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THE BUCKLING OF STIFFENED PLATES IN SHEAR

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THE BUCKLING OF STIFFENED PLATES IN SHEAR

(Das Ausknicken versteifter Bleche unter Schubbeanspruchung)

by

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INTRODUCTION

Thin plates stiffened by corrugations or attached ribs have an important place in airplane design. Therefore the method herein described of determining the buckling loads of plates thus stiffened in an extreme case will be of interest.

In Sections 1 and 2 we shall consider a strip of infinite length and of height 2h, stiffened by the attachment of transverse ribs (y-axis). Since it is assumed that these ribs are of infinite thinness, it is also assumed that they are spaced at infinitely small intervals (Fig. 1). In Sec. 3, longitudinal stiffeners will be considered in addition.

In setting up the differential equation, we shall confine ourselves to a symmetrical case, in which both sides of the plate are stiffened. The neutral axis will then coincide with the mid-plane of the plate. When only one side is stiffened, the only difference will be the significance of the constants entering into the differential equation (the moments of inertia to be referred to the neutral axis) so that an investigation of a symmetrical strip will entail no limitation of the method in general.

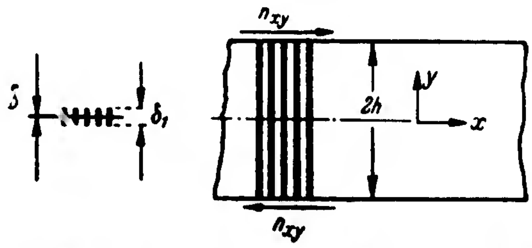

Figure 1. Strip of Plating of Infinite Length with Transverse Ribs Attached.

The limitations of infinite length of the strip and of infinitely close spacing of the stiffeners were selected for treatment because a stiffened plate of finite length with finite rib spacing is not susceptible to mathematical treatment in practice. Moreover it is safe to assume infinite length of the strip, since in this way we obtain the lower limit of the buckling load. For an otherwise similar strip of finite length the buckling load must be greater.

At the outset nothing is known regarding the dependence of the buckling load upon the spacing of the stiffeners, the same amount of material being used. However, considering the symmetrical case it may be reasonably stated that the curve of the buckling load as a function of the spacing t of the stiffeners has a horizontal tangent when t = 0, i.e., a true extreme. If this is true, the differences between the true buckling load and those calculated on the assumption t = 0 will be very small. For our present purpose we let the y-axis fall in any one of the ribs; then the neighboring ribs will have the x-coordinates +t, –t, etc. If we now interchange t and –t, this will be equivalent to reproducing a mirror image of the member at the y-axis. Since the member is of infinite length, this will change nothing, including the buckling load, so that the latter will be a linear function of t. Although it does not follow arbitrarily from this argument that the curve for t = 0 has a real extreme, it is highly improbable that there will be a peak in the curve at this point. Moreover it is true for reasons of continuity that the solution here derived gives a close approximation for small finite intervals between stiffeners also. In any case the spacing of the stiffeners must be close enough to preclude buckling of the plating between them.

The simplifications adopted ultimately yield extraordinarily simple formulas, which permit the buckling loads to be given almost entirely without calculation.

For the strip with cross stiffeners two assumptions are made: In Sec. 1 it is regarded as a thin plate, the calculation being carried out according to the formulas of the classic plate theory, while in Sec. 2 Wagner's# theory of diagonal tension fields is used as the basis. In Sec. 3 we consider a plate with transverse as well as longitudinal stiffeners with the same assumption as for the plate with only transverse stiffeners in Sec. 1. In Sec. 4 a sample calculation for each case is carried out.

1. THE STRIP AS A THIN PLATE

As usual in the plate theory, we adopt the premises implied in the assumption that the plate thickness δ is small, and the deflections w are also small with respect to the plate thickness. Finally we make the additional assumption for the present case that the plate thickness is also small with respect to the height of the ribs (height of ribs δ_1). Then, for a point in the plate at a distance z from the midplane+ we will find:

The deflection w gives the displacements

$$u = -z\frac{\partial w}{\partial x} \qquad v = -z\frac{\partial w}{\partial y}.$$

From the displacements we get for the strains and the specific displacement

$$\varepsilon_x = -z\frac{\partial^2 w}{\partial x^2} \qquad \varepsilon_y = -z\frac{\partial^2 w}{\partial y^2} \qquad \gamma_{xy} = -2z\frac{\partial^2 w}{\partial x\,\partial y}.$$

In the sense of the classic plate theory we get from these the stresses

$$\sigma_x = -\frac{Ez}{1-\nu^2}\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right); \quad \sigma_y = -\frac{Ez}{1-\nu^2}\left(\nu\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right); \quad \tau_{xy} = -2Gz\frac{\partial^2 w}{\partial x\,\partial y}$$

and from these, by integration over the breadth of the plate we get the moments

$$m_x = -N\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) \quad m_y = -N\left(\nu\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) \qquad m_{xy} = -(1-\nu)N\frac{\partial^2 w}{\partial x\,\partial y},$$

H. Wagner: Ebene Blechwandträger mit sehr dünnem Stegblech, (Flat Plate Girders with very Thin Webs), ZFM XX. 1929, No. 8, et seq. In this paper approximations of buckling loads for any given stiffener spacing are also derived.

