

AD A953153

REPRODUCTION QUALITY NOTICE

This document is the best quality available. The copy furnished to DTIC contained pages that may have the following quality problems:

- **Pages smaller or larger than normal.**
- **Pages with background color or light colored printing.**
- **Pages with small type or poor printing; and or**
- **Pages with continuous tone material or color photographs.**

Due to various output media available these conditions may or may not cause poor legibility in the microfiche or hardcopy output you receive.

☐

If this block is checked, the copy furnished to DTIC contained pages with color printing, that when reproduced in Black and White, may change detail of the original copy.

A Method for Calculating the Pressure Distribution of a Body of Revolution Moving in a Circular Path through a Perfect Incompressible Fluid

By

F. VANDREY

*Reports and Memoranda No. 3139**

December, 1953

Summary.—A method is developed for calculating the pressure distribution of a body of revolution moving along a circular path in an infinite inviscid fluid otherwise still.

The method is based on the use of source distributions on the surface of the body. Their strength is determined by solving three integral equations corresponding to the elementary cases of a longitudinal translation, a lateral translation and a rotation of the body about a lateral axis. The distributions of the relative velocity on the surface are then obtained by integrations. The complete solution for any particular case is obtained by a linear combination of the three elementary cases.

An arrangement of the numerical work is given by which it can be carried out fairly automatically by a computer of average experience; several computers may share the work between them. Tables are given facilitating the calculation of the kernels of the integrals which forms the major part of the numerical work.

Introduction.—Unlike the problems of calculating the pressure distribution along a given body of revolution in motion along a straight line through a perfect fluid with or without an angle of incidence for which numerous solutions have been given, the case of a body of revolution moving along a circular path appears to have been treated so far only for a few special cases. The exact solution of this problem is known for an ellipsoid and its various degenerations from the classical investigations of the nineteenth century which give also the lines of approach to the general problem. An approximate solution for very elongated bodies was given by M. M. Munk in connection with his work on the theory of airships. Very few experimental results are available owing to the difficulty of measuring sufficiently accurately the pressure distribution of a model attached to a whirling arm.

It is easy to see that the motion of a body along a circle can be obtained by a superposition of a longitudinal translation, a lateral translation and a rotation of the body about its centre (= centre of its axis). These elementary motions can be treated separately and, methods being available for calculating the effect of the translations, there remains only the question of calculating the relative flow caused by a rotation of the body about its centre.

This problem is closely related to the problem of calculating the transverse flow past the body, and any of the methods developed for this purpose (e.g. by Th. v. Karman, C. Kaplan or Mrs. Flügge-Lotz) can readily be extended to the case of a rotating body.

The present investigation was undertaken with a view to developing a method for calculating the pressure distribution for a torpedo. This means in mathematical language that the shape of the body may be composed of arcs of different curves (often straight lines and arcs of circles) and that its curvature may have isolated discontinuities.

* A.R.L. Report ARL/R3/G/HY/12/2, received March, 1954.

The methods of v. Karman and Kaplan which use either explicitly or implicitly a generation of the body by a distribution of singularities on its axis, will then fail in principle, because they require the possibility of an analytic continuation of the flow into the interior of the body which is free from singularities except on its axis. The method of Mrs. Flügge-Lotz which uses a generation of the body by a distribution of sources on its surface, is free from this restriction, but has the practical disadvantage of requiring an excessive amount of numerical work.

The present report is an attempt to reduce the work involved in the method of Mrs. Flügge-Lotz. The integrations which have to be carried out for this method are performed with the aid of a mechanical integration formula which takes account of the singularities of the integrands. Tables are given facilitating the numerical calculation of the kernels of the integrals and the whole numerical work is reduced to the filling of certain forms which can be done fairly automatically by a computer of average experience. The numerical calculation is arranged in such a way that several computers can be engaged in the work simultaneously. If at all possible, the work should be divided up among several persons, as it is still so laborious that a single computer is likely to get tired of it. It may be worth investigating if the present method can serve as a basis for the application of high-speed electronic calculating equipment to the problem in question.

With a view to the application of the method to the case of a torpedo the arrangement of the numerical work was made under the assumption that the shape of the body is 'normal', i.e., that it can be given in cylindrical co-ordinates by an equation $r = r(x)$, where $r(x)$ is a single-valued function (not necessarily the same analytical expression in different parts of the body) with a continuous derivative $r'(x)$ and a piecewise continuous second derivative $r''(x)$. The shape is supposed to have a finite radius of curvature at the ends of the body. The case of pointed ends can be treated in the same way after rounding off the ends with a small radius of curvature; this has only a local effect on the pressure distribution which remains practically unaltered elsewhere. The case of an abnormal shape of the body, e.g., of a body with a flat nose can be treated in a similar manner, using the same integration formula but integrating along the arc of the shape instead of along the axis of the body. It was further assumed in setting up the arrangement for the numerical work that the equations defining the shape have a comparatively simple structure so that it is not too difficult and not too laborious to calculate r , r' and r'' for a great number of points.

Throughout the present report, the maximum radius of the body is used as unit of length. This is convenient as the singularities generating the body under the various flow conditions have the form of source rings with either constant (longitudinal translation) or sinusoidal (latitudinal translation or rotation) distribution of their strength per unit length of arc of the ring. When dealing with such source rings, the radius of the ring is a natural unit of length and, the body being supposed to be elongated, its maximum radius will be of the order of the local radius almost everywhere on the axis of the body except very near to its ends.

The author regrets that, due to circumstances beyond his control, he has to present his results in a comparatively condensed form.

PART A.—GENERAL THEORY

1. *Systems of Co-ordinates and their Transformation.*—Consider a body of revolution revolving with a constant angular velocity Ω about a fixed axis in an inviscid fluid otherwise still. All its points will then move with constant velocity ΩR along circles of radius R about points of the axis of rotation. Let C be the circle of radius R_0 and centre O_0 along which the centre O_1 of the body ($=$ centre of its axis) is moving with the velocity ΩR_0 .

The complete motion can then be described by using a system of co-ordinates (X, Y, Z) ; unit vectors $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ with the origin O_0 , the axis of X coincident with the line O_0O_1 at the time $t = 0$,

the axis of Y perpendicular to the axis of X , coincident with the position of O_0O_0 at the time $t = \pi/2\Omega$, the axis of Z perpendicular to both the axes of X and Y and having the direction of the vector Ω . These co-ordinates will be referred to as absolute co-ordinates.

The use of the absolute co-ordinates has the practical disadvantage that the position of the body and the pattern of the flow are time-dependent. It is easier to use a system of relative co-ordinates (x, y, z) with the unit vectors $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ attached to the body in which the flow pattern is independent of the time t . On the other hand, the flow is a time-dependent potential flow in (X, Y, Z) for which the pressure is readily obtained from Kelvin's formula. In (x, y, z) , the flow is not a potential flow but has a constant vorticity. It is, however, possible to transform Kelvin's formula into the system (x, y, z) and to obtain in this way a convenient formula for the calculation of the pressure distribution.

In order to perform the transformation $(X, Y, Z) \rightleftharpoons (x, y, z)$, consider first an auxiliary system of co-ordinates (T, N, B) with the origin at the centre O_0 of the body. Let the axis of T be tangent to the circle C , pointing into the direction in which the body is moving. Let further the axis of N coincide with the line O_0O_0 , pointing into the interior of C and let finally the axis of B be perpendicular to the plane of C and point into the direction of Ω (Fig. 1).

The absolute co-ordinates of O_0 at the time t are obviously given by

$$\mathbf{O}_0\mathbf{O}_0 = R_0 \cos \Omega t \mathbf{X} + R_0 \sin \Omega t \mathbf{Y} + \mathbf{OZ} \dots \dots \dots (1)$$

The angles between the axes (X, Y, Z) and (T, N, B) are easily seen to be

$$\left. \begin{aligned} (X, T) &= \frac{\pi}{2} + \Omega t & (X, N) &= \pi + \Omega t & (X, B) &= \frac{\pi}{2} \\ (Y, T) &= \Omega t & (Y, N) &= \frac{\pi}{2} + \Omega t & (Y, B) &= \frac{\pi}{2} \\ (Z, T) &= \frac{\pi}{2} & (Z, N) &= \frac{\pi}{2} & (Z, B) &= 0 \end{aligned} \right\} \dots \dots \dots (2)$$

This gives the transformation

$$\begin{bmatrix} X - R_0 \cos \Omega t \\ Y - R_0 \sin \Omega t \\ Z \end{bmatrix} = \begin{bmatrix} -\sin \Omega t & \cos \Omega t & 0 \\ \cos \Omega t & \sin \Omega t & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \dots \dots \dots (3a)$$

and the inverse transformation

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} -\sin \Omega t & \cos \Omega t & 0 \\ -\cos \Omega t & -\sin \Omega t & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X - R_0 \cos \Omega t \\ Y - R_0 \sin \Omega t \\ Z \end{bmatrix} \dots \dots \dots (3b)$$

Consider now the system of co-ordinates (x, y, z) with the unit vectors $(\mathbf{x}, \mathbf{y}, \mathbf{z})$. Its origin is again the centre O_0 of the body. The axis of x is the axis of symmetry of the body, pointing into the direction in which the body is moving and forming an angle α (= angle of pitch) with the plane of C . The axis of y is perpendicular to the axis of x , in the plane of C , pointing into its interior and forming an angle β (= angle of yaw) with the axis of N . The axis of z is perpendicular to both the axes of x and y , pointing into the direction of the vector product $\mathbf{x} \times \mathbf{y}$.

The angles between the axes (T, N, B) and (x, y, z) are then easily seen to be given by

$$\left. \begin{aligned} \cos(T, x) &= \cos \alpha \cos \beta & \cos(T, y) &= \sin \beta & \cos(T, z) &= -\cos \beta \sin \alpha \\ \cos(N, x) &= -\cos \alpha \sin \beta & \cos(N, y) &= \cos \beta & \cos(N, z) &= \sin \alpha \sin \beta \\ \cos(B, x) &= \sin \alpha & \cos(B, y) &= 0 & \cos(B, z) &= \cos \alpha \end{aligned} \right\} \dots \dots \dots (4)$$



UNANNOUNCED

Availability Codes	
Dist	Avail and/or Special
A-1	

This gives the transformation

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} \cos \alpha \cos \beta & \sin \beta & -\cos \beta \sin \alpha \\ -\cos \alpha \sin \beta & \cos \beta & \sin \alpha \sin \beta \\ \sin \alpha & 0 & \cos \alpha \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \dots \dots \dots (5a)$$

and inversely

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos \alpha \cos \beta & -\cos \alpha \sin \beta & \sin \alpha \\ \sin \beta & \cos \beta & 0 \\ -\sin \alpha \cos \beta & \sin \alpha \sin \beta & \cos \alpha \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \quad \dots \dots \dots (5b)$$

The transformation $(X, Y, Z) \rightleftharpoons (x, y, z)$ is then obtained by combining (3) and (5)

$$\begin{bmatrix} X - R_0 \cos \Omega t \\ Y - R_0 \sin \Omega t \\ Z \end{bmatrix} = \begin{bmatrix} -\cos \alpha \sin (\Omega t - \beta) & -\cos (\Omega t - \beta) & \sin (\Omega t - \beta) \sin \alpha \\ \cos \alpha \cos (\Omega t - \beta) & -\sin (\Omega t - \beta) & -\cos (\Omega t - \beta) \sin \alpha \\ \sin \alpha & 0 & \cos \alpha \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad (6a)$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\cos \alpha \sin (\Omega t - \beta) & \cos \alpha \cos (\Omega t - \beta) & \sin \alpha \\ -\cos (\Omega t - \beta) & -\sin (\Omega t - \beta) & 0 \\ \sin \alpha \sin (\Omega t - \beta) & -\sin \alpha \cos (\Omega t - \beta) & \cos \alpha \end{bmatrix} \begin{bmatrix} X - R_0 \cos \Omega t \\ Y - R_0 \sin \Omega t \\ Z \end{bmatrix}, \quad \dots \dots (6b)$$

Instead of using the relative cartesian co-ordinates (x, y, z) , it is more convenient to use relative cylindrical co-ordinates (x, r, θ) for the practical calculation :

$$x = x \quad y = r \cos \theta \quad z = r \sin \theta, \quad \dots \dots \dots (7a)$$

$$x = x \quad r = \sqrt{y^2 + z^2} \quad \theta = \tan^{-1} \frac{z}{y}, \quad \dots \dots \dots (7b)$$

For the calculation of the relative velocity at a point P of the surface of the body (co-ordinates x_0, r_0, θ_0), it is necessary to introduce a further system of co-ordinates (n, t, b) ; unit vectors n, t, b attached to this point. The axis of n is normal to the surface of the body, pointing into its exterior; the axis of t is tangential to the meridian curve of the body passing through P and is pointing from its head ($x = +l/2$) to its tail ($x = -l/2$). The axis of b is perpendicular to the axes of n and t , pointing into the direction of increasing values of the angular co-ordinate θ (= direction of the vector product $n \times t$) (Fig. 2).

Only the angles between the axes (x, y, z) and (n, t, b) will be needed, as the transformation $(x, y, z) \rightleftharpoons (n, t, b)$ will only be used for velocity components, i.e., for vectors.

If the shape of the body is given by an equation

$$r = r(x) \quad \dots \dots \dots (8)$$

in the cylindrical co-ordinates, the cosines of these angles (= scalar products of the corresponding unit vectors) are easily seen to be

$$\begin{aligned} n \cdot x &= -\frac{r'}{\sqrt{(1+r'^2)}} & n \cdot y &= \frac{\cos \theta}{\sqrt{(1+r'^2)}} & n \cdot z &= \frac{\sin \theta}{\sqrt{(1+r'^2)}} \\ t \cdot x &= -\frac{1}{\sqrt{(1+r'^2)}} & t \cdot y &= \frac{-r' \cos \theta}{\sqrt{(1+r'^2)}} & t \cdot z &= \frac{-r' \sin \theta}{\sqrt{(1+r'^2)}} \\ b \cdot x &= 0 & b \cdot y &= -\sin \theta & b \cdot z &= \cos \theta \end{aligned} \quad \dots \dots \dots (9)$$

2. *Absolute Velocity of a Point of the Body* [Co-ordinates (X, Y, Z) ; (x, y, z) and (x, r, θ)].—The absolute velocity of a point P of the body, i.e., of a point with fixed co-ordinates (x, y, z) , is

$$\mathbf{V}_{abs} = \boldsymbol{\Omega} \times \mathbf{O}_a \mathbf{P}, \quad \dots \dots \dots (10)$$

where

$$\boldsymbol{\Omega} = \Omega \mathbf{Z} \quad \dots \dots \dots (11)$$

is the vector of the angular velocity and O_a the origin of the absolute co-ordinates. If, at the time t , P has the absolute co-ordinates X, Y, Z , the vector $\mathbf{O}_a \mathbf{P}$ is

$$\mathbf{O}_a \mathbf{P} = X\mathbf{X} + Y\mathbf{Y} + Z\mathbf{Z}. \quad \dots \dots \dots (12)$$

The absolute velocity is then

$$\mathbf{V}_{abs} = \begin{vmatrix} \mathbf{X} & \mathbf{Y} & \mathbf{Z} \\ 0 & 0 & \Omega \\ X & Y & Z \end{vmatrix} = \Omega(-Y\mathbf{X} + X\mathbf{Y}). \quad \dots \dots \dots (13)$$

Introducing the relative co-ordinates (x, y, z) , $\mathbf{V}_{abs}$ becomes with respect to (6)

$$\begin{aligned} \mathbf{V}_{abs} = \Omega[(R_0 \cos \beta - y) \cos \alpha \mathbf{x} + (R_0 \sin \beta + x \cos \alpha - z \sin \alpha) \mathbf{y} \\ - (R_0 \cos \beta - y) \sin \alpha \mathbf{z}]. \quad \dots \dots \dots (14) \end{aligned}$$

This gives

$$V_{abs}^2 = \Omega^2[(R_0 \cos \beta - y)^2 + (R_0 \sin \beta + x \cos \alpha - z \sin \alpha)^2] \quad \dots \dots \dots (15)$$

$$= \Omega^2[(R_0 \cos \beta - r \cos \theta)^2 + (R_0 \sin \beta + x \cos \alpha - r \sin \theta \sin \alpha)^2]. \quad \dots \dots (15a)$$

Equation (14) can also be written in the form

$$\begin{aligned} \mathbf{V}_{abs} = \Omega[R_0 \cos \alpha \cos \beta \mathbf{x} + R_0 \sin \beta \mathbf{y} - R_0 \sin \alpha \cos \beta \mathbf{z} \\ + \cos \alpha(xy - yx) + \sin \alpha(yz - zy)]. \quad \dots \dots \dots (14a) \end{aligned}$$

This means that the complete movement of the body can be considered as the result of a superposition of the following elementary movements:

- | | | |
|--|-----|------|
| (i) Translation in the direction of x with the velocity $V_x = \Omega R_0 \cos \alpha \cos \beta$ | } . | (16) |
| (ii) Translation in the direction of y with the velocity $V_y = \Omega R_0 \sin \beta$ | | |
| (iii) Translation in the direction of z with the velocity $V_z = -\Omega R_0 \sin \alpha \cos \beta$ | | |
| (iv) Rotation about the axis of x with the angular velocity $\Omega_x = \Omega \sin \alpha$ | | |
| (v) Rotation about the axis of z with the angular velocity $\Omega_z = \Omega \cos \alpha$ | | |

No rotation occurs about the axis of y , this axis being chosen in a plane perpendicular to the vector $\boldsymbol{\Omega}$.

3. *Relative Particle Velocity of the Fluid at a Point of the Body* [Co-ordinates (X, Y, Z) and (x, y, z)].—If P is a point on the surface of the body or outside the body in a fixed relative position (constant co-ordinates x, y, z) and if $\phi(X, Y, Z, t)$ is the time-dependant potential of the flow, the relative particle velocity of the fluid at P will obviously be

$$\mathbf{V}_{rel} = (\phi_x - \mathbf{V}_{abs} \cdot \mathbf{x})\mathbf{x} + (\phi_y - \mathbf{V}_{abs} \cdot \mathbf{y})\mathbf{y} + (\phi_z - \mathbf{V}_{abs} \cdot \mathbf{z})\mathbf{z}. \quad \dots \dots \dots (17)$$

The derivatives ϕ_x, ϕ_y, ϕ_z can be obtained from $\phi(X, Y, Z, t)$ by the transformation equation (6); these expressions are, however, of no particular interest for the present purpose. Introducing the components of $\mathbf{V}_{abs}$ from (14), the relative velocity becomes

$$\begin{aligned} \mathbf{V}_{rel} = & [\phi_x - \Omega \cos \alpha (R_0 \cos \beta - y)]\mathbf{x} \\ & + [\phi_y - \Omega (R_0 \sin \beta + x \cos \alpha - z \sin \alpha)]\mathbf{y} \\ & + [\phi_z - \Omega \sin \alpha (R_0 \cos \beta - y)]\mathbf{z} \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (18) \end{aligned}$$

and consequently with respect to equation (15)

$$\begin{aligned} V_{rel}^2 = & V_{abs}^2 - 2\Omega[(\phi_x \cos \alpha - \phi_z \sin \alpha)(R_0 \cos \beta - y) \\ & + \phi_y(R_0 \sin \beta + x \cos \alpha - z \sin \alpha)] + [\phi_x^2 + \phi_y^2 + \phi_z^2]. \quad \dots \quad \dots \quad (19) \end{aligned}$$

4. *Calculation of the Pressure.*—The absolute flow caused by the revolving body is a time-dependant potential flow; the pressure is therefore to be calculated from Kelvin's formula

$$p - p_0 = -\rho\phi_t - \frac{1}{2}\rho(\phi_x^2 + \phi_y^2 + \phi_z^2) \quad \dots \quad \dots \quad \dots \quad \dots \quad (20)$$

p_0 will in general depend on t . In the present case p_0 can, however, be assumed as constant. This equation can be given a more convenient form by introducing the relative co-ordinates, as the relative flow is independent of the time t .

In the system (x, y, z) the derivatives of ϕ with respect to X, Y, Z and t are

$$\phi_x = \phi_x \frac{\partial x}{\partial X} + \phi_y \frac{\partial y}{\partial X} + \phi_z \frac{\partial z}{\partial X} \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (21)$$

and correspondingly for ϕ_y, ϕ_z and ϕ_t . The derivatives of x, y, z with respect to X, Y, Z are obviously the elements of the matrix in equation (6b)

$$\frac{\partial x}{\partial X} = -\cos \alpha \sin (\Omega t - \beta), \text{ etc. ;} \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (22)$$

the derivatives of x, y, z with respect to t are the components of $-\mathbf{V}_{abs}$

$$\frac{\partial x}{\partial t} = -\Omega(R_0 \cos \beta - y) \cos \alpha, \text{ etc.} \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (23)$$

This gives

$$\begin{aligned} p - p_0 = & +\rho\Omega[(R_0 \cos \beta - y)(\phi_x \cos \alpha - \phi_z \sin \alpha) \\ & + (R_0 \sin \beta + x \cos \alpha - z \sin \alpha)\phi_y] - \frac{1}{2}\rho(\phi_x^2 + \phi_y^2 + \phi_z^2), \quad \dots \quad (24) \end{aligned}$$

or with respect to (19)

$$p - p_0 = \frac{1}{2}\rho(V_{abs}^2 - V_{rel}^2) \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (25)$$

This equation is formally identical with Bernoulli's equation for a stationary translation of the body. It is, however, to be noted that V_{abs} is not constant but varies with the distance of the point P from the axis of Z .

