{2} (1 + \nu) \frac{3}{4} pa^2/h^2 \qquad (r = 0, \ z = \pm \frac{1}{2} h), \tag{35}$$

and the corresponding representative stress points move out along OA and OD, respectively. However the points A and D are not reached before the points G and H are reached.

The discussion is now based upon general physical arguments, and only qualitative conclusions may be reached. The spreading inwards of the edge plastic regions results in a relative weakening of the plate, and therefore accelerates the initiation of central plastic regions. It is now clear from Eq. (35) that the spreading outwards of plastic regions from the points $r = 0$, $z = \pm \frac{1}{2} h$ must in fact occur quite soon and certainly before

$$p/\sigma_o = \frac{2}{1 + \nu} (p/\sigma_o)_{p\ell} . \tag{36}$$

Yielding at the plate center will occur as soon as the points A and D are reached. Note that the center is always an isotropic stress point.

In proceeding we shall first recall a result of Ref. [9] based upon plastic-rigid bending theory. This result is that hinge circles occur at the center and edge of the plate. In the present context it is immediately concluded that, until membrane action is significant, the tendency must be for the stress points to approximate this situation and also to be generally consistent with the plastic moment distributions found in Ref. [9]. Thus for $r = 0, a$ the representative stress points must tend towards

A,D or C,F according as $z \gtrless 0$, respectively. Moreover as the edge and central plastic regions grow the plastic stress points may be expected to occupy increasing parts of regimes AB, CB and DE, FE.

As p continues to increase the increasing importance of membrane action must disturb the symmetry of the plastic regimes about the center of the hexagon. The development of asymmetry is accompanied by unloading followed by reversed loading of plate elements. Such effects will spread out across outer annular and central regions in the lower and upper halves of the plate, respectively. Now when the displacements are quite large, membrane action must predominate in regions away from the plate edge but bending action will always predominate at the plate edge. Ultimately there must be a narrow boundary layer region at the plate edge across which there is a sharp transition from bending to membrane action.

It may be noted that the growth of plastic regions when the plate is simply-supported has been studied in detail by Sokolovsky [20]. This analysis is confined to elastic-plastic membrane action, and assumptions a), b) and e) are made. Although the inadmissible Hencky equations for the von Mises yield condition are used, it seems probable that the results would closely approximate those determined by the present analysis.

Note finally that probably the most significant limiting factor governing the applicability of the analysis will be the onset of high shear stresses in elastic regions which are neglected here.

5). Non-dimensional notation

It is necessary to introduce non-dimensional quantities so that the basic equations may be written in a form more appropriate for computation.

Note first that typical lengths parallel and normal to the plane of the plate are a and h, respectively. Further it is envisaged in the present work that h and the central deflection at the onset of yielding are of the same order. Thus a second typical length normal to the plane of the plate is

$$d = a^2 h^2 \sigma_o / 48D \tag{37}$$

(see Eq. 34). It is also supposed here that $\partial u'/\partial r$ and $\partial w/\partial r \times \partial w'/\partial r$ are of the same order. These considerations suggest the introduction of the following non-dimensional notation:

$$\rho = r/a, \quad \zeta = z/h; \quad W = w/d. \quad U = au/d^2. \tag{38}$$

The uniform load p is most conveniently expressed as a multiple of $\frac{4}{3}\frac{h^2}{a^2}\sigma_o$ (see Eq. 33), i.e. let t be defined by

$$p = \frac{4}{3}\frac{h^2}{a^2}\sigma_o t, \tag{39}$$

so that according to bending theory the plate will first yield when t = 1.