+ See Nadai, Elastic Plates, Section 10.

where $N = \frac{E\delta^3}{12(1-\nu^2)}$ is the plate rigidity. From the moments we get the shear forces of the plate

$$p_x = -N\frac{\partial \Delta w}{\partial x} \qquad p_y = -N\frac{\partial \Delta w}{\partial y}.$$

With the foregoing assumptions we get as shear force of the ribs

$$p_y{}^* = -EJ^*\frac{\partial^3 w}{\partial y^3},$$

where J* is the moment of inertia per unit of length of the stiffeners referred to the mid-plane of the plate. If p is the external load, then the equilibrium condition will be

$$\frac{\partial p_x}{\partial x} + \frac{\partial p_y}{\partial y} + \frac{\partial p_y{}^*}{\partial y} + p = 0,$$

or

$$N\Delta\Delta w + EJ^*\frac{\partial^4 w}{\partial y^4} = p,$$

If we stress the longitudinal edges of the plate by shearing forces n_{xy} per unit of length, then, in consequence of the deflection of the plate, these forces will have components perpendicular to the tangential plane, the value of which is given by+

$$p = 2n_{xy}\frac{\partial^2 w}{\partial x\,\partial y},$$

so that we get as the differential equation for the deflection

$$N\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2\partial y^2}\right) + (N + EJ^*)\frac{\partial^4 w}{\partial y^4} = 2n_{xy}\frac{\partial^2 w}{\partial x\,\partial y} \qquad (1)$$

If we write

$$\frac{N}{N+EJ^*} = A \quad \frac{n_{xy}h^2}{N+EJ^*} = B \quad \frac{x}{h} = \xi \quad \frac{y}{h} = \eta \qquad (2)$$

the differential equation becomes

$$A\left(\frac{\partial^4 w}{\partial \xi^4} + 2\frac{\partial^4 w}{\partial \xi^2\partial \eta^2}\right) + \frac{\partial^4 w}{\partial \eta^4} = 2B\frac{\partial^2 w}{\partial \xi\,\partial \eta} \qquad (1a)$$

Since the strip is of infinite length, we must set the following boundary conditions:

1. w must be purely periodic in ξ.

2a. $w = \frac{\partial^2 w}{\partial \eta^2} = 0$ when $\eta = \pm 1$ when the strip is simply supported.

2b. $w = \frac{\partial w}{\partial \eta} = 0$ when $\eta = \pm 1$ for a built-in strip.

To solve (1a) we set down the following expression#:

$$w = e^{i\lambda\xi}\,Y(\eta) \qquad (3)$$

+ Nadai, l.c. Sec. 57.

See Southwell-Skan: Stability under Shearing Forces of a flat elastic Strip. Proc. Royal Soc. London. 105 (1924), p. 582 et seq. Subsequently to the completion of my calculation there was published a paper by MM. Reissner and Bergmann (ZFM XX, p. 475 et seq.) which treats of the buckling of corrugated sheet metal, in which the authors likewise make use of the Southwell-Skan method. Wherever my calculations coincide in form with those in that paper, I have therefore presented them as briefly as possible.

where λ is a real number, and get the equation for Y (the primes denote derivatives with respect to η)

$$A(\lambda^4 Y - 2\lambda^2 Y'') + Y^{(IV)} = 2 i \lambda B Y' \tag{4}$$

With $Y = e^{i\mu\eta}$ we get the characteristic equation

$$\mu^4 + 2A\lambda^2\mu^2 + A\lambda^4 + 2\lambda\mu B = 0 \tag{4a}$$

Boundary condition (1) is satisfied directly since λ is to be a real number. We get for Y

$$Y = b_1 e^{i\mu_1\eta} + b_2 e^{i\mu_2\eta} + b_3 e^{i\mu_3\eta} + b_4 e^{i\mu_4\eta},$$