5. *Absolute Velocity on the Surface of the Body* [Co-ordinates (n, t, b)].—For the calculation of the flow conditions, the absolute velocity on the surface of the body has to be determined in the systems (n, t, b) of the surface co-ordinates attached to the points P of the surface. This is easily performed with the aid of the scalar products of the unit vectors $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ and $(\mathbf{n}, \mathbf{t}, \mathbf{b})$ which are given in equation (9). Introducing

$$\mathbf{x} = (\mathbf{n} \cdot \mathbf{x})\mathbf{n} + (\mathbf{t} \cdot \mathbf{x})\mathbf{t} + (\mathbf{b} \cdot \mathbf{x})\mathbf{b} \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (26)$$

and the corresponding expressions for $\mathbf{y}$ and $\mathbf{z}$ into (14a), the absolute velocity becomes

$$\begin{aligned}\mathbf{V}_{ab} = & V_x \left[-\frac{r'}{\sqrt{(1+r'^2)}} \mathbf{n} - \frac{1}{\sqrt{(1+r'^2)}} \mathbf{t} + 0 \mathbf{b} \right] \\ & + V_y \left[\frac{\cos \theta}{\sqrt{(1+r'^2)}} \mathbf{n} - \frac{r' \cos \theta}{\sqrt{(1+r'^2)}} \mathbf{t} - \sin \theta \mathbf{b} \right] \\ & + V_z \left[\frac{\sin \theta}{\sqrt{(1+r'^2)}} \mathbf{n} - \frac{r' \sin \theta}{\sqrt{(1+r'^2)}} \mathbf{t} + \cos \theta \mathbf{b} \right] \\ & + \Omega_x (0 \mathbf{n} + 0 \mathbf{t} + r \mathbf{b}) \\ & + \Omega_z \left[\frac{(x + rr') \cos \theta}{\sqrt{(1+r'^2)}} \mathbf{n} - \frac{(xr' - r) \cos \theta}{\sqrt{(1+r'^2)}} \mathbf{t} - x \sin \theta \mathbf{b} \right], \quad \dots \quad (27)\end{aligned}$$

with $V_x, V_y, V_z, \Omega_x, \Omega_z$ from (16).

6. *Relative Velocity on the Surface of the Body [Co-ordinates (n, t, b)].*—6.1. *General Remarks.*—It was shown in equations (14a) and (16) that the complete movement of the body can be considered as the result of the superposition of a longitudinal translation, two lateral translations and two rotations. The potential equation being linear, these elementary movements can be considered separately.

The following investigation starts from the general equations for the calculation of the relative velocity on the surface of a body moving arbitrarily in an inviscid fluid otherwise still. These equations will then be specialised so as to give the solutions for the elementary translations and rotations (16). At last the solution of the complete problem will be obtained as a linear combination of the elementary solutions.

6.2. *General Equations for $\mathbf{V}_{rel}$ in Vector Notation.*—Consider a solid body moving in an inviscid fluid otherwise still, so that a point $P = (x, r, \theta)$ of its surface S has the absolute velocity $\mathbf{V}_{abs}$ at a certain time t . In general $\mathbf{V}_{abs}$ will be different for different points of S , although subject to certain restrictions expressing the condition that the form of the body remains unaltered. Only in the case of a simple translation of the body, $\mathbf{V}_{abs}$ is constant for all points of S .

The surface of the body being impervious for the fluid, the relative particle velocity on S must be in the tangential plane of the body. The moving body will therefore generate a velocity field in the fluid with a normal component on S equal to the normal component of $\mathbf{V}_{abs}$. This velocity field can be produced by a suitable source distribution D on S .

A single elementary source of strength dg at the point H of S induces the velocity

$$d\mathbf{V}_D = \frac{\mathbf{R}}{R^3} dg \quad \dots \quad (28)$$

at the point P , where $\mathbf{R}$ is the vector $\mathbf{HP}$. The total velocity $\mathbf{V}_D$ induced at P by a distribution of the strength

$$\frac{dg}{d\sigma} = f(x, r, \theta) \quad \dots \quad (29)$$

per unit area of S is obtained by an integration, taking into account that the source at the point P gives a finite contribution to the normal component of $\mathbf{V}_D$ which equals half of its total output per unit of area.

$$\mathbf{V}_D(x, r, \theta) = 2\pi f(x, r, \theta) \mathbf{n} + \int_S f(\xi, \rho, \Theta) \frac{\mathbf{R}}{R^3} d\sigma. \quad \dots \quad (30)$$

The condition of equal normal components of the two vectors $\mathbf{V}_D$ and $\mathbf{V}_{abs}$ is then

$$\mathbf{V}_{abs} \cdot \mathbf{n} = \mathbf{V}_D \cdot \mathbf{n}, \quad \dots \dots \dots (31)$$

or

$$\mathbf{V}_{abs} \cdot \mathbf{n} = 2\pi f(x, r, \theta) + \int_S f(\xi, \rho, \Theta) \frac{\mathbf{R} \cdot \mathbf{n}}{R^3} d\sigma, \quad \dots \dots \dots (32)$$

For a given distribution of $\mathbf{V}_{abs}$ on S , this is a linear integral equation of the second kind for the strength f of the distribution D .

If f has been determined from (32), the relative velocity on the surface is easily obtained as the difference of $\mathbf{V}_D$ and $\mathbf{V}_{abs}$,

$$\mathbf{V}_{rel} = -\mathbf{V}_{abs} + \mathbf{V}_D, \quad \dots \dots \dots (33)$$

or, the normal components of $\mathbf{V}_{abs}$ and $\mathbf{V}_D$ being equal,

$$\begin{aligned} \mathbf{V}_{rel} = & \left[-\mathbf{V}_{abs} \cdot \mathbf{t} + \int_S f(\xi, \rho, \Theta) \frac{\mathbf{R} \cdot \mathbf{t}}{R^3} d\sigma \right] \mathbf{t} \\ & + \left[-\mathbf{V}_{abs} \cdot \mathbf{b} + \int_S f(\xi, \rho, \Theta) \frac{\mathbf{R} \cdot \mathbf{b}}{R^3} d\sigma \right] \mathbf{b}. \quad \dots \dots \dots (34) \end{aligned}$$

6.3. *Introduction of Co-ordinates into the Kernels of the General Equations.*—For the practical calculation, the expressions

$$\frac{\mathbf{R} \cdot \mathbf{n}}{R^3} d\sigma, \quad \frac{\mathbf{R} \cdot \mathbf{t}}{R^3} d\sigma \quad \text{and} \quad \frac{\mathbf{R} \cdot \mathbf{b}}{R^3} d\sigma \quad \dots \dots \dots (35)$$

in the kernels of equations (32) and (34) have to be expressed in terms of the co-ordinates.

Equations (32) and (34) are quite general and hold for any form of the moving body. With a view to the problem under consideration, the body will now be assumed to be a body of revolution.

With $P = (x, y, z)$ and $\Pi = (\xi, \eta, \zeta)$, the vector $\mathbf{R}$ is obviously

$$\mathbf{R} = (x - \xi) \mathbf{x} + (y - \eta) \mathbf{y} + (z - \zeta) \mathbf{z}. \quad \dots \dots \dots (36)$$

Its absolute value is

$$R = \sqrt{\{(x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2\}}. \quad \dots \dots \dots (37)$$

Introducing cylindrical co-ordinates, $\mathbf{R}$ and R become

$$\mathbf{R} = (x - \xi) \mathbf{x} + (r \cos \theta - \rho \cos \Theta) \mathbf{y} + (r \sin \theta - \rho \sin \Theta) \mathbf{z}, \quad \dots \dots (38)$$

$$R = \sqrt{\{(x - \xi)^2 + r^2 + \rho^2 - 2r\rho \cos(\theta - \Theta)\}}. \quad \dots \dots \dots (39)$$

The scalar products of $\mathbf{R}$ and the unit vectors $\mathbf{n}$, $\mathbf{t}$, $\mathbf{b}$ are then with respect to (9)

$$\begin{aligned} \mathbf{R} \cdot \mathbf{n} &= (x - \xi) (\mathbf{n} \cdot \mathbf{x}) + (r \cos \theta - \rho \cos \Theta) (\mathbf{n} \cdot \mathbf{y}) + (r \sin \theta - \rho \sin \Theta) (\mathbf{n} \cdot \mathbf{z}) \\ &= \frac{r - r'(x - \xi) - \rho \cos(\theta - \Theta)}{\sqrt{1 + r'^2}}, \quad \dots \dots \dots (40) \end{aligned}$$

and correspondingly

$$\mathbf{R} \cdot \mathbf{t} = -\frac{(x - \xi) + r'[r - \rho \cos(\theta - \Theta)]}{\sqrt{1 + r'^2}}, \quad \dots \dots \dots (41)$$

$$\mathbf{R} \cdot \mathbf{b} = \rho \sin(\theta - \Theta). \quad \dots \dots \dots (42)$$

The element of area $d\sigma$ being

$$d\sigma = \rho d\Theta \sqrt{1 + \rho'^2} d\xi \quad \dots \dots \dots (43)$$

in cylindrical co-ordinates, equations (32) and (34) can now be written in the forms

$$\mathbf{V}_{abs} \cdot \mathbf{n} = 2\pi f(x, r, \theta) + \int_{-1/2}^{+1/2} \int_0^{2\pi} f(\xi, \rho, \Theta) \frac{[r - r'(x - \xi) - \rho \cos(\theta - \Theta)]\rho}{[\sqrt{\{(x - \xi)^2 + r^2 + \rho^2 - 2r\rho \cos(\theta - \Theta)\}}]^3} \sqrt{\left(\frac{1 + \rho'^2}{1 + r'^2}\right)} d\theta d\xi, \quad \dots \quad (44)$$

or

$$g(x, r, \theta) = r\sqrt{1 + r'^2} \mathbf{V}_{abs} \cdot \mathbf{n} - \int_{-1/2}^{+1/2} \int_0^{2\pi} g(\xi, \rho, \Theta) \frac{r[r - r'(x - \xi) - \rho \cos(\theta - \Theta)]}{2\pi[\sqrt{\{(x - \xi)^2 + r^2 + \rho^2 - 2r\rho \cos(\theta - \Theta)\}}]^3} d\theta d\xi, \quad \dots \quad (45)$$

with

$$g(x, r, \theta) = 2\pi r \sqrt{1 + r'^2} f(x, r, \theta), \quad \dots \quad (46)$$

and correspondingly

$$\mathbf{V}_{rel} = \mathbf{t} \left[-\mathbf{V}_{abs} \cdot \mathbf{t} - \frac{1}{\sqrt{1 + r'^2}} \int_{-1/2}^{+1/2} \int_0^{2\pi} g(\xi, \rho, \Theta) \frac{(x - \xi) + r'[r - \rho \cos(\theta - \Theta)]}{2\pi[\sqrt{\{(x - \xi)^2 + r^2 + \rho^2 - 2r\rho \cos(\theta - \Theta)\}}]^3} d\theta d\xi \right] + \mathbf{b} \left[-\mathbf{V}_{abs} \cdot \mathbf{b} + \int_{-1/2}^{+1/2} \int_0^{2\pi} g(\xi, \rho, \Theta) \frac{\rho \sin(\theta - \Theta)}{2\pi[\sqrt{\{(x - \xi)^2 + r^2 + \rho^2 - 2r\rho \cos(\theta - \Theta)\}}]^3} d\theta d\xi \right], \quad \dots \quad (47)$$

Equations (45) and (47) will now be specialised in order to obtain the solutions for the elementary movements (16).

6.4(a). *Calculation of $\mathbf{V}_{rel}$ for a longitudinal translation ($\mathbf{V}_{abs} = V_x \mathbf{x}$).*—For reasons of symmetry of the flow, the strength g of the source distribution is a function of the only variable x and $\mathbf{V}_{rel}$ has the direction of $\mathbf{t}$ on the surface.

Introducing into the general equation (45)

$$\mathbf{V}_{abs} = V_x \mathbf{x}, \quad \mathbf{x} \cdot \mathbf{n} = -\frac{r'}{\sqrt{1 + r'^2}}, \quad \theta - \Theta = x \text{ and } g(x) = g_1(x) V_x, \quad \dots \quad (48)$$

the integral equation becomes

$$g_1(x) = -rr' - \int_{-1/2}^{+1/2} g_1(\xi) \left[\int_0^{2\pi} \frac{r[r - r'(x - \xi) - \rho \cos x]}{2\pi[\sqrt{\{(x - \xi)^2 + r^2 + \rho^2 - 2r\rho \cos x\}}]^3} d\chi \right] d\xi. \quad \dots \quad (49)$$

The integration with respect to χ can be carried out with the aid of complete elliptic integrals.

The result is

$$g_1(x) = -rr' - \int_{-1/2}^{+1/2} g_1(\xi) \left[\frac{r[r - \rho - r'(x - \xi)]}{(x - \xi)^2 + (r - \rho)^2} G_1(k^2) + G_2(k^2) \right] \frac{d\xi}{\sqrt{\{(x - \xi)^2 + (r + \rho)^2\}}} \quad (50)$$

$$= -rr' - \int_{-1/2}^{+1/2} g_1(\xi) K_{10}(x, \xi) d\xi, \quad \dots \quad (50a)$$

with

$$k^2 = \frac{4r\rho}{(x - \xi)^2 + (r + \rho)^2}, \quad \dots \quad (51)$$

$$G_1(k^2) = \frac{2}{\pi} E(k^2), \quad \dots \quad (52)$$

$$G_2(k^2) = \frac{1}{\pi} (K(k^2) - E(k^2)). \quad \dots \quad (53)$$

The relative velocity is then obtained from (47)

$$\frac{\mathbf{V}_{\text{rel long}}}{V_*} = \frac{t}{\sqrt{(1+r'^2)}} \left[1 - \int_{-1/2}^{+1/2} g_1(\xi) \left\{ \int_0^{2\pi} \frac{[x - \xi + r'(r - \rho \cos \chi)] d\chi}{2\pi \sqrt{\{(x - \xi)^2 + r^2 + \rho^2 - 2r\rho \cos \chi\}}} \right\} d\xi \right] \quad (54)$$

or, after integrating with respect to χ

$$\begin{aligned} \frac{\mathbf{V}_{\text{rel long}}}{V_*} &= \frac{t}{\sqrt{(1+r'^2)}} \left[1 - \int_{-1/2}^{+1/2} g_1(\xi) \left\{ \frac{(x - \xi) + r'(r - \rho)}{(x - \xi)^2 + (r - \rho)^2} G_1(k^2) \right. \right. \\ &\quad \left. \left. + \frac{r'}{r} G_2(k^2) \right\} \frac{d\xi}{\sqrt{\{(x - \xi)^2 + (r - \rho)^2\}}} \right] \quad \dots \dots \dots (55) \end{aligned}$$

$$= \frac{t}{\sqrt{(1+r'^2)}} \left[1 - \int_{-1/2}^{+1/2} g_1(\xi) \left\{ K_{11}(x, \xi) + \frac{r'}{r} K_{12}(x, \xi) \right\} d\xi \right] \quad \dots \dots (55a)$$

Appendix to Section 6.4(a).—Potential of the Source Distribution for the Longitudinal Translation

The potential of a single elementary source of strength dq at the point $H = (\xi, \rho, \Theta)$ of the surface is given by

$$d\Phi = -\frac{dq}{R^2} \quad \dots \dots \dots (56)$$

The source distribution on the surface will then have the potential

$$\Phi = - \int_s f(\xi, \rho, \Theta) \frac{1}{R^2} d\sigma \quad \dots \dots \dots (57)$$

This gives

$$\Phi_{\text{long}} = -V_* \int_{-1/2}^{+1/2} g_1(\xi) \left[\int_0^{2\pi} \frac{d\chi}{\sqrt{\{(x - \xi)^2 + r^2 + \rho^2 - 2r\rho \cos \chi\}}} \right] d\xi \quad (58)$$

or, after integrating with respect to χ

$$\Phi_{\text{long}} = -V_* \int_{-1/2}^{+1/2} g_1(\xi) G_1(k^2) \frac{d\xi}{\sqrt{\{(x - \xi)^2 + (r - \rho)^2\}}} \quad \dots \dots (59)$$

$$= -V_* \int_{-1/2}^{+1/2} g_1(\xi) K_{11}(x, \xi) d\xi, \quad \dots \dots (59a)$$

with

$$G_1(k^2) = \frac{c}{\pi} K(k^2), \quad \dots \dots (60)$$

6.4(b). Calculation of $\mathbf{V}_{\text{rel}}$ for a Lateral Translation.

$$\begin{bmatrix} \mathbf{V}_{\text{rel}} \\ \mathbf{V}_* \end{bmatrix} = \begin{bmatrix} V_y \mathbf{y} \\ V_z \mathbf{z} \end{bmatrix}.$$

In the case of a lateral translation of the body in the directions of y or z , it is easily seen that the solution of the integral equation (45) must be the product of a function of x and $\cos \theta$ or $\sin \theta$, respectively. In fact, let

$$\mathbf{V}_{\text{rel}} = \begin{bmatrix} V_y \mathbf{y} \\ V_z \mathbf{z} \end{bmatrix}, \quad \mathbf{y} \cdot \mathbf{n} = \frac{\cos \theta}{\sqrt{(1+r'^2)}} \quad \text{and} \quad g(x, r, \theta) = \frac{V_y g_1(x) \cos \theta}{V_* \sin \theta} \quad \dots \dots (61)$$

The integral equation can then be written in the form

$$g_2(x) = r - \int_{-1/2}^{+1/2} g_2(\xi) \left[\int_0^{2\pi} \frac{\cos \theta [r - r'(x - \xi) - \rho \cos(\theta - \Theta)]}{\sin \theta [2\pi \sqrt{\{(x - \xi)^2 + r^2 + \rho^2 - 2r\rho \cos(\theta - \Theta)\}}]} d\theta \right] d\xi. \quad (62)$$

Taking account of the relations

$$\frac{\cos \theta}{\sin \theta} = \frac{\cos [\theta - (\theta - \Theta)]}{\sin [\theta - (\theta - \Theta)]} = \frac{\cos \theta \cos(\theta - \Theta) \pm \sin \theta \sin(\theta - \Theta)}{\sin \theta \cos(\theta - \Theta)} \quad \dots \quad (63)$$

and

$$\int_0^{2\pi} F(\cos \chi) \sin \chi d\chi = 0, \quad \chi = \theta - \Theta, \quad \dots \quad (64)$$

the kernel of (62) is reduced to

$$\int_0^{2\pi} \frac{r[r - r'(x - \xi) - \rho \cos \chi] \cos \chi}{2\pi [\sqrt{\{(x - \xi)^2 + r^2 + \rho^2 - 2r\rho \cos \chi\}}]} d\chi. \quad \dots \quad (65)$$

The integration with respect to χ can again be carried out with the aid of complete elliptic integrals. The result is

$$g_2(x) = r - \int_{-1/2}^{+1/2} g_2(\xi) \left[\frac{r[r - r'(x - \xi) - \rho \cos \chi]}{(x - \xi)^2 + (r - \rho)^2} H_1(k^2) + H_2(k^2) \right] \frac{d\xi}{\sqrt{\{(x - \xi)^2 + (r + \rho)^2\}}} \quad (66)$$

$$= r - \int_{-1/2}^{+1/2} g_2(\xi) K_{30}(x, \xi) d\xi, \quad \dots \quad (66a)$$

with k^2 as given by (51) and

$$H_1(k^2) = \frac{2(1 + k'^2)E(k^2) - 2k'^2K(k^2)}{\pi k^2} \quad \dots \quad (67)$$

$$H_2(k^2) = \frac{1(1 + 3k'^2)K(k^2) - (3 + k'^2)E(k^2)}{\pi k^2} \quad \dots \quad (68)$$

$$k'^2 = 1 - k^2 \quad \dots \quad (69)$$

The relative velocity is then obtained from (47). This gives with respect to (9), (61), (63) and (64)

$$\mathbf{V}_{\text{relat}} = \frac{V_y}{V_x} \left[\frac{\cos \theta}{\sqrt{(1 + r'^2)}} \left\{ r' - \int_{-1/2}^{+1/2} g_2(\xi) \int_0^{2\pi} \frac{[(x - \xi) + r'(r - \rho \cos \chi)] \cos \chi}{2\pi [\sqrt{\{(x - \xi)^2 + r^2 + \rho^2 - 2r\rho \cos \chi\}}]} d\chi d\xi \right\} \right. \\ \left. \pm \frac{\sin \theta}{\cos \theta} b \left\{ 1 + \int_{-1/2}^{+1/2} g_2(\xi) \int_0^{2\pi} \frac{\rho \sin^2 \chi}{2\pi [\sqrt{\{(x - \xi)^2 + r^2 + \rho^2 - 2r\rho \cos \chi\}}]} d\chi d\xi \right\} \right] \quad (70)$$

and finally by integration with respect to χ

$$\mathbf{V}_{\text{relat}} = \frac{V_y}{V_x} \left\{ \frac{\cos \theta}{\sqrt{(1 + r'^2)}} \left[r' - \int_{-1/2}^{+1/2} g_2(\xi) \left\{ \frac{(x - \xi) + r'(r - \rho)}{(x - \xi)^2 + (r - \rho)^2} H_1(k^2) \right. \right. \right. \\ \left. \left. \left. + \frac{r'}{r} H_2(k^2) \right\} \frac{d\xi}{\sqrt{\{(x - \xi)^2 + (r + \rho)^2\}}} \right] \right. \\ \left. \pm \frac{\sin \theta}{\cos \theta} b \left[1 + \frac{1}{r} \int_{-1/2}^{+1/2} g_2(\xi) H_2(k^2) \frac{d\xi}{\sqrt{\{(x - \xi)^2 + (r + \rho)^2\}}} \right] \right\}, \quad \dots \quad (71)$$

with

$$H_2(k^2) = \frac{2(1 + k'^2)K(k^2) - 2E(k^2)}{\pi k^2}, \quad \dots \quad (72)$$

or with the use of symbols for the kernels

$$\mathbf{V}_{\text{relat}} = \frac{V_y}{V_z} \left\{ \frac{\cos \theta}{\sin \theta} \left[r' - \int_{-l/2}^{+l/2} g_1(\xi) K_{11}(x, \xi) d\xi - \frac{r'}{r} \int_{-l/2}^{+l/2} g_1(\xi) K_{12}(x, \xi) d\xi \right] + \sin \theta \ln \left[1 + \frac{1}{r} \int_{-l/2}^{+l/2} g_2(\xi) K_{22}(x, \xi) d\xi \right] \right\} \quad \dots \quad (71a)$$

