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Multiple Scattering Of Waves

By

Peter C. Waterman and Rohn Truell

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Division of Applied Mathematics

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TABLE OF CONTENTS

PAGE

I. Introduction.....1

II. The Governing Equations of Multiple Scattering.....3

2.1 The Statistical Approach.....3

2.2 The Single Scatterer.....4

2.3 The Averaged External Field.....7

2.4 The Averaged Total Field.....20

2.5 Energy and Measurement.....24

III. Application to Problems of Physics.....30

3.1 Scattering by a Random Distribution of Isotropic Point Scatterers.....30

3.2 Transmission of Plane Waves by a Semi-Infinite Uniform Array of Spherical Obstacles.....39

IV. Discussion.....63

Appendix I: The Multipole Representation.....67

Appendix II: The Matrix Equations of Multiple Scattering with Mode Conversion.....70

Bibliography.....73

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ABSTRACT

Governing equations are obtained for the problem of multiple scattering of waves in a homogeneous isotropic medium containing a statistical array of scattering regions. The equations are applicable to sound waves in solids, liquids and gases, electromagnetic waves, and stationary quantum-mechanical problems. From these equations the macroscopic properties of the "scattering medium" may be obtained in terms of the scattering properties of a single scatterer. Questions of energy and measurement are discussed, and several examples are considered. An extension of the theory to the elastic problem with mode conversion is included.

MULTIPLE SCATTERING OF WAVES

I. INTRODUCTION

The problem of scattering of a plane wave by a single obstacle has been considered rather thoroughly in the literature, beginning with Rayleigh's work in fluids, and continuing to the present day with various quantum mechanical, electromagnetic, and elastic cases.

The literature on multiple scattering, on the other hand, is not so extensive. Regular arrays of scatterers were treated by Huyghen's principle or various perturbation schemes, mainly with an eye to obtaining their "strong" filtering properties arising due to periodicity. Out of this work came x-ray diffraction and the band theory of solids.

The first work involving statistical ensembles of scatterers was done by L.L. Foldy[1], in 1945. He treated the problem of scattering by an ensemble of isotropic point scatterers from a classical wave viewpoint, introducing rigorous statistical techniques. M. Lax[2] later generalized Foldy's work to arrays of point scatterers with quite general scattering properties, using a quantum mechanical formulation.

There are, however, certain limitations involved in considering only point scatterers. One must modify the computation of energy density, for example, in order that this quantity be finite. Further, the true functional nature of certain field quantities is either obscured or lost altogether. And of course, there is a large collection of physical phenomena for which satisfactory description requires the use of finite scattering regions.

The purpose of this paper is to derive the equations governing wave motion in a medium containing an array of finite scattering regions. It will be seen that within the limitations of the specific case considered, the treatment is equally applicable to classical and quantum mechanical problems.

The program is as follows. We first consider briefly the single-scatterer problem, in order to introduce some of the notation and concepts.

Next, using the statistical approach of Foldy, the first partial average of the external field is computed; this is simply the field "incident" on a scatterer known to be at a given position. By "incident" field is meant the entire field in the external neighborhood of a scatterer, less the outgoing scattered wave from the scatterer.

This first partial average is obtained in terms of the second partial average of the external field. The second partial average is that of the field incident on a scatterer known to be at a given position where in addition the position of a second scatterer is known.

Continuing this procedure, one obtains each time the field incident on a scatterer with n scatterer positions known in terms of the corresponding field with one additional scatterer position known--thus with a total of N scatterers there results a system of N equations in N unknowns. While this system of equations may in principle be solved, the labor is prohibitive. Hence an approximation scheme is introduced which enables one to obtain an integral equation for any one of the unknowns.

Next the averaged total field is computed. This is the field one would actually measure if no explicit information were available concerning scatterer positions, and is obtained in terms of the first partial average of the external field.

Questions of energy and measurement are then discussed, and finally two examples are considered. In the first, multiple scattering due to an array of isotropic point scatterers is treated, using the second approximation to the system of equations mentioned above. In the second example, a uniform

array of spherical obstacles of arbitrary size and scattering properties is considered, using the first approximation to the infinite set of equations. In both of these examples the complex propagation factor K is obtained, from which one gets in the usual manner the phase velocity and attenuation appropriate to a plane wave propagating in the"scattering medium".

II. THE GOVERNING EQUATIONS OF MULTIPLE SCATTERING

2.1 The Statistical Approach

There is a continuum of problems of physical interest within the framework of multiple scattering, ranging from the case of a regular array of scatterers to the other extreme, in which scatterers are stationed completely at random.

We shall focus attention mainly on the latter case, using the statistical methods developed by Foldy [1] and later extended by Lax, [2] although at any stage of the computations the results may be applied to a specific configuration of scatterers.

It is generally accepted practice in statistical mechanics, whenever one deals with a problem involving a large number of independent variables bearing no regular interrelation, to consider a specific set of values of these variables as a microstate in their ensemble of possible states. This microstate is then weighted by the probability of its occurrence, and the product averaged over all possible states. The results of this operation describe the solution of the statistically averaged problem, and it is then argued that this will be in good agreement with the "smoothed out" solution for a given microstate, provided only that the microstate of interest has a reasonably large probability of occurring.

"Smoothed out" solution, in the context of multiple scatttering, implies that the specific variations of field quantities due to immediate neighbor scattering regions are to be neglected. In many physical applications the measuring device averages over a region large compared with any of the lengths involved, thus measuring the "smoothed out" field directly. If one is interested in obtaining the field in more detail, this may also be done, as will be seen from the computations.

2.2 The Single Scatterer

Consider a medium homogeneous and isotropic with the exception of a simply-connected scattering region. Waves propagate in the medium external to this region according to the equation

$$(\Delta + k^2) \quad \psi(\underline{r}) = 0, \qquad (2.1)$$

where we have assumed a steady state solution. The time dependence of the potential is obtained by multiplying by exp(iwt). The propagation constant k is taken to be real in order to simplify the discussion.

The following problem is now appropriate: given a potential $\psi^E(\underline{r})$ regular (in the sense that it has no singularities, and all derivatives exist) in the finite domain, which describes the field everywhere in the medium if the scattering region is not present, and a set of linear conditions which the total potential must satisfy at the boundary of the scattering region, to determine the potential everywhere.

This problem has a unique solution, provided one specifies that the potential has no singularities inside the scattering region, and outside satisfies the radiation condition, that the potential differs from $\psi^E(\underline{r})$ by terms representing outgoing radiation from the scattering region.

The solution may be written explicitly as

$$\psi^E(\underline{r}) + \psi^S(\underline{r}) \tag{2.2}$$

outside of the scattering region, and

$$\psi^I(\underline{r}) \tag{2.3}$$

inside. Here the scattered potential $\psi^S(\underline{r})$ is a solution of Equ.(2.1) having the form of outgoing radiation sufficiently far from the scattering region, and the internal potential $\psi^I(\underline{r})$ is a solution of the "internal" wave equation regular inside the region. The problem may be completely characterized by introduction of the symbolic operators T^S, T^I, defined by

$$\begin{aligned} \psi^S(\underline{r}) &\equiv T^S \psi^E(\underline{r}) \\ \psi^I(\underline{r}) &\equiv T^I \psi^E(\underline{r}) \end{aligned} \tag{2.4}$$

The energy flow out through any closed surface S enclosing the scattering region is given by $W_E + W_{E,S} + W_S$, where W_E is the energy flow out due to the incident potential $\psi^E(\underline{r})$ alone, W_S is that due to the scattered potential $\psi^S(\underline{r})$ alone, and $W_{E,S}$ is contributed by interaction of the incident and scattered potential. This sum must be just the negative of the energy W_A absorbed per unit time within and on the surface of the scattering region.

Since W_E is not influenced by the presence or absence of the scattering region inside S, we may compute it assuming there is no scattering region. Clearly a wave motion in non-dissipative material with no singularities in a region can have no sources or sinks of energy in that region, and W_E must vanish identically.

Conservation of energy during the scattering process now requires that

$$W_{E,S} = -(W_A + W_S), \tag{2.5}$$

which states simply that the energy lost through interference of the scattered and incident waves is the sum of the energy absorbed within the scattering region and that carried off by the scattered wave alone.

This energy computation was originally done by Mie,[3] later simplified by Bateman,[4] for the case of plane electromagnetic radiation incident on a spherical obstacle. We have set it up here, in general form, as it will be useful later in interpreting results of the multiple scattering problem.

The incident wave of potential $\psi^E(\underline{r})$, as formulated above, is rather limited by the condition that it be regular in the finite region of space. Consider what would happen if the incident wave were generated by a source or sources in the finite domain: the wave emanating from the scattering region would in general not satisfy boundary conditions in the source region, hence multiple scattering would occur between source and scatterer.

In order to allow incident potentials with sources in the finite domain, and at the same time avoid the complications inherent in multiple scattering from source regions, we introduce the concept of the "controlled" source.

The controlled source, in the case of a volume source, for example, is one which has its outgoing radiation specified, rather than a conventional set of driving conditions. Since all secondary scattering at the source region of radiation from the scattering region would be of the form of outgoing radiation, we see that a controlled source is effectively invisible to scattered radiation and <u>does not scatter</u>.

It should be emphasized that the use of controlled sources does not modify the basic concepts of multiple scattering in any way, but only spares us the necessity of carrying source-scattered potentials along through the computations.

Clearly the single scatterer computation aboves goes through with only slight modification when controlled source incident potentials are used, and we pass over this point without further comment.

2.3 The Averaged External Field

Consider now the situation when there are N identical, similarly oriented scattering regions present. A convenient internal reference point is chosen in a similar manner for each region, and the array of regions is specified by the N position vectors $\underline{r}_1, \ldots, \underline{r}_N$ drawn to the appropriate reference points.

A known incident field $\psi^o(\underline{r})$, generated by controlled sources external to all scattering regions, is present, and we consider only the steady state situation, with the radiation condition prescribed for all scatterers. The field within source regions is unnecessary for the computation of the external field. The field outside of source and scattering regions is a solution of Equ. (2.1), and has the form $\psi^o(\underline{r})$ plus a sum of outgoing waves from all scattering regions. In particular, the external field acting on the j^{th} scatterer is defined to be $\psi^o(\underline{r})$ plus the sum of fields radiated from all scatterers other than the j^{th}. Thus in the terminology of the single-scatterer problem, the external field acting on the j^{th} scatterer is just the incident wave on that scatterer. To obtain the total field outside of the j^{th} scatterer, one adds to the external field the outgoing scattered wave from the j^{th} scatterer.

If we exclude arrays in which scattering regions have interpenetrated, then we see further that this external field will be a solution of Equ.(2.1) having no singularities in some open region containing the j^{th} scatterer.

Since all the conditions established in the previous section are satisfied —in particular the external field acting on the j^{th} scatterer

has the nature of a controlled incident wave, as it will not be modified by secondary scattering of the radiation from the j^{th} scatterer--we may now apply the results of the single scatterer computation to each of the N scattering regions separately.

The external field incident on the j^{th} scatterer may be written as

$$\psi^E(\underline{r}_j;\underline{r};\underline{r}_1,---\underline{r}_N) = \psi^o(\underline{r}) + \sum_{k\neq j}^{N} \psi^S(\underline{r}_k;\underline{r};\underline{r}_1,---,\underline{r}_N) \qquad (2.6)$$

where the notation reads as follows: the first independent variable in ψ^E refers to the scattering region in whose neighborhood the external field is being computed; the second gives the dependence of the external field upon spatial coordinates in that neighborhood; the remaining variables $\underline{r}_1---,\underline{r}_N$ exhibit the dependence of ψ^E upon the positions of all N scattering regions. $\psi^E(\underline{r}_j;\underline{r};\underline{r}_1 m---,\underline{r}_N)$ is a solution of Equ. (2.1) in $\underline{r}$, with no singularities in some open region containing the j^{th} scattering region. The exact nature of this open region will of course depend upon how source regions are specified. $\psi^S(\underline{r}_k;\underline{r};\underline{r}_1---\underline{r}_N)$ is the scattered potential from the k^{th} scattering region evaluated at the point $\underline{r}$, and is a solution of Equ. (2.1) in $\underline{r}$ with no singularities outside of the k^{th} region.

Further, in terms of the symbolic operators of the previous section, we may write

$$\psi^S(\underline{r}_k;\underline{r};\underline{r}_1,---\underline{r}_N) = T^S(\underline{r}_k)\ \psi^E(\underline{r}_k;\underline{r};\underline{r}_1,---,\underline{r}_N)$$

$$\psi^I(\underline{r}_k;\underline{r};\underline{r}_1,---,\underline{r}_N) = T^I(\underline{r}_k)\ \psi^E(\underline{r}_k;\underline{r};\underline{r}_1---,\underline{r}_N) \qquad (2.7)$$

While $\psi^E(\underline{r}_k;\underline{r};\underline{r}_1,---\underline{r}_N)$ is only defined for $\underline{r}$ in some neighborhood of $\underline{r}_k$, $T^S(\underline{r}_k)\ \psi^E(\underline{r}_k;\underline{r};\underline{r}_1,---,\underline{r}_N)$ represents a solution of Equ. (2.1)

regular everywhere outside of the $k^{\underline{th}}$ scattering region. The first of Equ.'s (2.7) is to be interpreted in this light. $\psi^{I}(\underline{r}_k;\underline{r};\underline{r}_1,---,\underline{r}_N)$ is a solution of the "internal"wave equation evaluated at the point $\underline{r}$ inside the $k^{\underline{th}}$ region. $T^{S}(\underline{r}_k)$computes the scattered potential at $\underline{r}$, due to a scatterer situated at $\underline{r}_k$.

Now, following Foldy[1] we let

$$p\,(\underline{r}_1,---,\underline{r}_N)\; d\,\tau_1---d\tau_N \tag{2.8}$$

be the probability of finding the first scattering region with its "center" (i,e.reference point) in the volume element $d\,\tau_1$ containing the point $\underline{r}_1$, at the same time that the second scattering region has its "center" in the volume element $d\,\tau_2$ containing the point $\underline{r}_2$, and so forth.

We restrict $p(\underline{r}_1,---,\underline{r}_N)$ at the outset by the following conditions: its integral over all configurations of scatterers is normalized to unity; interpenetration of scattering regions is forbidden, hence $p(\underline{r}_1,---\underline{r}_N)$ vanishes for all these situations; scattering regions may not penetrate source regions, giving another set of excluded configurations; and lastly, the N scattering regions are to be considered indistinguishable, which allows position vectors to be interchanged at will in Equ. (2.8) without changing the numerical value of the expression.

The probability of finding a particular scatterer, say the first, with "center" in the volume element $d\,\tau_1$ with no further information available, is

$$p(\underline{r}_1)\; d\,\tau_1 \;=\; d\,\tau_1 \int ---- \int d\,\tau_2---d\,\tau_N p(\underline{r}_1,---\underline{r}_N) \tag{2.9}$$

Since each scatterer has equal likelihood of occupying a given volume element the density $n(\underline{r}_1)$ of scatterers at $\underline{r}_1$ is then given by

$$n(\underline{r}_1) \;=\; Np(\underline{r}_1)$$

In like manner, the joint probability of finding the first scatterer in $d\tau_1$ and the second in $d\tau_2$ is

$$p(\underline{r}_1,\underline{r}_2)d\tau_1 d\tau_2 = d\tau_1 d\tau_2 \int\cdots\int d\tau_3\cdots d\tau_N p(\underline{r}_1,\cdots\underline{r}_N), \tag{2.11}$$

where again in terms of densities

$$p(\underline{r}_1,\underline{r}_2) = \frac{n(\underline{r}_1)n(\underline{r}_2 \mid \underline{r}_1)}{N(N-1)} \tag{2.12}$$

Here $n(\underline{r}_2 \mid \underline{r}_1)$ is the density at $\underline{r}_2$ if a scattering region is known to be at $\underline{r}_1$.

The configurational average of a function of space coordinates and the N position vectors is now obtained by multiplying the function by the expression of Equ. (2.8) and integrating over all space accessible to the scattering regions. Thus,

$$\langle f(\underline{r})\rangle \equiv \int\cdots\int d\tau_1\cdots d\tau_N f(\underline{r};\underline{r}_1,\cdots\underline{r}_N)\, p(\underline{r}_1,\cdots d\underline{r}_N) \tag{2.13}$$

The partial average with one or more scatterers held fixed is defined to be the result obtained by omitting the integration over the fixed scatterers and dividing by the joint probability distribution for the fixed configuration taken. For example, the partial average of $f(\underline{r};\underline{r}_1,\cdots,\underline{r}_N)$ with the first scatterer fixed at $\underline{r}_1$, the second fixed at $\underline{r}_2$, is given by

$$p(\underline{r}_1,\underline{r}_2)\langle f(\underline{r})\rangle_{\underline{r}_1,\underline{r}_2} \equiv \int\cdots\int d\tau_3\cdots d\tau_N f(\underline{r};\underline{r}_1,\cdots,\underline{r}_N)p(\underline{r}_1\cdots,\underline{r}_N) \tag{2.14}$$

and so forth.