Where the u_i are roots of (4a). Boundary condition (2) yields four linear, homogeneous equations for determining the b_i terms, which can be satisfied if the determinant of the coefficients vanishes. Thus we get as the "buckling equation" for a simply supported strip

$$\text{(Ia)} \quad \begin{vmatrix} e^{i\mu_1} & e^{i\mu_2} & e^{i\mu_3} & e^{i\mu_4} \\ e^{-i\mu_1} & e^{-i\mu_2} & e^{-i\mu_3} & e^{-i\mu_4} \\ \mu_1^2 e^{i\mu_1} & \mu_2^2 e^{i\mu_2} & \mu_3^2 e^{i\mu_3} & \mu_4^2 e^{i\mu_4} \\ \mu_1^2 e^{-i\mu_1} & \mu_2^2 e^{-i\mu_2} & \mu_3^2 e^{-i\mu_3} & \mu_4^2 e^{-i\mu_4} \end{vmatrix} \equiv \begin{vmatrix} \sin\mu_1 & \sin\mu_2 & \sin\mu_3 & \sin\mu_4 \\ \cos\mu_1 & \cos\mu_2 & \cos\mu_3 & \cos\mu_4 \\ \mu_1^2\sin\mu_1 & \mu_2^2\sin\mu_2 & \mu_3^2\sin\mu_3 & \mu_4^2\sin\mu_4 \\ \mu_1^2\cos\mu_1 & \mu_2^2\cos\mu_2 & \mu_3^2\cos\mu_3 & \mu_4^2\cos\mu_4 \end{vmatrix} = 0.$$

and similarly for a built-in strip

$$\text{(Ib)} \quad \begin{vmatrix} \sin\mu_1 & \sin\mu_2 & \sin\mu_3 & \sin\mu_4 \\ \cos\mu_1 & \cos\mu_2 & \cos\mu_3 & \cos\mu_4 \\ \mu_1\sin\mu_1 & \mu_2\sin\mu_2 & \mu_3\sin\mu_3 & \mu_4\sin\mu_4 \\ \mu_1\cos\mu_1 & \mu_2\cos\mu_2 & \mu_3\cos\mu_3 & \mu_4\cos\mu_4 \end{vmatrix} = 0$$

By expanding these determinants we will get after several conversions

$$\text{(IIa)} \quad (\mu_1^2-\mu_3^2)(\mu_2^2-\mu_4^2)\sin(\mu_1-\mu_2)\sin(\mu_3-\mu_4) = (\mu_1^2-\mu_2^2)(\mu_3^2-\mu_4^2)\sin(\mu_1-\mu_3)\sin(\mu_2-\mu_4)$$

$$\text{(IIb)} \quad (\mu_1-\mu_3)(\mu_2-\mu_4)\sin(\mu_1-\mu_2)\sin(\mu_3-\mu_4) = (\mu_1-\mu_2)(\mu_3-\mu_4)\sin(\mu_1-\mu_3)\sin(\mu_2-\mu_4)$$

It follows from the characteristic equation that the sum of the four roots must vanish. Then if we write

$$\mu_{1,2} = \alpha \pm \beta \qquad \mu_{3,4} = -\alpha \pm \gamma,$$

where α must be real, and β and γ may be either real or purely imaginary, we will obtain by equating the coefficients

$$\left.\begin{aligned} \sum_{i \neq k} \mu_i\mu_k &= -2\alpha^2 - \beta^2 - \gamma^2 = 2A\lambda^2 \\ \sum_{i \neq k \neq l} \mu_i\mu_k\mu_l &= -2\alpha(\beta^2-\gamma^2) = 2\lambda B \\ \mu_1\mu_2\mu_3\mu_4 &= (\alpha^2-\beta^2)(\alpha^2-\gamma^2) = A\lambda^4 \end{aligned}\right\} \tag{5}$$

By substituting the u_i in (IIa) and (IIb) we get

$$\text{(IIIa)} \quad 8\alpha^2\beta\gamma\left[\cos 2\beta\cos 2\gamma - \cos 4\alpha\right] = \left[4\alpha^2(\beta^2+\gamma^2) - (\beta^2-\gamma^2)^2\right]\sin 2\beta\sin 2\gamma$$

(IIIb) $$2\beta\gamma\left[\cos 2\beta\cos 2\gamma - \cos 4\alpha\right] = \left[4\alpha^2 - \beta^2 - \gamma^2\right]\sin 2\beta\sin 2\gamma$$

Since A is of the order of magnitude of $\left(\frac{\delta}{\delta_1}\right)^3$, i.e., it is very small by assumption, the obvious thing to do is to cancel out the factor A in (5), thus solving the system

$$-\alpha_0(\beta_0^2 - \gamma_0^2) = \lambda B \quad 2\alpha_0^2 + \beta_0^2 + \gamma_0^2 = 0 \quad (\alpha_0^2 - \beta_0^2)(\alpha_0^2 - \gamma_0^2) = 0$$