The fundamental equations (29) and boundary conditions (30) now take the form:

$$\left.\begin{aligned}
&\left.\begin{aligned}
&\frac{\partial}{\partial\rho}\left(\rho\,\frac{\partial U'}{\partial\rho}\right) - \frac{U'}{\rho} + \left\{(1-\nu) + \rho\,\frac{\partial}{\partial\rho}\right\}\frac{\partial W}{\partial\rho}\,\frac{\partial W'}{\partial\rho} + \Phi = 0, \\
&\Phi = -\,\frac{\partial}{\partial\rho}\left\{\rho\int_{-\frac{1}{2}}^{\frac{1}{2}}(\alpha+\beta\nu)\Lambda' d\zeta\right\} + \int_{-\frac{1}{2}}^{\frac{1}{2}}(\beta+\alpha\nu)\Lambda' d\zeta,
\end{aligned}\right\} \\
&\left.\begin{aligned}
&\frac{\partial}{\partial\rho}\left\{\frac{1}{\rho}\,\frac{\partial}{\partial\rho}\left(\rho\frac{\partial W'}{\partial\rho}\right)\right\} - 12\,\frac{d^2}{h^2}\left(n_r\,\frac{\partial W'}{\partial\rho} + \frac{\partial W}{\partial\rho}\,n_r'\right) + 12\,\frac{d}{h}\,\Psi - 32\rho = 0, \\
&\rho\Psi = \frac{\partial}{\partial\rho}\left\{\rho\int_{-\frac{1}{2}}^{\frac{1}{2}}(\alpha+\beta\nu)\Lambda'\zeta d\zeta\right\} - \int_{-\frac{1}{2}}^{\frac{1}{2}}(\beta+\alpha\nu)\Lambda'\zeta d\zeta,
\end{aligned}\right\}
\end{aligned}\right\} \tag{40}$$

$$W' = \frac{\partial W'}{\partial\rho} = U' = 0 \quad \text{at} \quad \rho = 1 \quad \text{for} \quad t \geq 0, \tag{41}$$

where

$$n_r' = \frac{\partial U'}{\partial\rho} + \frac{\partial W}{\partial\rho}\,\frac{\partial W'}{\partial\rho} + \nu\,\frac{U'}{\rho} - \int_{-\frac{1}{2}}^{\frac{1}{2}}(\alpha+\beta\nu)\Lambda' d\zeta, \tag{42}$$

$$\begin{aligned}(\alpha^2+\beta^2+2\alpha\beta\nu)\Lambda' &= (\alpha+\beta\nu)\left\{\frac{\partial U'}{\partial\rho} + \frac{\partial W}{\partial\rho}\,\frac{\partial W'}{\partial\rho} - \left(\frac{h}{d}\right)\frac{\partial^2 W'}{\partial\rho^2}\,\zeta\right\} \\
&\quad + (\beta+\alpha\nu)\left\{U' - \left(\frac{h}{d}\right)\frac{\partial W'}{\partial\rho}\,\zeta\right\}/\rho\,;\end{aligned} \tag{43}$$

Λ' having replaced λ',

$$\Lambda' = \frac{a^2}{d^2}\,\lambda'. \tag{44}$$

Note also that

$$\left.\begin{aligned}
\sigma_r'/\sigma_o &= \tfrac{1}{4}\left(\tfrac{d}{h}\right)\left[\frac{\partial U'}{\partial\rho} + \frac{\partial W}{\partial\rho}\,\frac{\partial W'}{\partial\rho} - \left(\frac{h}{d}\right)\frac{\partial^2 W'}{\partial\rho^2}\,\zeta + \nu\left\{U' - \left(\frac{h}{d}\right)\frac{\partial W'}{\partial\rho}\,\zeta\right\}/\rho - (\alpha+\beta\nu)\Lambda'\right], \\
\sigma_t'/\sigma_o &= \tfrac{1}{4}\left(\tfrac{d}{h}\right)\left[\left\{U' - \left(\frac{h}{d}\right)\frac{\partial W'}{\partial\rho}\,\zeta\right\}/\rho + \nu\left\{\frac{\partial U'}{\partial\rho} + \frac{\partial W}{\partial\rho}\,\frac{\partial W'}{\partial\rho}\right.\right. \\
&\qquad \left.\left. - \left(\frac{h}{d}\right)\frac{\partial^2 W'}{\partial\rho^2}\,\zeta\right\} - (\beta+\alpha\nu)\Lambda'\right].
\end{aligned}\right\} \tag{45}$$