In this formalism we now proceed to take the partial average of Equ. (2.6) with the $j^{\underline{th}}$ scatterer held fixed at $\underline{r}_j$, to obtain the average external field acting on the $j^{\underline{th}}$ scatterer when it is at $\underline{r}_j$. Application of the

definition implied in Equ. (2.14) gives

$$p(\underline{r}_j) \quad \langle \psi^E(\underline{r}_j;\underline{r}) \rangle_{r_j} = p(\underline{r}_j)\, \psi^o(\underline{r}) + \sum_{k\neq j}^{N} \int d\tau_k p(\underline{r}_k,\underline{r}_j) \langle \psi^S(\underline{r}_k;\underline{r}) \rangle_{\underline{r}_k,\underline{r}_j} \tag{2.15}$$

where

$$\langle \psi^S(\underline{r}_k;\underline{r}) \rangle_{\underline{r}_k,\underline{r}_j} = \langle T^S(\underline{r}_k) \quad \psi^E(\underline{r}_k;\underline{r}) \rangle_{\underline{r}_k,\underline{r}_j} = T^S(\underline{r}_k) \langle \psi^E(\underline{r}_k;\underline{r}) \rangle_{\underline{r}_k,\underline{r}_j} \tag{2.16}$$

where the first equality follows from the definition of $T^S(\underline{r}_k)$, and the second from the fact that since the scattering is linear, it is immaterial whether we compute the scattered field from the external field and then average, or average the external field first.

Upon further noticing that Equs. (2.10) and(2.12) enable us to eliminate an $n(\underline{r}_j)$, and by virtue of indistinguishability each term in the summation is identical, we may finally write Equ. (2.15) in the form

$$\langle \psi^E(\underline{r}_1;\underline{r}) \rangle_{\underline{r}_1} = \psi^o(\underline{r}) + \int d\tau' \, n(\underline{r}' \mid \underline{r}_1) \quad T^S(\underline{r}') \langle \psi^E(\underline{r}';\underline{r}) \rangle_{\underline{r}',r_1} \tag{2.17}$$

A brief discussion of Equ. (2.17) is in order. It was natural to seek an implicit relationship for the averaged external field with one scatterer fixed. Instead we have obtained an expression for the averaged external field with one scatterer fiexed in terms of the averaged external field with two scatterers fixed.

The underlying cause of this difficulty is as follows: we are after the external field acting on a scatterer, averaged over all configurations for which the scatterer is fixed at $\underline{r}_1$. The integral of Equ. (2.17) represents the sum over contributions to this external field from all other scatterers.

The contribution from any point $\underline{r}'$ in space, however, must be restricted to cases where there is both a scatterer at the point $\underline{r}'$ to send out the contribution, and a scatterer at the reception point $\underline{r}_1$ to receive it.

We may now proceed to compute all the higher partial averages of the external field in the neighborhood of any scatterer at say, $\underline{r}_1$, starting in each case from Equ. (2.6). The final result is the system of N-1 sets of equations

$$\langle \psi^E(\underline{r}_1;\underline{r}) \rangle_{\underline{r}_1} = \psi^o(\underline{r}) + \int d\tau' \, n(\underline{r}'|\underline{r}_1) \, T^S(\underline{r}') \langle \psi^E(\underline{r}';\underline{r}) \rangle_{\underline{r}',\underline{r}_1}$$

$$\langle \psi^E(\underline{r}_1;\underline{r}) \rangle_{\underline{r}_1,\underline{r}_2} = \psi^o(\underline{r}) + \int d\tau' \, n(\underline{r}'|\underline{r}_1,\underline{r}_2) \, T^S(\underline{r}') \langle \psi^E(\underline{r}';\underline{r}) \rangle_{\underline{r}',\underline{r}_1\underline{r}_2}$$

$$\vdots$$

$$+ T^S(\underline{r}_2) \, \langle \psi^E(\underline{r}_2;\underline{r}) \rangle_{\underline{r}_1,\underline{r}_2} \qquad (2.18)$$

$$\vdots$$

$$\langle \psi^E(\underline{r}_1;\underline{r}) \rangle_{\underline{r}_1,---,\underline{r}_{N-1}} = \psi^o(\underline{r}) + \int d\tau' \, n(\underline{r}'|\underline{r}_1,---,\underline{r}_{N-1}) T^S(\underline{r}') \langle \psi^E(\underline{r}';\underline{r};\underline{r}_1,---,\underline{r}_N) \rangle$$

$$+ \sum_{j=2}^{N-1} T^S(\underline{r}_j) \quad \psi^E(\underline{r}_j;\underline{r})_{\underline{r}_1,---,\underline{r}_{N-1}}$$

where $n(\underline{r}'|\underline{r}_1,\underline{r}_2)$ is the density of scatterers at $\underline{r}'$ if scatterers are known to be at $\underline{r}_1$ and $\underline{r}_2$, and so forth. That each of the partial averages of the external field is a solution of the wave equation governing propagation external to scattering regions is easily verified.

Equations similar to Equs. (2.18), but incomplete in that fixed scatterer contributions were omitted, were obtained by Lax [2] for the case of aniso-

tropic quantum-mechanical point scatterers.

The system of Equs. (2.18) may in principle be solved exactly by solving Equs. (2.6), (2.7) explicitly for $\psi^E(\underline{r}';\underline{r};\underline{r}_1,---,\underline{r}_N)$, then starting with the last of Equs. (2.18) and working up from the bottom. However, the computational difficulties of such a program are severe, so we seek an alternative scheme.

One might try an iterative procedure, substituting each equation in the preceding one and letting the number of scatterers increase without limit, to obtain for the first partial average of the external field the infinite series

$$\langle \psi^E(\underline{r}_1;\underline{r}) \rangle_{\underline{r}_1} = \psi^o(\underline{r}) + \int d\tau' \, n(\underline{r}'|\underline{r}_1)\, T^S(\underline{r}')\, \psi^o(\underline{r}) + \int d\tau' \, n(\underline{r}'\,\underline{r}_1)\, T^S(\underline{r}') \int d\tau'' n(\underline{r}''|\underline{r}',\underline{r}_1) T^S(\underline{r}'')\, \psi^o(\underline{r}) + \ldots. \tag{2.19}$$

where for simplicity we have omitted the terms in Equs. (2.18) representing scattered waves from fixed scatterers. This series is easily recognizable from the "multiple orders of scattering" viewpoint. The first term gives the contribution to the field at the point $\underline{r}$ due to the incident wave alone. The second term gives the terms contributed by all other scatterers, each scattering the original incident wave independently, the so-called "primary" scattering. The next term contributes the secondary scattering, resulting from two scatterings of the incident wave, and so forth. One notes that since the iteration procedure commutes with the averaging process, the same result might have been obtained by first expanding Equ. (2.6) in an infinite series by iteration, then averaging.

The multiple orders of scattering approach has been considered in the literature by Twersky [5]. For purposes of computation it is only useful

when one is dealing with a small number of weakly-coupled scatterers, or when one is only interested in purely incoherent scattering.

The poor convergence inherent in the series of Equ. (2.19) may be observed by the following considerations. Take, for example, the primary scattering term, and suppose the incident wave is a plane wave. Since the propagation constant associated with scattered radiation is the same as that for the incident wave, the scattered waves from any two scatterers, one nearly in the "shadow" of the other, will be nearly in phase along their common "shadow" direction. A similar situation arises for each succeeding term in the series. This result is rather undesirable when one is interested in coherent effects of large numbers of scatterers, as a great number of terms must be taken in order to obtain convergence. In particular, for an infinite array of scatterers every term in the series, with the exception of the first, becomes infinite along the "shadow" direction, where coherent effects dominate.

Fortunately, there is another technique available. By modifying the implicit representation of Equ. (2.6), and then averaging, it is possible to expand any one of the partially averaged external fields in a new representation involving the partially averaged external field with one less scatterer held fixed. By combining these results with Equs. (2.18), we may obtain integral equations for each of the partial external fields to any degree of accuracy desired.

Specifically, we consider the expansion of $\langle \psi^E(\underline{r}_j;\underline{r}) \rangle_{\underline{r}_j;\underline{r}_k}$ by this scheme. Purely for notational convenience, we shall compute $\langle \psi^E(\underline{r}_1;\underline{r}) \rangle_{\underline{r}_1,\underline{r}_N}$, with the partially averaged field at any fixed scatterer

with any other scatterer fixed of course following directly by virtue of indistinguishability.

Two separate problems, one involving N scatterers, the other N-1, will be carried through simultaneously. To avoid confusion, an "N" or"N-1" will be attached where necessary.

Consider a given configuration of scatterers in the N problem, and suppose the $N^{\underline{th}}$ scatterer is temporarily removed, leaving the (N-1) problem. The implicit solution for the external fields in the (N-1) problem may be written exactly as in Equ. (2.6), as

$$\psi^{E}_{N-1}(\underline{r}_j;\underline{r};\underline{r}_1,---,\underline{r}_{N-1}) = \psi^{o}(\underline{r})+\sum_{k\neq j}^{N-1} T^{S}(\underline{r}_k)\ \psi^{E}_{N-1}(\underline{r}_k;\underline{r};\underline{r}_1,----\underline{r}_{N-1}) \qquad (2.20)$$

The external field acting on, say, the first scatterer, in the N problem may now be expanded in the form

$$\begin{aligned}\psi^{E}_{N}(\underline{r}_1;\underline{r};\underline{r}_1,---,\underline{r}_N) = {} & \psi^{o}(\underline{r}) + \sum_{j=2}^{N-1} T^{S}(\underline{r}_j)\ \psi^{E}_{N-1}(\underline{r}_j;\underline{r};\underline{r}_1,--,\underline{r}_{N-1}) \\ & + T^{S}(\underline{r}_N)\quad \psi^{E}_{N}(\underline{r}_N;\underline{r};\underline{r}_1,---,\underline{r}_N) \qquad (2.21) \\ & + \sum_{j=2}^{N-1} T^{S}(\underline{r}_j)\ T^{S}(\underline{r}_N)\quad \psi^{E}_{N}(\underline{r}_N;\underline{r};\underline{r}_1,--\underline{r}_N) + ---\end{aligned}$$

Here the first two terms on the right give just $\psi^{E}_{N-1}(\underline{r}_1;\underline{r};\underline{r}_1,---,\underline{r}_{N-1})$, the contribution we would get if there were no $N^{\underline{th}}$ scatterer present. To this must be added the scattered wave from the $N^{\underline{th}}$ scatterer, and the effects of primary, secondary, and so on, scattering of this latter wave from all other scatterers. Terms up through primary scattering of the scattered wave from the $N^{\underline{th}}$ scatterer are explicitly included in Equ. (2.21).

To reiterate: Equ. (2.21) is a quasi-implicit representation. The first two terms give the external field contribution when one scatterer is temporarily removed, calculated implicitly from Equs. (2.20). The remaining infinite series gives the correction terms that must be added due to the presence of the N^{th} scatterer, computed by the "multiple orders of scattering" approach. The eventual goal of this process will be achieved by equating corresponding field quantities of the N and N-1 problems in the limit as N becomes infinite.

Defining the joint probability distribution function for the N-1 problem by

$$p_{N-1}(\underline{r}_1,---,\underline{r}_{N-1}) = p_N(\underline{r}_1,--,\underline{r}_{N-1}) \tag{2.22}$$

where $p_N(\underline{r}_1,---,\underline{r}_N)$ is the distribution function appropriate to the N problem, and dropping the subscripts since there can be no confusion, we may now proceed to average Equ. (2.21). Taking only the first three terms on the right, and taking the N problem average of both sides, keeping the first and N^{th} scatterers fixed, there results

$$\left\langle \psi^E_N(\underline{r}_1;\underline{r}) \right\rangle^N_{\underline{r}_1,\underline{r}_N} = \psi^o(\underline{r}) + \int d\tau' n(\underline{r}'|\underline{r}_1,\underline{r}_N)\, T^S(\underline{r}') \left\langle \psi^E_{N-1}(\underline{r}';\underline{r}) \right\rangle^N_{\underline{r}',\underline{r}_1,\underline{r}_N}$$

$$+ T^S(\underline{r}_N) \left\langle \psi^E_N(\underline{r}_N;\underline{r}) \right\rangle^N_{\underline{r}_N,\underline{r}_1} + --- \tag{2.23}$$

where, as in previous calculations, the summation has been replaced by a multiplying factor by virtue of indistinguishability.

Now consider the field quantity $\left\langle \psi^E_{N-1}(\underline{r}';\underline{r}) \right\rangle^N_{\underline{r}',\underline{r}_1\underline{r}_N}$. Since this

is an external field connected with the N-1 problem, the scatterer situated at $\underline{r}_N$ takes no part in the multiple scattering which generates this field; the scatterer at $\underline{r}_N$ only influences the field statistically in that its presence at $\underline{r}_N$ modifies the various configurations of the remaining scatterers.

Hence if scatterer positions are only " weakly" correlated, we can expect the field under discussion to be nearly equal to the corresponding field in the N-1 problem with no scatterer at $\underline{r}_N$, that is

$$\left\langle \psi^{E}_{N-1}(\underline{r}';\underline{r}) \right\rangle^{N}_{\underline{r}';\underline{r}_1\underline{r}_N} \approx \left\langle \psi^{E}_{N-1}(\underline{r}';\underline{r}) \right\rangle^{N-1}_{\underline{r}';\underline{r}_1} \tag{2.24}$$

This approximation becomes exact if scatterer positions are statistically independent.

To the same degree of approximation we may replace $n(\underline{r}'|\,\underline{r}_1,\underline{r}_N)$ by $n(\underline{r}'|\,\underline{r}_1)$ in the above equation, obtaining

$$\begin{aligned}\left\langle \psi^{E}_{N}(\underline{r}_1;\underline{r}) \right\rangle^{N}_{\underline{r}_1;\underline{r}_N} &\approx \psi^{o}(\underline{r}) + \int d\,\tau'\, n(\underline{r}'|\,\underline{r}_1)\, T^{S}(\underline{r}') \left\langle \psi^{E}_{N-1}(\underline{r}';\underline{r}) \right\rangle^{N-1}_{\underline{r}',\underline{r}_1} \\ &\quad + T^{S}(\underline{r}_N) \left\langle \psi^{E}_{N}(\underline{r}_N;\underline{r}) \right\rangle^{N}_{\underline{r}_N,\underline{r}_1} + \cdots \\ &= \left\langle \psi^{E}_{N-1}(\underline{r}_1;\underline{r}) \right\rangle^{N-1}_{\underline{r}_1} + T^{S}(\underline{r}_N) \left\langle \psi^{E}_{N}(\underline{r}_N;\underline{r}) \right\rangle^{N}_{\underline{r}_N\underline{r}_1} + \cdots\end{aligned} \tag{2.25}$$

where the last equality follows directly from the first of Equs. (2.18).

We now take as self-evident the statement that in the limit as $N \to \infty$, corresponding field quantities in the N and N-1 problems become equal. In the present derivation we need the statement

$$\lim_{N\to\infty}\left\{\left\langle\psi^{E}_{N}(\underline{r}_1;\underline{r})\right\rangle^{N}_{\underline{r}_1} - \left\langle\psi^{E}_{N-1}(\underline{r}_1;\underline{r})\right\rangle^{N-1}_{\underline{r}_1}\right\} = 0 \qquad (2.26)$$

Making this substitution in Equ. (2.25) and performing one iteration we obtain for large N

$$\left\langle\psi^{E}(\underline{r}_1;\underline{r})\right\rangle_{\underline{r}_1,\underline{r}_N} \approx \left\langle\psi^{E}(\underline{r}_1;\underline{r})\right\rangle_{\underline{r}_1} + T^{S}(\underline{r}_N)\left\langle\psi^{E}(\underline{r}_N;\underline{r})\right\rangle_{\underline{r}_N} \qquad (2.27)$$

The approximations involved in Equ. (2.27) are of three kinds. First are those concerned with the statistical description of scatterer positions, incurred with the approximation of Equ. (2.24)and the replacement of the conditional density $n(\underline{r}'|\underline{r}_1,\underline{r}_N)$ by $n(\underline{r}'|\underline{r}_1)$. Second are those due to leaving out terms involving two or more scatterings of the external field, both in the starting expansion of Equ. (2.21) and in the iteration of Equ. (2.25). We might call the region where these approximations are valid the "weak scattering interaction" region. Physically it is reasonable that weak scattering interaction obtains whenever the amplitude of a scattered wave, evaluated at a distance from its source equal to the average separation distance between scatterers, is small next to its amplitude at the surface of its source. Thus to obtain weak scattering interaction one may use either weakly scattering obstacles or low densities of obstacles or a suitable combination of these factors. The third approximation is that of equating corresponding field quantities in the N and N-1 problems, according to Equ.(2.26), and is valid for large N.

To sum up: Equ. (2.27) should hold provided there is (1)"weak" correlation of scatterer positions, (2) "weak" scattering interaction, (3) a large number of scatterers.