Taking α_0 as parameter, we get

$$\beta_0^2 = \alpha_0^2 \quad \gamma_0^2 = -3\alpha_0^2$$

and the transcendental equations become

$$\cos 2\alpha_0 \mathfrak{Cof}\, 2\sqrt{3}\alpha_0 - \cos 4\alpha_0 - \sqrt{3}\sin 2\alpha_0 \mathfrak{Sin}\, 2\sqrt{3}\alpha_0 = 0$$
$$\cos 2\alpha_0 \mathfrak{Cof}\, 2\sqrt{3}\alpha_0 - \cos 4\alpha_0 + \sqrt{3}\sin 2\alpha_0 \mathfrak{Sin}\, 2\sqrt{3}\alpha_0 = 0.$$

The smallest root of the first equation gives $2\alpha_0 = 3.6661$. In this approximation, therefore, λB = constant. Thus there will be no minimum buckling load, since the λB curve is in the form of an equilateral hyperbola. In second approximation we retain the original equations ($A \neq 0$) and write

$$\alpha = \alpha_0 + \alpha_1 \quad \beta = \beta_0 + \beta_1 \quad \gamma = \gamma_0 + \gamma_1,$$

where $\alpha_1, \beta_1, \gamma_1$ are assumed to be small with respect to $\alpha_0, \beta_0, \gamma_0$. Then we can expand (III) in a Taylor series and break it down into its linear terms. Thus we get three linear equations for $\alpha_1, \beta_1, \gamma_1$ and may then substitute the improved roots in

$$-\alpha(\beta^2 - \gamma^2) \equiv f(\lambda) = \lambda B$$

By differentiation of this expression with respect to λ and equating the derivative to zero in order to get the minimum of B, it follows that

$$A\lambda_{min}^4 = \text{const} \gg 1$$

With this λ_{min}, however, $\alpha_1, \beta_1, \gamma_1$ will by no means be small with respect to $\alpha_0, \beta_0, \gamma_0$, so that this method of improving the roots is not practicable. Nevertheless this result suggests that the critical λ-value is proportional to $\frac{1}{\sqrt[4]{A}}$. Then if we write

$$\lambda = \frac{\lambda_1}{\sqrt[4]{A}} \quad B = B_1\sqrt[4]{A}$$

we will get

$$\left.\begin{aligned} -2\alpha^2 - \beta^2 - \gamma^2 &= 2\sqrt{A}\lambda_1^2 \\ -\alpha(\beta^2 - \gamma^2) &= \lambda_1 B_1 \\ (\alpha^2 - \beta^2)(\alpha^2 - \gamma^2) &= \lambda_1^4 \end{aligned}\right\} \qquad (5a)$$

and may then once more attempt the boundary transition $A \to 0$. Thus we obtain

$$\left.\begin{aligned} 2\alpha_0^2 + \beta_0^2 + \gamma_0^2 &= 0 \\ -\alpha_0(\beta_0^2 - \gamma_0^2) &= \lambda_1 B_1 \\ (\alpha_0^2 - \beta_0^2)(\alpha_0^2 - \gamma_0^2) &= \lambda_1^4 \end{aligned}\right\} \qquad (5b)$$

together with the equations (III). The simplest method of solving this system of equations is as follows: We assume λ_1^4 to be fixed and calculate β_0 and γ_0 from the first and third equations as functions of α_0. These values are inserted into the transcendental equation, which now is merely a function of α_0, and by one of the usual methods of solving such equations we calculate the smallest root of α_0. From the first three equations we see that either β_0 or γ_0 must be purely imaginary, or perhaps both. In either case we choose $\beta_0^2 > \gamma_0^2$, so that α_0 must be negative in any case. If we compute β_0 and γ_0 from the first and third equation (5b), we will get

$$\beta_0^2 + \gamma_0^2 = -2\alpha_0^2;\ \beta_0^2\gamma_0^2 = \lambda_1^4 - 3\alpha_0^4;\ \beta_0^2 = -\alpha_0^2 + \sqrt{4\alpha_0^4 - \lambda_1^4};\ \gamma_0^2 = -\alpha_0^2 - \sqrt{4\alpha_0^4 - \lambda_1^4}.$$

So that β_0 and γ_0 may be real, $4\alpha_0^4$ must be greater than λ_1^4. If $3\alpha_0^4 < \lambda_1^4$, then $\beta_0^2\gamma_0^2$ will be positive, β_0 and γ_0 purely imaginary. It can be demonstrated, however, that there is no root in the interval $\frac{\lambda_1^4}{4} < \alpha_0^4 < \frac{\lambda_1^4}{3}$, so that β_0 will be real, and γ_0 purely imaginary.