Appendix to Section 6.4(b).—Potential of the Source Distribution for a Lateral Translation

Introducing into the general formula (57) for the potential of a source distribution its strength in the case of a lateral translation

$$f(x, r, \theta) = \frac{V_y}{V_z} \left(\frac{g_1(x)}{2\pi r \sqrt{(1+r^2)}} \right) \frac{\cos \theta}{\sin \theta} \quad \dots \quad (73)$$

the potential becomes

$$\Phi_{\text{lat}} = - \frac{V_y}{V_z} \int_{-l/2}^{+l/2} g_1(\xi) \left[\int_0^{2\pi} \frac{\cos \theta}{\sin \theta} \frac{d\theta}{2\pi \sqrt{[(x-\xi)^2 + r^2 + \rho^2 - 2r\rho \cos(\theta - \Theta)]}} d\theta \right] d\xi \quad (74)$$

and after integrating with respect to θ

$$\Phi_{\text{lat}} = - \frac{V_y \cos \theta}{V_z \sin \theta} \int_{-l/2}^{+l/2} g_1(\xi) H_2(k^2) \frac{d\xi}{\sqrt{[(x-\xi)^2 + (r+\rho)^2]}} \quad \dots \quad (75)$$

$$= - \frac{V_y \cos \theta}{V_z \sin \theta} \int_{-l/2}^{+l/2} g_1(\xi) K_{11}(x, \xi) d\xi \quad \dots \quad (75a)$$

6.4(c). Calculation of $\mathbf{V}_{\text{rot}}$ for a Rotation about the Axes of y or z .

$$\mathbf{V}_{\text{rot}} = \frac{\Omega_y}{\Omega_z} \begin{bmatrix} z \cdot \mathbf{x} & x \cdot \mathbf{z} \\ x \cdot \mathbf{y} & y \cdot \mathbf{x} \end{bmatrix}.$$

As in the case of a lateral translation, the solution of the integral equation (45) is easily seen to be the product of a function of x and either $\sin \theta$ or $\cos \theta$. With

$$\mathbf{V}_{\text{rot}} = \frac{\Omega_y}{\Omega_z} \begin{bmatrix} z \cdot \mathbf{x} & x \cdot \mathbf{z} \\ x \cdot \mathbf{y} & y \cdot \mathbf{x} \end{bmatrix} \quad \dots \quad (76)$$

$$\mathbf{V}_{\text{rot}} \cdot \mathbf{n} = \mp \frac{x + rr'}{\sqrt{(1+r^2)}} \frac{\Omega_y \sin \theta}{\Omega_z \cos \theta} \quad \dots \quad (77)$$

$$g(x, r, \theta) = \mp \frac{l \Omega_y}{2 \Omega_z} g_2(x) \frac{\sin \theta}{\cos \theta} \quad \dots \quad (78)$$

equation (45) becomes

$$g_2(x) = \frac{r(x + rr')}{l/2} - \int_{-l/2}^{+l/2} g_2(\xi) \int_0^{2\pi} \frac{r[r - r'(x - \xi) - \rho \cos \chi] \cos \chi}{2\pi \sqrt{[(x-\xi)^2 + r^2 + \rho^2 - 2r\rho \cos \chi]}} d\chi \quad \dots \quad (79)$$

The kernel of this equation is the same as in the case of a lateral translation (cf. (65)). The final form of the integral equation is therefore

$$g_2(x) = \frac{r(x + rr')}{l/2} - \int_{-l/2}^{+l/2} g_2(\xi) \left[\frac{r[r - \rho - r'(x - \xi)]}{(x - \xi)^2 + (r - \rho)^2} H_1(k^2) + H_2(k^2) \right] \frac{d\xi}{\sqrt{\{(x - \xi)^2 + (r + \rho)^2\}}}, \quad \dots \dots \dots (80)$$

$$= \frac{r(x + rr')}{l/2} - \int_{-l/2}^{+l/2} g_2(\xi) K_{20}(x, \xi) d\xi. \quad \dots \dots \dots (80a)$$

The relative velocity is then obtained from (47) with

$$\mathbf{V}_{\text{abs.}} \cdot \mathbf{t} = \begin{matrix} + \Omega_y \sin \theta \\ - \Omega_z \cos \theta \end{matrix} \left(\frac{r'x - r}{\sqrt{(1 + r'^2)}} \right) \quad \dots \dots \dots (81)$$

$$\mathbf{V}_{\text{abs.}} \cdot \mathbf{b} = \begin{matrix} - \Omega_y \cos \theta \\ - \Omega_z \sin \theta \end{matrix} (x), \quad \dots \dots \dots (82)$$

which gives in the same way as in equations (70) to (72) above the final result:

$$\mathbf{V}_{\text{rel rot y, z}} = \frac{l}{2} \Omega_y \left\{ \begin{matrix} - \sin \theta \mathbf{t} \\ + \cos \theta \mathbf{b} \end{matrix} \left[\frac{r'x - r}{\sqrt{(1 + r'^2)}} \left[\frac{r'x - r}{l/2} - \int_{-l/2}^{+l/2} g_2(\xi) \left\{ \frac{x - \xi + r'(r - \rho)}{(x - \xi)^2 + (r - \rho)^2} H_1(k^2) + \frac{r'}{r} H_2(k^2) \right\} \frac{d\xi}{\sqrt{\{(x - \xi)^2 + (r + \rho)^2\}}} \right] \right. \right. \right. \dots \dots \dots (83)$$

$$\left. + \frac{\cos \theta \mathbf{b}}{\sin \theta} \left[\frac{x}{l/2} + \frac{1}{r} \int_{-l/2}^{+l/2} g_2(\xi) H_2(k^2) \frac{d\xi}{\sqrt{\{(x - \xi)^2 + (r + \rho)^2\}}} \right] \right\} \\ = \frac{l}{2} \Omega_y \left\{ \begin{matrix} - \sin \theta \mathbf{t} \\ + \cos \theta \mathbf{b} \end{matrix} \left[\frac{r'x - r}{\sqrt{(1 + r'^2)}} \left[\frac{r'x - r}{l/2} - \int_{-l/2}^{+l/2} g_2(\xi) K_{20}(x, \xi) d\xi \right. \right. \right. \dots \dots \dots (83a)$$

Appendix to Section 6.4(c).—Potential of the Source Distribution for a Rotation about the Axes of y or z

The potential is obtained in the same way as for the lateral translation (equations (73) to (75)):

$$f(x, r, \theta) = \mp \frac{l}{2} \frac{\Omega_y}{\Omega_z} \frac{g_2(x)}{2\pi r \sqrt{(1 + r'^2)}} \frac{\sin \theta}{\cos \theta} \quad \dots \dots \dots (84)$$

$$\Phi_{\text{rot y, z}} = \pm \frac{l}{2} \frac{\Omega_y}{\Omega_z} \int_{-l/2}^{+l/2} g_2(\xi) \left[\int_0^{2\pi} \frac{\sin \theta d\theta}{2\pi \sqrt{\{(x - \xi)^2 + r^2 + \rho^2 - 2r\rho \cos(\theta - \Theta)\}}} \right] d\xi \quad \dots \dots \dots (85)$$

$$\Phi_{\text{rot y, z}} = \pm \frac{l}{2} \frac{\Omega_y}{\Omega_z} \frac{\sin \theta}{\cos \theta} \int_{-l/2}^{+l/2} g_2(\xi) H_2(k^2) \frac{d\xi}{\sqrt{\{(x - \xi)^2 + (r + \rho)^2\}}} \quad \dots \dots \dots (86)$$

$$= \pm \frac{l}{2} \frac{\Omega_y}{\Omega_z} \frac{\sin \theta}{\cos \theta} \int_{-l/2}^{+l/2} g_2(\xi) K_{20}(x, \xi) d\xi. \quad \dots \dots \dots (86a)$$

6.4(d). $\mathbf{V}_{rel}$ for a rotation about the axis of x

$$[\mathbf{V}_{abs} = \Omega_x \cdot (y \cdot \mathbf{z} - z \cdot \mathbf{y})].$$

It is obvious that a rotation of the body about the axis of x cannot create a velocity field in a perfect fluid. $\mathbf{V}_{rel}$ is therefore simply $-\mathbf{V}_{abs}$.

$$\mathbf{V}_{rel \text{ on } x} = -\Omega_x(y \cdot \mathbf{z} - z \cdot \mathbf{y}) \quad \dots \quad (87)$$

$$= -\frac{l}{2} \Omega_x \cdot \frac{r}{l/2} \cdot \mathbf{b} \quad \dots \quad (88)$$

7. Summary of Sections 1 to 6.—The preceding results can be summarised as follows :

(1) If a body of revolution is revolving with a constant angular velocity Ω about a fixed axis in an inviscid fluid otherwise still, the pressure at any point P of its surface is given by

$$p - p_0 = \frac{1}{2} \rho (V_{abs}^2 - V_{rel}^2) \quad \dots \quad (89)$$

(2) $\mathbf{V}_{abs}$ is given by

$$V_{abs}^2 = \Omega^2 [(R_0 \cos \beta - r \cos \theta)^2 + (R_0 \sin \beta + x \cos \alpha - r \sin \theta \sin \alpha)^2] \quad \dots \quad (90)$$

(3) In order to calculate V_{rel}^2 , the motion of the body is considered as the result of a superposition of the following elementary motions :

- (a) Translation in the direction of x with the velocity $V_x = \Omega R_0 \cos \alpha \cos \beta$
- (b) Translation in the direction of y with the velocity $V_y = \Omega R_0 \sin \beta$
- (c) Translation in the direction of z with the velocity $V_z = -\Omega R_0 \sin \alpha \cos \beta \quad \dots \quad (91)$
- (d) Rotation about the axis of x with the angular velocity $\Omega_x = \Omega \sin \alpha$
- (e) Rotation about the axis of z with the angular velocity $\Omega_z = \Omega \cos \alpha$.

These elementary motions can be treated separately.

(4) $\mathbf{V}_{rel}$ for the elementary motions. The condition of zero normal velocity at the surface S is satisfied by a source distribution D on S . Its strength is determined from an integral equation. $\mathbf{V}_{rel}$ is then obtained as the difference $\mathbf{V}_D - \mathbf{V}_{abs}$ where $\mathbf{V}_D$ is the velocity induced by the distribution at the point P .

(a) Longitudinal translation.

Integral equation :

$$g_1(x) = -rr' - \int_{-l/2}^{+l/2} g_1(\xi) \left[\frac{r[r - \rho - r'(x - \xi)]}{(x - \xi)^2 + (r - \rho)^2} G_1(k^2) + G_2(k^2) \right] \frac{d\xi}{\sqrt{\{(x - \xi)^2 + (r + \rho)^2\}}} \quad \dots \quad (92)$$

Relative velocity :

$$\mathbf{V}_{rel \text{ long}} = V_x \frac{t}{\sqrt{(1 + r'^2)}} \left[1 - \int_{-l/2}^{+l/2} g_1(\xi) \left\{ \frac{x - \xi + r'(r - \rho)}{(x - \xi)^2 + (r - \rho)^2} G_1(k^2) + \frac{r'}{r} G_2(k^2) \right\} \frac{d\xi}{\sqrt{\{(x - \xi)^2 + (r + \rho)^2\}}} \right] \quad \dots \quad (93)$$

Potential of the source distribution :

$$\Phi_{\text{long}} = -V_* \int_{-l/2}^{+l/2} g_1(\xi) G_2(k^2) \frac{d\xi}{\sqrt{\{(x-\xi)^2 + (r+\rho)^2\}}} \quad \dots \quad (94)$$

(b), (c) Lateral translations.

$$g_2(x) = r - \int_{-l/2}^{+l/2} g_2(\xi) \left[\frac{r[r-\rho-r'(x-\xi)]}{(x-\xi)^2 + (r-\rho)^2} H_1(k^2) + H_2(k^2) \right] \frac{d\xi}{\sqrt{\{(x-\xi)^2 + (r+\rho)^2\}}} \quad (95)$$

$$\begin{aligned} \mathbf{V}_{\text{rel lat}} = \frac{V_y}{V_*} \left\{ \frac{\cos \theta}{\sqrt{(1+r'^2)}} \left[r' - \int_{-l/2}^{+l/2} g_2(\xi) \left\{ \frac{x-\xi+r'(r-\rho)}{(x-\xi)^2 + (r-\rho)^2} H_1(k^2) \right. \right. \right. \\ \left. \left. \left. + \frac{r'}{r} H_2(k^2) \right\} \frac{d\xi}{\sqrt{\{(x-\xi)^2 + (r+\rho)^2\}}} \right] \right. \\ \left. \pm \frac{\sin \theta}{\cos \theta} b \left[1 + \frac{1}{r} \int_{-l/2}^{+l/2} g_2(\xi) H_2(k^2) \frac{d\xi}{\sqrt{\{(x-\xi)^2 + (r+\rho)^2\}}} \right] \right\} \quad \dots \quad (96) \end{aligned}$$

$$\Phi_{\text{lat}} = -\frac{V_y \cos \theta}{V_* \sin \theta} \int_{-l/2}^{+l/2} g_2(\xi) H_2(k^2) \frac{d\xi}{\sqrt{\{(x-\xi)^2 + (r+\rho)^2\}}} \quad \dots \quad (97)$$

(d) Rotation about the longitudinal axis.

$$\mathbf{V}_{\text{rel rot } x} = -\frac{l}{2} \Omega_x \frac{r}{l/2} b \quad \dots \quad (98)$$

$$\Phi_{\text{rot } x} = 0 \quad \dots \quad (99)$$

(e) Rotation about the lateral axes.

$$\begin{aligned} g_3(x) = \frac{r(x+rr')}{l/2} - \int_{-l/2}^{+l/2} g_3(\xi) \left[\frac{r[r-\rho-r'(x-\xi)]}{(x-\xi)^2 + (r-\rho)^2} H_1(k^2) \right. \\ \left. + H_2(k^2) \right] \frac{d\xi}{\sqrt{\{(x-\xi)^2 + (r+\rho)^2\}}} \quad \dots \quad (100) \end{aligned}$$

$$\begin{aligned} \mathbf{V}_{\text{rel rot } y, z} = \frac{l}{2} \Omega_y \left\{ \frac{-\sin \theta}{\sqrt{(1+r'^2)}} \left[\frac{r'x-r}{l/2} - \int_{-l/2}^{+l/2} g_3(\xi) \left\{ \frac{x-\xi+r'(r-\rho)}{(x-\xi)^2 + (r-\rho)^2} H_1(k^2) \right. \right. \right. \\ \left. \left. \left. + \frac{r'}{r} H_2(k^2) \right\} \frac{d\xi}{\sqrt{\{(x-\xi)^2 + (r+\rho)^2\}}} \right] \right. \\ \left. + \frac{\cos \theta}{\sin \theta} b \left[\frac{x}{l/2} + \frac{1}{r} \int_{-l/2}^{+l/2} g_3(\xi) H_2(k^2) \frac{d\xi}{\sqrt{\{(x-\xi)^2 + (r+\rho)^2\}}} \right] \right\} \quad \dots \quad (101) \end{aligned}$$

$$\Phi_{\text{rot } y, z} = \pm \frac{l}{2} \Omega_y \sin \theta \int_{-l/2}^{+l/2} g_3(\xi) H_2(k^2) \frac{d\xi}{\sqrt{\{(x-\xi)^2 + (r+\rho)^2\}}} \quad \dots \quad (102)$$

PART B.—ADAPTATION OF THE THEORY FOR THE NUMERICAL WORK

1. *Discussion of the Integral Equations and the Formulae for $\mathbf{V}_{rel}$ and Φ .*—1.1. *Structure of the Kernels.*—The kernels of the integrals for the lateral translations are identical with the kernels for the rotations about the axes of y and z . The kernels for the longitudinal translation have a very similar structure. The only differences are that the elliptic integrals H , are replaced by the corresponding G , and that the b -component of $\mathbf{V}_{rel\,long}$ is zero.

The kernels are built up from the following expressions

$$r_1 = \sqrt{\{(x - \xi)^2 + (r - \rho)^2\}} \quad \dots \quad (103)$$

$$r_2 = \sqrt{\{(x - \xi)^2 + (r + \rho)^2\}} \quad \dots \quad (104)$$

$$b = (r - \rho) - r'(x - \xi) \quad \dots \quad (105)$$

$$c = (x - \xi) + r'(r - \rho) \quad \dots \quad (106)$$

A geometric interpretation of these quantities is given in Fig. 3. Introducing the abbreviations

$$A(x, \xi) = \frac{1}{r_1} \quad \dots \quad (107)$$

$$B(x, \xi) = \frac{r b}{r_1^3} \quad \dots \quad (108)$$

$$C(x, \xi) = \frac{c}{r_1^3} \quad \dots \quad (109)$$

the kernels become

$$\left. \begin{aligned} K_{11} &= (BG_1 + G_2)A & K_{22} &= (BH_1 + H_2)A \\ K_{11} &= CG_1A & K_{21} &= CH_1A \\ K_{12} &= G_2A & K_{22} &= H_2A \\ K_{12} &= G_2A & K_{22} &= H_2A \end{aligned} \right\} \quad \dots \quad (110)$$

The modulus k^2 (equation (51)) of the functions G , and H , can also be expressed in terms of r_1 and r_2

$$k^2 = \frac{r_2^2 - r_1^2}{r_2^2} \quad \dots \quad (111)$$

For the numerical work it has some advantages to use in general the complementary modulus

$$k'^2 = \frac{r_1^2}{r_2^2} \quad \dots \quad (112)$$

This will be done in the following :

The equations for the g , $\mathbf{V}_{rel}$ and Φ are then

$$g_1(x) = -rr' - \int_{-1/2}^{+1/2} g_1(\xi) K_{11}(x, \xi) d\xi \quad \dots \quad (113)$$

$$\mathbf{V}_{rel\,long} = V_s \cdot \frac{t}{\sqrt{(1 + r'^2)}} \left[1 - \int_{-1/2}^{+1/2} g_1(\xi) K_{11}(x, \xi) d\xi - \frac{r'}{r} \int_{-1/2}^{+1/2} g_1(\xi) K_{12}(x, \xi) d\xi \right] \quad \dots \quad (114)$$

$$\Phi_{long} = -V_s \int_{-1/2}^{+1/2} g_1(\xi) K_{12}(x, \xi) d\xi \quad \dots \quad (115)$$

$$g_2(x) = r - \int_{-1/2}^{+1/2} g_2(\xi) K_{22}(x, \xi) d\xi \quad \dots \quad (116)$$

$$\begin{aligned} \mathbf{V}_{\text{rel lat}} = \frac{V_y}{V_z} \left\{ \frac{\sin \theta}{\sqrt{(1+r'^2)}} \left[r' - \int_{-1/2}^{+1/2} g_2(\xi) K_{21}(x, \xi) d\xi - \frac{r'}{r} \int_{-1/2}^{+1/2} g_2(\xi) K_{22}(x, \xi) d\xi \right] \right. \\ \left. \pm \frac{\sin \theta}{\cos \theta} b \left[1 + \frac{1}{r} \int_{-1/2}^{+1/2} g_2(\xi) K_{22}(x, \xi) d\xi \right] \right\} \quad \dots \quad (117) \end{aligned}$$

$$\Phi_{\text{lat}} = \frac{-V_y \cos \theta}{V_z \sin \theta} \int_{-1/2}^{+1/2} g_2(\xi) K_{22}(x, \xi) d\xi \quad \dots \quad (118)$$

$$g_2(x) = \frac{r(x+rr')}{l/2} - \int_{-1/2}^{+1/2} g_2(\xi) K_{22}(x, \xi) d\xi \quad \dots \quad (119)$$

$$\begin{aligned} \mathbf{V}_{\text{rel rot } y, z} = \frac{l}{2} \frac{\Omega_y}{\Omega_z} \left\{ \frac{-\sin \theta}{\sqrt{(1+r'^2)}} \left[\frac{r'x-r}{l/2} - \int_{-1/2}^{+1/2} g_2(\xi) K_{21}(x, \xi) d\xi \right. \right. \\ \left. \left. - \frac{r'}{r} \int_{-1/2}^{+1/2} g_2(\xi) K_{22}(x, \xi) d\xi \right] + \frac{\cos \theta}{\sin \theta} b \left[\frac{x}{l/2} + \frac{1}{r} \int_{-1/2}^{+1/2} g_2(\xi) K_{22}(x, \xi) d\xi \right] \right\} \quad \dots \quad (120) \end{aligned}$$

$$\Phi_{\text{rel rot } y, z} = \pm \frac{l}{2} \frac{\Omega_y}{\Omega_z} \frac{\sin \theta}{\cos \theta} \int_{-1/2}^{+1/2} g_2(\xi) K_{22}(x, \xi) d\xi \quad \dots \quad (121)$$

$$\mathbf{V}_{\text{rel rot } x} = -\frac{l}{2} \Omega_x \frac{r}{l/2} b \quad \dots \quad (122)$$

$$\Phi_{\text{rot } x} = 0 \quad \dots \quad (123)$$

1.2. *The Functions G, and H.*—The functions G , and H , were defined in equations (52, 53, 60, 67, 68 and 72) in terms of the standard integrals,

$$E(k^2) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \psi} d\psi \quad \text{and} \quad K(k^2) = \int_0^{\pi/2} \frac{d\psi}{\sqrt{1 - k^2 \sin^2 \psi}} \quad (124)$$

Their developments for $k^2 \ll 1$ and $k'^2 \ll 1$ are easily obtained from the well-known developments E and K

$$\left. \begin{aligned} k^2 \ll 1 : \\ G_1 &= 1 - \frac{1}{4} k^2 - \frac{3}{64} k^4 - \frac{5}{256} k^6 \dots \\ G_2 &= \frac{1}{4} k^2 + \frac{3}{32} k^4 + \frac{15}{256} k^6 \dots \\ G_3 &= 1 + \frac{1}{4} k^2 + \frac{9}{64} k^4 + \frac{25}{256} k^6 \dots \\ H_1 &= \frac{3}{8} k^2 + \frac{3}{32} k^4 + \frac{45}{1024} k^6 \dots \\ H_2 &= -\frac{1}{8} k^2 + \frac{15}{1024} k^4 \dots \\ H_3 &= \frac{1}{8} k^2 + \frac{3}{32} k^4 + \frac{75}{1024} k^6 \dots \end{aligned} \right\} \quad \dots \quad (125)$$

$k'^2 \ll 1$ Let $\Lambda = \ln \frac{4}{k'}$

$$\left. \begin{aligned} G_1 &= \frac{2}{\pi} \left[1 + \frac{1}{2}(\Lambda - \frac{1}{2})k'^2 + \frac{3}{16} \left\{ \Lambda - \frac{13}{12} \right\} k'^4 \dots \right] \\ G_2 &= \frac{1}{\pi} \left[(\Lambda - 1) - \frac{1}{2}\Lambda k'^2 - \frac{3}{64} \left\{ \Lambda - \frac{5}{6} \right\} k'^4 \dots \right] \\ G_3 &= \frac{2}{\pi} \left[\Lambda + \frac{1}{2}(\Lambda - 1)k'^2 + \frac{9}{64} \left\{ \Lambda - \frac{7}{6} \right\} k'^4 \dots \right] \\ H_1 &= \frac{2}{\pi} \left[1 - \frac{3}{2} \left\{ \Lambda - \frac{7}{6} \right\} k'^2 - \frac{21}{16} \left\{ \Lambda - \frac{115}{84} \right\} k'^4 \dots \right] \\ H_2 &= \frac{1}{\pi} \left[(\Lambda - 3) + \frac{11}{4} \left\{ \Lambda - \frac{14}{11} \right\} k'^2 + \frac{165}{64} \left\{ \Lambda - \frac{91}{66} \right\} k'^4 \dots \right] \\ H_3 &= \frac{2}{\pi} \left[(\Lambda - 2) + \frac{5}{4} \left\{ \Lambda - \frac{7}{5} \right\} k'^2 + \frac{81}{64} \left\{ \Lambda - \frac{25}{18} \right\} k'^4 \dots \right] \end{aligned} \right\} \quad (126)$$

For $k'^2 = 0$, the functions G_1 and H_1 are finite, whereas G_2 , G_3 , H_2 and H_3 have a logarithmic singularity.

