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ONE DIMENSIONALIZED AERO-THERMODYNAMIC THEORY OF TURBULENT FLAME PROPAGATION IN FLAME TUBES

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PROJECT SQUID

A PROGRAM OF FUNDAMENTAL RESEARCH

ON LIQUID ROCKET AND PULSE JET PROPULSION

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ONE DIMENSIONALIZED AERO-THERMODYNAMIC THEORY OF TURBULENT FLAME PROPAGATION IN FLAME TUBES

By

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PHASE I

In connection with pulsating jet engines: to undertake theoretical and experimental investigations of (1) flame motions with controlled initial turbulence, (2) stationary flames with controlled turbulence, (3) suitable theoretical models based on the above observations, and (4) statistical mechanics of non-uniform gases.

SUMMARY

General one-dimensional aero-thermodynamical equations and methods for solving them are obtained in forms suitable for use in connection with eddy-turbylent flame propagation in flame tubes. Such flames resemble those in pulse jets and are being studied partly to provide data on combustion parameters for use in the theoretical treatment of pulsejets. Solutions obtained by use of characteristics and finite differences are compared with explicit analytical solutions of approximating linear differential equations. Good agreement is found between the solutions for typical cases. The analytical solutions are general enough to cover almost any practical case, and they will be applied to analyse forthcoming experimental data on flame tubes.

INTRODUCTION

It is possible in principle to construct macroscopic equations for energies, momenta and concentrations of active species in fuel-air mixtures through which flames are propagating three-dimensionally. However, even when only laminar motions and smooth flame fronts are involved, the detailed mechanisms are so incompletely known[1] that the equations must, at present, involve many unknown variables and coefficients. When, as in actual jet devices, there is large scale eddy turbulence and inhomogeneous mixing of the fuel and air, a detailed three dimensional treatment appears to be well beyond present techniques. Since *some* reasonably simple theoretical framework is needed for the analysis and design of pulsejets and other devices involving turbulent propagation of flames and pressure waves, it is natural to investigate one-dimensionalized formulations.

It would of course be desirable to obtain the one-dimensionalized equations for propagations in tubes by taking cross-sectional averages of the terms in the complete three dimensional equations. Unfortunately, as is indicated above, there is as yet insufficient knowledge to permit the development of a numerically applicable treatment along such lines. Instead it is apparently necessary at present to introduce rather idealized one-dimensional models in terms of which it is hoped that actual cross-sectional averages of pressures, densities, velocities, temperatures and energies may be represented to a reasonable degree of approximation.

From the viewpoint of such idealizations it is somewhat depressing to view high-speed motion pictures[2] taken through transparent walls in pulsejets and other devices in which highly turbulent flames appear. The combustion in such devices seems to proceed inhomogeneously in cluster form rather than in the relatively slowly moving 'sheet' form to which the LeChatelier theory and its recent extensions apply[1]. In fact it appears that such cluster-like flames spread in the form of turbulently distorted and interpenetrating thin shells making entire regions glow with combustion for several milliseconds. Presumably the fine grain components of the turbulence increase the effective rate of diffusion of heat and of chain carriers, while the large scale eddies in the turbulent flow carry the flame along locally, thus possibly accounting for the observed[2] high rates of propagation of such flames. A cross-sectional average of a pressure (or other physical quantity) might be subject to considerable fluctuation, but a sufficiently strong eddy turbulence might be quite effective in reducing large scale inhomogeneities across the cross-section. In any event such fluctuations have to be ignored in a simple one-dimensionalized treatment.

Because of the present lack of knowledge of the intermediate species involved in the combustion of common fuel-air mixtures used in jet devices, a highly idealized molecular model will be used here to represent the reaction processes. The mixture will be treated as a perfect gas with a gas constant unaffected by the combustion (a reasonably good approximation for common hydrocarbon-air mixtures in the burned and the unburned states); furthermore the chemical energy will be represented by an additional term in the internal energy of the molecule. Combustion then will be represented by the release of this extra energy over a period of time corresponding to the effective duration of burning observed at typical cross-sections for three-dimensional cluster-type burning.

Even when the above-mentioned idealization is employed, the resulting aero-thermodynamic equations are non-linear and, in general, must be solved by finite difference numerical methods (separately for each numerical case) unless some approximate explicit analytical solutions are found. This paper is mainly concerned with the finding of such explicit approximations, and with a determination of their accuracy for flame tubes. This paper is also concerned with a comparison of idealized theoretical computations and some experimental results for flame tubes obtained by a group under Dr. M. W. Evans of Project SQUID at New York University. Such comparisons provide data on combustion and flame parameters for use in the theoretical treatment of jet devices along lines indicated in a previous report.[2] (A general discussion of the experimental methods and results will appear in a separate report.)

ONE-DIMENSIONALIZED EQUATIONS FOR FLAME TUBES

Consider a tube of uniform cross-section and of length L, closed at one end ($x = 0$) and open at the other ($x = L$), and initially ($t = 0$) filled with a homogeneous combustible gas at atmospheric pressure p_0 and absolute temperature θ_0. For a layer of gas at distance x (cm.) from the closed end at time t (sec.), let

u = stream velocity (cm./sec.)

p = pressure (dynes/cm.^{2})

τ = specific volume (cm.^{3}/gm.)

ρ = density (gm./cm.^{3})

θ = temperature ($^\circ$K.)

c_p = specific heat at constant pressure (ergs/gm.$^\circ$K.)

c_v = specific heat at constant volume (ergs/gm.$^\circ$K.)