Substituting Equ.(2.27) into the first of Equs. (2.18) we obtain for the first partial average of the external field the integral equation

$$\langle \psi^E(\underline{r}_1;\underline{r}) \rangle_{\underline{r}_1} \approx \psi^o(\underline{r}) + \int d\tau'\, n(\underline{r}'|\underline{r}_1)\, T^S(\underline{r}')\, \langle \psi^E(\underline{r}';\underline{r}) \rangle_{\underline{r}'} \qquad (2.28)$$

$$+ \int d\tau'\, n(\underline{r}'\underline{r}_1)\; T^S(\underline{r}')\; T^S(\underline{r}_1)\, \langle \psi^E(\underline{r}_1;\underline{r}) \rangle_{\underline{r}_1}$$

While $\langle \psi^E(\underline{r}';\underline{r}) \rangle_{\underline{r}'}$ is a function of two sets of space variables, the knowledge that it is a solution of the unperturbed Helmholz equation in $\underline{r}$ will in general be sufficient to reduce Equ. (2.28) to a set of simultaneous ordianry integral equations.

Continuing in this fashion one could develop Equ. (2.28) to as many terms as were required for the specific problem being considered(provided of course that no convergence difficulties were encountered). Also, any of the higher partial averages of the external field could be expanded along the above lines.

While Equ. (2.28) bears strong resemblance to Equ. (2.19) which gave a series representation for the external field by way of the pure multiple orders of scattering approach, the convergence difficulty mentioned there has now been overcome. Note in the second term on the right of Equ. (2.28), for example, that scattering contributions from various points $\underline{r}'$ still propagate with the unperturbed medium velocity, but are generated by the external field, rather than the incident wave. Hence, for an incident plane wave, assuming the external field to propagate with a modified velocity, contributions from various scatterers will have their phases "mixed-up" even along the shadow direction.

At this point we temporarily cease the discussion of the external field in order to consider the total field, and the role played by the first partially averaged external field in generating it.

2.4 The Averaged Total Field

The expression for the total field is somewhat more involved than that for the external field, as it will depend on whether the point of evaluation falls inside or outside of a scattering region.

The computation of the averaged external fields did not depend on the existence of the internal potentials ψ^I. However, these potentials are essential to the computation of the averaged total potential. The viewpoint we adopt is as follows: if a wave equation exists for the interior of scattering regions, then the potential is defined in these regions and there is no trouble. If the scattering regions cannot sustain wave motion, but may be considered as limiting cases of media which permit wave motion, then the computation is carried through with internal potentials, with the understanding that the appropriate limiting process must be employed in order to obtain a final answer.

If neither of these conditions is fulfilled, then it is meaningless to speak of the total field. This situation is satisfactory for our purposes, because any measurement which did not disturb the ensemble would necessarily be based on a partial averaged external field. That is, in order to guarantee that a measurement be possible at any particular point, we would have the alternatives of either modifying the ensemble so that the point in question was never in the interior of a scattering region, which seems rather unsatisfactory, or agreeing to measure, say, the partial average of the external field acting on a scatterer fixed at the point.

Assuming that internal potentials are defined, the total field at any point is made up of the incident potential plus contributions from all scatterers, should the point of evaluation lie ouside of all scattering regions. The total field within or on the boundary of any scatterer is made up of only the contribution from the internal field of that scatterer.

We first define the function $\alpha(\underline{\rho})$ as

$$\alpha(\underline{\rho}) = \begin{cases} 0 & \underline{\rho} \text{ "inside" 0} \\ 1 & \underline{\rho} \text{ "outside" 0,} \end{cases} \tag{2.29}$$

where " $\underline{\rho}$ inside 0" states that the point $\underline{\rho}$ is in the interior or on the boundary of a scattering region with " center" at the origin of coordinates, and " $\underline{\rho}$ outside 0" is the complementary statement.

In terms of $\alpha(\underline{\rho})$, the total field at any point is given by

$$\psi(\underline{r};\underline{r}_1,\cdots,\underline{r}_N) = \left\{\psi^0(\underline{r}) + \sum_{j=1}^{N} \psi^S(\underline{r}_j;\underline{r};\underline{r}_1,\cdots,\underline{r}_N)\right\}\prod_{k=1}^{N} \alpha(\underline{r}-\underline{r}_k)$$

$$+ \sum_{j=1}^{N} \psi^I(\underline{r}_j;\underline{r};\underline{r}_1,\cdots,\underline{r}_N)\left[1 - \alpha(\underline{r}-\underline{r}_j)\right] \tag{2.30}$$

Observe that the product function eliminates "outside point" contributions whenever the point of evaluation is in the interior of any scattering region, and the $1-\alpha$ terms pick out only the appropriate internal potential for the "inside point " contribution.

Now consider the identity

$$1 \equiv \prod_{k=1}^{N} \left[\alpha(\underline{r}-\underline{r}_k) + 1- \alpha(\underline{r}-\underline{r}_k)\right]$$

$$= \prod_{k=1}^{N} \alpha(\underline{r}-\underline{r}_k) + \sum_{j=1}^{N} \left[1- \alpha(\underline{r}-\underline{r}_j)\right] \prod_{k\neq j}^{N} \alpha(\underline{r}-\underline{r}_j)$$

$$+ \sum_{k=1}^{N} \sum_{j\neq k}^{N} \left[1- \alpha(\underline{r}-\underline{r}_k)\right]\left[1- \alpha(\underline{r}-\underline{r}_j)\right] \prod_{\ell\neq j,k}^{N} \alpha(\underline{r}-\underline{r}_\ell)$$

$$+ - - - + \prod_{k=1}^{N} \left[1- \alpha(\underline{r}-\underline{r}_k)\right]$$

Multiplying the extreme sides through by $p(\underline{r}_1,---,\underline{r}_N)$, we see that all terms on the right after the second vanish identically, since products of the form $\left[1- \alpha(\underline{r}-\underline{r}_k)\right]\left[1- \alpha(\underline{r}-\underline{r}_j)\right]$ vanish except when the $k^{\underline{th}}$ and $j^{\underline{th}}$ scattering regions have interpenetrated, at which time $p(\underline{r}_1,---,\underline{r}_N)$ vanishes.

Consequently we may write

$$p(\underline{r}_1,---,\underline{r}_N) \prod_{k=1}^{N} \alpha(\underline{r}-\underline{r}_k) = p(\underline{r}_1,---,\underline{r}_N)\left\{1-\sum_{j=1}^{N} \left[1-\alpha(\underline{r}-\underline{r}_j)\right]\prod_{k\neq j}^{N} \alpha(\underline{r}-\underline{r}_k)\right\}$$

$$= p(\underline{r}_1,---\underline{r}_N)\left\{1-\sum_{j=1}^{N} \left[1- \alpha(\underline{r}-\underline{r}_j)\right]\right\} \qquad (2.31)$$

where the last equality follows from the fact that the (N-1) - termed product contributes no information that is not contained in the product of the probability distribution and the summation.

The operation of taking the configurational average of Equ. (2.30) is straightforward when Equ. (2.31) is employed, and one finally obtains for the averaged total field

$$\langle \psi(\underline{r}) \rangle = \psi^{o}(\underline{r}) + \int_{\underline{r} \text{ "outside" } \underline{r}'} d\tau' \, n(\underline{r}') \, T^{S}(\underline{r}') \, \langle \psi^{E}(\underline{r}';\underline{r}) \rangle_{\underline{r}'}$$

(2.32)

$$+ \int_{\underline{r} \text{ "inside" } \underline{r}'} d\tau' \, n(\underline{r}') \left[T^{I}(\underline{r}') - 1 \right] \langle \psi^{E}(\underline{r}';\underline{r}) \rangle_{\underline{r}'} ,$$

where the statement "$\underline{r}$ outside $\underline{r}'$" means, as before, that the first integration is to be carried over all points $\underline{r}'$ such that $\underline{r}$ is outside of the scattering region with center $\underline{r}'$, and so forth.

The interpretation of Equ.(2.32) is as follows: the first two terms represent the "outside point" contributions, i.e. the terms contributed to the averaged total field by those configurations of scatterers for which the evaluation point $\underline{r}$ is external to all scattering regions. The first term in the second integral gives the "inside point" contributions, and the second term serves to kill the "outside point" contributions as the likelyhood of $\underline{r}$ being an "outside point" diminishes.

The derivation of the exact equations governing multiple scattering is now complete. The averaged total field is given by Equ. (2.32) in terms of the first partial average of the external field. This latter quantity is given implicitly by an integral equation approximated by Equ.(2.28) or alternatively in terms of $\psi^{E}(\underline{r}_j;\underline{r};\underline{r}_1,---,\underline{r}_N)$, the external field acting on a scatterer for a specified array of scatterer positions, through the chain of Equs. (2.18). Finally, Equs. (2.6) and the first of Equs. (2.7) may in principle be solved to give $\psi^{E}(\underline{r}_j;\underline{r};\underline{r}_1,---\underline{r}_N)$ explicitly.

2.5 Energy and Measurement

Since the energy density at any point in the medium results from performing some non-linear operation on the potential, and this non-linear operation will not in general commute with the statistical averaging process, it is immediately clear that the result will depend on how the computation is done. The aim of this section is to present the alternative expressions for energy density, and describe their relation to physical measurement.

We take the square of the magnitude of the potential as a measure of energy density. On this basis, the energy density associated with the averaged total field $\langle \psi(\underline{r}) \rangle$ follows directly from Equ. (2.32):

$$
\begin{aligned}
\left| \langle \psi(\underline{r}) \rangle \right|^2 = & \left| \psi^o(\underline{r}) + \int_{\underline{r} \text{ "outside" } \underline{r}'} d\tau' \, n(\underline{r}') \, T^S(\underline{r}') \, \langle \psi^E(\underline{r}';\underline{r}) \rangle_{\underline{r}'} \right|^2 \\
& + \left| \int_{\underline{r} \text{ "inside" } \underline{r}'} d\tau' \, n(\underline{r}') \left[T^I(\underline{r}') - 1 \right] \langle \psi^E(\underline{r}';\underline{r}) \rangle_{\underline{r}'} \right|^2 \qquad (2.33) \\
& + 2\mathrm{Re} \left\{ \left[\psi^{o*}(\underline{r}) + \int_{\underline{r} \text{ "outside "} \underline{r}'} d\tau' \, n(\underline{r}') \, T^{S*}(\underline{r}') \, \langle \psi^E(\underline{r}';\underline{r}) \rangle_{\underline{r}'} \right] \int_{\underline{r} \text{ "inside" } \underline{r}'} d\tau' \, n(\underline{r}') \left[T^I(\underline{r}') - 1 \right] \langle \psi^E(\underline{r}';\underline{r}) \rangle_{\underline{r}'} \right\}
\end{aligned}
$$

The first squared quantity and part of the second respectively give rise to the "outside" and "inside" point contributions to the energy density. The remainder of the terms appear erroneous, as they represent for the most part interactions between potentials which are never present simultaneously in any volume element of space. For example, one term represents a contribution to the energy due to interaction between an averaged external and averaged internal

field, while physically it is clear that these two fields never interact. The presence of these apparently erroneous terms will be discussed shortly.

Alternatively, we might write down the energy density associated with a specified configuration of scatterers, and statistically average this expression to obtain the averaged energy density. Observing that the function $\alpha(\underline{\rho})$ satisfies

$$\alpha^2(\underline{\rho}) = \alpha(\underline{\rho}) , \tag{2.34}$$

we may obtain the square of the magnitude of the total field from Equ. (2.30). The result, true in any case for which scatterers have not interpenetrated, is

$$\begin{aligned} \left|\psi(\underline{r};\underline{r}_1,---,\underline{r}_N)\right|^2 = \Big\{ & \left|\psi^o(\underline{r})\right|^2 + 2\mathrm{Re}\ \psi^{o*}(\underline{r}) \sum_{j=1}^{N} \psi^S(\underline{r}_j;\underline{r};\underline{r}_1,---,\underline{r}_N) \\ & + \sum_{j,k=1}^{N}{}' \psi^{S*}(\underline{r}_j;\underline{r};\underline{r}_1,---,\underline{r}_N)\ \psi^S(\underline{r}_k;\underline{r};\underline{r}_1,---,\underline{r}_N) \\ & + \sum_{j=1}^{N} \left|\psi^S(\underline{r}_j;\underline{r};\underline{r}_1,---,\underline{r}_N)\right|^2 \Big\} \prod_{m=1}^{N} \alpha(\underline{r}-\underline{r}_m) \\ & + \sum_{j=1}^{N} \left|\psi^I(\underline{r}_j;\underline{r};\underline{r}_1,---,\underline{r}_N)\right|^2 \left[1-\alpha(\underline{r}-\underline{r}_j)\right] , \end{aligned} \tag{2.35}$$

where the prime on the summation indicates that terms with $j=k$ are to be omitted. The four terms in curly brackets are "outside" point contributions, and represent respectively the self-energy of the incident field, interaction terms between incident and scattered field, and finally the coherent and incoherent scattered terms. The final summation gives the appropriate "inside" point contribution.

We now define $f(\underline{r})$, the locally averaged fractional volume of scattering material, by

$$f(\underline{r}) = \int_{\underline{r}\ \text{"inside"}\ \underline{r}'} d\tau'\, n(\underline{r}') \tag{2.36}$$

Since scatterers are not allowed to interpenetrate, one notes that $f(\underline{r})$ is restricted to the range $0 \leqslant f(\underline{r}) \leqslant 1$, and indeed the upper limit could be achieved only if scatterers were capable by virtue of their geometry of complete close-packing, or if one knows with certainty that a scatterer was situated at a specified point, in which event $n(\underline{r})$ would assume a delta-function behavior.

Again, the statistical averaging of $|\psi(\underline{r};\underline{r}_1,---,\underline{r}_N)|^2$ is straightforward and the result, preserving the order of terms in Equ. (2.35), is

$$\langle|\psi(\underline{r})|^2\rangle = |\psi^o(\underline{r})|^2 \left[1-f(\underline{r})\right]$$

$$+ 2\mathrm{Re}\left\{\psi^{o*}(\underline{r})\left[\int_{\underline{r}\ \text{"outside"}\ \underline{r}'} d\tau'\, n(\underline{r}')\langle\psi^S(\underline{r}';\underline{r})\rangle_{\underline{r}'} - \iint_{\underline{r}\ \text{"inside"}\ \underline{r}'} d\tau'\, d\tau''\, n(\underline{r}',\underline{r}'')\langle\psi^S(\underline{r}'';\underline{r})\rangle_{\underline{r}',\underline{r}''}\right]\right\}$$

$$+ \iint_{\underline{r}\ \text{"outside"}\ \underline{r}',\underline{r}''} d\tau'\, d\tau''\, n(\underline{r}',\underline{r}'')\,\langle\psi^{S*}(\underline{r}';\underline{r})\,\psi^S(\underline{r}'';\underline{r})\rangle_{\underline{r}',\underline{r}''}$$

$$- \iiint_{\underline{r}\ \text{"inside"}\ \underline{r}'} d\tau'\, d\tau''\, d\tau'''\, n(\underline{r}',\underline{r}'',\underline{r}''')\,\langle\psi^{S*}(\underline{r}'';\underline{r})\,\psi^S(\underline{r}''';\underline{r})\rangle_{\underline{r}',\underline{r}'',\underline{r}'''}$$

$$+ \int_{\underline{r}\ \text{"outside"}\ \underline{r}'} d\tau'\, n(\underline{r}')\langle|\psi^S(\underline{r}';\underline{r})|^2\rangle_{\underline{r}'} - \iint_{\underline{r}\ \text{"inside"}\ \underline{r}'} d\tau'\, d\tau''\, n(\underline{r}',\underline{r}'')\langle|\psi^S(\underline{r}'';\underline{r})|^2\rangle_{\underline{r}',\underline{r}''}$$

$$+ \int_{\underline{r}\ \text{"inside"}\ \underline{r}'} d\tau'\, n(\underline{r}')\,\langle|\psi^I(\underline{r}';\underline{r})|^2\rangle_{\underline{r}'}$$

The partially averaged quantities appearing above may be computed by the methods of Section (2.3), using either implicit or quasi-implicit starting representations. Because of the tremendous complexity of the computation, further details will not be presented here, even though results would unquestionably be of great physical interest, particularly in problems involving strongly incoherent excitation. Instead, the reader is referred to Foldy [1], who has considered $\langle |\psi(\underline{r})|^2 \rangle$ in some detail for an ensemble of isotropic point scatterers. His approximations, it might be mentioned, are equivalent to taking the first terms in the expansions obtained in this paper by the quasi-implicit technique.

The first four odd terms on the right side of Equ. (2.37) specify the averaged values of the first four terms on the right side of Equ. (2.37), respectively, in the event that the evaluation point $\underline{r}$ is on the average an "outside" point, i.e. $f(\underline{r}) \approx 0$. The last term gives the averaged contribution of the internal field to the energy. Should $\underline{r}$ be nearly an "inside" point, whence $f(\underline{r}) \approx 1$, each even term on the right serves to kill the preceding odd term, leaving only the "inside" point contribution.

We saw above that $|\langle \psi(\underline{r}) \rangle|^2$, the energy associated with the averaged total field, contained terms representing interactions between fields which physically <u>can never interact</u>. We are now confronted with a second paradox. $|\langle \psi(\underline{r}) \rangle|^2$ is the energy density one would obtain using an amplitude-measuring device to measure $\langle \psi(\underline{r}) \rangle$, while $\langle |\psi(\underline{r})|^2 \rangle$ could be measured directly, using an energy-sensitive device. In general <u>the two results will not agree</u>.