Then if we let $i\gamma_0' = \gamma_0$, we will get from (III)

(IVa) $\alpha_0^2\sqrt{3\alpha_0^4 - \lambda_1^4}\,(\cos 2\beta_0 \operatorname{Cof} 2\gamma_0' - \cos 4\alpha_0) = -\frac{6\alpha_0^4 - \lambda_1^4}{2}\sin 2\beta_0 \operatorname{Sin} 2\gamma_0'$ simply supported

(IVb) $\sqrt{3\alpha_0^4 - \lambda_1^4}\,(\cos 2\beta_0 \operatorname{Cof} 2\gamma_0' - \cos 4\alpha_0) = 3\alpha_0^2 \sin 2\beta_0 \operatorname{Sin} 2\gamma_0'$ built-in

For numerical solution it is advisable to write

$$\alpha_0 = \alpha_1\pi;\ \beta_0 = \beta_1\pi;\ \gamma_0' = \gamma_1'\pi;\ \lambda_1 = \lambda_1'\pi,\ B_1^* = B_1'\pi^2.$$

In solving the transcendental equations it is found that in the first case $2\beta_0$ lies in the first quadrant, and in the second case in the third quadrant. The corresponding γ_0' will be so large that $\cos 4\alpha_0$ can be neglected with respect to $\cos 2\beta_0 \operatorname{Cof} 2\gamma_0'$, and that $\operatorname{Sin} 2\gamma_0'$ and $\operatorname{Cof} 2\gamma_0'$ can be substituted for each other when it is desired to calculate $\alpha_0, \beta_0, \gamma_0$ to four decimal places. Thus the equations (IV) are simplified as follows:

$$\text{a)}\quad \alpha_0^2\sqrt{3\alpha_0^4 - \lambda_1^4}\cos 2\beta_0 = -\frac{6\alpha_0^4 - \lambda_1^4}{2}\sin 2\beta_0$$

$$\text{b)}\quad \sqrt{3\alpha_0^4 - \lambda_1^4}\cos 2\beta_0 = 3\alpha_0^2 \sin 2\beta_0.$$

The quantities compiled in the following table were obtained by solving these equations. Only in the case of the quantity designated by * was the complete equation used, the numbers obtained from the simplified equation being placed opposite in brackets. It is evident that even in this unfavorable case the deviations are very small (see also Fig. 2).

Only that part of the solutions has been tabulated which contains the minimum of B_1'. Outside this field B_1' uniformly approaches infinity. With these values we get in the limiting case where $A \to 0$

1. For a simply supported strip

(Va) $$\lambda_k = \frac{0.548\,\pi}{\sqrt[4]{A}} \qquad P_k = \frac{4\,(N + EJ^*)\,\pi^2}{(2\,h)^2}\,0.84\,\sqrt[4]{A}.$$

2. For a built-in strip

(Vb) $$\lambda_k = \frac{0{,}740\,\pi}{\sqrt[4]{A}} \qquad P_k = \frac{4\,(N + EJ^*)\,\pi^2}{(2\,h)^2}\,1{,}53\,\sqrt[4]{A}.$$

I. SIMPLY SUPPORTED STRIP.

$\lambda_1^{4\prime}$	λ_1'	$-\alpha_1$	β_1	γ_1'	$\lambda_1' B_1'$	B_1'
*0.009	0.3080	0.4295	0.4150	0.7357	0.3064	0.9948
		(0.4313)	(0.4164)	(0.7382)		
0.03	0.4126	0.4598	0.4174	0.7725	0.3545	0.8518
0.09	0.5477	0.5174	0.4192	0.8433	0.4589	0.8378
0.15	0.6263	0.5585	0.4210	0.8950	0.5464	0.8780

II. BUILT-IN STRIP.

0.15	0.6223	0.6495	0.5724	0.0823	0.9736	0.5645
0.30	0.7401	0.6988	0.5659	1.1388	1.1300	1.5269
0.50	0.8409	0.7506	0.5603	1.2003	1.3170	1.5662
1.00	1.0000	0.8448	0.5521	1.3161	1.7208	1.7208

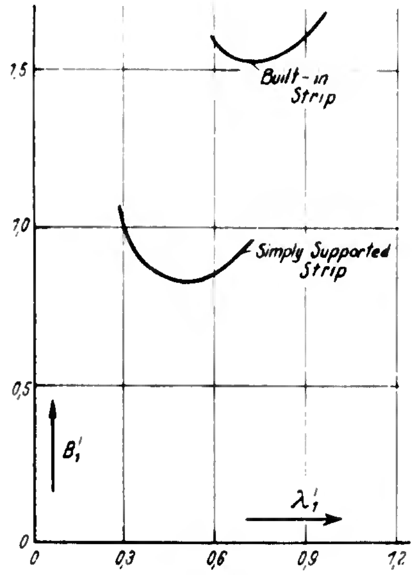

Figure 2.