If tables of the complete elliptic integrals

$$\left. \begin{aligned} B(k^2) &= \int_0^{\pi/2} \frac{\cos^2 \psi \, d\psi}{\sqrt{(1 - k^2 \sin^2 \psi)}}, & C(k^2) &= \int_0^{\pi/2} \frac{\sin^2 \psi \cos^2 \psi \, d\psi}{\{\sqrt{(1 - k^2 \sin^2 \psi)}\}^3} \\ D(k^2) &= \int_0^{\pi/2} \frac{\sin^2 \psi \, d\psi}{\sqrt{(1 - k^2 \sin^2 \psi)}} \end{aligned} \right\} \quad (127)$$

are available (*e.g.*, Jahnke-Emde, *Tafeln höherer Funktionen*, 4 Aufe., Leipzig, 1948), the following alternative formulae for G_2 , H_1 , H_2 , H_3 will be found useful for the numerical calculation of these functions:

$$G_2 = \frac{1}{\pi} k^2 D(k^2) \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (128)$$

$$\left. \begin{aligned} H_1 &= \frac{2}{\pi} [2B(k^2) - E(k^2)] \\ &= \frac{2}{\pi} k^2 [D(k^2) - C(k^2)] \end{aligned} \right\} \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (129)$$

$$\left. \begin{aligned} H_2 &= \frac{1}{\pi} [D(k^2) + E(k^2) - 3B(k^2)] \\ &= \frac{1}{\pi} k^2 [2C(k^2) - D(k^2)] \end{aligned} \right\} \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (130)$$

$$\left. \begin{aligned} H_3 &= \frac{2}{\pi} k^2 C(k^2) \\ &= \frac{2}{\pi} [2D(k^2) - K(k^2)] \end{aligned} \right\} \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (131)$$

A three-figure table of the G , and H , is given at the end of this report; its accuracy will in general be sufficient for the present purpose.

1.3. *Singularities of the Kernels and Limiting Cases.*—In order to develop a suitable method for the evaluation of the integrals occurring in the numerical work, the singularities of the kernels have to be considered. A special treatment is necessary for the ends of the body where the elliptic integrals degenerate.

1.3(a). *Singularities of the kernels for $x \neq \pm l/2$.*—Only the case $\xi \rightarrow x$ has to be considered. Introducing the development

$$\rho = r - r'(x - \xi) + \frac{1}{2}r''(x - \xi)^2 - \dots, \quad \dots \quad (132)$$

the expressions A , B , C and k'^2 become

$$A = 1/\sqrt{(x - \xi)^2 + (r + \rho)^2} \rightarrow 1/2r \quad \dots \quad (133)$$

$$B = \frac{r[r - \rho - r'(x - \xi)]}{(x - \xi)^2 + (r - \rho)^2} = \frac{r}{1 + \left(\frac{r - \rho}{x - \xi}\right)^2} \frac{\frac{r - \rho}{x - \xi} - r'}{x - \xi} \rightarrow -\frac{rr''}{2(1 + r'^2)} \quad \dots \quad (134)$$

$$C = \frac{(x - \xi) + r'(r - \rho)}{(x - \xi)^2 + (r - \rho)^2} = \frac{1 + r'\left(\frac{r - \rho}{x - \xi}\right)}{1 + \left(\frac{r - \rho}{x - \xi}\right)^2} \frac{1}{x - \xi} \rightarrow \frac{1}{x - \xi} \quad \dots \quad (135)$$

$$k'^2 = \frac{(x - \xi)^2 + (r - \rho)^2}{(x - \xi)^2 + (r + \rho)^2} = \frac{1 + \left(\frac{r - \rho}{x - \xi}\right)^2}{(x - \xi)^2 + (r + \rho)^2} \rightarrow \frac{1 + r'^2}{4r^2} (x - \xi)^2. \quad (136)$$

The behaviour of the functions G , and H , is obtained from the first terms of their developments (126) for $k'^2 \ll 1$ with

$$A = \ln \frac{4}{k'} \rightarrow \ln \frac{8r}{\sqrt{(1 + r'^2)} |x - \xi|} \quad \dots \quad (137)$$

The kernels become then

$$\frac{K_{10}}{K_{20}} = \left(B \frac{G_1}{H_1} + \frac{G_2}{H_2} \right) A \rightarrow \frac{1}{2\pi} \left(-\frac{rr''}{1 + r'^2} + \ln \frac{8r}{\sqrt{(1 + r'^2)} |x - \xi|} - \frac{1}{3} \right) \quad \dots \quad (138)$$

$$\frac{K_{11}}{K_{21}} = C \frac{G_1}{H_1} A \rightarrow \frac{1}{\pi r(x - \xi)} \quad \dots \quad (139)$$

$$\frac{K_{12}}{K_{22}} = \frac{G_2}{H_2} A \rightarrow \frac{1}{2\pi r} \left(\ln \frac{8r}{\sqrt{(1 + r'^2)} |x - \xi|} - \frac{1}{3} \right) \quad \dots \quad (140)$$

$$\frac{K_{13}}{K_{23}} = \frac{G_3}{H_3} A \rightarrow \frac{1}{\pi r} \left(\ln \frac{8r}{\sqrt{(1 + r'^2)} |x - \xi|} - \frac{0}{2} \right) \quad \dots \quad (141)$$

K_{11} and K_{21} have a singularity of the type $(x - \xi)^{-1}$; the other kernels have a logarithmic singularity for $\xi \rightarrow x$.

1.3(b). *Ends of the body.*—The ends of the body require a special treatment, as the elliptic integrals G , and H , degenerate on the axis. It is sufficient to consider the head ($x = +l/2$), as the corresponding formulae for the stern are easily obtained by a change of the direction of the axis of x . Practically, the calculation for the stern has but little importance, as the flow conditions there differ too much from the behaviour of a perfect fluid.

Let

$$r\left(+\frac{l}{2}\right) = 0, \quad r'\left(+\frac{l}{2}\right) = -\infty \text{ in such a way that } \lim_{x \rightarrow l/2} rr' \text{ is finite.} \quad \dots \quad (142)$$

Near $\eta = 0$, ρ is then given by

$$\rho(\xi) \approx \sqrt{\left\{ 2 \lim_{x \rightarrow l/2} (-rr') \left(\frac{l}{2} - \xi \right) \right\}}. \quad \dots \quad (143)$$

The integral equation for $g_1(x)$ becomes then

$$g_1\left(\frac{l}{2}\right) = \lim_{x \rightarrow l/2} (-rr') - \int_{-l/2}^{+l/2} g_1(\xi) \lim_{x \rightarrow l/2} \left\{ \frac{r[r - \rho - r'(x - \xi)]}{\sqrt{\{(x - \xi)^2 + (\rho - \rho')^2\}}} G_1 + G_2 \right\} d\xi, \quad \dots \quad (144)$$

or

$$g_1\left(\frac{l}{2}\right) = \lim_{x \rightarrow l/2} (-rr') \left(1 - \int_{-l/2}^{+l/2} g_1(\xi) \left[\sqrt{\left\{ \left(\frac{l}{2} - \xi \right)^2 + \rho^2 \right\}} \right]^{-1/2} d\xi \right) \quad \dots \quad (145)$$

with respect to

$$G_1 \approx 1, \quad G_2 \approx \frac{k^2}{4} \text{ for } k^2 = \frac{4\rho}{(x - \xi)^2 + (\rho - \rho')^2} \ll 1. \quad \dots \quad (146)$$

The kernel of this equation has a singularity of the type

$$\frac{l/2 - \xi}{\left[\sqrt{\left\{ \left(\frac{l}{2} - \xi \right)^2 + \rho^2 \right\}} \right]^2} \rightarrow \frac{1}{\left[\sqrt{\left\{ 2 \lim_{x \rightarrow l/2} (-rr') \right\}} \right]^2} \left(\frac{l}{2} - \xi \right)^{-1/2} \text{ for } \xi \rightarrow \frac{l}{2}. \quad \dots \quad (147)$$

The values of $g_1(l/2)$ and $g_2(l/2)$ are easily seen to be zero. It is, however, advisable to determine the limits of $g_1(x)/r(x)$ and $g_2(x)/r(x)$ which can be obtained in a similar manner as $g_1(l/2)$. The results are

$$\lim_{x \rightarrow l/2} \frac{g_1(x)/r(x)}{g_2(x)/r(x)} = \frac{1}{1 + \frac{\lim_{x \rightarrow l/2} rr'}{l/2}} - \int_{-l/2}^{+l/2} \frac{g_2(\xi)/\rho}{g_1(\xi)/\rho} \frac{\rho^2}{2 \left[\sqrt{\left\{ \left(\frac{l}{2} - \xi \right)^2 + \rho^2 \right\}} \right]^2} \left[\frac{3 \left(\frac{l}{2} - \xi \right) \lim_{x \rightarrow l/2} (-rr')}{\left(\frac{l}{2} - \xi \right)^2 + \rho^2} - 1 \right] d\xi \quad (148)$$

with the singularity of the kernel

$$\frac{\rho^2}{2 \left[\sqrt{\left\{ \left(\frac{l}{2} - \xi \right)^2 + \rho^2 \right\}} \right]^2} \left[\frac{3 \left(\frac{l}{2} - \xi \right) \lim_{x \rightarrow l/2} (-rr')}{\left(\frac{l}{2} - \xi \right)^2 + \rho^2} - 1 \right] \rightarrow \frac{1}{4 \sqrt{\left\{ 2 \lim_{x \rightarrow l/2} (-rr') \right\}}} \left(\frac{l}{2} - \xi \right)^{-1/2} \text{ for } \xi \rightarrow \frac{l}{2}. \quad \dots \quad (149)$$

Consider now the equations for V_{rel} . In longitudinal translation, V_{rel} will obviously be zero at the ends of the body where the relative flow past the body has stagnation points

$$\lim_{x \rightarrow l/2} V_{rel \text{ long}}(x) = 0. \quad \dots \quad (150)$$

In the case of a lateral translation, V_{rel} has a finite limit at the ends

$$\lim_{x \rightarrow l/2} V_{rel \text{ lat}}(x) = \frac{V_\infty}{V_s} \left(-\frac{\cos \theta}{\sin \theta} t \pm \frac{\sin \theta}{\cos \theta} b \right) \left(1 + \int_{-l/2}^{+l/2} \frac{g_1(\xi)}{\rho} \frac{\rho^2 d\xi}{2 \left[\sqrt{\left\{ \left(\frac{l}{2} - \xi \right)^2 + \rho^2 \right\}} \right]^2} \right). \quad (151)$$

The singularity of the kernel is

$$\frac{\rho^2}{2 \left[\sqrt{\left\{ \left(\frac{l}{2} - \xi \right)^2 + \rho^2 \right\}} \right]^2} \rightarrow 2 \sqrt{2 \lim_{r \rightarrow r'} (-rr')} \left(\frac{l}{2} - \xi \right)^{-1/2} \text{ for } \xi \rightarrow \frac{l}{2}. \quad \dots (152)$$

For a rotation of the body about its longitudinal axis, the relative velocity is obviously zero at the ends

$$\lim_{x \rightarrow l/2} \mathbf{V}_{\text{rel}, \text{rot}, x}(x) = 0. \quad \dots (153)$$

Finally, the relative velocity for a rotation about the axes of y or z is obtained as

$$\lim_{x \rightarrow l/2} \mathbf{V}_{\text{rel}, \text{rot}, y, z}(x) = \frac{l}{2} \frac{\Omega_y}{\Omega_z} \left(\begin{matrix} + \sin \theta \mathbf{t} + \cos \theta \mathbf{b} \\ - \cos \theta \mathbf{t} + \sin \theta \mathbf{b} \end{matrix} \right) \left(\frac{x}{l/2} + \int_{-l/2}^{+l/2} \frac{g_1(\xi)}{\rho} \frac{\rho^2 d\xi}{2 \left[\sqrt{\left\{ \left(\frac{l}{2} - \xi \right)^2 + \rho^2 \right\}} \right]^2} \right) \quad (154)$$

with the same kernel as (151) and its singularity (152). Consider at last the equations for the potential. The limit of Φ_{long} is

$$\lim_{x \rightarrow l/2} \Phi_{\text{long}} = -V_x \int_{-l/2}^{+l/2} g_1(\xi) \frac{d\xi}{\sqrt{\left\{ \left(\frac{l}{2} - \xi \right)^2 + \rho^2 \right\}}}, \quad \dots (155)$$

with

$$\frac{1}{\sqrt{\left\{ \left(\frac{l}{2} - \xi \right)^2 + \rho^2 \right\}}} \rightarrow \sqrt{2 \lim_{r \rightarrow r'} (-rr')} \left(\frac{l}{2} - \xi \right)^{-1/2} \text{ for } \xi \rightarrow \frac{l}{2}. \quad \dots (156)$$

The limits of Φ_{lat} and Φ_{rot} are zero for reasons of symmetry.

2. *An Integration Formula for the Numerical Evaluation of the Integrals.*—The numerical evaluation of the integrals occurring in the integral equations and in the equations for $\mathbf{V}_x$ and Φ is most conveniently carried out by using an integration formula of the type

$$I(x) = \int_{-l/2}^{+l/2} g(\xi) K(x, \xi) d\xi = \sum g(\xi_n) p_n K(x, \xi_n). \quad \dots (157)$$

As the kernels of the integrals have singularities, the points ξ_n and the corresponding weights p_n have to be determined in such a way that the singularities are automatically taken account of. This can be achieved by a suitable transformation of the variable of integration ξ .

The singularities can be of the types

$$\left. \begin{aligned} (a) \quad K(x, \xi) &\sim \log |x - \xi| && \text{for } \xi \rightarrow x \\ (b) \quad K(x, \xi) &\sim (x - \xi)^{-1} && \text{for } \xi \rightarrow x \\ (c) \quad K\left(\pm \frac{l}{2}, \xi\right) &\sim \left| \pm \frac{l}{2} - \xi \right|^{-1/2} && \text{for } \xi \rightarrow \pm \frac{l}{2} \end{aligned} \right\}. \quad \dots (158)$$

Consider

$$h(\xi); \quad d\xi = \frac{1}{h'(\xi)} dh(\xi). \quad \dots (159)$$

Then

$$I(x) = \int_{h(-l/2)}^{h(+l/2)} g(\xi) K(x, \xi) \frac{1}{h'(\xi)} dh(\xi). \quad \dots (160)$$

In order to remove the singularities (158) as far as possible, $h(\xi)$ has to satisfy the following conditions:

$$\left. \begin{aligned} (a) \quad h(x - \xi) &= -h(\xi - x), \quad h(x) = 0 \\ (b) \quad \lim_{\xi \rightarrow x} K(x, \xi) \frac{1}{h'(\xi)} &= \text{finite or zero in the cases 158 (a) and (c)} \end{aligned} \right\}. \quad \dots (161)$$

The first condition is necessary with respect to the singularities of the type 158 (b) of the kernels K_{11} and K_{21} . The corresponding integrals have to be calculated in the form of their so-called principal values; this requires that the points ξ_v have to be distributed symmetrically with respect to x , at least in the neighbourhood of x . This gives

$$\frac{1}{2}(\xi_v + \xi_{-v}) = x \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (162)$$

and together with 161 (a)

$$p_{-v} = p_v \dots \dots \dots \dots \dots \dots \dots \dots \dots (163)$$

The second condition is required in order to remove the singularities 158 (a) and (c).

A transformation satisfying these conditions can obviously be chosen in many different ways. A comparatively simple transformation of this kind is

$$h(\xi) = \{\operatorname{sgn}(\xi - x)\} \sqrt{\left(\frac{|\xi - x|}{|\xi - x| + 1}\right)} \dots \dots \dots \dots (164)$$

$$\frac{1}{h'(\xi)} = 2\sqrt{\{|\xi - x|(|\xi - x| + 1)^2\}} \dots \dots \dots \dots (165)$$

The condition 161 (a) is obviously satisfied. 161 (b) is also satisfied since

$$\lim_{\xi \rightarrow x} K(x, \xi) \frac{1}{h'(\xi)} = 2 \lim_{\xi \rightarrow x} \{K(x, \xi) \sqrt{(|\xi - x|)}\}, \dots \dots \dots \dots (166)$$

which is zero in the case 158 (a) and finite in the case 158 (c). The transformation has further the properties

$$\lim_{\xi \rightarrow \infty} h(\xi) = 1 \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (167)$$

and

$$\lim_{\xi \rightarrow \infty} \frac{1}{\xi^2} \frac{1}{h'(\xi)} = 2, \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (168)$$

which are convenient when dealing with the practically most important case of an elongated body.

Introducing (164) and (165) into (160), the integral becomes.

$$I(x) = \int_{h(-l/2)}^{h(+l/2)} g(\xi) K(x, \xi) 2[\sqrt{\{|\xi - x|(|\xi - x| + 1)^2\}}] d\left[\{\operatorname{sgn}(\xi - x)\} \sqrt{\left(\frac{|\xi - x|}{|\xi - x| + 1}\right)}\right] \dots \dots \dots \dots (169)$$

This integral can be evaluated approximately by using the trapezoidal rule.

Let first $x \neq \pm l/2$, and let

$$\begin{aligned} -\frac{l}{2} &= \xi_{-n-1} = \xi_{-n} < \xi_{-n+1} < \dots < \xi_{-1} < \xi_0 \\ &= x < \xi_1 < \dots < \xi_{m-1} < \xi_m = \xi_{m+1} = +\frac{l}{2} \end{aligned} \quad \dots \quad \dots \quad (170)$$

be a set of points on the axis of the body satisfying the condition

$$x - \xi_{-v} = \xi_v - x \text{ for } v < \operatorname{Min}(n, m) \quad \dots \quad \dots \quad \dots \quad (171)$$

[cf. equation (162)].

The general trapezoidal rule for non-equidistant points gives then

$$I(x) \approx \sum_{v=-n}^{+m} g(\xi_v) K(x, \xi_v) p_v \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (172)$$

with the weights

$$p_v = \frac{h(\xi_{v+1}) - h(\xi_{v-1})}{2h'(\xi_v)} \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (173)$$

or, with respect to (164) and (165)

$$p_v = \sqrt{(|\xi_v - x|(|\xi_v - x| + 1))^2} \left[\{\operatorname{sgn}(\xi_{v+1} - x)\} \sqrt{\left(\frac{|\xi_{v+1} - x|}{|\xi_{v+1} - x| + 1}\right)} - \{\operatorname{sgn}(\xi_{v-1} - x)\} \sqrt{\left(\frac{|\xi_{v-1} - x|}{|\xi_{v-1} - x| + 1}\right)} \right] \quad \dots \quad (174)$$

Σ' in (172) indicates that the term corresponding to $v = 0$ is zero, as the singularity of K is of the types 158 (a) or 158 (b).