The latter difficulty is doubtless not all unexpected to those familiar

with the concepts of coherence and incoherence. Both paradoxes are intimately connected with the concept of measurement, and in order to resolve them we now define precisely the various physical measurements which are permissible.

From the statistical nature of the problem with which we are dealing, clearly it is imperative to make an averaged measurement of some sort. There are in general two ways in which this may be done.

The first is a space averaging, appropriate to problems having a high degree of symmetry. Suppose we know from the symmetry of the problem that the averaged total field is a plane, spherical, or cylindrical wave, for example. If now several measurements of the field are made at different points on a surface of equal phase (that is, what should be a surface of equal phase purely from symmetry considerations) and the average value of these measurements taken, this average value will give the averaged total field, $\langle \psi(\underline{r}) \rangle$, at all points on the chosen surface of equal phase (any discussion of the error incurred in measurements of this sort falls purely in the realm of probability theory, and will not be carried further in this paper.)

This explains why the energy density associated with the averaged total field contains extraneous interactions - the measured $\langle \psi(\underline{r}) \rangle$ is a sum of terms, some containing internal fields, some external, and the computation of $|\langle \psi(\underline{r}) \rangle|^2$ will introduce the cross-terms in question.

If on the other hand one desired to measure the averaged energy density, the energy associated with each of the individual measurments mentioned above is obtained, using either an amplitude or energy-sensitive device to make the actual measurements, and the average of these results gives $\langle |\psi(\underline{r})|^2 \rangle$ at all points on the appropriate surface of equal phase. Since the process of averaging experimental data will not commute with the operation of com-

puting energy, it is now clear why one could not expect equality between $|\langle\psi(\underline{r})\rangle|^2$ and $\langle|\psi(\underline{r})|^2\rangle$.

Should the excitation in any region of space be purely incoherent, and the configuration of scatters truly random, then measurement in this region would yield $\langle\psi(\underline{r})\rangle = |\langle\psi(\underline{r})\rangle|^2 = 0$, while $\langle|\psi(\underline{r})|^2\rangle$ would be non-vanishing. For this reason $\langle\psi(\underline{r})\rangle$ is sometimes referred to as the "coherent" field[2]. However, it is important to note that these measurements would only result in the case of true randomness. As knowledge of the positions of the various scatterers increases, the concept of $\langle\psi(\underline{r})\rangle$ as coherent becomes less and less valid, and in the limit when the positions of all scatterers are known with certainty, $\langle\psi(\underline{r})\rangle$ becomes exact and $|\langle\psi(\underline{r})\rangle|^2 = \langle|\psi(\underline{r})|^2\rangle$.

The alternative scheme for making measurements involves sample averages. One imagines a large bin filled with samples, each containing an array of scatterers representing various microstates, and collectively forming the ensemble of interest.

We choose several samples from the bin and perform the experiment with each individually, each time making a measurement at the same field point. The analysis of the data now proceeds exactly as above, with, of course, the advantage that no symmetry is required of the problem.

There are many other quantities whose measurement may be of physical interest. For example, one may inquire about the field in the immediate external neighborhood of a scatterer, knowing that scatterer is there. This quantity is of particular interest if scatterers cannot support wave motion in their interior, in which case the averaged total field is, of course, undefined. The technique of measuring further quantities, along the same lines described above, is fairly apparent. For this reason, we dispense with any further discussion of the question of measurement.

III. APPLICATION TO PROBLEMS OF PHYSICS

3.1 Scattering by a Random Distribution of Isotropic Point Scatterers

In order to demonstrate the usefulness of the approximation consisting of taking only the first two terms on the right side of Equ. (2.28), the integral equation for the first partial average of the external field, we consider an array of isotropic point scatterers. The mathematical treatment of such a case is considerably simpler than it is if obstacles have arbitrary scattering properties, enabling us to carry along the third term on the right side of Equ. (2.28) and show that it makes negligible contributions to the results in many cases of physical interest.

The equations simplify considerably in the case of point scatterers. Consider first $\langle \psi^{E}(\underline{r}_1;\underline{r}) \rangle_{\underline{r}_1}$. As scatterer volumes become smaller, the region in which this field quantity is a regular solution of the Helmholtz equation in $\underline{r}$ diminishes, in the limit reducing to the point $\underline{r}_1$. Hence we simplify the notation by writing

$$\langle \psi^{E}(\underline{r}_1;\underline{r}) \rangle_{\underline{r}_1} \rightarrow \langle \psi^{E}(\underline{r}_1) \rangle \tag{3.1}$$

The internal field is no longer of interest. For the scattered field we may write

$$T^{S}(\underline{r}_1) \langle \psi^{E}(\underline{r}_1;\underline{r}) \rangle_{\underline{r}_1} \rightarrow T^{S}(\underline{r};\underline{r}_1) \langle \psi^{E}(\underline{r}_1) \rangle . \tag{3.2}$$

$T^{S}(\underline{r}_1)$ was originally defined as an operation on a regular solution of the Helmholtz equation. In the case of point scatterers, where this functional behavior has been lost, we take Equation (3.2) as the modified definition of $T^{S}(\underline{r}_1)$. That is, the right side of the equation is defined to be the limit

of the left side, if this limit exists. The notation for the scattering operator has been expanded to include the field point of evaluation, since this quantity can no longer be contained in the operand.

We simplify the discussion further by taking the scatterers to be statistically independent in the limit. Then all conditional densities become equal, and we have

$$n(\underline{r}) = n(\underline{r}|\underline{r}_1) = n(\underline{r}|\underline{r}_1,\underline{r}_2) = - - -$$

As we are now dealing with a problem involving complete randomness of scatterer positions, it is appropriate to invoke the approximate integral equation (2.28), for the lowest partial average of the external field, obtaining

$$\begin{aligned}\langle \psi^E(\underline{r}_1)\rangle \to \psi^o(\underline{r}_1) &+ \int d\tau'\, n(\underline{r}')\, T^S(\underline{r}_1;\underline{r}')\, \langle \psi^E(\underline{r}')\rangle \\ &+ \int d\tau'\, n(\underline{r}')\, T^S(\underline{r}_1;\underline{r}')\, T^S(\underline{r}';\underline{r}_1)\, \langle \psi^E(\underline{r}_1)\rangle\end{aligned} \tag{3.3}$$

Also, from Equ. (2.34), we obtain for the averaged total field,

$$\langle \psi(\underline{r})\rangle \to \psi^o(\underline{r}) + \int d\tau'\, n(\underline{r}')\, T^S(\underline{r};\underline{r}')\, \langle \psi^E(\underline{r}')\rangle. \tag{3.4}$$

Comparision with the previous equation gives finally

$$\langle \psi^E(\underline{r}_1)\rangle \to \langle \psi(\underline{r}_1)\rangle + \int d\tau'\, n(\underline{r}')\, T^S(\underline{r}_1;\underline{r}')\, T^S(\underline{r}';\underline{r}_1)\, \langle \psi(\underline{r}_1)\rangle \tag{3.5}$$

and

$$\begin{aligned}\langle \psi(\underline{r})\rangle \to \psi^o(\underline{r}) &+ \int d\tau'\, n(\underline{r}')\, T^S(\underline{r};\underline{r}')\, \langle \psi(\underline{r}')\rangle \\ &+ \int d\tau'\, n(\underline{r}')\, T^S(\underline{r};\underline{r}') \int d\tau''\, n(\underline{r}'')\, T^S(\underline{r}';\underline{r}'')\, T^S(\underline{r}'';\underline{r}')\, \langle \psi(\underline{r}')\rangle\end{aligned} \tag{3.6}$$

where one iteration was performed to obtain Equation (3.5).

The interpretation of Equ.(3.5) is straightforward. The first term on the right, $\langle \psi(\underline{r}_1) \rangle$, is the contribution to the external field at $\underline{r}_1$ computed as if there were no scatterer there. The next term is the correction obtained when a scatterer is inserted at $\underline{r}_1$, allowed to scatter the"incident" wave $\langle \psi(\underline{r}_1) \rangle$, and the resulting scattered wave then rescattered once from all other scatterers back to $\underline{r}_1$.

Foldy, for the case of isotropic point scatterers, used the approximation (1)

$$\langle \psi^E(\underline{r}_1) \rangle_{\underline{r}_1} \approx \langle \psi(\underline{r}_1) \rangle ,$$

while Lax took for the anisotropic point scatterers the approximation (2)

$$\langle \psi^E(\underline{r}_1) \rangle_{\underline{r}_1,\underline{r}_2} \approx \langle \psi^E(\underline{r}_1) \rangle_{\underline{r}_1} .$$

It is seen that both of these assumptions lead to the integral equ.(3.6) with only the first two terms on the right and therefore agreed to within "square and higher powers of the scattering act".

Now in the special case of isotropic scatterers, the scattered wave is proportional to the incident wave, and the scattering operator reduces to the factor

$$T^S(\underline{r};\underline{r}') = B_o h_o(k|\underline{r}-\underline{r}'|) \equiv -\frac{i B_o}{k} \frac{e^{ik|\underline{r}-\underline{r}'|}}{|\underline{r}-\underline{r}'|}, \qquad (3.7)$$

where B_o is as yet an arbitrary proportionality constant, in general complex, and $h_o(z)$ is the spherical Hänkel function of the first kind of order zero.

For this form of the scattering operator, Equ. (3.6) for the averaged total field becomes, to within terms involving three successive scatterings,

$$\langle\psi(\underline{r})\rangle = \psi^{o}(\underline{r}) + B_{o}\int d\tau' \, n(\underline{r}') \, h_{o}(k|\underline{r}-\underline{r}'|) \, \langle\psi(\underline{r}')\rangle$$

(3,8)

$$+ B_{o}^{3}\int d\tau' n(\underline{r}') h_{o}(k|\underline{r}-\underline{r}'|) \int d\tau'' n(\underline{r}'') h_{o}^{2}(k|\underline{r}'-\underline{r}''|)\langle\psi(\underline{r}'')\rangle$$

Following Foldy[1], we note that, due to the Green's function nature of the scattering operator, operating on $T^{S}(r;r')$ with the unperturbed (i.e. matrix material) Helmholtz operator yields a δ-function:

$$(\Delta_{\underline{r}} + k^{2}) \, T^{S}(\underline{r};\underline{r}') = + \frac{4\pi \, iB_{o}}{k} \, \delta(\underline{r}'-\underline{r}) \qquad (3.9)$$

Hence operating on both sides of Equ. (3.8) with $(\Delta + k^{2})$, and assuming this operator may be carried inside the integral signs on the right side, the result may be written

$$\left[\Delta + K^{2}(\underline{r})\right]\langle\psi(\underline{r})\rangle = 0 \qquad (3.10)$$

where the "scattering medium" propagation factor $K(\underline{r})$ is given by

$$K(\underline{r}) = k\left[1 - \frac{4\pi i B_{o} n(\underline{r})}{k^{3}}\left(1 - \frac{B_{o}}{k^{2}}\int d\tau'' \, n(\underline{r}'') \, \frac{e^{2ik|\underline{r}-\underline{r}''|}}{|\underline{r}-\underline{r}''|^{2}}\right)\right]^{\frac{1}{2}} .$$

If one neglects the third term in the radical, this is the result obtained by Foldy.

Since $K(\underline{r})$ is only correct to within terms of second order in the scatterer density, we expand the radical correct to this order, obtaining finally (3.11)

$$K(\underline{r}) = k\left[1 - \frac{2\pi i B_{o} n(\underline{r})}{k^{3}}\left(1 - \frac{B_{o}^{2}}{k^{2}}\int d\tau'' n(\underline{r}'') \, \frac{e^{2ik|\underline{r}-\underline{r}''|}}{|\underline{r}-\underline{r}''|^{2}}\right) - 2\left(\frac{\pi i B_{o} n(\underline{r})}{k^{3}}\right)^{2}\right]$$

For a homogeneous scattering medium, where $n(\underline{r}) = n_o$ = constant, we have

$$\int d\tau'' \, n(\underline{r}'') \frac{e^{2ik|\underline{r}-\underline{r}''|}}{|\underline{r}-\underline{r}''|^2} = 4\pi n_o \int_o^\infty dp e^{2ikp} = + \frac{2\pi i n_o}{k},$$

where the finitely oscillating part of the integral has been discarded, invoking the usual arguments (one might, for example, endow the matrix material with a small but non- vanishing dissipation). This yields for the propagation factor the value

$$K = k\left[1 - 2\frac{\pi i B_o n_o}{k^3} - 2\left(\frac{\pi i B_o n_o}{k^3}\right)^2 (1 - 2B_o)\right] \qquad (3.12)$$

Hence for a plane wave, we may write

$$\langle \psi(z) \rangle = e^{iKz} \equiv e^{i\frac{\omega}{v'}z} e^{-\alpha z} \qquad (3.13)$$

With attenuation and modified phase velocity given by

$$\alpha = -\frac{2\pi n_o}{k^2} \mathrm{Re}\left\{B_o\left[1 + \frac{\pi i B_o n_o}{k^3}(1 - 2B_o)\right]\right\}$$

$$v' = v\left[1 + \frac{2\pi n_o}{k^3} \mathrm{Im}\left\{B_o\left[1 + \frac{\pi i B_o n_o}{k^3}(1 - 2B_o)\right\}\right]\right]^{-1}. \qquad (3.14)$$

Let us apply this result to the case of scattering by an array of spherical obstacles, each of radius a. Taking the Rayleigh limit where $ka \ll 1$, the spheres will scatter nearly isotropically (while in general dipole and quadrupole radiation must also be considered, even in the Raleigh limit, the modifications thus introduced do not materially affect the present discussion) and if we further restrict the fractional volume of scattering material to moderately small values, the point scatterer computation might be expected to apply.

The simplest boundary condition for non-dissipative obstacles is the Neumann condition, that the normal gradient of the potential must vanish on the surface of the obstacle, which gives (6a)

$$B_o = - \frac{j'_o (ka)}{h_o' (ka)} \xrightarrow[ka \to 0]{} \frac{-i (ka)^3}{3} \left[1 - \frac{i (ka)^3}{3} \right] ,$$

where we have kept the terms of lowest order in (ka) in the real and imaginary parts of B_o separately. Here $j_o (z)$ is the spherical Bessel function of zero order. This choice of B_o corresponds to the case of an array of rigid fixed spherical obstacles in a perfect fluid, having the boundary condition that the normal component of fluid velocity must vanish on the surfaces of the obstacles.

Using this value of B_o in Equs.(3.14) and noting that the fractional volume of scattering material is given by

$$f = \frac{4}{3} \pi a^3 n_o ,$$

we obtain for the attenuation and modified phase velocity of a plane wave

$$\alpha = \frac{n_o \gamma_s}{2}$$

$$v' = v \left[1 - \frac{f}{2} \right]^{-1}$$

where γ_s is the single scatterer cross-section for an incident plane wave, given by (6b)

$$\gamma_s = \frac{4}{9} a^2 (ka)^4 .$$

The attenuation is just that which one would obtain by assuming that the fractional energy lost by the propagated wave per unit volume was given simply by the product of scattering cross-section and scatterer density.

Thus if we consider a slab of material bounded by the planes z and z + dz, the energy-dI(z) lost by the propagated wave per unit area while traversing the slab is to a first approximation the intensity I(z) of the wave incident on the slab, times the scattering cross-section γ_s of one obstacle, times the number of obstacles per unit area in the slab, $n_o dz$.

Now

$$\frac{dI(z)}{I(z)} = -n_o \gamma_s dz$$

and by integration we obtain

$$I(z) = I_o e^{-n_o \gamma_s z}$$

which gives

$$\alpha = n_o \gamma_s / 2.$$

The use of scattering cross-sections implies that the obstacles respond as they would to an incident plane wave. While the present discussion is restricted to plane waves, it could just as well have been carried on more generally, as a consideration of real and imaginary parts of K regardless of the type of wave being propagated. The appearance of scattering cross-sections would then have been by virtue of the Rayleigh nature of the individual obstacles. That is, in a region sufficiently small in comparision with wave-length, any regular wave motion approximates a plane wave.

Had we kept the next term in the imaginary part of B_o, we would have obtained a square law frequency dependence for the phase velocity. As these higher order terms in ka are contributed to by higher harmonics, however, and true isotropic scattering occurs rather infrequently in physical applications, we reserve any more detailed computations for a following example, in which scatterers will not be restricted to the Rayleigh condition.

The important point to be established here is the validity of the starting approximation, the integral equation (3.8). The effect of the third term on the right was only to cancel a term of order $(n_o Q_s/z)(f/2)$ in the expression for attenuation, not modifying the phase velocity at all when terms of order $(ka)^2$ are neglected in comparision with one.

A little consideration shows that each additional term kept on the right side of the approximate integral equation for $\langle \Psi(\underline{r}) \rangle$ will contain one higher power of the fractional volume of scatterers. Hence for small to moderate values of f the scheme seems to yield rather good convergence. For large values of f the approximation that scatterer positions were statistically independent breaks down, and other techniques must be employed.

In connection with multiple scattering fr an assembly of isotropic scatterers, we might also consider the cases of dissipative and resonant obstacles. Here the model chosen is an array of spherical fluid obstacles embedded in a perfect fluid.