Naturally this solution is valid rigorously only in the limiting case. In practice, however, it will also be permissible to apply it for small finite values of A, since obviously the algebraic law applies that the roots of an equation are continous functions of its coefficients. A slight change in α, β, γ, moreover, will cause only a slight change in the corresponding P_k, and consequently the P_k calculated by our formula will differ very slightly from the actual P_k. Thus we find as essential result of Part 1: The buckling load of a strip of plating with transverse stiffeners loaded by shearing forces approaches zero with $\sqrt[4]{A}$, where $A = \frac{N}{N + EJ^*}$.

However, since experience has shown that girders of this type have a high buckling strength, and moreover, that in buckling the plate is first deformed in a diagonal tension field, the whole system buckling only after an additional increase of load, Sec. 2 shall be devoted to determining the buckling load of a girder the plating of which is assumed to be a diagonal tension field.

2. THE PLATE AS A DIAGONAL TENSION FIELD

The nomenclature used in Sec. 1 will be retained. The plating will now be stressed in one direction. Let the tensile stress along a diagonal be σ, the slope of a diagonal α, and the spacing between two ribs t, for the present. (In the final result t will approach zero). Then (Fig. 3) will apply.

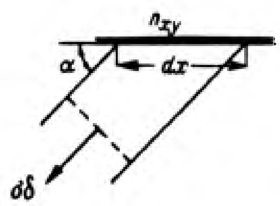

Figure 3.

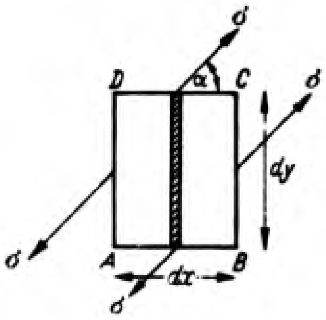

Figure 4.

$$\left.\begin{aligned} &\int_0^x \sigma\delta\sin\alpha\cos\alpha\, dx = n_{xy}\cdot x \qquad \sigma = \frac{n_{xy}}{\delta\sin\alpha\cos\alpha} \\ &\int_{-t/2}^{+t/2} \sigma\delta\sin^2\alpha\, dx = \sigma\cdot\delta\cdot t\sin^2\alpha = n_{xy}\cdot t\cdot \mathrm{tg}\,\alpha \end{aligned}\right\} \tag{6}$$

For an element $d_x d_y$ of the plate we will have (Fig. 4)

	y–components	x–components
On AB(CD)	$\sigma\sin^2\alpha\, dx$	$\sigma\sin\alpha\cos\alpha\, dx$
On BC(DA)	$\sigma\sin\alpha\cos\alpha\, dy$	$\sigma\cos^2\alpha\, dy.$

Due to the deflection w of the strip, we get components perpendicular to the plane of the strip

	on AB	on CD
of x–components	$-\sigma\delta\sin\alpha\cos\alpha\left(\frac{\partial w}{\partial x}\right)_{x_0} dx$	$+\sigma\delta\sin\alpha\cos\alpha\left(\frac{\partial w}{\partial x}\right)_{x_0+dy} dx$
of y–components	$-\sigma\delta\sin^2\alpha\left(\frac{\partial w}{\partial y}\right)_{x_0} dx$	$+\sigma\delta\sin^2\alpha\left(\frac{\partial w}{\partial y}\right)_{x_0+dy} dx$

	on DA	on BC
of x–components	$-\sigma\delta\cos^2\alpha\left(\frac{\partial w}{\partial x}\right)_{x_0} dy$	$+\sigma\delta\cos^2\alpha\left(\frac{\partial w}{\partial x}\right)_{x_0+dx} dy$
of y–components	$-\sigma\delta\sin\alpha\cos\alpha\left(\frac{\partial w}{dy}\right)_{x_0} dy$	$+\sigma\delta\sin\alpha\cos\alpha\left(\frac{\partial w}{\partial x}\right)_{x_0+dx} dy$

with the resultant

$$\sigma\delta\, dx\, dy\left(\sin 2\alpha\,\frac{\partial^2 w}{\partial x\,\partial y} + \sin^2\alpha\,\frac{\partial^2 w}{\partial y^2} + \cos^2\alpha\,\frac{\partial^2 w}{\partial x^2}\right).$$

Thus we get as the load per unit of area

$$p^* = n_{xy}\left(2\,\frac{\partial^2 w}{\partial x\,\partial y} + \mathrm{tg}\,\alpha\,\frac{\partial^2 w}{\partial y^2} + \mathrm{ctg}\,\alpha\,\frac{\partial^2 w}{\partial x^2}\right).$$