The gas constant, $R = c_p - c_v$ (ergs/gm. $^\circ$K), appears in the assumed equation of state.

$$p\tau = R\theta = c_v(\gamma - 1)\theta \tag{1}$$

where $\gamma = c_p/c_v$ is the ratio of specific heats, possibly a function of θ. The *total* specific internal energy i(ergs/gm.) is assumed to have the form

$$i = \int_0^\theta c_v \, d\theta + e. \tag{2}$$

Here the integral term corresponds to the usual internal energy of a "quasi-perfect" gas with c_v a function of θ alone; the term e (ergs/gm.) is the "effective specific internal chemical energy" characteristic of the model concerned, and the value of e drops from an initial value e_0 to zero during the combustion of the layer which has an effective duration of burning t'(sec.) Since e is supposed to provide a one-dimensionalized representation for energies involved in turbulent exchanges in three-dimensional combustion, it probably should be regarded as including heat energy transferred from adjacent layers, as well as chemical energy. The functional forms for e and for effective one-dimensional flame speeds relative to the stream may be regarded as determinable from a comparison of experimental and theoretical results for explicit cases.

To discuss flame and gas motions in the theory of the flame tube it appears advantageous in many respects to use independent variables (a,t) of the Lagrange type rather than (x,t) of the Euler type.[3] Here for a given layer at $x = x\,(a,t)$ at time t the coordinate a is the value of x at time $t = 0$ for the same layer. This definition is illustrated in Figure 1, which also indicates the definitions of x^* and a^* for the flame front, and of x^{**} and a^{**} for the flame tail.

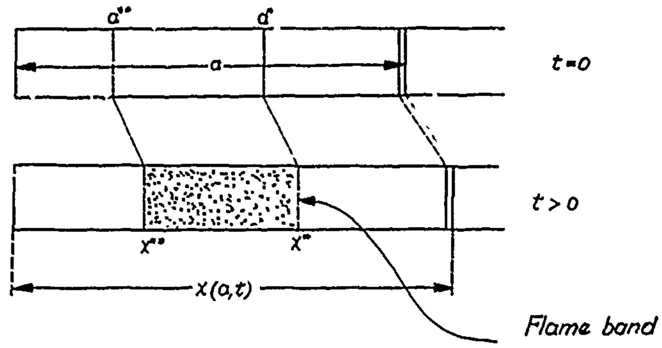

Figure 1.

Motion of typical layers of gas in the flame tubes in terms of Lagrange variables (a,t). The flame band at time t lies in $x^{**} \leqq x \leqq x^*$, where $x^{**} = x(a^{**},t)$ and $x^* = x(a^*,t)$ in terms of the initial positions $a^{**} = a^{**}(t)$ and $a^* = a^*(t)$ of the layers which bound the band at the time t.

The main advantages of the Lagrangian (a,t) over the Euler (x,t) are that in terms of (a,t): (i) the aero-thermodynamic equations have simpler forms; (ii) fewer and in general smaller terms have to be dropped in order to linearize the differential equations so that explicit approximate solutions may be obtained; (iii) it is easier to represent intrinsic effective flame velocities and to keep track of the course of combustion of a given layer. One disadvantage of (a,t) over (x,t) is that the boundary condition at $x = L$ strictly demands that a corresponding a be determined as a function of time compatible with the solution, but this apparent disadvantage turns out to be relatively unimportant in most cases of interest, since the boundary condition at $x = L$ can be well approximated by one at $a = L$.

In connection with advantages (i) and (ii) above, it is of interest to compare the Euler and Lagrange forms for gas dynamics in one dimension. In gas dynamics p is a function of ρ and, by calling $dp/d\rho = c^2$, the equations for conservation of mass and momentum[3] may be written in the following forms:

(3) (Euler) $p_t + u\,p_x + \rho c^2 u_x = 0, \quad u_t + u\,u_x + p_x/\rho = 0,$

(4) (Lagrange) $p_t + \rho^2 c^2\, u_a/\rho_o = 0, \quad u_t + p_a/\rho_o = 0,$

where the subscripts x,t and a,t refer to partial derivatives, and the o to (uniform) initial values. By neglecting $u\,p_x$, $(\rho c^2 - \rho_o c_o^2)u_x$, $u\,u_x$, and $(1/\rho - 1/\rho_o)p_x$ in the Eulerian forms, the acoustic approximation is obtained in the form of the following linear equations with constant coefficients:

(5) $$\text{(Euler)} \quad p_t + \rho_o c_o^2\, u_x = 0, \quad u_t + p_x/\rho_o = 0.$$

On the other hand it is sufficient to neglect the one term $(\rho^2 c^2 - \rho_o^2 c_o^2)\, u_a/\rho_o$ in the Lagrangian forms in order to obtain the linear equations

(6) $$\text{(Lagrange)} \quad p_t + \rho_o c_o^2\, u_a = 0, \quad u_t + p_a/\rho_o = 0.$$

It therefore might be expected that in many problems (6) would provide better approximations than (5) for the solutions of (3) and (4), especially for cases involving high velocities with moderate gradients. Note that (6) is exactly (4) for constant 'Lagrangian sound speed' $\rho\, c/\rho_o$.