Consider first the case of a dissipative obstacle, with $ka \ll 1$. This might correspond to an array of viscous oil bubbles in water, for example. The single scatterer computation with plane wave incident has been done by Morse and Feshbach[6c]. From their results one obtains the value

$$B_o \xrightarrow[ka \to 0]{} \frac{-i}{3}(ka)^3 \left[1 - \frac{\rho}{\rho_s}(p + iq)^2\right]$$

for the amplitude of the scattered potential, where ρ and ρ_s are the densities of matrix and scattering material, respectively, and $p + iq$ is the complex index of refraction of the scattering material, in terms of which the scattering material propagation constant k_s is given by

$$k_s = (p + iq)\, k.$$

Now from Equ. (3.14) the attenuation and modified phase velocity of a plane wave propagating in the "scattering medium" are given by

$$\alpha \approx \frac{\rho \beta q f}{\rho_s} k$$

$$v' \approx v\left[1 - \frac{f}{2}\left(1 - \frac{\rho}{\rho_s}\right)(p^2 - q^2)\right]^{-1}$$

Furthermore, from Morse and Feshbach's computation, one may obtain for the absorption cross-section

$$\gamma_A = \frac{8\pi a^3 \rho \beta q k}{3 \rho_s},$$

in terms of which the attenuation may be written

$$\alpha \approx \frac{n_o \gamma_A}{2}$$

again a result which can be obtained by a simple intuitive scheme. The change in phase velocity is proportional to the fractional volume of scatterers, and the phase velocity may either increase or decrease depending on the ratio of densities of matrix to scattering material and so forth.

We have dropped contributions from the double integral of the approximate integral equation for $\langle \psi(\underline{r}) \rangle$, because in this case they are negligible in comparison with the above results. Hence the approximation scheme is extremely simple in dealing with an array of dissipative obstacles small compared to the wavelength of incident radiation, with only the first two terms on the right of Equ. (3.8) required for high accuracy.

Finally, we mention the case of resonant air bubbles in water. Applying the results of Morse and Feshbach, who computed B_o for the single bubble case, (6d)

we again obtain negligible contributions to the attenuation and phase velocity due to the B_o^3 terms in Equs. (3.14).

From these examples we conclude that the approximation involved in taking only three terms in the integral equation for $\langle \psi(\underline{r}) \rangle$, as done in Equ.(3.8), is in general quite good. Note that we have obtained results equivalent to Foldy's, with small correction terms in a few cases. Thus Foldy's result, given by

$$K(\underline{r}) = k\left[1 - \frac{4\pi i B_o\, n(\underline{r})}{k^3}\right]^{\frac{1}{2}},$$

is in agreement with ours, provided the radical is expanded and only the first three terms kept. The terms beyond these will be modified by higher terms neglected in Equ. (3.8).

3.2. Transmission of Plane Waves by a Semi-Infinite Uniform Array of Spherical Obstacles

In this examples we are concerned with the transmission properties of a medium containing a semi-infinite homogeneous distribution of identical spherical obstacles with general scattering and absorption properties. Because of the complexity of the computations, we restrict ourselves to the case of a normally incident plane wave.

Each obstacle is located by specifying the position of its center, and we consider the simplest statistical problem in which the positions of obstacles are statistically independent with the exception of interpenetration, which is excluded.

The densities are then given by

$$n(\underline{r}) = \begin{cases} n(z) = n_o \text{(constant)} & \text{for } z \geq 0 \\ 0 & \text{for } z < 0 \end{cases}$$

$$n(\underline{r}|\underline{r}_1) = \begin{cases} n_o \text{ for } z \geq 0 \text{ and } |r - r_1| \geq 2a \\ 0 \text{ otherwise} \end{cases} \qquad (3.15)$$

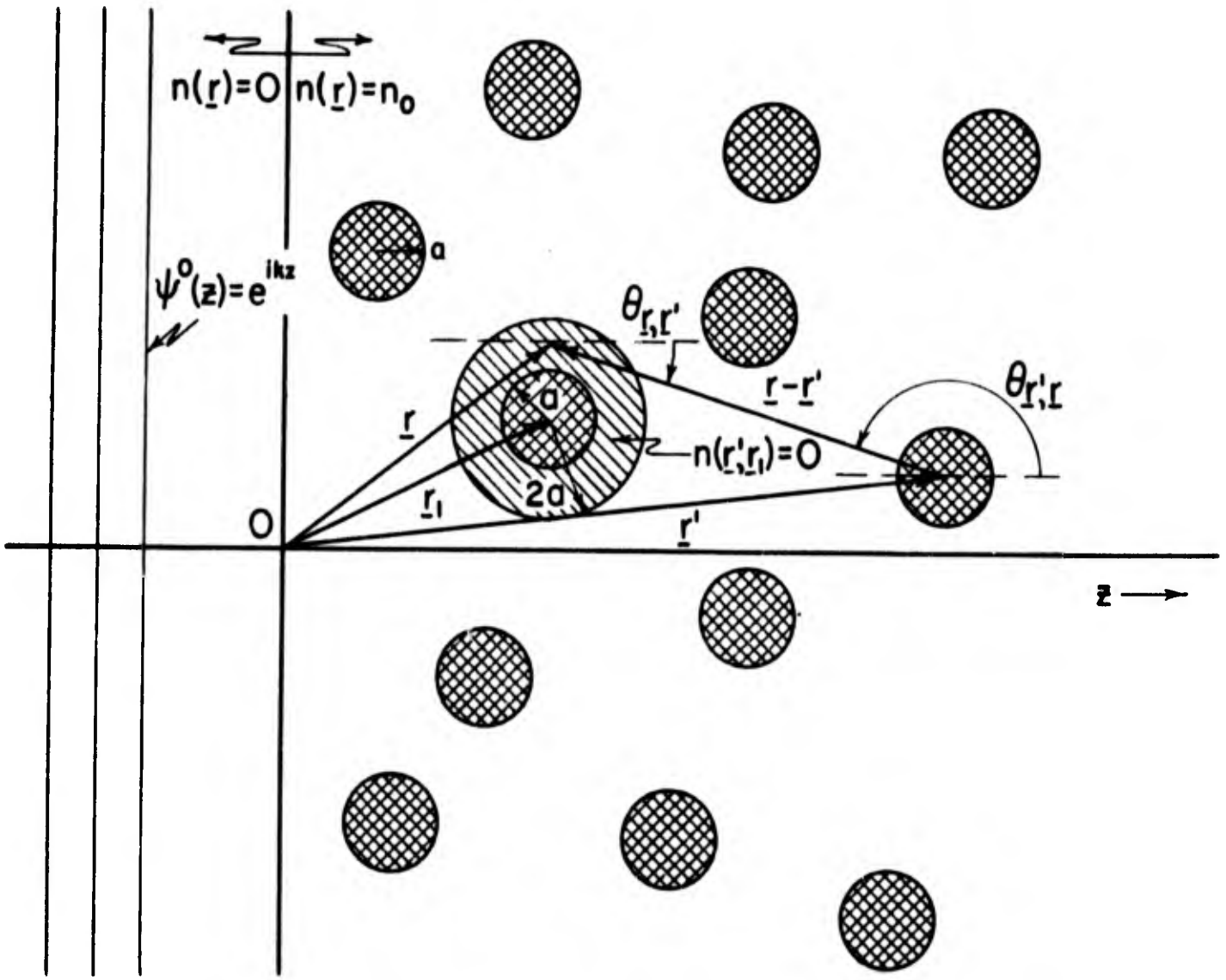

Fig. 1. The geometry of a semi-infinite array of identical spherical obstacles, showing a specified configuration. For an array of obstacles which cannot interpenetrate, the external field acting on the obstacle at $\underline{r}_1$ can have no singularities at any point $\underline{r}$ in the shaded region shown, which is defined by $|\underline{r} - \underline{r}_1| < 2a$.

and so forth, where a is the radius of each obstacle. The fractional volume of obstacles is restricted to small to moderate values, in order that the assumption of statistical independence be valid, and we utilize the approximate integral equation (2.28) for the first partial average of the external field in the form

$$\langle \psi^E(\underline{r}_1;\underline{r}) \rangle_{\underline{r}_1} = \psi^o(\underline{r}) + \int d\tau' n(\underline{r}'|\underline{r}_1)\ T^s(\underline{r}')\ \langle \psi^E(\underline{r}';\underline{r}) \rangle_{\underline{r}'} . \quad (3.16)$$

The incident potential is taken to be

$$\psi(\underline{r}) = \psi^o(z) = e^{ikz} \quad (3.17)$$

and having solved Equ. (3.16) the averaged total field may then be obtained from Equ. (2.32) by integration. The geometry is shown in Fig. (1) for a given configuration of the obstacles.

The validity of the approximation involved in using only two terms on the right side of Equ. (3.16) has been established in several cases in the low-frequency limit, through the discussion of isotropic point scatterers. We shall attempt to further justify Equ. (3.16) later in the discussion.

Now $\langle \psi^E(\underline{r}_1;\underline{r}) \rangle_{\underline{r}_1}$ is known to be a solution of the Helmholtz equation in $\underline{r}$, and further can have no singularities in the open region $|\underline{r}-\underline{r}_1| < 2a$. This latter condition follows simply from the fact that with interpenetration forbidden, the distance from $\underline{r}_1$ to the center of an adjacent obstacle (and hence to a singularity in the external field acting on $\underline{r}_1$) must be at least twice the radius of the obstacle.

Utilizing spherical harmonics and noting that the z-axis is an axis of symmetry for $\langle \psi^E(\underline{r}_1;\underline{r}) \rangle_{\underline{r}_1}$ this quantity must have the form

$$\langle \psi^E(\underline{r};\underline{r}) \rangle_{\underline{r}_1} = \sum_{n=0}^{\infty} A_n(z_1)\, j_n(k|\underline{r}-\underline{r}_1|)\, P_n(\cos \vartheta_{\underline{r}_1,\underline{r}}) \qquad (3.18)$$

where the $j_n(z)$ and $P_n(\cos \vartheta)$ are spherical Bessel functions of the first kind of order n, and Legendre polynomials of order n, respectively [6e]. The undetermined expansion coefficients $A_n(z_1)$ express the dependence of the external field on the position of the obstacle "receiving" it and depend only on z_1, the z component of $\underline{r}_1$, as a further consequence of the symmetry of the problem. $\vartheta_{\underline{r}_1,\underline{r}}$ is the angle formed by the vector $\underline{r}-\underline{r}_1$ and the positive z-direction.

We further restrict each obstacle to axial symmetry about the z-axis in all its scattering properites. Specifying in addition the radiation condition, that all scattered radiation be outgoing, the most general form of the scattering operator is given by

$$T^S(\underline{r}_1)\, j_n(k|\underline{r}-\underline{r}_1|)\, P_n(\cos \vartheta_{\underline{r}_1,\underline{r}}) = \sum_{m=0}^{\infty} B^n_m h_m(k|\underline{r}-\underline{r}_1|)\, P_n(\cos \vartheta_{\underline{r}_1,\underline{r}})$$

where the $h_m(z)$ are spherical Hankel functions of the first kind of order m [6e] and the expansion coefficients B^n_m are determined from the boundary conditions of the single scatterer problem.

Since the scattering operation is assumed to be linear, applying $T^S(\underline{r})$ to Equ. (3.18) and making use of superposition we obtain

$$T^S(\underline{r}_1)\, \langle \psi^E(\underline{r}_1;\underline{r}) \rangle_{\underline{r}_1} = \sum_{m,n=0}^{\infty} B^n_m A_n(z_1)\, h_m(k|\underline{r}-\underline{r}_1|)\, P_m(\cos \vartheta_{\underline{r}_1,\underline{r}}) \qquad (3.19)$$

for the scattered potential resulting when the external field $\langle \psi^E(\underline{r}_1;\underline{r}) \rangle_{\underline{r}_1}$ inpinges on an obstacle with center at $\underline{r}_1$.

Making use of the expansion of a plane wave in spherical harmonics, an excellent treatment of which is given by Morse and Feshbach [6f], we may write the incident wave as

$$\begin{aligned}\psi^o(z) &= e^{ikz} \\ &= e^{ikz_1}\, e^{ik(z-z_1)} \\ &= e^{ikz_1} \sum_{n=0}^{\infty} a_n\, j_n(k|\underline{r}-\underline{r}_1|)\, P_n(\cos \vartheta_{\underline{r}_1,\underline{r}})\end{aligned} \tag{3.20}$$

where the expansion coefficients are given by $a_n = (2n+1)\,(i)^n$.

Inserting the expressions for the external field, scattered field, and incident field, as given by Equs. (3.18-20) in the integral equation (3.16) and bringing $\psi^o(z)$ over to the left side, we obtain

$$\sum_{n=0}^{\infty} \left[A_n(z_1) - a_n e^{ikz_1}\right] j_n(k|\underline{r}-\underline{r}_1|)\, P_n(\cos \vartheta_{\underline{r}_1,\underline{r}}) \tag{3.21}$$

$$= \sum_{m,n=0}^{\infty} B_m^n \int d\tau'\, n(\underline{r}'|\underline{r}_1)\, A_n(z')\, h_m(k|\underline{r}-\underline{r}'|)\, P_m(\cos \vartheta_{\underline{r}',\underline{r}}).$$

It is now convenient to re-expand the right side of Equ. (3.21) in terms of the functions involved on the left side, which will enable us to dispense with the unprimed coördinates ($\underline{r}$) entirely.

Noting from Fig. (1) that $\vartheta_{\underline{r}',\underline{r}} = \pi - \vartheta_{\underline{r},\underline{r}'}$ and using the well-known relation $P_m(-\mu) = (-1)^m P_m(\mu)$, we may write

$$h_m(k|\underline{r}-\underline{r}'|)\, P_m(\cos \vartheta_{\underline{r}',\underline{r}}) = (-1)^m h_m(k|\underline{r}'-\underline{r}|)\, P_m(\cos \vartheta_{\underline{r},\underline{r}'})$$

$$= (+i)^m P_m\left(\frac{1}{ik}\frac{\partial}{\partial z'}\right) h_o(k|\underline{r}'-\underline{r}|) \tag{3.22}$$

Here the first equality constitues a change in origin of the wave-functions from $\underline{r}'$ to $\underline{r}$, and follows from the two preceding relations. The second equality results from application of the multipole representation of $h_m(k\underline{r})\ P_m(\cos\vartheta)$, of which a derivation is presented in Appendix I.

The expansion of $h_o(k|\underline{r}'-\underline{r}|)$ in terms of spherical Bessel functions about the origin $\underline{r}$ is now performed. From Morse and Feshbach[6f] we obtain

$$h_o(k|\underline{r}'-\underline{r}|) = \sum_{n=0}^{\infty}(2n+1)\sum_{m=0}^{n}\varepsilon_m\frac{(n-m)!}{(n+m)!}\cos\left[m(\varphi_{\underline{r}_1,\underline{r}}-\varphi_{\underline{r}_1,\underline{r}'})\right] \tag{3.23}$$

$$\cdot P_n^m(\cos\vartheta_{\underline{r}_1,\underline{r}'})\ P_n^m(\cos\vartheta_{\underline{r}_1,\underline{r}})\ j_n(k|\underline{r}-\underline{r}_1|\ h_n(k|\underline{r}'-\underline{r}_1|)$$

$$\text{for}\ |\underline{r}-\underline{r}_1|<|\underline{r}'-\underline{r}_1|\ .$$

where

$$\varepsilon_m = \begin{cases}1 & m=0\\ 2 & m>0\end{cases}$$

The $P_n^m(\mu)$ are the associated Lengendre polynomials, and the subscript $\underline{r}_1,\underline{r}$, notation on angles follows that employed in Equ. (3.18) and following. $\varphi_{\underline{r}_1,\underline{r}}$, for example, is the azimuthal angle formed by the vector $\underline{r}-\underline{r}_1$ and the positive x-direction.

With reference to Fig. (1) we see that the shaded region containing the obstacle at $\underline{r}_1$ is effectively excluded from the range of integration in Equ. (3.21), since $n(\underline{r}'|\underline{r}_1)$ vanishes therein. Hence the range of integration is restricted by the condition $|\underline{r}'-\underline{r}_1| \geq 2a$, and for all points $\underline{r}$ with the shaded region the condition $|\underline{r}-\underline{r}_1|<|\underline{r}'-\underline{r}_1|$ is met, enabling us to use the expansion of $h_o(k|\underline{r}'-\underline{r}|)$ in Equ. (3.21).