In addition, the ribs per unit of length must also take up the longitudinal force n_{xy} tan α so that we get the differential equation for the deflection of the ribs

$$EJ\,\frac{\partial^4 w}{\partial y^4} = n_{xy}\left(2\,\frac{\partial^2 w}{\partial x\,\partial y} + \mathrm{ctg}\,\alpha\,\frac{\partial^2 w}{\partial x^2}\right) \tag{7}$$

Without loss of generality we can let $\alpha = \frac{\pi}{4}$, since with another α–value only the significance of the constants entering into (7) will change. The rest of the calculation will be exactly as in the first section. We again let

$$x = h\xi, \quad y = h\eta, \quad \frac{n_{xy}h^2}{EJ} = B, \quad w = e^{i\lambda\xi}\,Y(\eta) \tag{8}$$

and get

$$Y^{(IV)} - 2\,i\,\lambda\,B\,Y' + \lambda^2\,B\,Y = 0 \qquad (9)$$

The expression $Y = e^{i\mu\eta}$ yields the characteristic equation for the μ_i terms

$$\mu^4 + 2\,\lambda\,\mu\,B + \lambda^2\,B = 0 \qquad (9a)$$

But in form this equation agrees exactly with the "abbreviated" characteristic equation used in Sec. 1, so that every statement made in the foregoing applies word for word to the present case.

However, the numerical calculation must be carried somewhat further, since the minimum of B will occur only at higher values of α. The range of the roots has been compiled in the following table:

SIMPLY SUPPORTED STRIP

$-\alpha_1$	β_1	γ_1'	λ_1	B_1
0.6311	0.4244	0.9883	0.4108	1.777
0.6980	0.4280	1.0759	0.5343	1.752
0.8084	0.4337	1.2228	0.7349	1.851

BUILT-IN STRIP

$-\alpha_1$	β_1	γ_1'	λ_1	B_1
0.9136	0.5474	1.4032	0.7237	2.864
0.9480	0.5453	1.4473	0.7937	2.857
0.9689	0.5442	1.4744	0.8358	2.863

Thus we get the buckling loads

(VIa) $$P_k = 4 \cdot 2.857\,\frac{E\,J\,\pi^2}{(2\,h)^2} = 11.43\,P_E$$ built-in

(VIb) $$P_k = 4 \cdot 1.752\,\frac{E\,J\,\pi^2}{(2\,h)^2} = 7.01\,P_E$$ simply supported

The essential result of calculating with the diagonal tension field therefore is a well defined, finite buckling load for the stiffened, infinitely thin girder.

3. THE GIRDER WITH TRANSVERSE AND LONGITUDINAL STIFFENERS

In connection with the foregoing, the girder with transverse and longitudinal stiffeners will also be discussed briefly. Let the moment of inertia in transverse direction be J_1, the moment of inertia in longitudinal direction J_2, and the plate rigidity n, as in the foregoing. Then, exactly as in Sec. 1, we will get as differential equation for the girder stressed in shear along its edges by the forces n_{xy}

$$(N + E\,J_2)\,\frac{\partial^4 w}{\partial x^4} + 2\,N\,\frac{\partial^4 w}{\partial x^2\,\partial y^2} + (N + E\,J_1)\,\frac{\partial^4 w}{\partial y^4} = n_{xy}\,\frac{\partial^2 w}{\partial x\,\partial y} \qquad (10)$$

If this expression is compared with Expression (1), it will be noted that if we let

$$\frac{N + EJ_2}{N + EJ_1} = A \tag{11}$$

and retain the other notations of this equation, Equation (1a) will be retained. Now, in contrast to Sec. 1, we can pass directly to plating of vanishing thickness, and we get as buckling loads

(VIIa) $$P_k = \frac{3{,}4\,E\pi^2}{(2h)^2}\sqrt[4]{J_1{}^3 J_2} \quad \text{for a simply supported plate}$$

(VIIb) $$P_k = \frac{6{,}1\,E\pi^2}{(2h)^2}\sqrt[4]{J_1{}^3 J_2} \quad \text{for a built-in plate}$$

Also of interest here is the problem of greatest material economy in the stiffeners at a given load. This is obviously a problem of minimums with an accessory condition, which shall here be given special treatment for stiffeners of similar cross-section. For other cross-sections the calculation is essentially the same.