Similar remarks apply to the aero-thermodynamic equations which may be expressed in terms of Lagrangian coordinates suitable for application to flame tubes (see Figure 1) in the following forms·

(7) $x_a = \tau/\tau_o$ (conservation of mass), whence $u_a = \tau_t/\tau_o$;

(8) $u_t = -\tau_o p_a + g$ (momentum equation with g defined below);

(9) $i_t = -p\tau_t$ [whence, by (1) and (2)] $\gamma p \tau_t + \tau p_t + (\gamma - 1)\, e_t = 0$ (internal energy equation) [alternatively an enthalpy equation may be obtained, but when combined with (8) leads to (9)].

Equation (7) corresponds to the statement that a layer of thickness da and specific volume τ_o at time $t = 0$ acquires a thickness dx and specific volume τ at time $t > 0$. Equation (8) states that the acceleration of a layer is due to the pressure drop through the layer, plus the effect of a term g which represents (one-dimensionally) momentum transfer due to viscosity and large scale eddy motion in the three-dimensional flow. Equation (9) expresses the fact that the "ordinary internal energy" of a layer is increased by compression from adjacent layers as well as by the release of "chemical energy."

In addition to the uniform initial conditions

(10) $$p = p_o, \ \tau = \tau_o, \ u = 0 \text{ at } t = 0,$$

we require boundary conditions, say

(11) $$u = 0 \text{ at } x = 0, \text{ and } p = p_o \text{ at } x = L,$$

and flame band propagation velocities (relative to the stream)

(12) $$f^* = (x_{a^*})\,(da^*/dt) \text{ and } f^{**} = (x_{a^{**}})(da^{**}/dt)$$

in terms of the symbols defined in Figure 1. It is convenient to use the symbols t^* and t^{**} respectively for the times of ignition and of extinction of burning for the layer determined by a:

(13) $$a^*(t^*) = a \text{ and } a^{**}(t^{**}) = a,$$

with $t' = t^{**} - t^*$ = duration of burning of the layer a.

The functions g, e_t, t^*, t^{**} and t' (or g, e_t, a^* and a^{**}) in (8) - (13) may be regarded as determinable from a comparison of theoretical forms with experimental observations (of a type depending on available instrumentation). For sample calculations such functions may be regarded as constants.

The boundary condition $p = p_o$ at the open end of the flame tube has been investigated experimentally by means of pressure oscillograms, and has been found to be reasonably representative. A more refined condition may be used when available.

NUMERICAL SOLUTION OF EQUATIONS (7) TO (13)

If it is demanded that u, p and τ be continuous functions of a and t (i.e. no shock waves), but that the derivatives u_a, u_t, p_a, p_t, τ_a and τ_t be not necessarily continuous across certain "characteristic curves" (wave fronts) to be found in differential form below, then the following relations come from fundamental differential forms such as $du = u_a da + u_t dt$:

(14) $$u_a]\, da + u_t]\, dt = 0,\ p_a]\, da + p_t]\, dt = 0 \text{ and } \tau_a]\, da + \tau_t]\, dt = 0,$$

where, for any function q, q] denotes the jump in the value of q across a curve with slope dt/da in an (a,t) plane. Furthermore, if this curve does not correspond to a discontinuity in γ, g or e_t (say across a flame front or flame tail) it follows from (7) - (9) that

(15) $$u_t] + p_a]\tau_o = 0 \text{ and } u_a]\gamma p\tau_o + p_t]\tau = 0.$$

For a non-trivial solution to exist for the jumps in equations (14) and (15) the determinant of their coefficients must vanish, which leads to the differential forms

(16) $$da = 0 \text{ or } da = \pm \sqrt{\gamma p \tau_0{}^2/\tau}\, dt.$$

These are the differential equations for the characteristic curves across which discontinuities in the derivatives of u, p and τ may arise, and along which the following particularly simple relations may be shown to hold among the differentials:
Using the symbols $j = (\gamma - 1)\, e_t/\tau$ and $\mu = \sqrt{\gamma p/\tau}$ we have [4]

(17) $$dp \pm \mu\, du = (-j \pm g\mu)\, dt \text{ on } da = \pm \tau_0 \mu\, dt \text{ (respectively), and}$$

(18) $$dp + \mu^2 d\tau = -j\, dt \text{ on } da = 0.$$

These relations follow from $dq = q_a\, da + q_t\, dt$ for $q = u, p$ and τ, and from (7) - (9). For any (da, dt), dx is given by

(19) $$dx = x_a\, da + x_t\, dt = (\tau/\tau_0)\, da + u\, dt.$$

The initial and the boundary conditions (10) and (11) are represented by

(20) $$x = a,\ p = p_0,\ \tau = \tau_0,\ u = 0 \text{ at } t = 0;$$

(21) $$u = 0 \text{ on } da = 0 \text{ at } a = 0;$$

(22) $$p = p_0 \text{ on } dx = 0\ [\text{i.e. } (\tau/\tau_0)\, da + u\, dt = 0] \text{ at } x = L.$$

The flame conditions may be regarded as given by (12) or by explicit specification of a^* and a^{**} directly as functions of time demanded by the form of e_t.