Let now $x = +l/2$ and

$$-\frac{l}{2} = \xi_{-n+1} = \xi_{-n} < \xi_{-n+1} < \dots < \xi_{-1} < \xi_0 = \xi_1 = +\frac{l}{2} \quad \dots \quad (175)$$

In this case, K has a singularity of the type 158 (c). The trapezoidal rule gives then

$$I\left(\frac{l}{2}\right) = \sum_{v=-n}^n g(\xi_v) K\left(\frac{l}{2}, \xi_v\right) p_v + g\left(\frac{l}{2}\right) \lim_{\epsilon \rightarrow l/2} \left[K\left(\frac{l}{2}, \xi\right) \sqrt{\left(\frac{l}{2} - \xi\right)} \right] \sqrt{\left(\frac{l}{2} - \xi_{-1}\right)} \quad (176)$$

with the same values of the p as in equation (174). The limits of $K\sqrt{(l/2 - \xi)}$ in this formula are given in equations (147), (149), (152) and (156) for the various cases.

For the practical application of the integration formulae (172) and (176), it is convenient to use a standard set of values $|x - \xi_v|$ which correspond to approximately equal values of Δh and are numerically convenient. In the subsequent numerical examples, the standard set

$v = 0$	± 1	± 2	± 3	± 4	± 5	± 6	± 7	± 8	± 9	± 10	± 11	± 12	± 13	± 14
$x - \xi_v = 0$	∓ 0.01	∓ 0.04	∓ 0.1	∓ 0.2	∓ 0.4	∓ 0.7	∓ 1	∓ 1.5	∓ 2.5	∓ 4	∓ 6	∓ 9	∓ 15	$\mp \infty$
p_v	0.0199	0.0428	0.0774	0.134	0.245	0.320	0.376	0.658	1.24	1.85	2.46	4.02	12.72	—

$$p_0 = 0.0995 \text{ in (176)} \quad \dots \quad (177a)$$

was used. In general, the two ends of the body $\xi = \pm l/2$ will not coincide with one of the points ξ_v for any particular point x . The first two points $\xi_{-n+1} < \xi_{-n} < -l/2$ and the first two points $\xi_{n+1} > \xi_n > +l/2$ in (177) were then replaced by $-l/2$ or $+l/2$, respectively, and the new values of the weights p_{-n} , p_{-n+1} , p_{n-1} , p_n were calculated from (174).

Example: $l = 10$, $x = +3$

$$\xi_{-13} = \xi_{-12} = -5, \underbrace{\xi_{-11} = -3, \dots, \xi_0 = +3, \dots, \xi_8 = +4.5}_{\text{from (177)}}, \xi_9 = \xi_{10} = +5$$

The weights are then

$$p_{-13} = \sqrt{(8.9^2)} \left(\sqrt{\frac{8}{9}} - \sqrt{\frac{6}{7}} \right) = 0.451 \quad \dots \quad (178)$$

$$p_{-11} = \sqrt{(6.7^2)} \left(\sqrt{\frac{8}{9}} - \sqrt{\frac{4}{5}} \right) = 0.839$$

p_{-10} to p_7 as in (177)

$$p_8 = \sqrt{(1.5 \times 2.5^2)} \left(\sqrt{\frac{2}{3}} - \sqrt{\frac{1}{2}} \right) = 0.530$$

$$p_9 = \sqrt{(2.3^2)} \left(\sqrt{\frac{2}{3}} - \sqrt{\frac{1.5}{2.5}} \right) = 0.146.$$

The results of the subsequent numerical examples suggest that the set (177) gives a sufficient accuracy of the numerical integrations in most cases of practical interest, especially for the lateral translation and for the rotation. In the case of the longitudinal translation where the requirements on the accuracy are considerably greater, the set (177) gave quite satisfactory results for an ellipsoid. The results for a longitudinal translation of a torpedo with a hemispherical head were still fair but the velocity near the head of the body was found up to about 2 per cent less than the velocity which was obtained in a previous calculation with the use of the vortex-ring method. The latter results which were obtained with about twice the number of points ξ , were in excellent agreement with American experimental results. This suggests that it may be advisable to use more points ξ , for the calculation of the longitudinal flow which unfortunately will lead to a corresponding increase of the already considerable numerical work.

3. *Procedure for the Numerical Calculation.*—In the actual numerical work, the functions g , the relative velocity and the potential will be calculated for a certain number of points x (in the following referred to as pivotal points) along the axis of the body. The selection of these points is therefore the first problem for the computer. Generally speaking, the pivotal points should give sufficient guidance to draw the curves $g_r(x)$, $\nabla_{rel}(x) \cdot \mathbf{t}$, $\nabla_{rel}(x) \cdot \mathbf{b}$ and $\Phi(x)$. They should be spaced rather closely together in those parts of the body where irregularities (suction peaks, influence of discontinuities of the curvature, etc.), may be expected. The front end of the body ($x = +l/2$) should be a pivotal point, but it is in general not necessary to use $x = -l/2$ as one, because the flow conditions near the stern of the body will differ too much from the behaviour of a perfect fluid. It is advisable to start the work with about 12 to 15 pivotal points; more points can be added later in the course of the work wherever they may be required.

The subsequent procedure of the numerical work is given under the assumption that the shape of the body, although it may have isolated discontinuities of the curvature, is composed of arcs of relatively simple analytical curves (e.g., straight lines, circles, ellipses, parabolas, etc.), so that it is not difficult and not too laborious to obtain sufficiently accurate values of r , r' and r'' for every point x and of ρ and ρ' for every point ξ .

Consider first the case $x \neq l/2$.

By far the greatest part of the numerical work is required for the calculation of the eight kernels (110) for the points ξ . The calculation follows closely the pattern given in equations (103) to (112). The first step is the calculation of the expressions b , c , r_1^2 and r_2^2 (equations (103) to (106)) on Form I. The calculation of b requires some attention near $\xi = x$, as b is there of the order ϵ^2 and has to be calculated from r and ρ which are of the order 1. In order to obtain b with an accuracy of 3 significant decimals, r and ρ have to be calculated with about 7 significant decimals, r' with 5 to 6 significant decimals. Form II gives then the calculation of pA , $B^\dagger$, C , k^2 and the elliptic integrals $G_{1,1,1}$ and $H_{1,1,1}$, and Form III the final assembly of the kernels.

The calculation is then split up into three parts corresponding to the longitudinal translation (Forms IV and V), the lateral translation (Forms VI and VII) and the rotation of the body (Forms VIII and IX). All necessary instructions for the numerical work are given on these forms.

The solution of the three integral equations for the $g_r(x)$ is carried out by iteration. In the case of $g_1(x)$, the direct iteration according to the scheme

$$\left. \begin{aligned} h(x) &= f(x) + \int h(\xi) K(x, \xi) d\xi \\ h_0(x) &= f(x) \\ h_n(x) &= f(x) + \int h_{n-1}(\xi) K(x, \xi) d\xi \end{aligned} \right\} \dots \dots \dots (179)$$

[†] In some cases, the calculation of B can be simplified near $\xi = x$, especially if both x and ξ are on a part of the body where the shape is an arc of circle.

was found to oscillate and to converge only very slowly. A much quicker convergence was obtained by the scheme

$$h_0(x) = \frac{1}{2}f(x)$$

$$h_n^*(x) = f(x) + \int h_{n-1}(\xi) K(x, \xi) d\xi, \quad h_n(x) = \frac{1}{2}(h_{n-1}(x) + h_n^*(x)) \dots \dots (180)$$

$g_1(x)$ and $g_2(x)$ could be obtained from (179). Two or three steps of the iteration procedure were in general sufficient and a fourth step had only to be carried out for the pivotal points near to the ends of the body. After each step of the iteration, the results $g_{\nu\mu}(x)$ were represented by a curve from which the values $g_{\nu\mu}(\xi)$ were read for the subsequent step of the procedure.

Special forms are needed for the case $x = l/2$ where the elliptic integrals in the kernels degenerate (equations 142 to 156). These forms I (a) to IX (a) contain also all the necessary instructions for the numerical work.

After carrying out the calculation of $\mathbf{V}_{in}$ and Φ for the elementary translations and rotations of the body, the result for the complete motion can easily be obtained by a superposition (equation (16)).

4. *Formulae and Forms for the Numerical Work.—Body of Revolution in Translation or Rotation. Calculation of the Kernels for $x \neq l/2$.*

FORMULAE :

1. Longitudinal translation:

$$K_{10} = \left\{ \frac{r[r - \rho - r'(x - \xi)]}{(x - \xi)^2 + (r - \rho)^2} G_1(k^2) + G_2(k^2) \right\} \frac{1}{\sqrt{\{(x - \xi)^2 + (r + \rho)^2\}}}$$

$$K_{11} = \frac{(x - \xi) + r'(r - \rho)}{(x - \xi)^2 + (r - \rho)^2} G_1(k^2) \frac{1}{\sqrt{\{(x - \xi)^2 + (r + \rho)^2\}}}$$

$$K_{12} = G_2(k^2) \frac{1}{\sqrt{\{(x - \xi)^2 + (r + \rho)^2\}}}$$

$$K_{13} = G_2(k^2) \frac{1}{\sqrt{\{(x - \xi)^2 + (r + \rho)^2\}}}$$

2. Lateral translation and rotation:

$$K_{20} = \left\{ \frac{r[r - \rho - r'(x - \xi)]}{(x - \xi)^2 + (r - \rho)^2} H_1(k^2) + H_2(k^2) \right\} \frac{1}{\sqrt{\{(x - \xi)^2 + (r + \rho)^2\}}}$$

$$K_{21} = \frac{(x - \xi) + r'(r - \rho)}{(x - \xi)^2 + (r - \rho)^2} H_1(k^2) \frac{1}{\sqrt{\{(x - \xi)^2 + (r + \rho)^2\}}}$$

$$K_{22} = H_2(k^2) \frac{1}{\sqrt{\{(x - \xi)^2 + (r + \rho)^2\}}}$$

$$K_{23} = H_2(k^2) \frac{1}{\sqrt{\{(x - \xi)^2 + (r + \rho)^2\}}}$$

EXAMPLE 1.—*Ellipsoid* $b : a = 1 : 4$

In order to check the accuracy of the method, the calculation was first carried out for the case of an ellipsoid of fineness ratio $b : a = 1 : 4$. Figs. 4 to 6 give the distributions of the relative velocity in longitudinal translation, lateral translation and rotation about a lateral axis, respectively, in comparison with the known exact solutions. In general, the agreement is very good. Close to the ends of the body, the error of the present method tends to become greater, but is still small enough for practical purposes. The calculations were carried out with the standard set (177) of points ξ . If a greater accuracy near the ends of the body should be desirable, this is likely to be achieved by using more points ξ , for the points x near the ends.

EXAMPLE 2.—*Torpedo of Fineness Ratio 1 : 8*

As a second numerical example, the calculation was carried out for a torpedo of fineness ratio 1 : 8. Its shape, composed of arcs of circles and straight lines, was given by the equations :

$$1. \quad \frac{r}{r_{\max}} = \sqrt{1 - \left(\frac{x}{r_{\max}} - 7\right)^2} \text{ for } +7 \leq \frac{x}{r_{\max}} \leq +8$$

(Hemispherical head).

$$2. \quad \frac{r}{r_{\max}} = 1 \text{ for } -2.003 \leq \frac{x}{r_{\max}} \leq +7$$

(Straight cylindrical central part).

$$3. \quad \frac{r}{r_{\max}} = \sqrt{13.715^2 - \left(\frac{x}{r_{\max}} + 2.003\right)^2} - 12.715 \text{ for } -4.385 \leq \frac{x}{r_{\max}} \leq -2.003$$

(Transition to the tail).

$$4. \quad \frac{r}{r_{\max}} = 0.1763 \frac{x}{r_{\max}} + 1.5647 \text{ for } -8 \leq \frac{x}{r_{\max}} \leq -4.385$$

(Conical tail).

At the rear end of the body, $x/r_{\max} = -8$, $r/r_{\max}$ is not zero as required by the definition of a 'normal' shape. It was assumed for the calculation that the end was suitably rounded off. The rear end of the body was, however, not included in the number of the pivotal points as the result of the calculation was likely to have little practical value beyond $x = -6$ or -7 .

Body of Revolution in Translation or Rotation.—I. b, c, r_1^2, r_2^2

[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]	[11]	[12]	[13]
ξ^*	$x - \xi$	$\sqrt{x - \xi}$	ρ	$4r\rho$	$r - \rho$	$\sqrt{r(r - \rho)}$	$(r - \rho)^2$	$(x - \xi)^2$	$[9] - [3][2]$	$[7][8] + [9][12] + [5]$	r_1^2	r_2^2
	15							225				
	9							81				
	6							36				
	4							16				
	2.5							6.25				
	1.5							2.25				
	1							1				
	0.7							0.49				
	0.4		(4)					0.16				
	0.2		(5)					0.04				
	0.1		(6)					0.01				
	0.04		(7)					0.0016				
	0.01		(7)					0.0001				
(*)	0	0	(7)	(4*)	0	0	0	0	0	0	0	(4*)
	0.01		(7)					0.0001				
	0.04		(7)					0.0016				
	0.1		(6)					0.01				
	0.2		(6)					0.04				
	0.4		(5)					0.16				
	0.7		(4)					0.49				
	1							1				
	1.5							2.25				
	2.5							6.25				
	4							16				
	6							36				
	8							64				
	15							225				

Constants of the
pivotal point :

$x =$ ($\neq 1/2$)

$r =$ (7 dec)

$r' =$ (5)

$r'' =$ (3)

The numbers in
brackets give the
number of signif-
icant decimals
where more than
3 are required.

* Replace the last $\xi < -1/2$ by $-1/2$ and the first $\xi > +1/2$ by $1/2$ and correct [2] and [9] correspondingly.

Body of Revolution in Translation or Rotation.—III. Kernels

[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]	[11]	[12]	[13]	[14]	[15]
ϵ	BG_1	BH_1	CG_1	CH_1	$[2] + G_1$	$[3] + H_1$	$[5] - pA$	$[4] pA$	$G_2 pA$	$G_2 pA$	$[7] pA$	$[5] pA$	$H_2 pA$	$H_2 pA$
(5)	$\left(\frac{-\eta''}{s(1+\eta^2)} \right)$	$\left(\frac{-\eta''}{s(1+\eta^2)} \right)$	$\pm \infty$	$\pm \infty$	∞	∞	-	-	-	-	-	-	-	-

$$x \neq (2/1)$$

Body of Revolution in Longitudinal Translation.—Calculation of g_1 , $V_{\text{rel long}}$ and Φ_{long} for $x \neq l/2$

FORMULAE :

1. *Integral Equation:*

$$g_1(x) = -rr' - \int_{-l/2}^{+l/2} g_1(\xi) K_{10}(x, \xi) d\xi$$

Solution:

$$g = -\frac{1}{2}rr'$$

$$g_{11}^* = -rr' - \int_{-l/2}^{+l/2} g_{10}(\xi) K_{11}(x, \xi) d\xi, \quad g_{11} = \frac{1}{2}(g_{10} + g_{11}^*)$$

$$g_{12}^* = -rr' - \int_{-l/2}^{+l/2} g_{11}(\xi) K_{12}(x, \xi) d\xi, \quad g_{12} = \frac{1}{2}(g_{11} + g_{12}^*)$$

2. *Relative Velocity:*

$$V_{\text{rel long}} = V_s \frac{t}{\sqrt{1+t'^2}} \left[1 - \int_{-l/2}^{+l/2} g_1(\xi) K_{11}(x, \xi) d\xi - \frac{r'}{r} \int_{-l/2}^{+l/2} g_1(\xi) K_{12}(x, \xi) d\xi \right]$$

3. *Potential:*

$$\Phi_{\text{long}} = -V_s \int_{-l/2}^{+l/2} g_1(\xi) K_{13}(x, \xi) d\xi$$

82(5)

$$x = (x_1, x_2) \in \mathbb{R}^2$$

Body of Revolution in Translation or Rotation.—V. Relative Velocity and Potential for Longitudinal Translation

[illegible]

$$x = (\neq 1/2)$$

$$\frac{1}{2} =$$

$$\frac{1}{\sqrt{1+r'^2}} =$$

$$\frac{V_{\text{red long}}}{V_s} = \frac{1 - \Sigma[4] - \frac{\gamma'}{\gamma} \Sigma[6]}{\sqrt{1 + \gamma'^2}} t = \dots t$$

$$V_{\text{long}} = -V_1 \Sigma[8] = \dots, V_n$$

Body of Revolution in Lateral Translation. Calculation of g_2 , V_{rel} and Φ_{lat} for $x \neq l/2$

FORMULAE :

1. Integral Equation:

$$g_2(x) = r - \int_{-l/2}^{+l/2} g_2(\xi) K_{20}(x, \xi) d\xi$$

Solution:

$$g_{20} = r$$

$$g_{21} = r - \int_{-l/2}^{+l/2} g_{20}(\xi) K_{20}(x, \xi) d\xi$$

$$g_{22} = r - \int_{-l/2}^{+l/2} g_{21}(\xi) K_{20}(x, \xi) d\xi$$

2. Relative Velocity:

$$V_{rel} = \frac{V_y}{V_x} \left\{ \frac{\cos \theta}{\sin \theta} \left[r' - \int_{-l/2}^{+l/2} g_2(\xi) K_{21}(x, \xi) d\xi - \frac{r'}{r} \int_{-l/2}^{+l/2} g_2(\xi) K_{22}(x, \xi) d\xi \right] \right. \\ \left. \pm \frac{\sin \theta}{\cos \theta} b \left[1 + \frac{1}{r} \int_{-l/2}^{+l/2} g_2(\xi) K_{22}(x, \xi) d\xi \right] \right\}$$

3. Potential:

$$\Phi_{lat} = - \frac{V_y \cos \theta}{V_x \sin \theta} \int_{-l/2}^{+l/2} g_2(\xi) K_{22}(x, \xi) d\xi.$$

520(2)

[illegible]

(= 1/2)

17

Body of Revolution in Translation or Rotation. —VII. Relative Velocity and Potential for Lateral Translation

[illegible]

$$\mathbf{V}_{\text{relat}} = \frac{V_y}{V_x} \left[\frac{r' - \Sigma[4] - \frac{r'}{r} \Sigma[6]}{\sqrt{(1+r'^2)}} \cos \theta \pm \left(1 + \frac{\Sigma[8]}{r} \right) \frac{\sin \theta}{\cos \theta} \right]$$

$$= \frac{V_s}{V_a} \left(\dots \frac{\cos}{\sin} \theta t \pm \dots \frac{\sin}{\cos} \theta b \right)$$

$$\phi_{\text{int}} = -\Sigma[\delta] \frac{V'_i \cos \theta}{V_i \sin \theta} = \dots \frac{V'_i \cos \theta}{V_i \sin \theta}.$$

Body of Revolution in Rotation.—Calculation of g_3 , $\mathbf{V}_{\text{rel rot}}$ and Φ_{rot} for $x \neq l/2$

FORMULAE :

1. Integral Equation:

$$g_3(x) = \frac{r(x + rr')}{l/2} - \int_{-l/2}^{+l/2} g_3(\xi) K_{33}(x, \xi) d\xi$$

Solution:

$$g_{30} = \frac{r(x + rr')}{l/2}$$

$$g_{31} = \frac{r(x + rr')}{l/2} - \int_{-l/2}^{+l/2} g_{30}(\xi) K_{33}(x, \xi) d\xi$$

$$g_{32} = \frac{r(x + rr')}{l/2} - \int_{-l/2}^{+l/2} g_{31}(\xi) K_{33}(x, \xi) d\xi.$$

2. Relative Velocity:

$$\begin{aligned} \mathbf{V}_{\text{rel rot } y, z} = & \frac{l}{2} \Omega_y \left\{ \frac{-\sin \theta}{\sqrt{(1+r'^2)}} \left[\frac{r'x - r}{l/2} - \int_{-l/2}^{+l/2} g_3(\xi) K_{31}(x, \xi) d\xi \right. \right. \\ & \left. \left. - \frac{r'}{r} \int_{-l/2}^{+l/2} g_3(\xi) K_{32}(x, \xi) d\xi \right] \right. \\ & \left. + \frac{\cos \theta}{\sin \theta} b \left[\frac{x}{l/2} + \frac{1}{r} \int_{-l/2}^{+l/2} g_3(\xi) K_{32}(x, \xi) d\xi \right] \right\} \end{aligned}$$

$$\mathbf{V}_{\text{rel rot } x} = -\frac{l}{2} \Omega_x \frac{r}{l/2} \mathbf{b}.$$

3. Potential:

$$\Phi_{\text{rot } y, z} = \pm \frac{l}{2} \Omega_x \frac{\sin \theta}{\cos \theta} \int_{-l/2}^{+l/2} g_3(\xi) K_{33}(x, \xi) d\xi$$

$$\Phi_{\text{rot } x} = 0.$$

Body of Revolution in Translation or Rotation.—VIII. Solution of the Integral Equation for Rotation

$\xi_m(\xi)$

[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]	[11]	[12]
ξ	$\frac{\rho(\xi + \rho\rho')}{l/2}$	pX_m	[2] · [3]	$\xi_m(\xi)$	[3] · [5]	$\xi_m(\xi)$	[3] · [7]	$\xi_m(\xi)$	[3] · [9]	$\xi_m(\xi)$	[3] · [11]

$$x = (\neq l/2)$$

$$\frac{r(x + \rho\rho')}{l/2} =$$

$\xi_m = \frac{r(x + \rho\rho')}{l/2}$	$\Sigma[4] =$	$\Sigma[6] =$	$\Sigma[8] =$	$\Sigma[10] =$	$\Sigma[12] =$
$=$	$\xi_m = \frac{r(x + \rho\rho')}{l/2}$	$\xi_m = \frac{r(x + \rho\rho')}{l/2}$	$\xi_m = \frac{r(x + \rho\rho')}{l/2}$	$\xi_m = \frac{r(x + \rho\rho')}{l/2}$	$\xi_m = \frac{r(x + \rho\rho')}{l/2}$
	$-\Sigma[4] =$	$-\Sigma[6] =$	$-\Sigma[8] =$	$-\Sigma[10] =$	$-\Sigma[12] =$

Body of Revolution in Translation or Rotation.—IX. *Relative Velocity and Potential for Rotation*

[illegible]

$$\nabla_{\text{rel rot } y, z} = \frac{l}{2} \frac{\Omega_p}{\Omega_i} \left\{ \frac{\frac{r'x-r}{l/2} - \Sigma[4]}{\sqrt{(1+r'^2)}} \begin{pmatrix} -\sin \theta \\ +\cos \theta \end{pmatrix} \mathbf{t} + \left(\frac{x}{l/2} + \frac{\Sigma[8]}{r} \right) \sin \theta \mathbf{b} \right\}$$

$$= \frac{1}{2} \Omega_r \left[\dots \begin{pmatrix} -\sin \theta \\ +\cos \theta \end{pmatrix} \mathbf{t} + \dots \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \mathbf{b} \right]$$

$$\mathbf{V}_{\text{rel rot } x} = -\frac{l}{2} \Omega_x \frac{r}{\sqrt{2}} \mathbf{b} = -\frac{l}{2} \Omega_x \dots \mathbf{b}$$

$$\phi_{1st, r} = \pm \frac{l}{2} \frac{\Omega_r \sin \theta}{\Omega_r \cos \theta} 2[8] = \pm \frac{l}{2} \frac{\Omega_r \sin \theta}{\Omega_r \cos \theta} \dots$$

$$\Phi_{\text{rot}} = 0.$$

Body of Revolution in Translation or Rotation.—Calculation of the Kernels for $x = l/2$

FORMULAE: Let $\lim_{x \rightarrow l/2} (-rr') = a$.