For similar cross-sections the moments of inertia are to each other as the squares of the sectional areas. Since P_k is given, $J_1{}^3J_2$ is equal to a constant obtained from (VII). Thus the same is true for the magnitude $F_1{}^3F_2$ (F_1 being the sectional area of the transverse stiffeners, F_2 the sectional area of the longitudinals). Then the problem is

$$F_1 + F_2 = \text{Minimum} \tag{12}$$

with the accessory condition

$$F_1{}^3F_2 = \text{constant} \tag{12a}$$

Calculating F_2 from (12a), substituting in (12) and letting the differential quotient of this equation with respect to F_1 be zero, it follows that

$$1 - \frac{3C}{F_1{}^4} = 0, \quad F_1 = \sqrt[4]{3C}, \quad F_2 = \sqrt[4]{\frac{C}{27}}.$$

By this method the most favorable case is found to be when the areas of the stiffeners of similar cross-section are in the ratio of 3:1, the moments of inertia, therefore, of 9:1.

It should be expressly noted that the case of the plate with only longitudinal stiffeners is not included in our solution since it was obtained on the assumption that the moments of inertia, longitudinal as well as transverse, should be finite, while the plate thickness was to be infinitely small.

4. NUMERICAL EXAMPLE

Let there be given a girder of plate thickness δ = 1.4mm, elastic modulus E = 700,000 kg/cm² and a depth of 2h = 100 cm, stressed transversely by a force Q = 10 t. This load entails an edge shear force of $n_{xy} = \frac{Q}{2h}$ = 100 kg/cm. The required moments of inertia of the stiffeners are to be calculated for the three cases treated in the foregoing.

The formulas may be somewhat simplified for this computation by letting $\nu = 0$ and $\pi^2 = 10$. Then it will follow from (V) that

$$J_{\text{тр}} = \sqrt[3]{\frac{P_k^4 (2h)^8}{a^4 E^4 \pi^8 \frac{N}{E}}},$$

where

$$a = \begin{cases} 3{,}4 & \text{simply supported strip} \\ 6{,}1 & \text{built-in strip} \end{cases}$$

Making all substitutions we get

$$J_{\text{тр}} \sim \sqrt[3]{\frac{10^8 \cdot 10^{16}}{a^4 \cdot 10^4 \cdot 0{,}25 \cdot 10^{24} \cdot 2{,}5 \cdot 10^{-4}}} = \sqrt[3]{\frac{1}{0{,}62\, a^4}} = \begin{cases} 0{,}23 \\ 0{,}105 \end{cases}$$

If we assume a diagonal tension field we will get

$$J_{\text{тр}} = \frac{P_k \cdot (2h)^2}{E \pi^2 a},$$

where

$$a = \begin{cases} 7{,}01 & \text{simply supported strip} \\ 11{,}43 & \text{built-in strip} \end{cases}$$

or

$$J_{\text{тр}} \sim \frac{10^2 \cdot 10^4}{0{,}7 \cdot 10^6 \cdot 10 \cdot a} = \frac{1}{7a} = \begin{cases} 0{,}02 \\ 0{,}0125. \end{cases}$$

Finally let us consider the girder with transverse and longitudinal stiffeners. If we take the most favorable ratio of moments of inertia for similar sectional areas of the stiffeners, it follows from (VII) and $J_2 = \frac{J_1}{9}$

$$J_{1\,\text{тр}} = \frac{P_k (2h)^2 \sqrt[4]{9}}{a \cdot E \pi^2},$$

where

$$a = \begin{cases} 3{,}4 & \text{simply supported strip} \\ 6{,}1 & \text{built-in strip} \end{cases}$$

or

$$J_{1\,\text{тр}} = \frac{10^2 \cdot 10^4 \cdot \sqrt[4]{9}}{a \cdot 0{,}7 \cdot 10^6 \cdot 10} = \frac{2{,}47}{10\, a} = \begin{cases} 0{,}073 \\ 0{,}041 \end{cases} \quad J_2 = \begin{cases} 0{,}008 \\ 0{,}0046. \end{cases}$$

On the other hand, if we assume the moments of inertia to be equal it follows that

$$J_{\text{тр}} = \frac{P_k \cdot (2h)^2}{E \pi^2 a},$$

where

$$a = \begin{cases} 3{,}4 & \text{simply supported strip} \\ 6{,}1 & \text{built-in strip} \end{cases}$$

or

$$J_{\text{тр}} = \frac{10^2 \cdot 10^4}{0{,}7 \cdot 10^6 \cdot 10 \cdot a} = \frac{1}{7a} = \begin{cases} 0{,}042 \\ 0{,}023. \end{cases}$$

If we work out the equation

$$\frac{\sqrt{J_1} + \sqrt{J_2}}{2\sqrt{J}} = \begin{cases} \frac{0{,}360}{0{,}410} = 0{,}88 \\ \frac{0{,}270}{0{,}304} = 0{,}89, \end{cases}$$

which with similar cross-sections is a measure of the ratio of the cross-sectional areas required in the two last cases, we see that there is no great difference in material economy in the two cases.

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