For numerical calculations (17) - (22) are approximated by difference equations obtained by replacing dx, da, dt, du, dp, $d\tau$, by corresponding finite differences between adjacent values in a lattice of values $(x, a, t, u, p, \tau) = (x_n, a_n, t_n, u_n, p_n, \tau_n)$ $n = 1, 2, \ldots$ The lattice points (values) may be calculated graphically or numerically starting, for example, with the initial set $(x_n, a_n, t_n, u_n, p_n, \tau_n) = (nL/N, nL/N, 0, 0, p_0, \tau_0)$ $(0 \leq n \leq N = 5$ is adequate in typical cases), and then proceeding by steps of the type described below. (Here, and in the remainder of this sections, subscripts will denote labels and not differentiation.) Suppose that values of $(x_n, a_n, t_n, u_n, p_n, \tau_n)$ with corresponding $j = j_n$, $g = g_n$, $\mu = \mu_n$ are known at two adjacent lattice points denoted by $n = k$ and $n = l$ respectively, with

$a_k < a_l$. Then a new adjacent point denoted, say, by $n = m$ is determined by use of the difference equations corresponding to (17) - (22) as follows:[4]

$(a, t) = (a_m, t_m)$ is the point of intersection (determined graphically or algebraically) of the line $(a - a_k) = (\tau_o \mu_k)(t - t_k)$ (if $a_k \neq 0$, or of the line $a = 0$ if $a_k = 0$) and the line $(a - a_l) = (-\tau_o u_l)(t - t_l)$ [if $x_l \neq L$, or the line $(\tau_l/\tau_o)(a - a_l) + u_l (t - t_l) = 0$ if $x_l = L$]. The value of x_m is determined by $x_m - x_l = (\tau_l/\tau_o)(a_m - a_l) + u_l (t_m - t_l)$ corresponding to (19). Further $(u, p) = (u_m, p_m)$ is the point of intersection of the line

(23) $$(p - p_k) + (u - u_k)\mu_k = (-j_{km} + g_{km}\mu_k)(t_m - t_k)$$
(if $a_k \neq 0$, or of the line $u = 0$ if $a_k = 0$)

and the line

(24) $$(p - p_l) - (u - u_l)\mu_l = (-j_{lm} - g_{lm}\mu_l)(t_m - t_l)$$
(if $x_l \neq L$, or the line $p = p_o$ if $x_l = L$).

In (23) and (24) the symbols j_{km}, g_{km}, j_{lm}, and g_{lm} denote j_k, g_k, j_l and g_l if j and g are nearly constant functions along the line segments concerned, and are suitable approximate averages (with respect to t, say,) if the functions are very variable over the interval concerned between the points of the lattice. The use of such averages may permit the use of fewer points to achieve a desired accuracy in the calculations.

To determine τ_m if (a_m, t_m) is in an *unburned region* (i.e. with $j = 0$ at $a = a_m$ for $t < t_m$) (18) may be integrated to yield $p_m\tau_m^\gamma = p_o\tau_o^\gamma$ if γ = constant, or to yield $\log_e (p_m/p_o) = \int_{p_o\tau_o/R}^{p_m\tau_m/R} \frac{\gamma(\theta)\, d\theta}{[\gamma(\theta) - 1]\,\theta}$ if $\gamma = \gamma(\bar{\theta})$ is a function of the absolute temperature θ. If, however, (a_m, t_m) is in a burning or burned-over region, more involved forms must be used to calculate τ_m: If $p = p_n$, $\tau = \tau_n$, $\mu = \mu_n$ are known at $(a, t) = (a_n, t_n)$, and if $a_m = a_n$, then, corresponding to (18), τ_m is determined by

(25) $$(p_m - p_n) + \mu_n^2 (\tau_m - \tau_n) = -j_{mn}(t_m - t_n),$$

where j_{mn} is analogous to the terms used in (23) and (24). Therefore, in connection with the above mentioned points denoted by k, l and m, it suffices to interpolate between the known values of p, τ, μ at (a_k, t_k) and (a_l, t_l), to find p_n, τ_n, μ_n at (a_n, t_n) where $a_n = a_m$ and, for q denoting t, p, τ or u, $q_n = \frac{[(a_l - a_m)\, q_k + (a_m - a_k)\, q_l]}{(a_l - a_k)}$.

Finally, the values of j_m, μ_m and g_m required for the calculation of other lattice values, may calculated as known empirical functions of the values of (a_m, t_m).

The accuracy of the finite differences method depends on the mesh size in the lattice and on the number of significant figures retained in the calculations. The effects of these two factors can be investigated numerically in any given case.

Some results of calculations by this method are given in a later section.

APPROXIMATE ANALYTICAL SOLUTION OF EQUATIONS (7) TO (13)

Although the numerical methods described in the preceeding section can yield sufficiently accurate solutions, they demand a separate detailed calculation for each numerical case. An analytical formula, on the other hand, may permit many general conclusions to be drawn without requiring much numerical work to be performed. Riemann, Love, and Pidduck[5] have obtained exact analytical solutions of the gas dynamics equations in one dimension by use of a Legendre transformation which linearizes the equations exactly. Unfortunately for flows involving non-uniform cross-sections or combustion, no such exact linearization seems to be available. In the present state of mathematical techniques, linearization appears to be practically essential in order to obtain analytical solutions.

It should be noted that equation (7) is linear, and (8) is linear if g is a known function of (a, t) (or linear in p, τ and u), but (9) is definitely not linear. In this section these equations will be approximated by the linear system

(26) $$x_a = \tau/\tau_o,\ x_t = u,\ u_t + \tau_o p_a = g(a, t),\ \text{and}\ u_a + k^2\tau_o p_t = h(a, t),$$

where $g(a,t)$, k and $h(a,t)$ are some chosen approximations for g, $1/\mu\tau_o$ and $[-jk^2\tau_o + (u_a)(1 - \mu^2\tau_o^2k^2)]$ respectively in terms of the symbols used in (8), (9) and (17). The quantity $(1/k)$ may be regarded as an effective sound speed in Lagrangian variables. The $h(a,t)$ involves the rate of combustion through the term j, and the other term is small, in general, because $1 - \mu^2\tau_o^2k^2$ is small.