Now Equs. (3.22) and (3.23) are inserted in Equ. (3.21), and the cylindrical coordinates (ρ', φ', z') are chosen for the volume integration, where

$$\begin{aligned} \rho' &= \left[|\underline{r}'-\underline{r}_1|^2 - |z' - z_1|^2 \right]^{\frac{1}{2}} \\ \varphi' &= \varphi_{\underline{r}_1,\underline{r}'} \\ z' &= z' \end{aligned} \tag{3.24}$$

Upon noting that the terms in $\cos\left[m(\varphi_{\underline{r}_1,\underline{r}} - \varphi_{\underline{r}_1,\underline{r}'})\right]$ vanish under the φ' integration unless m=0, Equ. (3.21) reduces to

$$\sum_{n=0}^{\infty} j_n(k|\underline{r}-\underline{r}_1|)\, P_n(\cos\vartheta_{\underline{r}_1,\underline{r}}) \Big\{ A_n(z_1) - a_n e^{ikz_1}$$

$$-2\pi(2n+1)\sum_{m,j=0}^{\infty} (i)^{j-m} B_j \int dz'\, A_m(z') \int \rho' d\rho'\, n(\underline{r}'|\underline{r}_1)\; P_j\left(\frac{1}{ik}\frac{\partial}{\partial z'}\right)\left[h_n(k|\underline{r}'-\underline{r}_1|)\right.$$

$$\left.\cdot P_n(\cos\vartheta_{\underline{r}_1,\underline{r}'})\right]\Big\} = 0.$$

This equation is valid for $|\underline{r}-\underline{r}_1| < 2a$, and from the orthogonality properties of the $P_n(\cos\vartheta_{\underline{r}_1,\underline{r}})$ we see that the term in brackets must vanish for each value of n, yielding the set of simultaneous ordinary integral equations

$$A_n(z_1) - a_n e^{ikz_1}$$

$$= (2\pi)(2n+1)(-i)^n \sum_{m,j=0}^{\infty} (i)^{j-m} B_j \int dz'\, A_m(z') \int \rho' d\rho'\, n(\underline{r}'|\underline{r}_1)\; P_j\left(\frac{1}{ik}\frac{\partial}{\partial z'}\right)$$

$$P_n\left(\frac{1}{ik}\frac{\partial}{\partial z'}\right) h_o(k|\underline{r}'-\underline{r}_1|) \tag{3.26}$$

for n=0,1,2----

where we have employed the multipole representation of $h_n(k|\underline{r}'-\underline{r}_1|)\cdot P_n(\cos\vartheta_{\underline{r}_1,\underline{r}'})$. The limits of integration in Equ. (3.26) are determined by the conditional density $n(\underline{r}'|\underline{r}_1)$. Notice that while the Legendre operators

$P_j(\partial/ik\,\partial z')$ and $P_n(\partial/ik\,\partial z')$ are independent of the integration variable ρ', they cannot be taken outside of the ρ' integration since they do not commute with $n(\underline{r}'|\underline{r}_1)$.

For $z_1 \geq 2a$, the conditional density $n(\underline{r}'|\underline{r}_1)$ prescribes integration over the half-space $z' \geq 0$, with the spherical region $|\underline{r}'-\underline{r}_1| \leq 2a$ deleted, as may be seen from Equs.(3.15). However, for $z_1 < 2a$ the deleted region is the truncated sphere given by $|\underline{r}'-\underline{r}_1| \leq 2a$ and $z' \geq 0$. Hence it appears that there are "surface" effects in the transition region $0 \leq z_1 \leq 2a$, in which the external field behavesin a more complicated manner than it does for $z_1 > 2a$.

While the presence of such "surface" effects is not surprising the exact physical explanation is not apparent. One does notice, however, that the transition region thickness $2a$ is just the distance an obstacle must be moved into the scattering medium before complete "shielding" (i.e. lying directly in the "shadow" or positive z-direction with respect to an adjacent obstacle) is possible.

Excluding this transition region, we now assume that the only functional variation of the external field acting on an obstacle when the position of the obstacle is varied is a simple harmonic phase behavior and an overall attenuation as the obstacle is moved further into the scattering medium.

Thus we take

$$A_n(z_1) = A^o_n e^{iKz_1} \qquad n=0,1,2,\cdots \quad \text{for} z_1 \geq 2a \tag{3.27}$$

where the expansion coefficients A^o_n and the complex propagation constant K are to be determined. The justification of this assumption lies in the fact that it will satisfy Equs. (3.26) exactly, in the process giving expressions for K and the A^o_n. The presence of the transition region

manifests itself mathematically in the fact that the assumption of Equs.(3.27) will not satisfy Equs. (3.26) for $0 \leq z_1 \leq 2a$, and hence one must resort to handling the latter as an infinite set of simultaneous integral equations. As our present concern is with the "medium" properties of the array of obstacles, we dispense with any further consideration of this transition region.

Inserting Equs. (3.27) in Equs. (3.26) and splitting the volume integration into an integration over the entire half-space $z' \geq 0$ less an integration over the deleted spherical region, we obtain

$$A^o_n e^{iKz_1} - a_n e^{ikz_1} = 2\pi n_o (2n+1)(-i)^n \sum_{m,j=0}^{\infty} (+1)^j B^m_j A^o_m \left[I^1_{jn}(z_1) - I^2_{jn}(z_1) \right]$$

where we have written

$$I^1_{jn}(z_1) \equiv \int_0^{\infty} dz' e^{iKz'} \int_0^{\infty} \rho' d\rho' P_j\left(\frac{1}{ik}\frac{\partial}{\partial z'}\right) P_n\left(\frac{1}{ik}\frac{\partial}{\partial z'}\right) h_o\left(k\sqrt{(\rho')^2+|z'-z_1|^2}\right)$$

$$I^2_{jn}(z_1) \equiv \int_{z_1-2a}^{z_1+2a} dz' e^{iKz'} \int_0^{\sqrt{4a^2-(z'-z_1)^2}} \rho' d\rho' P_j\left(\frac{1}{ik}\frac{\partial}{\partial z'}\right) P_n\left(\frac{1}{ik}\frac{\partial}{\partial z'}\right) h_o\left(k\sqrt{(\rho')^2+|z'-z_1|^2}\right)$$

The Legendre operators may be commuted with the ρ' integration in the expression for $I^1_{jn}(z_1)$. Hence, performing the ρ' integration, discarding the finitely oscillating part of the result (as discussed in the previous example) and noting that trivially

$$P_n\left(\frac{1}{ik}\frac{\partial}{\partial z'}\right) e^{\pm ikz'} = P_n(\pm 1)\, e^{\pm ikz'} = (\pm 1)^n e^{\pm ikz'},$$

we may finally perform the z' integration to obtain

$$I^{1}_{jn}(z_1) = -\frac{(-1)^{j+n}}{i(K-k)\,k^2}\, e^{ikz_1} + \frac{1}{k^2}\left\{\frac{(-1)^{j+n}}{i\,(K-k)} - \frac{1}{i(K+k)}\right\} e^{iKz_1} \tag{3.29}$$

Here in order to ensure convergence of the z'-integration, it was necessary to assume the imaginary part of K to be greater than zero, corresponding to a positive attenuation in the scattering medium. Again, the results will verify this assumption.

The evaluation of $I^{2}_{jn}(z_1)$ is not quite so simple. The ρ'-integration proceeds independently of the Legendre operators, but these operators must be applied <u>before</u> substituting in the end points of the integration.

This difficulty may be overcome by the substitution $u^2=(\rho')^2+|z'-z_1|^2$, which transforms the Legendre operators into

$$P_n\left(\frac{1}{ik}\,\frac{\partial}{\partial z'}\right) = P_n\left(\frac{1}{ik}\,\frac{\partial u}{\partial z'}\,\frac{\partial}{\partial u}\right) = P_n\left[\frac{(z'-z_1)}{ik}\,\frac{\partial}{u\partial u}\right]$$

Finally, the dependence of the z'-integration on z_1 may be removed by the substitution $\xi = z'-z_1$, giving

$$I^{2}_{jn}(z_1) = \frac{(-1)^{j+n}}{i(K-k)k^2}\left[1 - e^{-2i(K-k)a}\right]e^{iKz_1}$$

$$+ \frac{1}{i(K+k)k^2}\left[e^{+2i(K+k)a} - 1\right] e^{iKz_1} \tag{3.30}$$

$$- \frac{1}{k^2}\, e^{iKz_1} \int_{-2a}^{+2a} d\xi\, e^{iK\xi}\, P_j\left(\frac{\xi}{4ik}\,\frac{\partial}{a\partial a}\right) P_n\left(\frac{\xi}{4ik}\,\frac{\partial}{a\partial a}\right) e^{2ika}\,.$$

While the ξ - integration could be done explicitly, invoking the polynomial forms of the Legendre operators, no immediate purpose would be served. Hence we leave the expression in its more compact integral form for the present.

Putting the above expressions for $I^{1}_{jn}(z_1)$ and $I^{2}_{jn}(z_1)$ in Equs. (3.28), we see that they are satisfied provided that the coefficients of e^{ikz_1} and e^{iKz_1} vanish separately, giving

$$K = k\left[1 + \frac{2\pi n_o}{ik^3}\sum_{m,j=0}^{\infty}(-i)^j B^m{}_j A^o{}_m\right]$$

$$A^o{}_n = a_n e^{-2i(K-k)a} - a_n^* e^{+2i(K+k)a}\frac{2\pi n_o}{i(K+k)k^2}\sum_{m,j=0}^{\infty}(+1)^j B^m{}_j A^o{}_m$$

$$+ a_n^*\frac{2\pi n_o}{k^2}\sum_{m,j=0}^{\infty}(+1)^j B^m{}_j A^o{}_m G_{jn}$$

where we have noted from Equ. (4.16) that $(-1)^n a_n = a^*_n$, and the quantity G_{jn} is given by

$$G_{jn} \equiv \int_{-a}^{+a} d\xi\, e^{iK\xi} P_j\left(\frac{\xi}{4ik}\frac{\partial}{a\partial a}\right) P_n\left(\frac{\xi}{4ik}\frac{\partial}{a\partial a}\right) e^{+2ika}. \qquad (3.32)$$

Consider first the results for the expansion coefficients of the external field. It follows from Equ. (3.20) that the a^*_n are simply the expansion coefficients of a plane wave travelling in the back (negative z) direction. Hence putting the $A_n(z_1) = A^o{}_n e^{iKz_1}$ in Equ. (3.18), the spherical harmonic expansion of the external field, we obtain

$$\left\langle \psi^E(\underline{r}_1;\underline{r})\right\rangle_{r_1} = e^{iK(z_1-2a)} e^{ik[z-(z_1-2a)]} \qquad (3.33)$$

$$- e^{iK(z_1+2a)} e^{-ik[z-(z_1+2a)]}\frac{2\pi n_o}{i(K+k)k^2}\sum_{m,j=0}^{\infty}(+1)^j B^m{}_j A^o{}_m$$

$$- e^{iKz_1}\frac{2\pi n_o}{k^2}\sum_{n,m,j=0}^{\infty} a^*_n (+1)^j B^m{}_j A^o{}_m G_{jn} j_n(k|\underline{r}-\underline{r}_1|) P_n(\cos\theta_{\underline{r}_1,\underline{r}}).$$

The external field incident on an obstacle fixed at z_1 is made up of a plane wave travelling in the forward (positive z) direction, plus a plane wave travelling in the back (negative z) direction, plus a contribution which cannot be simply described in terms of plane waves.

Now the external field incident on an obstacle is just the sum of the original incident wave and the scattered waves from all the obstacles. Examination of the computations shows that the three terms in the external field are generated by scattered waves from obstacles situated in three different regions of space, as shown in Fig. (2).

The scattered waves from obstacles located in Region I, the region between the planes $z=0$ and $z=z_1 - 2a$, add up to give two forward-travelling plane waves, the first of which just cancels the original incident wave, while the second contributes the forward-travelling plane wave in Equ. (3.33). Notice that the phase and amplitude of this wave are those of a unit amplitude plane wave starting out at the interface $z = 0$, propagating with the "scattering medium" phase velocity and attenuation up to the surface $z=z_1-2a$, and from there on propagatingaccording to the scatterer-free medium propagation constant k.

While this forward plane wave should be proportional to the "multiple-scattering strength" of the "scattering medium"--by which we mean the attenuation and fractional difference velocity due to scatterers---it will also depend inversely on this quantity, essentially due to the fact that as the fractional difference velocity tends to zero (through letting $n_o \rightarrow 0$, for example) the contributions at z_1 due to obstacles at different depths, z in Region I will add up more nearly in phase. Thus as the effects

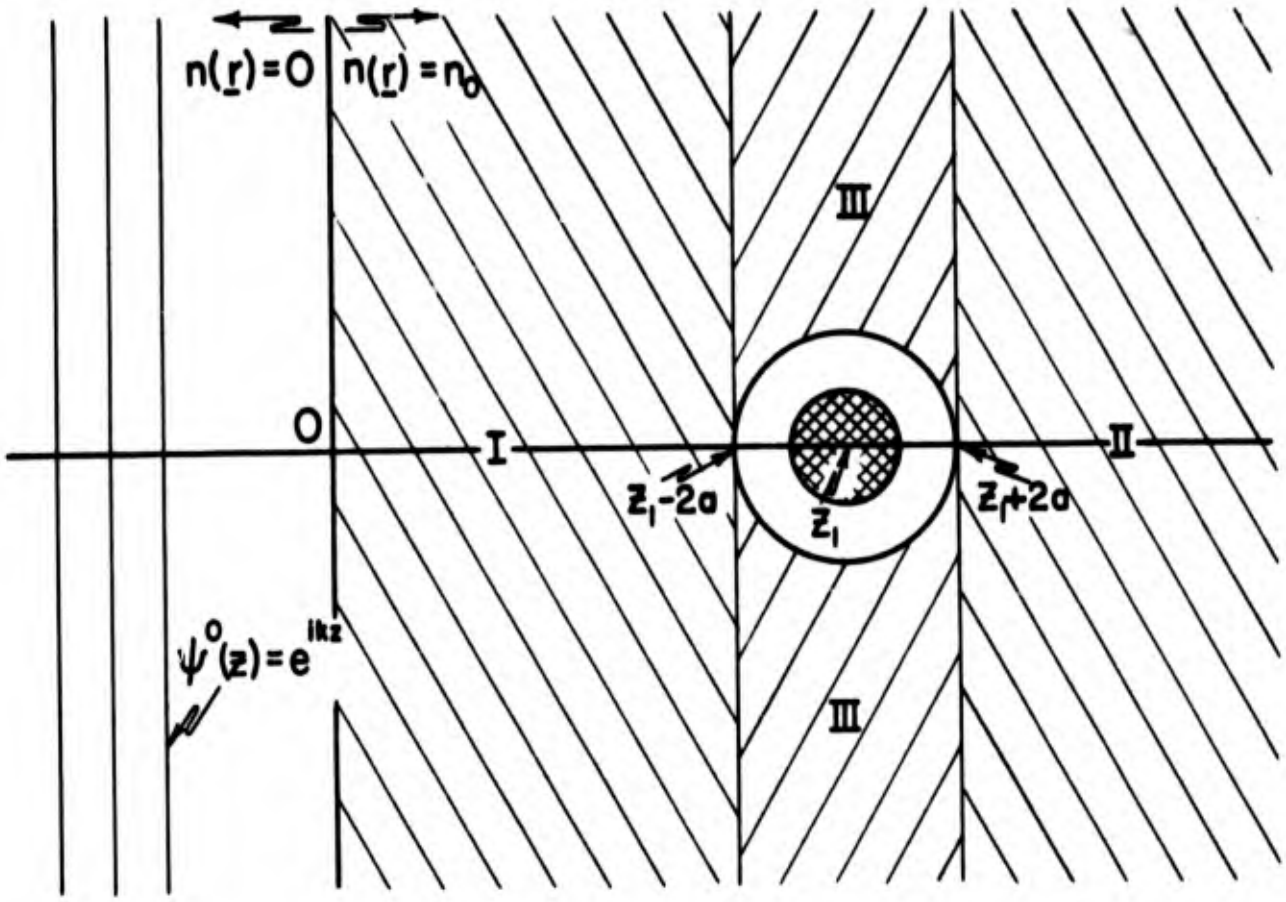

Fig. 2. Contributions to the external field acting on an obstacle situated at z_1, due to the scattered waves from all other obstacles, can be divided into three types. The contributions from Region I serve to extinguish the incident wave $\psi^0(z)$, replacing it by a new plane wave travelling in the forward (positive z) direction. Region II contributes a plane wave travelling in the back (negative z) direction. The contributions from Region III cannot be simply described in terms of plane waves. There are no contributions from the spherical region $|\underline{r}-\underline{r}_1|<2a$ containing the obstacle, as this would require interpenetration.

of scatterers are removed completely, $K - k \to 0$ and the forward plane wave reduces to e^{ikz}, the original incident wave.

The back plane wave represented by the second term on the right of Equ. (4.29) is just the integrated amplitude of waves back-scattered from obstacles in Region II, defined by $z \geqslant z_1 + 2a$. The remaining term is contributed by scattered waves from Region III, defined by $|z - z_1| \leqslant 2a$ and $|\underline{r} - \underline{r}_1| \geqslant 2a$, and cannot be conveniently written in terms of plane waves.

With this picture of the external field in mind, we now consider the first of Equs. (3.31) for the complex propagation constant K. The attenuation and modified phase velocity of the external field (as a function of the location of the obstacle on which the external field is acting) may be derived in the usual way from the imaginary and real parts of K, respectively. The important point to note is this: the form of the summation is such that the attenuation is <u>not</u> expressed in terms of the energy carried off by the scattered wave.