1. *Longitudinal Translation:*

$$K_{10} \left(\frac{l}{2}, \xi \right) = \frac{l/2 - \xi}{\left[\sqrt{\left\{ \left(\frac{l}{2} - \xi \right)^2 + \rho^2} \right\}} \right]^3}$$

with $\lim_{\xi \rightarrow l/2} \left[K_{10} \left(\frac{l}{2}, \xi \right) \sqrt{\left(\frac{l}{2} - \xi \right)} \right] = \frac{1}{[\sqrt{(2a)}]^3}$

$$K_{12} \left(\frac{l}{2}, \xi \right) = \frac{1}{\sqrt{\left\{ \left(\frac{l}{2} - \xi \right)^2 + \rho^2 \right\}}}$$

with $\lim_{\xi \rightarrow l/2} \left[K_{12} \left(\frac{l}{2}, \xi \right) \sqrt{\left(\frac{l}{2} - \xi \right)} \right] = \frac{1}{\sqrt{(2a)}}.$

2. *Lateral Translation and Rotation:*

$$K_{20} \left(\frac{l}{2}, \xi \right) = \frac{\rho^2}{2 \left[\sqrt{\left\{ \left(\frac{l}{2} - \xi \right)^2 + \rho^2} \right\}} \right]^3} \left[\frac{3a \left(\frac{l}{2} - \xi \right)}{\left(\frac{l}{2} - \xi \right)^2 + \rho^2} - 1 \right]$$

with $\lim_{\xi \rightarrow l/2} \left[K_{20} \left(\frac{l}{2}, \xi \right) \sqrt{\left(\frac{l}{2} - \xi \right)} \right] = \frac{1}{4\sqrt{(2a)}}$

$$K_{22} \left(\frac{l}{2}, \xi \right) = \frac{\rho^2}{2 \left[\sqrt{\left\{ \left(\frac{l}{2} - \xi \right)^2 + \rho^2} \right\}} \right]^3}$$

with $\lim_{\xi \rightarrow l/2} \left[K_{22} \left(\frac{l}{2}, \xi \right) \sqrt{\left(\frac{l}{2} - \xi \right)} \right] = \frac{1}{2\sqrt{(2a)}}$

Body of Revolution in Translation or Rotation.—I (a) to III (a). Kernels for $x = l/2$

[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]	[11]	[12]	[13]	[14]
ξ^*	$\frac{l}{2} - \xi$	ρ	ρ^2	$\left(\frac{l}{2} - \xi\right)^2$	$[4] + [5]$	p^{**}	$[7]/\sqrt{[6]}$	$[8]/[6]$	$[2] \cdot [9]$	$\frac{1}{4} \cdot [4] \cdot [9]$	$3a \cdot \frac{[2]}{[6]}$	$[12] - 1$	$[11] \cdot [13]$
	15			225		12.72							
	9			81		4.02							
	6			36		2.46							
	4			16		1.85							
	2.5			6.25		1.24							
	1.5			2.25		0.638							
	1			1		0.376							
	0.7			0.49		0.320							
	0.4			0.16		0.245							
	0.2			0.04		0.134							
	0.1			0.01		0.0774							
	0.04			0.0016		0.0428							
	0.01			0.0001		0.0199							
(l/2)							$\frac{0.0704}{\sqrt{a}}$	—	$\frac{0.0352}{\sqrt{a^2}}$	$\frac{0.0352}{\sqrt{a}}$			$\frac{0.0176}{\sqrt{a}}$
	0	0	0	0	0	—	=	—	=	=	1.500	0.500	=

$$x =$$

$$a = \lim_{s \rightarrow l/2} (-rr') =$$

* Replace the last $\xi < -l/2$ by $-l/2$ and correct [2] and [5] correspondingly.

** Correct according to (178).

Body of Revolution in Longitudinal Translation.— g_1 , $\mathbf{V}_{\text{rel long}}$ and Φ_{long} for $x = l/2$

FORMULAE :

1. *Integral equation:*

$$g_1(l/2) = \lim_{x \rightarrow l/2} (-rr') \left[1 - \int_{-l/2}^{+l/2} g_1(\xi) K_{10} \left(\frac{l}{2}, \xi \right) d\xi \right]$$

Solution:

$$g_{10}(l/2) = \frac{1}{2} \lim_{x \rightarrow l/2} (-rr')$$

$$g_{11}^*(l/2) = \lim_{x \rightarrow l/2} (-rr') \left[1 - \int_{-l/2}^{+l/2} g_{10}(\xi) K_{10} \left(\frac{l}{2}, \xi \right) d\xi \right], \quad g_{11} = \frac{1}{2}(g_{10} + g_{11}^*)$$

2. *Relative Velocity:*

$$\mathbf{V}_{\text{rel long}}(l/2) = 0$$

3. *Potential:*

$$\Phi_{\text{long}}(l/2) = -V_s \int_{-l/2}^{+l/2} g_1(\xi) K_{10} \left(\frac{l}{2}, \xi \right) d\xi.$$

for Longitudinal Translation

11

$$a = \lim_{n \rightarrow \infty} \left(-r^n \right) =$$

Body of Revolution in Translation or Rotation.—V (a). Relative Velocity and Potential for Longitudinal Translation

[1]	[2]	[3]	[4]
ξ	$E_1(\xi)$	pK_{10}	$[2] \cdot [3]$
(1/2)		$\frac{0.0704}{\sqrt{u}}$	

$$x = \frac{1}{2}$$

$$a = \lim_{x \rightarrow 1/2} (-rr') =$$

2[4] -

$$\mathbf{V}_{\text{rel long}}(l/2) = 0$$

$$\Phi_{\text{mag}}(l/2) = -V_s \Sigma[4] = \dots V_s$$

Body of Revolution in Lateral Translation.— $\lim_{x \rightarrow l/2} g_2(x)/r(x)$, $\mathbf{V}_{rel lat}$ and ϕ_{lat} for $x = l/2$

FORMULAE :

1. *Integral Equation:*

$$\lim_{x \rightarrow l/2} \frac{g_2(x)}{r(x)} = 1 - \int_{-l/2}^{+l/2} \frac{g_2(\xi)}{\rho} K_{20}(l/2, \xi) d\xi$$

Solution:

$$\lim_{x \rightarrow l/2} \frac{g_{20}(x)}{r(x)} = 1$$

$$\lim_{x \rightarrow l/2} \frac{g_{2n}(x)}{r(x)} = 1 - \int_{-l/2}^{+l/2} \frac{g_{2(n-1)}(\xi)}{\rho} K_{2n}(l/2, \xi) d\xi$$

2. *Relative Velocity:*

$$\mathbf{V}_{rel lat}(l/2) = \frac{V_y}{V_x} \begin{bmatrix} -\cos \theta & \sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} t & b \end{bmatrix} \left[1 + \int_{-l/2}^{+l/2} \frac{g_2(\xi)}{\rho} K_{21}(l/2, \xi) d\xi \right]$$

3. *Potential:*

$$\phi_{lat}(l/2) = 0.$$

Body of Revolution in Translation or Rotation.—VII (a). Relative Velocity and Potential for Lateral Translation

[1]	[2]	[3]	[4]
ξ	$g_2(\xi)/\rho$	ρK_M	[2] · [3]
(l/2)		$\frac{0.0352}{\sqrt{a}}$	
$\Sigma[4] =$			

$$x = \quad = l/2$$

$$a = \lim_{x \rightarrow l/2} (-rr') =$$

$$V_{rel}(l/2) = \frac{V_y}{V_x} \left[-\cos \theta t + \sin \theta b \right] [1 + \Sigma[4]]$$

$$= \frac{V_y}{V_x} \left[-\cos \theta t + \sin \theta b \right] \dots$$

$$\phi_{rel}(l/2) = 0.$$

Body of Revolution in Rotation.— $\lim_{x \rightarrow l/2} g_s(x)/r(x)$, $\mathbf{V}_{\text{rel tot}}$ and Φ_{tot} for $x = l/2$

FORMULAE :

1. *Integral Equation:*

$$\lim_{x \rightarrow l/2} \frac{g_s(x)}{r(x)} = 1 + \frac{\lim rr'}{l/2} - \int_{-l/2}^{+l/2} \frac{g_s(\xi)}{\rho} K_{ss}(l/2, \xi) d\xi.$$

Solution:

$$\lim_{x \rightarrow l/2} \frac{g_{ss}(x)}{r(x)} = 1 + \frac{\lim rr'}{l/2}$$

$$\lim_{x \rightarrow l/2} \frac{g_{ss}(x)}{r(x)} = 1 + \frac{\lim rr'}{l/2} - \int_{-l/2}^{+l/2} \frac{g_{s, s-1}(\xi)}{\rho} K_{ss}(l/2, \xi) d\xi.$$

2. *Relative velocity:*

$$\mathbf{V}_{\text{rel tot}, s}(l/2) = 0$$

$$\mathbf{V}_{\text{rel tot}, y, z}(l/2) = \frac{l}{2} \frac{\Omega_y}{\Omega_s} \left[\begin{matrix} + \sin \theta & \mathbf{t} \\ - \cos \theta & \mathbf{b} \end{matrix} \right] \left[1 + \int_{-l/2}^{+l/2} \frac{g_s(\xi)}{\rho} K_{ss}(l/2, \xi) d\xi \right].$$

3. *Potential:*

$$\Phi_{\text{tot}, s}(l/2) = \Phi_{\text{tot}, y, z}(l/2) = 0.$$

$$= 1 - \frac{a}{l^2} - \frac{\sum 2n + 2}{r(l^2)}$$

$$1 - \frac{a}{l/2} = a = \lim_{s \rightarrow l/2} (-rr') = \frac{l}{2} = x$$

Body of Revolution in Translation or Rotation.—IX (a). Relative Velocity and Potential for Rotation

[1]	[2]	[3]	[4]
ξ	$g_a(\xi)/\rho$	pK_u	$[2] \cdot [3]$
$(l/2)$		$\frac{0.0352}{\sqrt{u}}$	

$$x = \quad = l/2$$

$$a = \lim_{x \rightarrow l/2} (-rr') =$$

$$V_{rel,rot}(l/2) = 0$$

$$\mathbf{V}_{\text{rel rot}, z}(l/2) = \frac{l}{2} \Omega_z \begin{bmatrix} +\sin \theta & \cos \theta \\ -\cos \theta & \sin \theta \end{bmatrix} (1 + 2\gamma) [4]$$

$$= \frac{1}{2} \Omega_p \left[\begin{matrix} + \sin \theta t + \cos \theta b \\ - \cos \theta t + \sin \theta b \end{matrix} \right] \dots$$

$$\Phi_{\text{tot}, x}(l/2) = \Phi_{\text{tot}, y, z}(l/2) = 0,$$

Tables of the Functions G_n and H_n

For $k^* < 0.0^*10$, the functions G_n and H_n are

$$G_1(0.0^{*+n}x) = 0.637$$

$$G_2(0.0^{*+n}x) = G_2(0.0^*x) + 0.366n$$

$$G_3(0.0^{*+n}x) = G_3(0.0^*x) + 0.732n$$

$$H_1(0.0^{*+n}x) = 0.637$$

$$H_2(0.0^{*+n}x) = H_2(0.0^*x) + 0.366n$$

$$H_3(0.0^{*+n}x) = H_3(0.0^*x) + 0.732n$$

k^2	G_1	G_2	G_3	H_1	H_2	H_3	k^2
0.0410	0.637	1.955 ₁₅	4.547 ₃₀	0.637	1.319 ₁₅	3.274 ₃₀	0.9490
11		1.940 ₁₄	4.517 ₂₈		1.304 ₁₄	3.244 ₂₈	89
12		1.928 ₁₂	4.489 ₂₄		1.290 ₁₂	3.216 ₂₅	88
13		1.914 ₁₂	4.465 ₂₄		1.278 ₁₂	3.191 ₂₄	87
14		1.902 ₁₁	4.441 ₂₂		1.266 ₁₁	3.167 ₂₂	86
15		1.891 ₁₀	4.419 ₂₀		1.255 ₁₀	3.145 ₂₀	85
16		1.881 ₁₀	4.399 ₂₀		1.245 ₁₀	3.125 ₂₀	84
17		1.871 ₉	4.374 ₁₈		1.235 ₉	3.105 ₁₈	83
18		1.862 ₉	4.361 ₁₈		1.226 ₉	3.087 ₁₇	82
19		1.853 ₈	4.343 ₁₆		1.217 ₈	3.070 ₁₇	81
0.0420	0.637	1.845 ₈	4.327 ₁₆	0.637	1.209 ₈	3.053 ₁₅	0.9480
21		1.837 ₇	4.311 ₁₄		1.201 ₇	3.038 ₁₅	79
22		1.830 ₇	4.297 ₁₄		1.194 ₇	3.023 ₁₄	78
23		1.823 ₇	4.283 ₁₄		1.187 ₇	3.009 ₁₄	77
24		1.816 ₆	4.269 ₁₂		1.180 ₇	2.995 ₁₃	76
25		1.810 ₇	4.257 ₁₄		1.173 ₆	2.982 ₁₂	75
26		1.803 ₇	4.243 ₁₂		1.167 ₆	2.970 ₁₂	74
27		1.797 ₆	4.231 ₁₂		1.161 ₆	2.958 ₁₂	73
28		1.791 ₅	4.219 ₁₀		1.155 ₅	2.946 ₁₁	72
29		1.786 ₅	4.209 ₁₀		1.150 ₆	2.935 ₁₁	71
0.0430	0.637	1.781 ₄	4.199 ₁₂	0.637	1.144 ₅	2.924 ₁₁	0.9470
31		1.775 ₅	4.187 ₁₀		1.139 ₅	2.913 ₁₀	69
32		1.770 ₅	4.177 ₁₀		1.134 ₅	2.903 ₁₀	68
33		1.765 ₄	4.167 ₈		1.129 ₅	2.893 ₉	67
34		1.761 ₅	4.159 ₁₀		1.124 ₅	2.884 ₉	66
35		1.756 ₅	4.149 ₁₀		1.119 ₄	2.875 ₉	65
36		1.751 ₄	4.139 ₈		1.115 ₄	2.866 ₉	64
37		1.747 ₄	4.131 ₈		1.111 ₄	2.857 ₈	63
38		1.743 ₄	4.123 ₈		1.107 ₅	2.849 ₈	62
39		1.739 ₄	4.115 ₈		1.102 ₃	2.841 ₈	61
0.0440	0.637	1.735 ₄	4.107 ₈	0.637	1.099 ₄	2.833 ₈	0.9460

k^2	G_1	G_2	G_3	H_1	H_2	H_3	k^2
0.040	0.637	1.735 ₄	4.107 ₈	0.637	1.099 ₄	2.833 ₈	0.940
41		1.731 ₄	4.099 ₈		1.095 ₄	2.825 ₇	59
42		1.727 ₄	4.091 ₈		1.091 ₄	2.818 ₈	58
43		1.723 ₃	4.083 ₆		1.087 ₃	2.810 ₇	57
44		1.720 ₄	4.077 ₈		1.084 ₄	2.803 ₈	56
45		1.716 ₄	4.069 ₈		1.080 ₄	2.795 ₇	55
46		1.712 ₃	4.061 ₆		1.076 ₃	2.788 ₆	54
47		1.709 ₃	4.055 ₆		1.073 ₃	2.782 ₇	53
48		1.706 ₄	4.049 ₈		1.070 ₃	2.775 ₆	52
49		1.702 ₃	4.041 ₆		1.067 ₄	2.769 ₇	51
0.0450	0.637	1.699 ₃	4.035 ₆	0.637	1.063 ₃	2.762 ₆	0.9450
51		1.696 ₃	4.029 ₆		1.060 ₃	2.756 ₆	49
52		1.693 ₃	4.023 ₆		1.057 ₃	2.750 ₇	48
53		1.690 ₃	4.017 ₆		1.054 ₃	2.743 ₆	47
54		1.687 ₃	4.011 ₆		1.051 ₃	2.737 ₆	46
55		1.684 ₃	4.005 ₆		1.048 ₂	2.731 ₅	45
56		1.681 ₃	3.999 ₆		1.046 ₃	2.726 ₆	44
57		1.678 ₂	3.993 ₄		1.043 ₃	2.720 ₅	43
58		1.676 ₃	3.989 ₆		1.040 ₃	2.715 ₆	42
59		1.673 ₃	3.983 ₆		1.037 ₃	2.709 ₅	41
0.0460	0.637	1.670 ₂	3.977 ₄	0.637	1.034 ₂	2.704 ₅	0.940
61		1.668 ₃	3.973 ₆		1.032 ₃	2.699 ₅	39
62		1.665 ₃	3.967 ₆		1.029 ₂	2.694 ₆	38
63		1.662 ₂	3.961 ₄		1.027 ₃	2.688 ₅	37
64		1.660 ₃	3.957 ₆		1.024 ₂	2.683 ₅	36
65		1.657 ₂	3.951 ₄		1.022 ₂	2.678 ₅	35
66		1.655 ₂	3.947 ₄		1.020 ₃	2.673 ₄	34
67		1.653 ₃	3.943 ₆		1.017 ₂	2.669 ₅	33
68		1.650 ₂	3.937 ₄		1.015 ₃	2.664 ₄	32
69		1.648 ₂	3.933 ₄		1.012 ₂	2.660 ₅	31
0.0470	0.637	1.648 ₃	3.929 ₆	0.637	1.010 ₂	2.655 ₄	0.9430

k^*	G_1	G_2	G_3	H_1	H_2	H_3	k^*
0.0470	0.637	1.646 ₃	3.929 ₆	0.637	1.010 ₂	2.655 ₄	0.9430
71		1.643 ₂	3.923 ₄		1.008 ₂	2.651 ₄	29
72		1.641 ₂	3.919 ₄		1.006 ₂	2.647 ₅	28
73		1.639 ₂	3.915 ₄		1.004 ₂	2.642 ₄	27
74		1.637 ₂	3.911 ₄		1.002 ₃	2.638 ₄	26
75		1.635 ₂	3.907 ₄		0.999 ₂	2.634 ₄	25
76		1.633 ₃	3.903 ₆		0.997 ₂	2.630 ₄	24
77		1.630 ₂	3.897 ₄		0.995 ₂	2.626 ₅	23
78		1.628 ₂	3.893 ₄		0.993 ₂	2.621 ₄	22
79		1.626 ₂	3.889 ₄		0.991 ₂	2.617 ₄	21
0.0480	0.637	1.624 ₂	3.885 ₄	0.637	0.989 ₂	2.613 ₄	0.9420
81		1.622 ₂	3.881 ₄		0.987 ₂	2.609 ₄	19
82		1.620 ₁	3.877 ₂		0.985 ₂	2.605 ₃	18
83		1.619 ₂	3.875 ₄		0.983 ₂	2.602 ₄	17
84		1.617 ₂	3.871 ₄		0.981 ₂	2.598 ₄	16
85		1.615 ₂	3.867 ₄		0.979 ₂	2.594 ₄	15
86		1.613 ₂	3.863 ₄		0.977 ₂	2.590 ₄	14
87		1.611 ₂	3.859 ₄		0.975 ₂	2.586 ₃	13
88		1.609 ₂	3.855 ₄		0.973 ₁	2.583 ₄	12
89		1.607 ₁	3.851 ₂		0.972 ₂	2.579 ₄	11
0.0490	0.637	1.606 ₂	3.849 ₄	0.637	0.970 ₂	2.575 ₃	0.9410
91		1.604 ₂	3.845 ₄		0.968 ₁	2.572 ₄	09
92		1.602 ₂	3.841 ₄		0.967 ₂	2.568 ₃	08
93		1.600 ₁	3.837 ₂		0.965 ₂	2.565 ₄	07
94		1.599 ₂	3.835 ₄		0.963 ₁	2.561 ₃	06
95		1.597 ₂	3.831 ₄		0.962 ₂	2.558 ₃	05
96		1.595 ₁	3.827 ₂		0.960 ₂	2.555 ₄	04
97		1.594 ₂	3.825 ₄		0.958 ₂	2.551 ₃	03
98		1.592 ₂	3.821 ₄		0.956 ₁	2.548 ₄	02
99		1.590 ₁	3.817 ₂		0.955 ₂	2.544 ₃	01
0.04100	0.637	1.589 ₁₅	3.815 ₃₀	0.637	0.953	2.541 ₃₀	0.9400