In order to obtain reasonably simple explicit solutions it is practically essential that k be constant (which involves less error in flame tube theory than assuming constant sound speed c) and that the boundary conditions be imposed at fixed values of a rather than of x thus:

(27) $$u = u_o(t)\ \text{at}\ a = 0,\ p = p_L(t)\ \text{at}\ a = L.$$

Here $u_o(t)$ and $p_L(t)$ are prescribed functions which may be based on physical observations. The form $u = u_o(t)$ is used in place of the $u = 0$ of (11), since in the flame tube work at N.Y.U. a grid is usually placed in the tube near the igniter at the closed end. The first burning causes unburned gases to flow through the grid as if a moving piston were present at $a = 0$ marking the layer which first carries the flame through the grid and establishes the "turbulent" flame which propagates very rapidly towards the open end of the tube. The function $p_L(t)$ is probably sufficiently well represented by p_o for most practical purposes, but the solution given below involves the general form.

The initial conditions will be taken as before to be

(28) $p = p_o$, $\tau = \tau_o$ and $u = 0$ at $t = 0$ [with $p_L(0) = p_o$ and $u_o(0) = 0$ in (27)].

The system (26) - (28) may be solved explicitly by means of Laplace transforms,[6] and the solutions may be expressed as sums of contributions from successive direct and reflected "waves" as follows:

It is convenient to introduce a sort of "stream function" w in terms of which

$$u = w_a \text{ and } \tau_o p = -w_t + \int_L^a g(a,t)\,da + \tau_o p_o \tag{29}$$

identically satisfying the third equation in (26), and when inserted in the fourth, yielding the differential equation

$$w_{aa} - k^2 w_{tt} = b(a,t) \text{ where } b(a,t) = h(a,t) - k^2 \int_L^a g_t(a,t)\,da. \tag{30}$$

In terms of w the boundary and the initial conditions are

(31) $w_a = u_o(t)$ at $a = 0$, $w_t = [p_o - p_L(t)]\tau_o$ at $a = L$; $w = w_a = w_t = 0$ at $t = 0$
[if $g(a,t) = 0$ at $t = 0$].

Using the tilde notation

$$\tilde{q} = \int_o^\infty e^{-st} q\,dt \tag{32}$$

for the Laplace transform of any function q of t (and possibly other variables), the transforms of (30) -(31) yield the following:

$$(d^2/da^2)\,\tilde{w} - k^2 s^2 \tilde{w} = \tilde{b}(a) \tag{33}$$

with $\tilde{b}(a)$ the transform of $b(a,t)$;

$$\tilde{w}_a = \tilde{u}_o \text{ at } a = 0, \text{ and } \tilde{w} = [p_o - s\tilde{p}_L]\, \tau_o/s^2 \text{ at } a = L. \tag{34}$$

The solution of (33) and (34) is

$$\tilde{w} = -(\tilde{u}_o/ks)\, \tilde{S}(L,a) + (p_o - s\tilde{p}_L)(\tau_o/s^2)\, \tilde{C}(L,a) + \tilde{w}^*, \tag{35}$$

where

$$\tilde{w}^* = \int_0^L [\, - e^{-ks|a-\alpha|} + e^{-ks(L-\alpha)}\, \tilde{C}(L,a) - e^{-ks\alpha}\, \tilde{S}(L,a)]\tilde{b}(\alpha)\, d\alpha/2ks \tag{36}$$

with

$$\tilde{S}(L,a) = [\sinh ks\,(L-a)]/[\cosh ksL] = \sum_{n=0} (-)^n \left\{\tilde{e}[2nL + a] - e[2nL + 2L - a]\right\} \tag{37}$$

and

$$\tilde{C}(L,a) = [\cosh ksa]/[\cosh ksL] = \sum_{n=0} (-)^n \left\{\tilde{e}[2nL + L - a] + \tilde{e}[2nL + L + a]\right\} \tag{37}$$

with $\tilde{e}[z] = e^{-ksz}$. Note that (35) is the sum of two boundary terms and a pure combustion term.

The following properties may be used to invert the transform (35) termwise: For any function $q(t)$ of t (and possibly other variables) with transform $\tilde{q}$:

$$\tilde{q}/s \text{ inverts to } \int_o^t q(\lambda)\, d\lambda; \tag{38}$$

$$\tilde{q}\,\tilde{e}(z) \text{ inverts to } q[t * z] = \begin{cases} q(t - kz) & \text{if } t - kz > 0 \\ 0 & \text{if } t - kz < 0 \end{cases} \quad \text{which defines } q[t*z] \tag{39}$$

The property (39) coupled with (37) inserted in (35), has the effect of introducing new non-zero terms as the time t increases, these new terms corresponding physically to the appearance of new direct and reflected "waves".