This is of course just an expression of the fact that since the energies of coexisting wave motions do not in general superpose, the energy associated with the scattered wave is devoid of physical meaning unless the scattered wave can be observed in some region of space free of other excitation. In the multiple scattering problem this is not possible.

While we could now proceed to calculate the averaged total field in terms of the first partial average of the external field, it is clear that $\langle \psi(\underline{r}) \rangle$ will be a plane wave propagating according to the complex propagation constant in the "scattering medium", and this is sufficient to yield the quantities of primary importance, attenuation and phase velocity.

Equs. (3.31) represent the exact solution of u. (3.16), the integral equation which was employed in this example, but we must keep in mind the fact that Equ. (3.16) was only an approximation to the exact equation governing multiple scattering.

We saw in the isotropic point scatterer computation that the effect of the last term taken in the integral equation for $\langle \psi(\underline{r}) \rangle$ (which would correspond to the first term omitted in Equ. (3.16)) was to provide a small correction to the propagation constant in the term proportional to the square of the scatterer density.

Hence, neglecting this correction term, the expression for the complex propagation constant in Equs. (3.31) is correct up to terms in n_o^2, at least at the low frequency limit.

Correct to this order K is given by

$$K = k\left[1 + \frac{2\pi n_o}{ik^3} \sum_{m,j=0}^{\infty} (-i)^j B_j^m A^o_j\right] \tag{3.34}$$

where

$$A^o_n = a_n\left[1 - 4\frac{\pi n_o\, ka}{k^3} \sum_{m,j=0}^{\infty} (-i)^j B_j^m a_m\right] - a^*_n \frac{\pi n_o}{ik^3} e^{+4ika} \sum_{m,j=0}^{\infty} (+i)^j B_j^m a_m$$

$$+ a^*_n \frac{2\pi n_o}{k^2} \sum_{m,j=0}^{\infty} (+i)^j B_j^m a_m G_{jn}$$

and in Equ. (3.32), the defining equation for G_{jn}, K is replaced by k.

We may immediately compare these results with the ones obtained in the isotropic point scatterer case, by taking $ka \ll 1$ and

$$B_j^m = \begin{cases} B_o & m=j=0 \\ 0 & \text{otherwise} \end{cases}$$

Neglecting terms of order $(ka)^2$ and higher in comparison with one, we obtain

$$A^o_o \approx 1 + \frac{\pi i B_o n_o}{k^3} ;$$

$$K = k\left[1 - \frac{2\pi i B_o n_o}{k^3} A^o_o\right]$$

$$\approx k\left[1 - \frac{2\pi i B_o n_o}{k^3} - 2\left(\frac{\pi i B_o n_o}{k^3}\right)^2\right]$$

which is in agreement with Equ. (3.12) for K, provided the term in $(B_o)^3$, which arose due to an additional term being taken in the starting integral equation, is neglected.

In order to completely establish the validity of Equs. (3.34) with regard to the terms in the complex propagation constant that vary as the square of the scatterer density n_o, one must include the next term in Equ. (3.16), the starting integral equation for the external field. This term is given explicitly in Equ. (2.28).

Because of the computational difficulties involved, we limit the remaining discussion to the behavior of K to first order in n_o. Equs. (3.34) give $A_n^o \approx a_n = (i)^n (2n+1)$, yielding for K the value

$$K = k\left[1 + \frac{2\pi n_o}{ik^3} \sum_{m,n=0}^{\infty} (2n+1) B_n^m\right]. \qquad (3.35)$$

The extinction cross-section γ_E of a spherical obstacle, which is defined as the sum of the scattering and absorption cross-sections, γ_S and γ_A respectively, is given by

$$\gamma_E = \gamma_S + \gamma_A = -\frac{4\pi}{k^2} \mathrm{Re}\left\{ \sum_{m,n=0}^{\infty} (2n+1)\, B_n^m \right\} \tag{3.36}$$

The reader is referred to the literature for an example of this computation[3,4]. There is unfortunately no simple physical interpretation of imaginary part of this summation. Equ. (3.35) now gives for the attenuation α and modified phase velocity v' of the averaged total field the values

$$\alpha = -\frac{2\pi n_o}{k^2} \mathrm{Re}\, \Sigma_1(ka) = \frac{1}{2}\, n_o\, \gamma_E$$
$$v' = v\left[1 + \frac{2\pi n_o}{k^3} \mathrm{Im}\, \Sigma_1(ka)\right]^{-1} \tag{3.37}$$

The B_n^m will in general be functions of ka, and we have written for brevity

$$\Sigma_1(ka) = \sum_{m,n=0}^{\infty} (2n+1)\, B_n^m(ka) \tag{3.38}$$

The B_n^m may be obtained from consideration of the single-scatterer problem, as mentioned proviously, and have been evaluated in the literature for quite a few physical problems in gases and fluids[6a,c,7a], isotropic elastic solids[8], electromagnetic theory[3,4,9], and quantum mechanics[9].

We now wish to apply the Neumann boundary condition to the array of spherical obstacles. The Neumann condition specifies that the normal gradient of the field vanish on the surface of each obstacle, and would apply to the problem of multiple scattering of pressure waves by an array of rigid fixed obstacles in a fluid or gas.

Morse and Feshbach have worked out the problem of scattering by a single spherical obstacle in the Neumann case, obtaining[6a]

$$B_n^m = \begin{cases} -\dfrac{j_n'(ka)}{h_n'(ka)} & \text{for } m=n \\ 0 & \text{for } m \neq n \end{cases}$$

and hence for Σ (ka) we obtain

$$\Sigma(ka) = -\sum_{n=0}^{\infty} (2n+1) \frac{j'_n(ka)}{h'_n(ka)}.$$

In the Rayleigh limit, where ka $\ll$ 1, only the first three terms of the series are important and we obtain, using the limiting forms of the spherical Bessel functions,[6e]

$$\Sigma(ka) \xrightarrow[ka\to 0]{} -\frac{7}{36}(ka)^6 \left[1 - \frac{2657}{2835}(ka)^2 + \cdots\right] + \frac{i(ka)^3}{6}\left[1 + \frac{127}{90}(ka)^2 + \cdots\right].$$

From Equs. (3.37), the attenuation and velocity in this case are given by

$$\frac{\alpha}{\pi a^2 n_o} \xrightarrow[ka\to 0]{} \frac{7}{18}(ka)^4 \left[1 - \frac{2657}{2835}(ka)^2 + \cdots\right]$$

$$\frac{v'}{v} \xrightarrow[ka\to 0]{} \left[1 + \frac{f}{4} + \frac{127}{360} f (ka)^2 + \cdots\right]^{-1}$$

where $f = 4\pi a^3 n_o/3$ is the fractional volume of obstacles.

While several authors have attempted to evaluate series of the same form as that for Σ (ka)[9,10] in the limiting case ka $\gg$ 1, results are not very accurate for ka $\lesssim$ 20. Thus we resort to numerical summation of the series. The imaginary part of Σ(ka) has been computed numerically and is given in Table I for several values of ka. The real part of Σ(ka) has been computed by Rubinow and Wu[9] and Morse and Feshbach[6b], and we use their values in computing attenuation.

ka	Im Σ' (ka)	ka	Im Σ' (ka)
0.2	0.00140	3.0	1.65
0.4	0.0116	4.0	2.76
0.5	0.0233	5.0	4.03
0.6	0.0410	6.0	5.45
0.7	0.0654	7.0	6.99
0.8	0.0964	8.0	8.63
1.0	0.174	9.0	10.38
2.0	0.768	10.0	12.20

TABLE I.

Fig. (3a) shows the frequency dependence of the normalized attenuation $\alpha / \pi a^2 n_o$. This quantity approaches one at high frequencies, in agreement with what one would expect on the basis of a simple geometrical optics picture.

In Fig. (3b) v'/v, the ratio of phase velocity in the medium with obstacles to that in the unperturbed medium, is plotted versus ka for several values of fractional volume. This ratio is seen to be everywhere less than one, assuming its minimum at ka $\approx$ 0.65, and then increasing monotonically toward one.

In order to indicate how the form of the results may change in more involved boundary value problems we might consider the case of multiple scattering by an array of spherical cavities in a homogenous elastic medium. Neglecting terms of second and higher order in the density of cavities n_o, Equs. (3.37) may be applied directly to compute phase velocity and attenuation. (If one wishes to obtain these quantities to terms in n_o^2 and higher the phenomenon of mode conversion must be considered in detail, as discussed later and in Appendix II.)

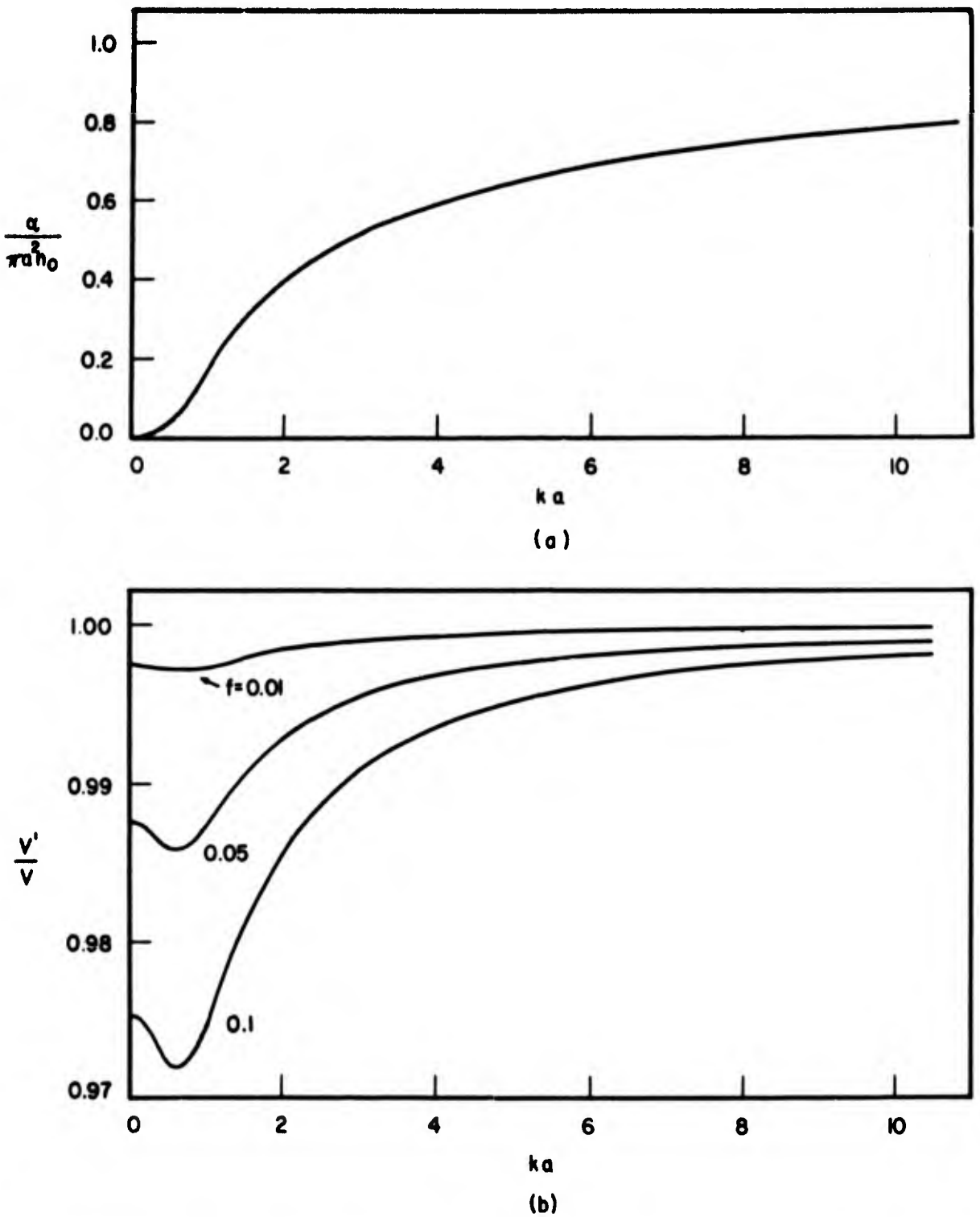

Fig. 3. The frequency dependence of the attenuation and modified phase velocity for a plane pressure wave in a perfect fluid containing a uniform random array of spherical obstacles of radius a. n_0 is the density of obstacles, and f the fractional volume occupied. (a) The normalized attenuation $\alpha/\pi a^2 n_0$ versus ka. This quantity approaches one at high frequencies. (b) The ratio v'/v of modified to unmodified phase velocity versus ka, for three values of fractional volume. This ratio approaches one at high frequencies.

Ying and Truell [8] have computed the elastic waves scattered by a spherical cavity of radius a when a plane compressional elastic wave is incident. In order to apply their results to the multiple scattering problem we need only make slight modifications due to the fact that Ying and Truell take $(1/ik)\exp(-ikz)$ as the scalar potential generating the incident wave (YT3), while we use $\exp(ikz)$.

From (YT 29a,b) we see that the B_n^m vanish for $m \neq n$, and Equ.(3.38) becomes

$$\Sigma(ka) = \sum_{m=0}^{\infty} (2m+1)\, B_m^m .$$

The expansion coefficient of the $m^{\underline{th}}$ partial wave in our notation is

$$(i)^m (2m+1)\, B_m^m,$$

while Ying and Truell use (YT 10)

$$A_{cm}.$$

Thus, modifying their result tc take into account the fact that different expressions were used for the incident wave potential, the correct form of the B_m^m is given by

$$(i)^m (2m+1)\, B_m^m = ik(-1)^m A_{cm} .$$

Making use also of the definition (YT 30a)

$$A_{cm} = (-i)^{m-1}(2m+1)\,\frac{1}{k}\,\frac{A'_{cm}}{\Delta_{cm}}$$

we may finally write

$$\Sigma(ka) = -\sum_{m=0} (2m+1)\,\frac{A'_{cm}}{\Delta_{cm}}$$

where (YT 31)

$$\begin{Bmatrix} A'_{cm} \\ \Delta_{cm} \end{Bmatrix} \equiv (ha)^2 \left[-(m-1)(2m+1) + \tfrac{1}{2}(ha)^2\right] \begin{Bmatrix} j_m(ka) \\ h_m(ka) \end{Bmatrix} h_m(ha)$$

$$+ (ha)\left[2m(m-1)(m+2) - (ha)^2\right] \begin{Bmatrix} j_m(ka) \\ h_m(ka) \end{Bmatrix} h_{m+1}(ha)$$

$$+ 2(ha)\left[(m^2-1)(m+2) - (ha)^2\right] \begin{Bmatrix} j_{m+1}(ka) \\ h_{m+1}(ka) \end{Bmatrix} h_m(ha)$$

$$-2(ka)(ha)(m-1)(m+2) \begin{Bmatrix} j_{m+1}(ka) \\ h_{m+1}(ka) \end{Bmatrix} h_{m+1}(ha)$$

Here k and h (in YT notation k_1 and K_1) are the propagation constants appropriate to compressional and transverse wave motion, respectively, and in terms of the Lame constants λ, μ and the density ρ of the medium are given by (YT 4,6)

$$k^2 = \frac{\omega^2\rho}{\lambda+2\mu}; \quad h^2 = \frac{\omega^2\rho}{\mu} .$$

Putting the above expression for $\Sigma'(ka)$ in Equs. (3.37), one may compute directly the attenuation and phase velocity of a plane compressional elastic wave in a medium containing an array of spherical cavities. For arbitrary values of ka numerical computation of the series is of course requried, but in the low frequency limit $ka \ll 1$ we may evaluate the series using limiting forms for small argument of the spherical Bessel functions.

Neglecting terms of order ka and higher in comparision with one, only the first three terms in the series $\Sigma'(ka)$ contribute to the results,

and we obtain

$$\alpha \underset{ka\to 0}{\longrightarrow} \frac{g_c\, f\, k}{6} (ka)^3$$

$$v' \underset{ka\to 0}{\longrightarrow} v\left\{1 + \frac{9}{2} f \left[\frac{1 - \frac{5}{4}\left(\frac{h}{k}\right)^2 + \frac{9}{16}\left(\frac{h}{k}\right)^4}{1 - \frac{9}{4}\left(\frac{h}{k}\right)^2}\right] - \frac{3}{2} f \left[1 - \frac{3}{4}\left(\frac{h}{k}\right)^2\right] \frac{k}{h} (ka)\right\}^{-1}$$

where f is the fractional volume of cavities, and g_c is given by (YT 36)

$$g_c = \frac{4}{3} + 40 \frac{2 + 3\left(\frac{h}{k}\right)^5}{\left[4 - 9\left(\frac{h}{k}\right)^2\right]^2} - \frac{3}{2}\left(\frac{h}{k}\right)^2 + \frac{2}{3}\left(\frac{h}{k}\right)^3 + \frac{9}{16}\left(\frac{h}{k}\right)^4.$$

Since the Lame constants are positive we see from the definition of the propagation constants that $(h/k)^2 \geq 2$. From this fact we conclude by inspection of the above expression for v' that the phase velocity, for sufficiently small ka, is greater than the unperturbed phase velocity v by a term proportional to the fractional volume of cavities f, and decreases linearly with ka. The attenuation follows the Rayleigh scattering law, varying directly with the fourth power of the frequency and the sixth power of the cavity radius. A plot of g_c versus (h/k) is available in the paper by Ying and Truell.