It is apparent now from the structure of the equations that the geometrical and physical characteristics of the plate enter only through the non-dimensional parameters

$$\left.\begin{aligned} \frac{d}{h} &= \frac{1}{4}\,(1 - \nu^2)\,\frac{\sigma_o a^2}{Eh^2}, \\ t &= \frac{3}{4}\,\frac{pa^2}{\sigma_o h^2}, \end{aligned}\right\} \qquad (46)$$

and of course Poisson's ratio ν. The procedure is therefore to integrate the equations for a range of values of these parameters.

Note finally that the neglect of shear deformation involves an error of order h^2/a^2, and that the use of approximate curvature formulas involves an error of order w^2/a^2.

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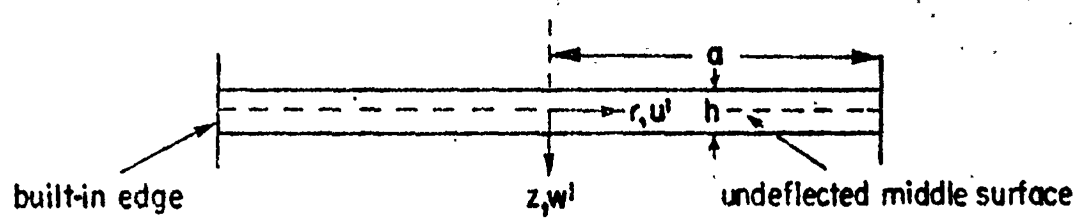
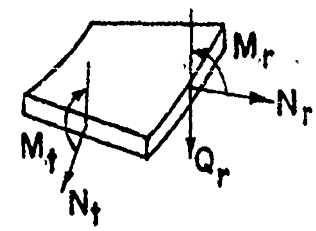


a) Vertical section through plate center



b) Element of plate

Fig. 1. Notation.

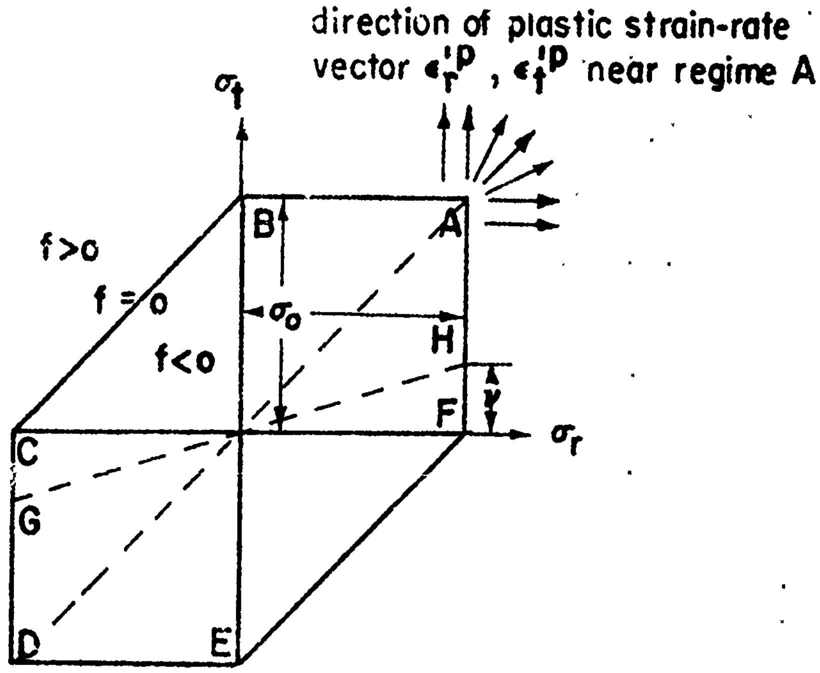


Fig. 2. Tresca Yield Condition.