In the special case of $b(a,t)$ in (30) having the form conveniently representable as

$$\begin{aligned} b(a,t) &= (f^{-2} - k^2)\, r_t(t - a/f) = 0 \text{ unless } 0 < t - a/f < t', \\ &\text{with } r(t - a/f) = \int_{a/f}^{t} b(a,t)\, dt/(f^{-2} - k^2), \end{aligned} \tag{40}$$

[corresponding in (13) to a flame front $a^*(t) = ft$ and a flame "tail" $a^{**}(t) = ft - ft'$,[7] with a uniform duration t' of burning for each layer, and with constant "intrinsic Lagrangian flame speed" f] then the $\widetilde{w}^*$ of (35) and (36) assumes the form

$$(41) \qquad \widetilde{w}^* = -\int_0^{t'} [-e^{-(\lambda + a/f)s} + e^{-(\lambda + L/f)s}\, \widetilde{C}(L,a) + e^{-\lambda s}\, \widetilde{S}(L,a)/fk]\; r_\lambda(\lambda)\; d\lambda/s^2.$$

In any case, for the determination of

$$(42) \qquad p = p_0 - \int_L^a g\,(a,t)\; da/\tau_0 - w_t/\tau_0, \quad u = w_a \text{ and } x = a + \int_0^t w_a\, dt,$$

the transforms $(\widetilde{w_t}) = s\widetilde{w}$ and $\widetilde{w}_a$ invert to

$$(43) \qquad w_t = w^*_t + \sum_{n=0} (-)^n \Big\{ -u_0[t*(2nL + a)]/k + u_0[t*(2nL + 2L - a)]/k \\ -\tau_0\Delta[t*(2nL + L - a)] - \tau_0\Delta[t*(2nL + L + a)] \Big\}$$

and

$$(44) \qquad w_a = w^*_a + \sum_{n=0} (-)^n \Big\{ u_0[t*\,(2nL + a)] + u_0[t*(2nL + 2L - a)] \\ -\tau_0\Delta[t*(2nL + L - a)] + \tau_0 k\Delta[t*(2nL + L + a)] \Big\}$$

where the notation of (39) and the symbol $\Delta(t) = p_L(t) - p_0$ have been used.

For the special form (41) based on (40), $s\widetilde{w}^*$ and $\widetilde{w}^*_a$ invert to

$$(45) \qquad w^*_t = r[t*(a/fk)] + \sum_{n=0} (-)^n \Big\{ -r[t*(2nL + L - a + L/fk)] - r[t*(2nL + L + a + L/fk)] \\ -r[t*(2nL + a)]/fk + r[t*(2nL + 2L - a)]/fk \Big\}$$

and

$$(46) \qquad w^*_a = -r[t*(a/fk)]/f + \sum_{n=0} (-)^n \Big\{ -kr[t*(2nL + L - a + L/fk)] + kr[t*(2nL + L + a + L/fk)] \\ + r[t*(2nL + a)]/f + r[t*(2nL + 2L - a)]/f \Big\}$$

The general form (36) may be inverted explicitly but it leads to rather lengthy expressions which will not be listed here.

On the basis of (42) - (46), formulas for p, u and x are readily obtained. In particular the flame front velocity relative to the tube is given by

$$(47) \qquad (d/dt)\; x^*(t) = f + fk^2\, w_t + w_a \text{ at } a = ft,$$

as may be shown by use of Figure 1, (30), (40) and (42).

Of special physical interest is the simple case of $p_L(t) = p_o$, $g(a,t) = 0$, $u_o(t) = 0$, with $b(a,t)$ as in (40), for which

$$p = p_o - w^*{}_t/\tau_o \text{ and } u = w^*{}_a$$

in terms of (45) and (46), and for which the flame speed relative to the tube is

$$(48) \quad (d/dt)\, x^*(t) = [f + fk^2 w^*{}_t(a,t) + w^*{}_a(a,t)] \text{ at } a = ft$$

$$= f + \sum_{n=0} (-)^n \Big\{ -(fk^2 + k)\, r[t + kft - 2nkL - kL - L/f]$$

$$-(fk^2 - k)\, r[t - kft - 2nkL - kL - L/f] - (k - f^{-1})\, r[t - kft - 2nkL]$$

$$+ (k + f^{-1})\, r[t + kft - 2nkL - 2kL] \Big\}$$

where $r[\lambda] \begin{cases} = 0 \text{ if } \lambda < 0 \\ = r(\lambda) \text{ if } \lambda > 0 \end{cases}$ with $r(\lambda) = r(t') \text{ if } \lambda > t'$. Here

$$(49) \quad r(t - a/f) = \int_{a/f}^{t} \frac{[-e_t\, k^2\, (\gamma - 1)(\tau_o/\tau)]\, dt}{(f^{-2} - k^2)}$$

in terms of the fundamental quantities used in (9). If in the above $u_o(t)$ were not identically zero, the following terms would have to be added to (48):

$$(50) \quad \sum_{n=0} (-)^n \Big\{ -(fk - 1)\, u_o[t - kft - 2nkL] + (fk + 1)\, u_o[t + kft - 2nkL - 2kL] \Big\}$$

where $u_o[\lambda] \begin{matrix} = 0 \\ = u_o(\lambda) \end{matrix}$ for $\lambda \begin{matrix} < \\ > \end{matrix} 0$.

NUMERICAL COMPARISONS OF SOLUTIONS OF LINEAR AND NON-LINEAR EQUATIONS

Figures 2 and 3 indicate how closely the analytical solutions obtained in the preceeding section can agree with the numerical solutions of the non-linear equations previously discussed.