k'^2	G_1	G_2	G_3	H_1	H_2	H_3	k^2
0.0 ² 10	0.637	1.589 ₁₅	3.815 ₃₀	0.637	0.953	2.541 ₃₀	0.9 ² 90
11		1.574 ₁₄	3.785 ₂₈	0.636	0.938	2.511 ₂₈	89
12		1.560 ₁₃	3.757 ₂₆	0.636	0.924	2.483 ₂₅	88
13		1.547 ₁₂	3.731 ₂₄	0.636	0.911	2.458 ₂₄	87
14		1.535 ₁₁	3.707 ₂₂	0.636	0.899	2.434 ₂₂	86
15		1.524 ₁₀	3.685 ₂₀	0.636	0.888	2.412 ₂₀	85
16		1.514 ₁₀	3.665 ₂₀	0.636	0.878	2.392 ₁₉	84
17		1.504 ₉	3.645 ₁₈	0.636	0.869	2.373 ₁₈	83
18		1.495 ₈	3.627 ₁₆	0.636	0.860	2.355 ₁₇	82
19		1.487 ₈	3.611 ₁₆	0.636	0.851	2.338 ₁₇	81
0.0 ² 20	0.637	1.479 ₈	3.595 ₁₆	0.636	0.843	2.321 ₁₅	0.9 ² 80
21		1.471 ₈	3.579 ₁₆	0.636	0.835	2.306 ₁₅	79
22		1.463 ₇	3.563 ₁₄	0.636	0.828	2.291 ₁₄	78
23		1.456 ₆	3.549 ₁₂	0.636	0.821	2.277 ₁₄	77
24		1.450 ₇	3.537 ₁₄	0.636	0.814	2.263 ₁₃	76
25		1.443 ₆	3.523 ₁₂	0.636	0.807	2.250 ₁₂	75
26		1.437 ₆	3.511 ₁₂	0.636	0.801	2.238 ₁₂	74
27		1.431 ₆	3.499 ₁₂	0.636	0.795	2.226 ₁₂	73
28		1.425 ₆	3.487 ₁₂	0.636	0.789	2.214 ₁₁	72
29		1.419 ₅	3.475 ₁₀	0.636	0.784	2.203 ₁₁	71
0.0 ² 30	0.637	1.414 ₅	3.465 ₁₀	0.636	0.778	2.192 ₁₁	0.9 ² 70
31		1.409 ₅	3.455 ₁₀	0.636	0.773	2.181 ₁₀	69
32		1.404 ₅	3.445 ₁₀	0.636	0.768	2.171 ₁₀	68
33		1.399 ₅	3.435 ₁₀	0.636	0.763	2.161 ₉	67
34		1.394 ₅	3.425 ₁₀	0.635	0.759	2.152 ₉	66
35		1.389 ₄	3.415 ₈	0.635	0.754	2.143 ₉	65
36		1.385 ₄	3.407 ₈	0.635	0.750	2.134 ₉	64
37		1.381 ₅	3.399 ₁₀	0.635	0.745	2.125 ₈	63
38		1.376 ₄	3.389 ₈	0.635	0.741	2.117 ₈	62
39		1.372 ₄	3.381 ₈	0.635	0.737	2.109 ₈	61
0.0 ² 40	0.637	1.368 ₄	3.373 ₇	0.635	0.733	2.101 ₈	0.9 ² 60

k^2	G_1	G_2	G_3	H_1	H_2	H_3	k^2
0.0240	0.637	1.368 ₄	3.373 ₇	0.635	0.733	2.101 ₈	0.9260
41		1.364 ₄	3.366 ₇	0.635	0.729	2.093 ₇	59
42		1.360 ₃	3.359 ₇	0.635	0.726	2.086 ₇	58
43		1.357 ₄	3.352 ₇	0.635	0.722	2.079 ₈	57
44		1.353 ₃	3.345 ₇	0.635	0.718	2.071 ₈	56
45		1.350 ₄	3.338 ₇	0.635	0.714	2.063 ₇	55
46		1.346 ₃	3.331 ₇	0.635	0.711	2.056 ₆	54
47		1.343 ₄	3.324 ₇	0.635	0.708	2.050 ₇	53
48		1.339 ₃	3.317 ₇	0.635	0.704	2.043 ₆	52
49		1.336 ₃	3.310 ₇	0.635	0.701	2.037 ₆	51
0.0250	0.637	1.333 ₃	3.303 ₆	0.635	0.698	2.031 ₆	0.9250
51	0.638	1.330 ₄	3.297 ₅	0.635	0.695	2.025 ₆	49
52		1.326 ₃	3.292 ₆	0.635	0.692	2.019 ₇	48
53		1.323 ₃	3.286 ₆	0.635	0.689	2.012 ₆	47
54		1.320 ₂	3.280 ₆	0.635	0.686	2.006 ₆	46
55		1.318 ₃	3.274 ₅	0.635	0.683	2.000 ₅	45
56		1.315 ₃	3.269 ₆	0.635	0.681	1.995 ₆	44
57		1.312 ₃	3.263 ₆	0.635	0.678	1.989 ₅	43
58		1.309 ₃	3.257 ₅	0.634	0.675	1.984 ₆	42
59		1.306 ₂	3.252 ₆	0.634	0.673	1.978 ₅	41
0.0260	0.638	1.304 ₃	3.246 ₅	0.634	0.670	1.973 ₅	0.9240
61		1.301 ₃	3.241 ₅	0.634	0.668	1.968 ₅	39
62		1.298 ₂	3.236 ₅	0.634	0.665	1.963 ₆	38
63		1.296 ₃	3.231 ₅	0.634	0.663	1.957 ₅	37
64		1.293 ₂	3.226 ₅	0.634	0.660	1.952 ₅	36
65		1.291 ₂	3.221 ₅	0.634	0.658	1.947 ₅	35
66		1.289 ₃	3.216 ₅	0.634	0.655	1.942 ₄	34
67		1.286 ₂	3.211 ₅	0.634	0.653	1.938 ₅	33
68		1.284 ₃	3.206 ₅	0.634	0.650	1.933 ₄	32
69		1.281 ₂	3.201 ₅	0.634	0.648	1.929 ₅	31
0.0270	0.638	1.279 ₂	3.196 ₄	0.634	0.645	1.924 ₄	0.9230

k^2	G_1	G_2	G_3	H_1	H_2	H_3	k^2
0.070	0.638	1.279 ₂	3.196 ₄	0.634	0.645	1.924 ₄	0.930
71		1.277 ₂	3.192 ₄	0.634	0.643	1.920 ₄	29
72		1.275 ₂	3.188 ₅	0.634	0.641	1.916 ₅	28
73		1.273 ₃	3.183 ₄	0.634	0.639	1.911 ₄	27
74		1.270 ₂	3.179 ₄	0.634	0.637	1.907 ₄	26
75		1.268 ₂	3.175 ₄	0.634	0.634	1.903 ₄	25
76		1.266 ₂	3.171 ₄	0.634	0.632	1.899 ₄	24
77		1.264 ₂	3.167 ₅	0.634	0.630	1.895 ₅	23
78		1.262 ₂	3.162 ₄	0.634	0.628	1.890 ₄	22
79		1.260 ₂	3.158 ₄	0.634	0.626	1.886 ₄	21
0.080	0.638	1.258 ₂	3.154 ₄	0.634	0.624	1.882 ₄	0.920
81		1.256 ₂	3.150 ₄	0.634	0.622	1.878 ₄	19
82		1.254 ₂	3.146 ₃	0.634	0.620	1.874 ₃	18
83		1.252 ₂	3.143 ₄	0.634	0.618	1.871 ₄	17
84		1.250 ₂	3.139 ₄	0.634	0.617	1.867 ₄	16
85		1.248 ₂	3.135 ₄	0.634	0.615	1.863 ₄	15
86		1.246 ₁	3.131 ₄	0.634	0.613	1.859 ₄	14
87		1.245 ₂	3.127 ₃	0.634	0.612	1.855 ₃	13
88		1.243 ₂	3.124 ₄	0.634	0.610	1.852 ₄	12
89		1.241 ₂	3.120 ₄	0.634	0.608	1.848 ₃	11
0.090	0.638	1.239 ₂	3.116 ₃	0.633	0.606	1.845 ₃	0.910
91		1.237 ₁	3.113 ₄	0.633	0.604	1.842 ₄	09
92		1.236 ₂	3.109 ₃	0.633	0.602	1.838 ₃	08
93		1.234 ₂	3.106 ₄	0.633	0.601	1.835 ₄	07
94		1.232 ₁	3.102 ₃	0.633	0.599	1.831 ₃	06
95		1.231 ₂	3.099 ₃	0.633	0.597	1.828 ₃	05
96		1.229 ₂	3.096 ₄	0.633	0.595	1.825 ₄	04
97		1.227 ₁	3.092 ₃	0.633	0.594	1.821 ₃	03
98		1.226 ₂	3.089 ₄	0.633	0.592	1.818 ₄	02
99		1.224 ₂	3.085 ₃	0.633	0.590	1.814 ₃	01
0.0100	0.638	1.222 ₁₅	3.082 ₃₀	0.633	0.589	1.811 ₃₀	0.900

k^2	G_1	G_2	G_3	H_1	H_2	H_3	k^2
0.0210	0.638	1.222 ₁₅	3.082 ₃₀	0.633	0.589	1.811 ₃₀	0.9990
11	0.638	1.207 ₁₄	3.052 ₂₈	0.633	0.574	1.781 ₂₈	89
12	0.638	1.193 ₁₃	3.024 ₂₅	0.633	0.560	1.753 ₂₅	88
13	0.638	1.180 ₁₂	2.999 ₂₄	0.632	0.548	1.728 ₂₃	87
14	0.639	1.168 ₁₁	2.975 ₂₂	0.632	0.537	1.705 ₂₂	86
15	0.639	1.157 ₁₀	2.953 ₂₀	0.632	0.526	1.683 ₂₀	85
16	0.639	1.147 ₁₀	2.933 ₂₀	0.631	0.516	1.663 ₂₀	84
17	0.639	1.137 ₉	2.913 ₁₈	0.631	0.507	1.643 ₁₇	83
18	0.639	1.128 ₈	2.895 ₁₇	0.631	0.498	1.626 ₁₇	82
19	0.639	1.120 ₉	2.873 ₁₆	0.630	0.490	1.609 ₁₆	81
0.0220	0.639	1.111 ₇	2.862 ₁₆	0.630	0.482	1.593 ₁₆	0.9980
21	0.639	1.104 ₈	2.846 ₁₄	0.630	0.474	1.577 ₁₅	79
22	0.639	1.096 ₇	2.832 ₁₄	0.629	0.467	1.562 ₁₃	78
23	0.640	1.089 ₇	2.818 ₁₄	0.629	0.460	1.549 ₁₄	77
24	0.640	1.082 ₆	2.804 ₁₃	0.629	0.453	1.535 ₁₃	76
25	0.640	1.076 ₇	2.791 ₁₂	0.629	0.447	1.522 ₁₂	75
26	0.640	1.069 ₆	2.779 ₁₂	0.628	0.441	1.510 ₁₁	74
27	0.640	1.063 ₅	2.767 ₁₂	0.628	0.436	1.499 ₁₂	73
28	0.640	1.058 ₆	2.755 ₁₁	0.628	0.430	1.487 ₁₁	72
29	0.640	1.052 ₅	2.744 ₁₁	0.628	0.424	1.476 ₁₁	71
0.0230	0.640	1.047 ₆	2.733 ₁₀	0.627	0.419	1.465 ₁₀	0.9970
31	0.640	1.041 ₅	2.723 ₁₀	0.627	0.414	1.455 ₁₀	69
32	0.641	1.036 ₅	2.713 ₁₀	0.627	0.409	1.445 ₉	68
33	0.641	1.031 ₄	2.703 ₉	0.627	0.405	1.436 ₉	67
34	0.641	1.027 ₅	2.694 ₉	0.626	0.401	1.427 ₁₀	66
35	0.641	1.022 ₅	2.685 ₁₀	0.626	0.396	1.417 ₉	65
36	0.641	1.017 ₄	2.675 ₈	0.626	0.391	1.408 ₈	64
37	0.641	1.013 ₄	2.667 ₉	0.626	0.387	1.400 ₈	63
38	0.641	1.009 ₄	2.658 ₈	0.625	0.383	1.392 ₈	62
39	0.641	1.005 ₅	2.650 ₈	0.625	0.379	1.384 ₈	61
0.0240	0.641	1.000 ₄	2.642 ₈	0.625	0.376	1.376 ₈	0.9960

k'^2	G_1	G_2	G_3	H_1	H_2	H_3	k^2
0.040	0.641	1.000 ₄	2.642 ₈	0.625	0.376	1.376 ₈	0.940
41	0.641	0.996 ₃	2.634 ₇	0.625	0.372	1.368 ₈	59
42	0.642	0.993 ₄	2.627 ₈	0.624	0.368	1.360 ₆	58
43	0.642	0.989 ₄	2.619 ₇	0.624	0.365	1.354 ₇	57
44	0.642	0.985 ₃	2.612 ₇	0.624	0.362	1.347 ₈	56
45	0.642	0.982 ₄	2.605 ₇	0.624	0.358	1.339 ₇	55
46	0.642	0.978 ₃	2.598 ₇	0.623	0.355	1.332 ₆	54
47	0.642	0.975 ₄	2.591 ₇	0.623	0.352	1.326 ₇	53
48	0.642	0.971 ₃	2.584 ₆	0.623	0.348	1.319 ₆	52
49	0.642	0.968 ₃	2.578 ₇	0.623	0.345	1.313 ₆	51
0.050	0.642	0.965 ₄	2.571 ₆	0.622	0.343	1.307 ₆	0.950
51	0.642	0.961 ₃	2.565 ₆	0.622	0.340	1.301 ₇	49
52	0.643	0.958 ₃	2.559 ₆	0.622	0.337	1.294 ₆	48
53	0.643	0.955 ₃	2.553 ₆	0.622	0.334	1.288 ₆	47
54	0.643	0.952 ₃	2.547 ₆	0.621	0.331	1.282 ₅	46
55	0.643	0.949 ₃	2.541 ₅	0.621	0.328	1.277 ₅	45
56	0.643	0.946 ₂	2.536 ₆	0.621	0.325	1.272 ₆	44
57	0.643	0.944 ₃	2.530 ₅	0.621	0.323	1.266 ₅	43
58	0.643	0.941 ₃	2.525 ₆	0.620	0.320	1.261 ₆	42
59	0.643	0.938 ₃	2.519 ₅	0.620	0.318	1.255 ₅	41
0.060	0.643	0.935 ₂	2.514 ₅	0.620	0.315	1.250 ₅	0.940
61	0.643	0.933 ₃	2.509 ₆	0.620	0.313	1.245 ₆	39
62	0.643	0.930 ₃	2.503 ₅	0.620	0.310	1.239 ₄	38
63	0.644	0.927 ₂	2.498 ₅	0.619	0.308	1.235 ₅	37
64	0.644	0.925 ₃	2.493 ₄	0.619	0.305	1.230 ₅	36
65	0.644	0.922 ₂	2.489 ₅	0.619	0.303	1.225 ₅	35
66	0.644	0.920 ₂	2.484 ₅	0.619	0.301	1.220 ₄	34
67	0.644	0.918 ₃	2.479 ₄	0.618	0.299	1.216 ₅	33
68	0.644	0.915 ₂	2.475 ₅	0.618	0.297	1.211 ₄	32
69	0.644	0.913 ₂	2.470 ₅	0.618	0.295	1.207 ₄	31
0.070	0.644	0.911 ₃	2.465 ₄	0.618	0.293	1.203 ₅	0.930

k^2	G_1	G_2	G_3	H_1	H_2	H_3	k^2
0.070	0.644	0.911 ₃	2.465 ₄	0.618	0.293	1.203 ₅	0.930
71	0.644	0.908 ₂	2.461 ₄	0.618	0.291	1.198 ₄	29
72	0.644	0.906 ₂	2.457 ₅	0.617	0.289	1.194 ₅	28
73	0.644	0.904 ₂	2.452 ₄	0.617	0.287	1.189 ₄	27
74	0.645	0.902 ₂	2.448 ₄	0.617	0.285	1.185 ₄	26
75	0.645	0.900 ₃	2.444 ₅	0.617	0.283	1.181 ₃	25
76	0.645	0.897 ₂	2.439 ₄	0.616	0.281	1.178 ₅	24
77	0.645	0.895 ₂	2.435 ₄	0.616	0.279	1.173 ₄	23
78	0.645	0.893 ₂	2.431 ₄	0.616	0.277	1.169 ₄	22
79	0.645	0.891 ₂	2.427 ₄	0.616	0.275	1.165 ₄	21
0.080	0.645	0.889 ₂	2.423 ₄	0.616	0.273	1.161 ₄	0.920
81	0.645	0.887 ₂	2.419 ₄	0.615	0.271	1.157 ₄	19
82	0.645	0.885 ₂	2.415 ₃	0.615	0.269	1.153 ₂	18
83	0.645	0.883 ₂	2.412 ₄	0.615	0.268	1.151 ₄	17
84	0.645	0.881 ₂	2.408 ₄	0.615	0.266	1.147 ₄	16
85	0.646	0.879 ₂	2.404 ₄	0.615	0.264	1.143 ₄	15
86	0.646	0.877 ₁	2.400 ₃	0.614	0.262	1.139 ₃	14
87	0.646	0.876 ₂	2.397 ₄	0.614	0.261	1.136 ₄	13
88	0.646	0.874 ₂	2.393 ₄	0.614	0.259	1.132 ₄	12
89	0.646	0.872 ₂	2.389 ₃	0.614	0.258	1.128 ₂	11
0.090	0.646	0.870 ₂	2.386 ₄	0.614	0.256	1.126 ₄	0.910
91	0.646	0.868 ₁	2.382 ₃	0.613	0.254	1.122 ₃	09
92	0.646	0.867 ₂	2.379 ₃	0.613	0.253	1.119 ₄	08
93	0.646	0.865 ₂	2.376 ₄	0.613	0.251	1.115 ₃	07
94	0.646	0.863 ₂	2.372 ₃	0.613	0.250	1.112 ₃	06
95	0.646	0.861 ₁	2.369 ₄	0.613	0.248	1.109 ₄	05
96	0.646	0.860 ₂	2.365 ₃	0.612	0.247	1.105 ₂	04
97	0.647	0.858 ₂	2.362 ₃	0.612	0.245	1.103 ₃	03
98	0.647	0.856 ₁	2.359 ₃	0.612	0.244	1.100 ₄	02
99	0.647	0.855 ₂	2.356 ₃	0.612	0.242	1.096 ₃	01
0.0100	0.647	0.853 ₁₅	2.353 ₃₀	0.612	0.241	1.093 ₂₈	0.900

k^2	G_1	G_2	G_3	H_1	H_2	H_3	k^2
0.010	0.647	0.853 ₁₅	2.353 ₃₀	0.612	0.241	1.093 ₂₈	0.990
11	0.648	0.838 ₁₄	2.323 ₂₈	0.610	0.228	1.065 ₂₆	89
12	0.648	0.824 ₁₃	2.295 ₂₅	0.608	0.216	1.039 ₂₄	88
13	0.649	0.811 ₁₃	2.270 ₂₃	0.606	0.205	1.015 ₂₃	87
14	0.650	0.798 ₁₂	2.247 ₂₂	0.604	0.194	0.992 ₂₁	86
15	0.651	0.787 ₁₀	2.225 ₂₀	0.602	0.185	0.971 ₁₉	85
16	0.652	0.777 ₁₀	2.205 ₁₉	0.600	0.176	0.952 ₁₈	84
17	0.653	0.767 ₉	2.186 ₁₈	0.599	0.168	0.934 ₁₇	83
18	0.653	0.758 ₉	2.168 ₁₇	0.597	0.160	0.917 ₁₆	82
19	0.654	0.749 ₉	2.151 ₁₆	0.596	0.153	0.901 ₁₅	81
0.020	0.655	0.740 ₈	2.135 ₁₅	0.594	0.146	0.886 ₁₄	0.980
21	0.656	0.732 ₇	2.120 ₁₄	0.592	0.140	0.872 ₁₄	79
22	0.656	0.725 ₈	2.106 ₁₄	0.591	0.134	0.858 ₁₃	78
23	0.657	0.717 ₇	2.092 ₁₄	0.589	0.128	0.845 ₁₂	77
24	0.658	0.710 ₆	2.078 ₁₂	0.588	0.123	0.833 ₁₂	76
25	0.659	0.704 ₇	2.066 ₁₂	0.586	0.118	0.821 ₁₁	75
26	0.659	0.697 ₆	2.054 ₁₂	0.585	0.113	0.810 ₁₁	74
27	0.660	0.691 ₆	2.042 ₁₂	0.583	0.108	0.799 ₁₀	73
28	0.661	0.685 ₆	2.030 ₁₀	0.582	0.1038	0.789 ₁₁	72
29	0.661	0.679 ₅	2.020 ₁₁	0.580	0.0994	0.778 ₁₀	71
0.030	0.662	0.674 ₆	2.009 ₁₀	0.579	0.0954	0.768 ₉	0.970
31	0.663	0.668 ₅	1.999 ₁₀	0.578	0.0916	0.759 ₉	69
32	0.663	0.663 ₅	1.989 ₉	0.576	0.0878	0.750 ₉	68
33	0.664	0.658 ₅	1.980 ₁₀	0.574	0.0842	0.741 ₈	67
34	0.665	0.653 ₅	1.970 ₉	0.572	0.0806	0.733 ₈	66
35	0.665	0.648 ₅	1.961 ₈	0.571	0.0771	0.725 ₈	65
36	0.666	0.643 ₄	1.953 ₉	0.570	0.0737	0.717 ₈	64
37	0.667	0.639 ₅	1.944 ₈	0.569	0.0705	0.709 ₈	63
38	0.667	0.634 ₄	1.936 ₈	0.568	0.0673	0.701 ₇	62
39	0.668	0.630 ₄	1.928 ₈	0.566	0.0643	0.694 ₇	61
0.040	0.669	0.626 ₄	1.920 ₇	0.565	0.0614	0.687 ₇	0.960