In the more involved case of an array of spherical elastic obstacles embedded in a homogeneous elastic solid of different density and moduli one may show from Ying and Truell's results that the modified phase velocity in the low frequency limit may be larger or smaller than the unperturbed phase velocity, depending on the values of the several parameters. The simplest case occurs when the moduli are equal ($\lambda_1 = \lambda_2$; $\mu_1 = \mu_2$) but the densities of scattering (ρ_2) and matrix (ρ_1)

material may differ. The obstacles behave as dipoles in this case in analogy with the fluid case,[7b] and from (YT 24, 25a, 26a) we obtain

$$v' \xrightarrow[\substack{ka \to 0 \\ \mu_1 = \mu_2 \\ \lambda_1 = \lambda_2}]{} v\left[1 + \frac{f}{2}\left(\frac{\rho_2 - \rho_1}{\rho_1}\right)\right]^{-1}$$

From this it is seen that the modified phase velocity is decreased or increased from the unperturbed phase velocity depending on whether the ratio of densities of scattering to matrix material is greater or less than one. This is of course a consequence of the fact that the phase of the dipole radiation relative to the phase of the incident wave depends on this ratio.

In the alternative case, when $\rho_1 = \rho_2$ but the elastic moduli may differ, there is no dipole radiation. This latter case is not completely analogous to the fluid situation, however, as in addition to monopole or simple source radiation the quadrupole radiation of the obstacles also influences the modified phase phase velocity in the low frequency limit.

Complete analogy with the fluid case[7b] obtains if the shear moduli of the scattering and matrix material are equal ($\mu_1 = \mu_2$), in which case the quadrupole radiation vanishes and

$$v' \xrightarrow[\substack{ka \to 0 \\ \rho_1 = \rho_2 \\ \mu_1 = \mu_2}]{} v\left[1 + \frac{f}{2}\left(1 - \frac{\lambda_1 + 2\mu}{\lambda_2 + 2\mu}\right)\right]^{-1}$$

Here the modified phase velocity is seen to be greater or less than the unperturbed phase velocity depending on whether the ratio of compressional moduli of the two media is greater or less than one.

Further evaluation of any of the elastic problems mentioned above is straightforward, and rests upon numerical summation of the various Σ (ka) involved.

IV. DISCUSSION

From the examples of the previous section we conclude that the statistical approach to the problem of multiple scattering of waves yields useful results in many cases. The limitations of the technique are those set forth in the derivation of an approximate integral equation for the external field: one must be dealing with a large number of scatterers "weakly" correlated in position and having "weak" scattering interaction. The last of these requirements will exhibit a dependence on the frequency, since the amplitude of any scattered wave in general increases with frequency.

Thus the requirement of "weak" scattering interaction may hold in some problems at low frequencies, and no longer be valid at high frequencies. In the examples dealing with isotropic scatterers we were able to establish the accuracy of the approximate integral equation employed because of the simplicity of the mathematics. This verification was not attempted in the example dealing with spherical obstacles possessing arbitrary scattering properties, since the amount of computation required appeared too formidable.

There are of course many other examples to which the theory might be applied. One might consider the effect of a point source of radiation in a medium containing a uniform distribution of scatterers. This introduces a new parameter having the dimensions of length, namely the radius of curvature of surfaces of equal phase of the excitation. The steady-state solution might be expected to be of the form $\exp\{i K(r) r\}/r$, where now the propagation factor $K(r)$ must exhibit a dependence on the distance from the source, in addition to the usual frequency dependence. The solution of this problem is of course of fundamental importance in the Green's function approach to boundary-value problems in such a "scattering medium".

While the radiation condition was used throughout the preceding calculations, this condition may be replaced by some other rule with trivial modifications. The only general requirement is that the scattering operator be defined, in order that one may compute the scattered potential appropriate to an arbitrary incident potential. In this connection, arguments are presented by Wheeler and Feynman[11] and Ekstein[12] for the use of "half-retarded half-advanced " potentials.

The extension of the theory to cases where to scatterers are not identical, but have a distribution in scattering properties, is not difficult. Foldy[1] has done this for isotropic point scatterers, and in the more general case the effects of such a distribution may usually be taken account of in the definition of the scattering operator T^S, thus divorcing them from the other aspects of multiple scattering.

The extension of the theory to problems involving multiple scattering in elastic solids is also straightforward. The new feature here, of course, is that the scattering will in general involve mode conversion. When a plane longitudinal compressional elastic wave is incident on a spherical obstacle, for example, Ying and Truell[8] have shown that both irrotational (compressional) and solenoidal (transverse) scattered waves result.

The treatment of the general equations in this case runs along the following lines. One introduces new scattering operators characterizing all the scatterings (longitudinal to transverse, transverse to transverse, transverse to longitudinal, and so forth) that can occur. Each mode of propagation is now thought of as one component of a column vector, the "vector" field (not to be confused with the usual concept of a vector field). The scattering operator becomes a scattering matrix, and to ob-

tain the scattered "vector" field due to a given external "vector" field, one operates on the latter with the scattering matrix, following the usual rules of matrix algebra. Thus all of the equations read exactly as before with the understanding that they are now matrix equations. A brief discussion of this extension is presented in Appendix II.

Fortunately in problems involving random distributions of scatterers this complicated formalism is not required in the first approximatfon . For concreteness consider the last example of the previous section, in which we dealt with a seimi-infinite uniform random distribution of spherical obstacles. Suppose the medium were an isotropic elastic solid, and the incident potential e^{ikz} that of a plane compressional elastic wave (in which case the vector particle displacement $\underline{u}$ is given by $\underline{u} = \underline{\nabla} e^{ikz}$). There will be three scattered waves from any obstacle; one compressional and two transverse modes. Correspondingly, the external field incident on any obstacle must contain all three modes. The transverse modes, however, can never build up to any appreciable order of magnitude, since the phases of scattered tranverse modes from any two obstacles add in a random manner. This is of course due to the fact that transverse modes propagate with a phase velocity distinct from that of a compressional mode.

Thus corrections to the complex propagation constant due to mode conversion come into only those terms which are quadratic or higher in the density of scatterers, and to first order terms in the density mode conversion may be ignored.

Application of the theory to problems involving elastic waves in solids is important in the following respect. Defects may be produced in single crystals by either chemical means or irradiation with fundamental particles.

As long as the material of which these defects are made up departs from the matrix material in any of its elastic properties, the defects will scatter elastic waves.

While direct experimental observation of the defects is not in general possible, due to the small dimensions involved, the macroscopic transmission properties of the crystal may easily be observed by ultrasonic techniques, and from these measurements the elastic properties of individual defects may be inferred from the multiple scattering theory.

APPENDIX I: The Multipole Representation

We wish to demonstrate that any axially symmetric solution of the wave equation in spherical coördinates may be written in the form

$$f_n(kr)\, P_n(\cos\vartheta) = (-i)^n\, P_n\left(\frac{1}{ik}\,\frac{\partial}{\partial z}\right) f_o(kr) \qquad (1)$$

where $f_n(kr)$ is one of the spherical Bessel functions $j_n(kr)$, $h_n(kr)$, or $n_n(kr)$, and $z = r\cos\vartheta$.

The proof of (1) by induction is straightforward, requiring only the well-known relations

$$(a)\quad P_{n+1}(\chi) = \left(\frac{2n+1}{n+1}\right)\chi\, P_n(\chi) - \frac{n}{n+1}\, P_{n-1}(\chi)$$

$$(b)\quad (1-x^2)\, P_n'(\chi) = (n+1)\,\chi\, P_n(\chi) - (n+1)\, P_{n+1}(\chi)$$

$$(c)\quad f_n'(\chi) = f_{n-1}(\chi) - (n+1)\,\frac{f_n(x)}{x}$$

$$(d)\quad (2n+1)\,\frac{f_n(\chi)}{\chi} = f_{n-1}(\chi) + f_{n+1}(\chi)\ .$$

Equ.(1) is easily verified for the first few integers. We next assume that it holds for $n \leqslant m$.

Then

$$(-i)^{m+1} P_{m+1}\left(\frac{1}{ik}\frac{\partial}{\partial z}\right) f_o(kr)$$

$$= (-i)^{m+1}\left[\left(\frac{2m+1}{m+1}\right)\frac{1}{ik}\frac{\partial}{\partial z} P_m\left(\frac{1}{ik}\frac{\partial}{\partial z}\right) - \frac{m}{m+1} P_{m-1}\left(\frac{1}{ik}\frac{\partial}{\partial z}\right)\right] f_o(kr) \text{ by (a)}$$

$$= -\frac{2m+1}{m+1}\frac{1}{k}\frac{\partial}{\partial z} f_m(kr)\, P_m(\cos\vartheta) + \frac{m}{m+1} f_{m-1}(kr)\, P_{m-1}(\cos\vartheta) \text{ by assumption}$$

$$= -\frac{2m+1}{m+1}\left[\cos\vartheta f_m'(kr) P_m(\cos\vartheta) + \frac{\sin^2\vartheta}{kr} f_m(kr) P_m'(\cos\vartheta)\right] + \frac{m}{m+1} f_{m-1}(kr)\, P_{m-1}(\cos\vartheta)$$

$$= -\frac{2m+1}{m+1}\left\{\cos\vartheta\left[f_{m-1}(kr) - (m+1)\frac{f_m(kr)}{kr}\right] P_m(\cos\vartheta) + \frac{(m+1)}{kr} f_m(kr)\left[\cos\vartheta P_m(\cos\vartheta) - P_{m+1}(\cos\vartheta)\right]\right\}$$
$$+ \frac{m}{m+1} f_{m-1}(kr)\, P_{m-1}(\cos\vartheta) \text{ by (b) and (c)}$$

$$= -\frac{2m+1}{m+1}\left[\cos\vartheta f_{m-1}(kr) P_m(\cos\vartheta) - \frac{(m+1)}{kr} f_m(kr)\, P_{m+1}(\cos\vartheta)\right]$$
$$+ \frac{m}{m+1} f_{m-1}(kr)\left[\frac{(2m+1)}{m}\cos\vartheta\, P_m(\cos\vartheta) - \frac{(m+1)}{m} P_{m+1}(\cos\vartheta)\right] \text{by (a)}$$

$$= (2m+1)\frac{f_m(kr)}{kr} P_{m+1}(\cos\vartheta) - f_{m-1}(kr)\, P_{m+1}(\cos\vartheta)$$

$$= \left[f_{m-1}(kr) + f_{m+1}(kr)\right] P_{m+1}(\cos\vartheta) - f_{m-1}(kr)\, P_{m+1}(\cos\vartheta) \text{ by (d)}$$

$$= f_{m+1}(kr)\, P_{m+1}(\cos\vartheta)$$

If in (1) one uses $h_n(kr)$, the spherical Hänkel function of the first kind, then the right hand side of the equation is a sum of derivatives with respect to z of $h_o(kr)$, the simple source. Each term of this sum is a multipole source oriented along the z-axis—terms involving the first derivative of $h_o(kr)$ representing dipole sources, terms involving second derivatives representing quadrupole sources, and so on. Thus in this case (1) gives an expansion of the axially symmetric wave function $h_n(kr)\ P_n(\cos\vartheta)$ in axially symmetric multipole sources.

To the best of the author's knowledge this expansion is not available in the literature, althought the special result obtained when ϑ is set equal to zero in Equ.(1) was previously obtained by Lord Rayleigh in his Theory of Sound, Vol.II (Dover, New York, 1945) pg. 259.

APPENDIX II: The Matrix Equations of Multiple Scattering with Mode Conversion

If one is concerned with the multiple scattering problem in an elastic medium, which may sustain more than one mode of propagation, then the phenomenon of mode conversion must be considered. A longitudinal mode incident on an obstacle will in general give rise to transverse scattered modes, and vice versa. Assuming a simple harmonic time dependence, the vector wave equation may be split into three vector Helmholz equations. A thorough discussion of the scalar potentials which may be used to generate solutions to the latter may be found in Chapter 13 of Morse and Feshbach's book, Methods of Theoretical Physics (McGraw-Hill, New York, 1953).

We now consier the problem of multiple scattering of elastic waves by an array of obstacles embedded in a homogeneous isotropic elastic solid, utilizing the scalar potentials mentioned above. There are three modes of propagation and we shall indicate which we are talking about by a subscript, maintianing as far as possible the notation employed previously. Thus, for example, $\langle \psi_1^E (\underline{r}, \underline{r}_i) \rangle_{\underline{r}_i}$ is that part of (the first partial average of) the external field which is mode 1.

Fortunately the statistical averaging processes used in obtaining equations for the various field quantities are not modified at all when more than one mode of propagation is introduced, and we need only generalize the equations themselves.

It is convenient to introduce a matrix notation at this point. The individual modes of any field quantity are taken to be components of a

column vector which completely describes the appropriate field. Thus for the incident wave we write

$$\underline{\Psi}^{o}(\underline{r}) = \begin{pmatrix} \psi_1^{o}(\underline{r}) \\ \psi_2^{o}(\underline{r}) \\ \psi_3^{o}(\underline{r}) \end{pmatrix} \tag{1}$$

with a corresponding expression for the "vector" external field $\langle \underline{\Psi}^{E}(\underline{r}_1;\underline{r}) \rangle_{\underline{r}_1}$, "vector" total field $\langle \underline{\Psi}(\underline{r}) \rangle$, and so forth. (Observe that these quantities are not vectors, and should not be confused with the usual concept of a vector field.)

Now consider the scattering operator. There are nine separate types of scattering which may occur, since we have three possible incident modes, each of which will in general generate three scattered modes. We may account for all nine situations by writing $T_{ij}^{S}(i,j = 1,2,3)$ for the operator which scatters the $j^{\underline{th}}$ incident mode into the $i^{\underline{th}}$ scattered mode. We define the scattering matrix T^{S} by

$$T^{S} = \begin{pmatrix} T_{11}^{S} & T_{12}^{S} & T_{13}^{S} \\ T_{21}^{S} & T_{22}^{S} & T_{23}^{S} \\ T_{31}^{S} & T_{32}^{S} & T_{33}^{S} \end{pmatrix} \tag{2}$$

It is easily verified using the rules of matrix algebra, that the scattered "vector" field $\langle \underline{\Psi}^{S}(\underline{r}_1;\underline{r}) \rangle_{\underline{r}_1}$ due to an incident external "vector" field $\langle \underline{\Psi}^{E}(\underline{r}_1;\underline{r}) \rangle_{\underline{r}_1}$ is given by

$$\langle \underline{\Psi}^{S}(\underline{r}_1;\underline{r}) \rangle_{\underline{r}_1} = T^{S}(\underline{r}_1) \langle \underline{\Psi}^{E}(\underline{r}_1;r) \rangle_{\underline{r}_1} \tag{3}$$

where we have made use of the fact that the scattering is linear, and consequently the various mode conversions proceed independently.

As mentioned above, the form of the equations for the various field quantities is unaltered when mode conversion is introduced (one may easily follow through the averaging process used to obtain an equation for the first partial average of the external field, for example) and we may now write down immediately any of the equations, simply extending them from scalar to matrix form. The two fundamental equations, giving the averaged total field and the first partial average of the external field, become

$$\langle \underline{\psi}(\underline{r}) \rangle = \underline{\psi}^{o}(\underline{r}) + \int_{\underline{r}\ \text{"outside"}} d\tau'\ n(\underline{r}')\ T^{S}(\underline{r}')\ \langle \underline{\psi}^{E}(\underline{r}';\underline{r}) \rangle_{\underline{r}'}$$

$$+ \int_{\underline{r}\ \text{"inside"}} d\tau'\ n(\underline{r}') \left[T^{I}(\underline{r}') - I \right] \langle \psi^{E}(\underline{r}';\underline{r}) \rangle_{\underline{r}'} \qquad (4)$$

and

$$\langle \underline{\psi}^{E}(\underline{r}_1;\underline{r}) \rangle_{\underline{r}_1} \approx \underline{\psi}^{o}(\underline{r}) + \int d\tau'\ n(\underline{r}'|\underline{r}_1)\ T^{S}(\underline{r}')\ \langle \underline{\psi}^{E}(\underline{r}';\underline{r}) \rangle_{\underline{r}'}$$

$$+ \int d\tau'\ n(\underline{r}'|\underline{r}_1)\ T^{S}(\underline{r}')\ T^{S}(\underline{r}_1)\ \langle \underline{\psi}^{E}(\underline{r}_1;\underline{r}) \rangle_{\underline{r}_1} . \qquad (5)$$

In equation (4), $T^{I}(r')$ is the internal scattering matrix, defined according to Equ. (2), and I is the identity matrix, given by

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (6)$$

Equs.(4) and (5) are the matrix forms of Equs. (2.32) and (2.23) respectively.

As mentioned in Section IV, the matrix formalism is not required in the elastic case of a plane wave incident on a uniform random array of obstacles, if one wishes to compute the attenuation and phase velocity of the propagated wave to within linear terms in the density of obstacles. In any other problem, of course, careful consideration should be given to the question of whether the matrix formalism is required.

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