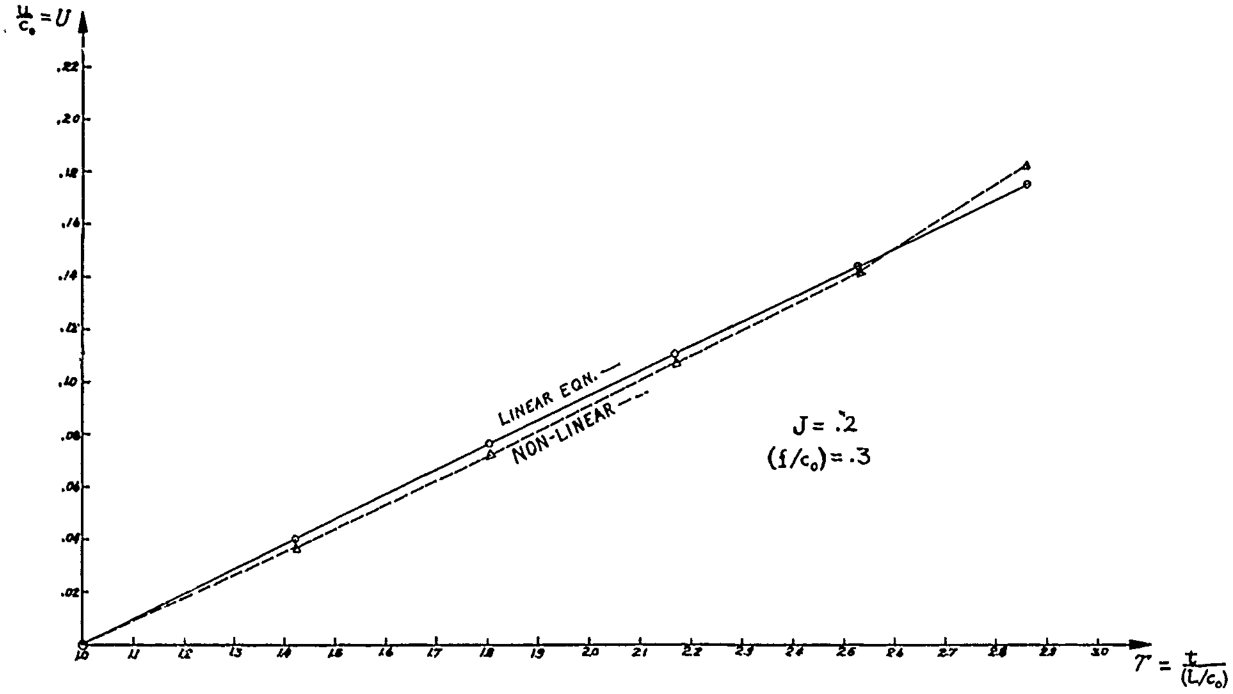

Figure 2.

Stream velocity u versus time t at open end of flame tube, calculated on the basis of the linear and on the basis of the non-linear equations with the same parameters.

These figures are based on the indicated values of the dimensionless "burning rate"[4] J and "intrinsic Lagrangian flame speed" f/c_0, with $k = 1/c_0$ and a burning duration $t' > 3L/c_0$ in connection with (40) to (47).

The close agreement indicated in Figures 2 and 3 would not in general be matched by cases involving much larger values for the parameters. However the linear system (26) is exactly equivalent to the non-linear system (7) - (10) if $g(a,t)$ and $h(a,t)$ in (26) are exact rather than approximate. Good agreement between the two systems therefore depends on good choices for these last-named functions. Such choices can be facilitated by use of (26) for iterations, but care should be taken to choose k at each stage so that $g(a,t)$ and $h(a,t)$ do not have the characteristics $t \pm ka$ = constants, as discontinuity lines, since otherwise discontinuities in p, u, and τ may appear.

COMPARISON WITH EXPERIMENTS

As was mentioned in the introduction. a general discussion of experimental results obtained with the flame tube at New York University will appear in a separate report which will contain some comparisons of theoretical with experimental results. At present really

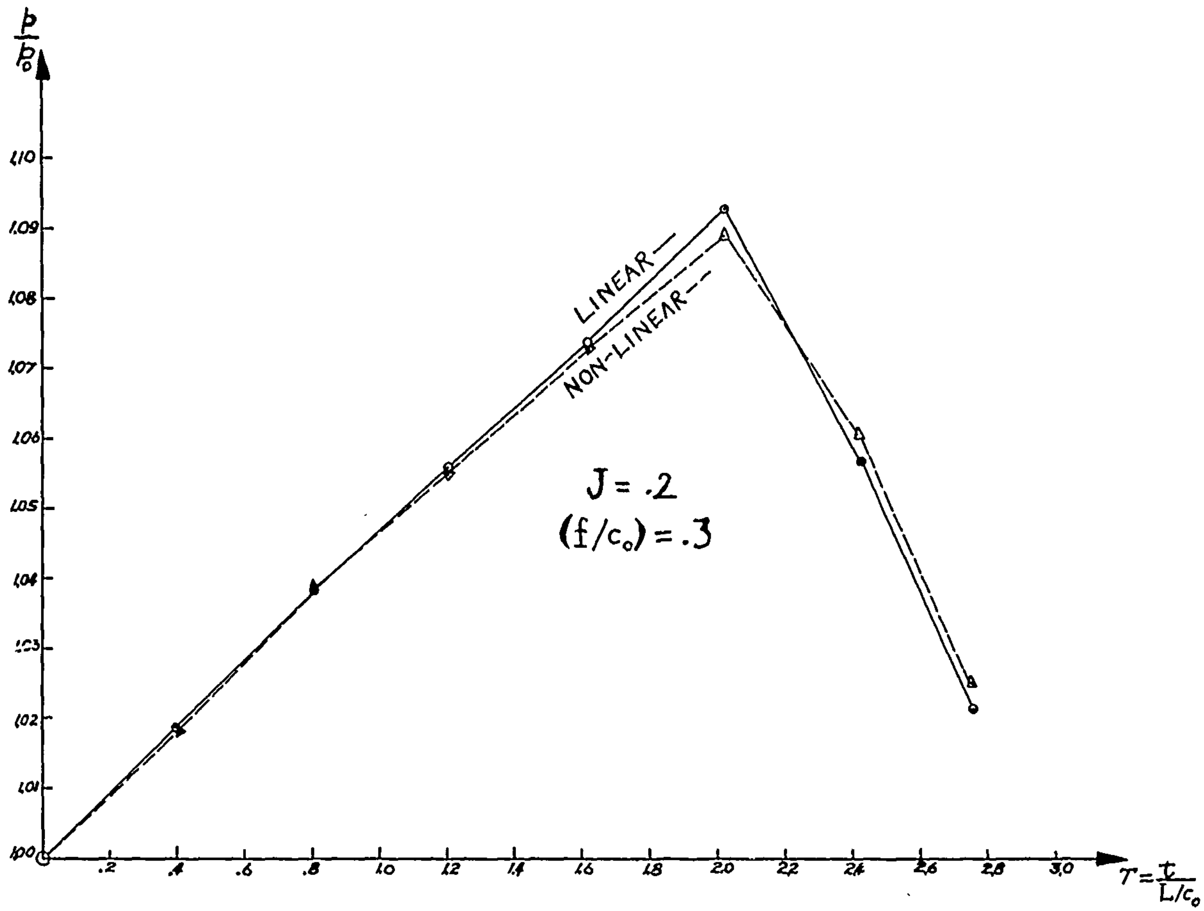

Figure 3.

Pressure p versus time t at closed end of flame tube, calculated for the same cases as in Figure 2.

accurate experimental results are limited to curves for flame front position (or velocity) versus time. It would appear that formulas (48) to (50) are general enough to fit almost any such results. When pressure and cross-sectional-average stream velocities and temperatures have been measured as functions of time and position, then, in particular, it should be possible to distinguish unambiguously between the contributions of (48) and (50). In this connection it is worth noting that the early variation (before reflected waves arrive from the open end) of the flame speed is represented in (48) and (50) by

$$(51) \qquad (d/dt)\, x^*(t) = f + (f^{-1} - k)\, r[t - kft] + (1 - kf)\, u_0[t - kft].^7$$

which may be used to obtain empirical forms for the functions γ and u_0 for use in (48) and (50).

REFERENCES AND NOTES

1. See *Combustion*, Project SQUID Field Survey Report Vol. 1 Part 1 (30 June 1947), Engineering Research Associates, discussion and references.

2. See *A Gas Dynamical Formulation for Waves and Combustion in Pulse-jets*, AMG-NYU Report No. 151 (July 1946), Institute for Math. and Mechanics, N.Y.U., 45 Fourth Ave., New York, N.Y.

3. See Lamb's *Hydrodynamics*, Ch. I (Dover, 1945): equations in Euler and Lagrange forms.

4. For numerical and scaling work it is probably best to use dimensionless variables $X = x/L$, $A = a/L$, $T = t/(L/c_0)(c_0 = \sqrt{\gamma_0 p_0 \tau_0}$ = initial sound speed), $U = u/c_0$, $P = p/p_0\gamma_0$, $V = \tau/\tau_0$, $M = \mu\tau_0/c_0$, $G = gL/c_0^2$, $J = jL/p_0 c_0 \gamma_0 = (\gamma - 1)\, e_t\, L/\tau p_0 c_0 \gamma_0$. Then in the forms in the text $(x,a,t,u,p,\tau,\mu,g,j,p_0,\tau_0,L)$ may be replaced by $(X,A,T,U,P,V,M,G,J,1/\gamma_0,1,1)$ respectively. Note that the $T = 1$ corresponds to the time required for a sound wave to travel the full length of the tube under the initial conditions.

5. Riemann, *Gesam. Math. Werke*, p. 145 (Leipzig, 1876); and Love and Pidduck, *On Lagrange's Ballistic Problem*, Trans. Roy. Soc. London, Vol. A 222, pp. 167-226 (1922).

6. See Churchill, *Modern Operational Mathematics in Engineering*, for some techniques in Laplace Transforms (McGraw Hill, 1944).

7. The case of ignition at $a = 0$ at time $t = t'' > 0$ [in line with the discussion below (27)] is most conveniently treated by shifting the time zero so that the ignition takes place at $t = 0$. Then the condition $u = u_0(t)$ applies for $t > 0$, and (40)-(51) hold but with $u_0[\lambda] = \begin{matrix} 0 \\ u_0(\lambda) \end{matrix}$ for $\lambda \begin{matrix} < \\ > \end{matrix} -t''$ in (50) and (51).

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(23) Jet engines; (23) Thermodynamics

A one-dimensional aero-thermodynamic theory of turbulent flame propagation in flame tubes is developed to provide data on combustion and flame parameters for use in the theoretical treatment of jet engines. General one-dimensional, aero-thermodynamical equations and methods for solving them are presented in forms suitable for use in connection with eddy-turbulent flame propagation in flame tubes. Solutions obtained by use of characteristics and finite differences are compared with explicit analytical solutions of approximating linear differential equations and show good agreement between the solutions of typical cases.

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