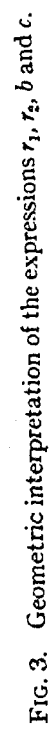
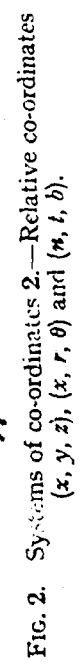
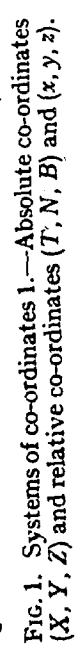
k'^2	G_1	G_2	G_3	H_1	H_2	H_3	k^2
0.040	0.669	0.626 ₄	1.920 ₇	0.565	0.0614	0.687 ₇	0.960
41	0.669	0.622 ₄	1.913 ₈	0.564	0.0586	0.680 ₇	59
42	0.670	0.618 ₄	1.905 ₈	0.562	0.0560	0.673 ₆	58
43	0.671	0.614 ₄	1.897 ₆	0.561	0.0534	0.667 ₇	57
44	0.671	0.610 ₄	1.891 ₇	0.559	0.0508	0.660 ₆	56
45	0.672	0.606 ₄	1.884 ₇	0.558	0.0483	0.654 ₆	55
46	0.673	0.602 ₃	1.877 ₇	0.557	0.0458	0.648 ₆	54
47	0.673	0.599 ₄	1.870 ₆	0.555	0.0434	0.642 ₆	53
48	0.674	0.595 ₃	1.864 ₆	0.554	0.0412	0.636 ₆	52
49	0.675	0.592 ₄	1.858 ₆	0.552	0.0390	0.630 ₅	51
0.050	0.675	0.588 ₃	1.852 ₆	0.551	0.0369	0.625 ₆	0.950
51	0.676	0.585 ₃	1.846 ₆	0.550	0.0349	0.619 ₅	49
52	0.676	0.582 ₄	1.840 ₆	0.549	0.0330	0.614 ₅	48
53	0.677	0.578 ₃	1.834 ₆	0.547	0.0311	0.609 ₅	47
54	0.678	0.575 ₃	1.828 ₆	0.546	0.0292	0.604 ₅	46
55	0.678	0.572 ₃	1.822 ₅	0.545	0.0274	0.599 ₅	45
56	0.679	0.569 ₃	1.817 ₅	0.544	0.0255	0.594 ₅	44
57	0.679	0.566 ₃	1.812 ₆	0.543	0.0236	0.589 ₄	43
58	0.680	0.563 ₃	1.806 ₅	0.541	0.0218	0.585 ₅	42
59	0.681	0.560 ₃	1.801 ₅	0.540	0.0201	0.580 ₄	41
0.060	0.681	0.557 ₂	1.796 ₅	0.539	0.0184	0.576 ₄	0.940
61	0.682	0.555 ₃	1.791 ₅	0.538	0.0168	0.572 ₄	39
62	0.682	0.552 ₃	1.786 ₅	0.537	0.0152	0.568 ₅	38
63	0.683	0.549 ₃	1.781 ₅	0.535	0.0136	0.563 ₄	37
64	0.684	0.546 ₂	1.776 ₄	0.534	0.0121	0.559 ₄	36
65	0.684	0.544 ₃	1.772 ₅	0.533	0.0107	0.555 ₄	35
66	0.685	0.541 ₂	1.767 ₄	0.532	0.0093	0.551 ₄	34
67	0.685	0.539 ₃	1.763 ₅	0.531	0.0079	0.547 ₄	33
68	0.686	0.536 ₂	1.758 ₅	0.529	0.0066	0.543 ₄	32
69	0.686	0.534 ₃	1.753 ₄	0.528	0.0052	0.539 ₄	31
0.070	0.687	0.531 ₂	1.749 ₄	0.527	0.0038	0.535 ₄	0.930

k^2	G_1	G_2	G_3	H_1	H_2	H_3	k^2
0.070	0.687	0.531 ₂	1.749 ₄	0.527	0.0038	0.535 ₄	0.930
71	0.688	0.529 ₃	1.745 ₄	0.526	0.0025	0.531 ₃	29
72	0.688	0.526 ₂	1.741 ₄	0.525	0.0012	0.528 ₄	28
73	0.689	0.524 ₂	1.737 ₅	0.524	0.0000	0.524 ₃	27
74	0.689	0.522 ₃	1.732 ₄	0.523	-0.0012	0.521 ₄	26
75	0.690	0.519 ₂	1.728 ₄	0.522	-0.0024	0.517 ₃	25
76	0.690	0.517 ₂	1.724 ₄	0.520	-0.0035	0.514 ₄	24
77	0.691	0.515 ₃	1.720 ₄	0.519	-0.0047	0.510 ₃	23
78	0.692	0.512 ₂	1.716 ₄	0.518	-0.0058	0.507 ₄	22
79	0.692	0.510 ₂	1.712 ₄	0.517	-0.0069	0.503 ₃	21
0.080	0.693	0.508 ₂	1.708 ₃	0.516	-0.0080	0.500 ₂	0.920
81	0.693	0.506 ₂	1.705 ₄	0.515	-0.0091	0.497 ₃	19
82	0.694	0.504 ₂	1.701 ₄	0.514	-0.0102	0.494 ₄	18
83	0.694	0.502 ₂	1.697 ₃	0.513	-0.0112	0.490 ₃	17
84	0.695	0.500 ₂	1.694 ₄	0.512	-0.0122	0.487 ₃	16
85	0.695	0.498 ₂	1.690 ₃	0.511	-0.0132	0.484 ₃	15
86	0.696	0.496 ₂	1.687 ₄	0.509	-0.0141	0.481 ₃	14
87	0.696	0.494 ₂	1.683 ₃	0.508	-0.0151	0.478 ₃	13
88	0.697	0.492 ₂	1.680 ₃	0.507	-0.0160	0.475 ₃	12
89	0.698	0.490 ₂	1.677 ₄	0.506	-0.0170	0.472 ₃	11
0.090	0.698	0.488 ₂	1.673 ₃	0.505	-0.0179	0.469 ₃	0.910
91	0.699	0.486 ₂	1.670 ₃	0.504	-0.0188	0.466 ₂	09
92	0.699	0.484 ₂	1.667 ₄	0.503	-0.0197	0.464 ₃	08
93	0.700	0.482 ₂	1.663 ₃	0.502	-0.0205	0.461 ₂	07
94	0.700	0.480 ₂	1.660 ₃	0.501	-0.0213	0.459 ₃	06
95	0.701	0.478 ₂	1.657 ₃	0.500	-0.0221	0.456 ₃	05
96	0.701	0.476 ₁	1.654 ₃	0.499	-0.0229	0.453 ₂	04
97	0.702	0.475 ₂	1.651 ₃	0.498	-0.0237	0.451 ₃	03
98	0.702	0.473 ₂	1.648 ₃	0.497	-0.0245	0.448 ₂	02
99	0.703	0.471 ₂	1.645 ₃	0.496	-0.0252	0.446 ₃	01
0.100	0.703	0.469	1.642	0.495	-0.0259	0.443	0.900

k^2	G_1	G_2	G_3	H_1	H_2	H_3	k^2
0.10	0.703 ₆	0.469 ₁₇	1.642 ₂₉	0.495 ₁₀	-0.0259 ₆₉	0.443 ₂₄	0.90
11	0.709 ₅	0.452 ₁₅	1.613 ₂₆	0.485 ₁₀	-0.0328 ₅₀	0.419 ₂₃	89
12	0.714 ₅	0.437 ₁₅	1.587 ₂₄	0.475 ₉	-0.0386 ₅₀	0.398 ₁₉	88
13	0.719 ₄	0.422 ₁₃	1.563 ₂₁	0.466 ₉	-0.0436 ₄₃	0.379 ₁₈	87
14	0.723 ₅	0.409 ₁₂	1.542 ₂₀	0.457 ₉	-0.0479 ₃₆	0.361 ₁₆	86
15	0.728 ₅	0.397 ₁₂	1.522 ₂₀	0.448 ₉	-0.0515 ₃₂	0.345 ₁₅	85
16	0.733 ₄	0.385 ₁₁	1.502 ₁₇	0.439 ₈	-0.0547 ₂₈	0.330 ₁₄	84
17	0.737 ₅	0.374 ₁₁	1.485 ₁₇	0.431 ₈	-0.0575 ₂₃	0.316 ₁₃	83
18	0.742 ₄	0.363 ₁₀	1.468 ₁₆	0.423 ₈	-0.0598 ₂₀	0.303 ₁₂	82
19	0.746 ₄	0.353 ₁₀	1.452 ₁₅	0.415 ₈	-0.0618 ₁₇	0.291 ₁₁	81
0.20	0.750 ₅	0.343 ₉	1.437 ₁₄	0.407 ₈	-0.0635 ₁₅	0.280 ₁₁	0.80
21	0.755 ₄	0.334 ₉	1.423 ₁₃	0.399 ₇	-0.0650 ₁₂	0.269 ₁₀	79
22	0.759 ₄	0.325 ₈	1.410 ₁₃	0.392 ₈	-0.0662 ₁₀	0.259 ₉	78
23	0.763 ₄	0.317 ₈	1.397 ₁₃	0.384 ₇	-0.0672 ₉	0.250 ₉	77
24	0.767 ₄	0.309 ₈	1.384 ₁₁	0.377 ₇	-0.0681 ₇	0.241 ₉	76
25	0.771 ₄	0.301 ₈	1.373 ₁₁	0.370 ₇	-0.0688 ₅	0.232 ₈	75
26	0.775 ₄	0.293 ₇	1.362 ₁₁	0.363 ₇	-0.0693 ₄	0.224 ₈	74
27	0.779 ₄	0.286 ₇	1.351 ₁₀	0.356 ₇	-0.0697 ₃	0.216 ₇	73
28	0.783 ₄	0.279 ₇	1.341 ₁₀	0.349 ₇	-0.0700 ₁	0.209 ₇	72
29	0.787 ₄	0.272 ₇	1.331 ₁₀	0.342 ₆	-0.0701 ₁	0.202 ₇	71
0.30	0.791 ₃	0.265 ₆	1.321 ₉	0.336 ₇	-0.0702 ₀	0.195 ₆	0.70
31	0.794 ₄	0.259 ₆	1.312 ₉	0.329 ₆	-0.0712 ₂	0.189 ₆	69
32	0.798 ₄	0.253 ₆	1.303 ₈	0.323 ₇	-0.0700 ₂	0.183 ₆	68
33	0.802 ₃	0.247 ₆	1.295 ₉	0.316 ₆	-0.0698 ₃	0.177 ₆	67
34	0.805 ₄	0.241 ₆	1.286 ₈	0.310 ₆	-0.0695 ₃	0.171 ₆	66
35	0.809 ₄	0.235 ₆	1.278 ₈	0.304 ₆	-0.0692 ₄	0.165 ₅	65
36	0.813 ₃	0.229 ₆	1.270 ₇	0.298 ₆	-0.0688 ₅	0.160 ₅	64
37	0.816 ₄	0.223 ₅	1.263 ₇	0.292 ₆	-0.0683 ₅	0.155 ₅	63
38	0.820 ₃	0.218 ₅	1.256 ₈	0.286 ₆	-0.0678 ₆	0.150 ₄	62
39	0.823 ₄	0.213 ₆	1.248 ₇	0.280 ₆	-0.0672 ₆	0.146 ₅	61
0.40	0.827 ₂	0.207 ₅	1.241 ₇	0.274 ₆	-0.0666 ₆	0.141 ₄	0.60

k^2	G_1	G_2	G_3	H_1	H_2	H_3	k^2
0.40	0.827 ₃	0.207 ₅	1.241 ₇	0.274 ₆	-0.0666 ₆	0.141 ₄	0.60
41	0.830 ₃	0.202 ₅	1.234 ₆	0.268 ₆	-0.0660 ₇	0.136 ₄	59
42	0.833 ₄	0.197 ₅	1.228 ₇	0.262 ₅	-0.0653 ₇	0.132 ₄	58
43	0.837 ₃	0.192 ₅	1.221 ₆	0.257 ₆	-0.0646 ₈	0.128 ₄	57
44	0.840 ₄	0.187 ₄	1.215 ₆	0.251 ₅	-0.0638 ₈	0.124 ₄	56
45	0.844 ₃	0.183 ₅	1.209 ₆	0.246 ₆	-0.0630 ₈	0.120 ₄	55
46	0.847 ₃	0.178 ₄	1.203 ₆	0.240 ₅	-0.0622 ₈	0.116 ₄	54
47	0.850 ₃	0.174 ₅	1.197 ₆	0.235 ₆	-0.0614 ₉	0.112 ₃	53
48	0.853 ₃	0.169 ₄	1.191 ₅	0.229 ₅	-0.0605 ₉	0.109 ₄	52
49	0.857 ₃	0.165 ₅	1.186 ₆	0.224 ₅	-0.0596 ₁₀	0.105 ₃	51
0.50	0.860	0.160	1.180	0.219	-0.0586	0.102	0.50
0.50	0.860 ₃	0.160 ₄	1.180 ₅	0.219 ₅	-0.0586 ₁₀	0.1017 ₃₄	0.50
51	0.863 ₃	0.156 ₄	1.175 ₅	0.214 ₆	-0.0576 ₉	0.0983 ₃₂	49
52	0.866 ₃	0.152 ₄	1.170 ₅	0.208 ₅	-0.0567 ₁₀	0.0951 ₃₁	48
53	0.869 ₄	0.148 ₄	1.165 ₅	0.203 ₅	-0.0557 ₁₀	0.0920 ₃₁	47
54	0.873 ₃	0.144 ₄	1.160 ₅	0.198 ₅	-0.0547 ₁₁	0.0889 ₃₀	46
55	0.876 ₃	0.140 ₄	1.155 ₅	0.193 ₅	-0.0536 ₁₀	0.0859 ₂₉	45
56	0.879 ₃	0.136 ₄	1.150 ₅	0.188 ₅	-0.0526 ₁₀	0.0830 ₂₈	44
57	0.882 ₃	0.132 ₄	1.145 ₄	0.183 ₅	-0.0516 ₁₁	0.0802 ₂₈	43
58	0.885 ₃	0.128 ₄	1.141 ₅	0.178 ₄	-0.0505 ₁₀	0.0774 ₂₇	42
59	0.888 ₃	0.124 ₄	1.136 ₄	0.174 ₅	-0.0495 ₁₁	0.0747 ₂₇	41
0.60	0.891 ₃	0.120 ₃	1.132 ₅	0.169 ₅	-0.0484 ₁₁	0.0720 ₂₆	0.40
61	0.894 ₃	0.117 ₄	1.127 ₄	0.164 ₅	-0.0473 ₁₂	0.0694 ₂₅	39
62	0.897 ₃	0.113 ₃	1.123 ₄	0.159 ₅	-0.0461 ₁₁	0.0669 ₂₅	38
63	0.900 ₃	0.110 ₄	1.119 ₄	0.154 ₄	-0.0450 ₁₁	0.0644 ₂₄	37
64	0.903 ₃	0.106 ₄	1.115 ₄	0.150 ₅	-0.0439 ₁₁	0.0620 ₂₃	36
65	0.906 ₃	0.1023 ₃₄	1.111 ₄	0.145 ₅	-0.0428 ₁₂	0.0597 ₂₄	35
66	0.909 ₃	0.0989 ₃₃	1.107 ₄	0.140 ₅	-0.0416 ₁₁	0.0573 ₂₂	34
67	0.912 ₂	0.0956 ₃₄	1.103 ₄	0.135 ₄	-0.0405 ₁₂	0.0551 ₂₂	33
68	0.914 ₃	0.0922 ₃₄	1.099 ₄	0.131 ₄	-0.0393 ₁₂	0.0529 ₂₂	32
69	0.917 ₃	0.0888 ₃₃	1.095 ₄	0.127 ₅	-0.0381 ₁₂	0.0507 ₂₁	31
0.70	0.920 ₃	0.0855 ₃₂	1.091 ₃	0.122 ₄	-0.0369 ₁₂	0.0486 ₂₁	0.30

k^2	G_1	G_2	G_3	H_1	H_2	H_3	k^2
0.70	0.920 ₃	0.0855 ₃₂	1.091 ₃	0.122 ₄	-0.0369 ₁₂	0.0486 ₂₁	0.30
71	0.923 ₃	0.0823 ₃₃	1.088 ₄	0.118 ₄	-0.0357 ₁₂	0.0465 ₂₁	29
72	0.926 ₃	0.0790 ₃₂	1.084 ₄	0.114 ₅	-0.0345 ₁₂	0.0444 ₂₀	28
73	0.929 ₂	0.0758 ₃₂	1.080 ₃	0.109 ₄	-0.0333 ₁₂	0.0424 ₂₀	27
74	0.931 ₃	0.0726 ₃₁	1.077 ₄	0.105 ₅	-0.0321 ₁₂	0.0404 ₁₉	26
75	0.934 ₃	0.0695 ₃₀	1.073 ₃	0.1004 ₄₂	-0.0309 ₁₂	0.0385 ₁₈	25
76	0.937 ₃	0.0665 ₃₁	1.070 ₃	0.0962 ₄₃	-0.0297 ₁₂	0.0367 ₁₉	24
77	0.940 ₃	0.0634 ₃₀	1.067 ₄	0.0919 ₄₃	-0.0285 ₁₂	0.0348 ₁₈	23
78	0.943 ₂	0.0604 ₃₀	1.063 ₃	0.0876 ₄₂	-0.0273 ₁₂	0.0330 ₁₈	22
79	0.945 ₃	0.0574 ₃₀	1.060 ₃	0.0834 ₄₂	-0.0261 ₁₂	0.0312 ₁₇	21
0.80	0.948 ₃	0.0544 ₃₀	1.057 ₃	0.0792 ₄₂	-0.0249 ₁₂	0.0295 ₁₈	0.20
81	0.951 ₂	0.0514 ₂₉	1.054 ₄	0.0750 ₄₁	-0.0237 ₁₃	0.0277 ₁₇	19
82	0.953 ₃	0.0485 ₂₉	1.050 ₃	0.0709 ₄₁	-0.0224 ₁₂	0.0260 ₁₆	18
83	0.956 ₃	0.0456 ₃₀	1.047 ₃	0.0668 ₄₂	-0.0212 ₁₃	0.0244 ₁₇	17
84	0.959 ₂	0.0426 ₂₈	1.044 ₃	0.0626 ₄₁	-0.0199 ₁₂	0.0227 ₁₆	16
85	0.961 ₃	0.0398 ₂₈	1.041 ₃	0.0585 ₄₀	-0.0187 ₁₂	0.0211 ₁₅	15
86	0.964 ₃	0.0370 ₂₈	1.038 ₃	0.0545 ₄₁	-0.0175 ₁₃	0.0196 ₁₆	14
87	0.967 ₂	0.0342 ₂₇	1.035 ₃	0.0504 ₄₀	-0.0162 ₁₂	0.0180 ₁₅	13
88	0.969 ₃	0.0315 ₂₈	1.032 ₃	0.0464 ₃₉	-0.0150 ₁₃	0.0165 ₁₅	12
89	0.972 ₃	0.0287 ₂₇	1.029 ₂	0.0425 ₄₀	-0.0137 ₁₂	0.0150 ₁₅	11
0.90	0.975 ₂	0.0260 ₂₇	1.027 ₃	0.0385 ₃₉	-0.0125 ₁₃	0.0135 ₁₄	0.10
91	0.977 ₃	0.0233 ₂₇	1.024 ₃	0.0346 ₄₀	-0.01124 ₁₂₅	0.0121 ₁₄	09
92	0.980 ₂	0.0206 ₂₇	1.021 ₃	0.0306 ₃₉	-0.00999 ₁₂₄	0.0106 ₁₄	08
93	0.982 ₃	0.0179 ₂₆	1.018 ₃	0.0267 ₃₈	-0.00875 ₁₂₅	0.0092 ₁₃	07
94	0.985 ₂	0.0153 ₂₅	1.015 ₂	0.0229 ₃₉	-0.00750 ₁₂₅	0.0079 ₁₄	06
95	0.987 ₃	0.0128 ₂₆	1.013 ₃	0.0190 ₃₈	-0.00625 ₁₂₅	0.0065 ₁₃	05
96	0.990 ₂	0.0102 ₂₆	1.010 ₂	0.0152 ₃₈	-0.00500 ₁₂₅	0.0052 ₁₄	04
97	0.992 ₃	0.0076 ₂₅	1.008 ₃	0.0114 ₃₉	-0.00375 ₁₂₅	0.0038 ₁₃	03
98	0.995 ₂	0.0051 ₂₆	1.005 ₂	0.0075 ₃₇	-0.00250 ₁₂₅	0.0025 ₁₂	02
99	0.997 ₃	0.0025 ₂₅	1.003 ₃	0.0038 ₃₈	-0.00125 ₁₂₅	0.0013 ₁₃	01
1.00	1.000	0.0000	1.000	0	0	0.0000	0.00



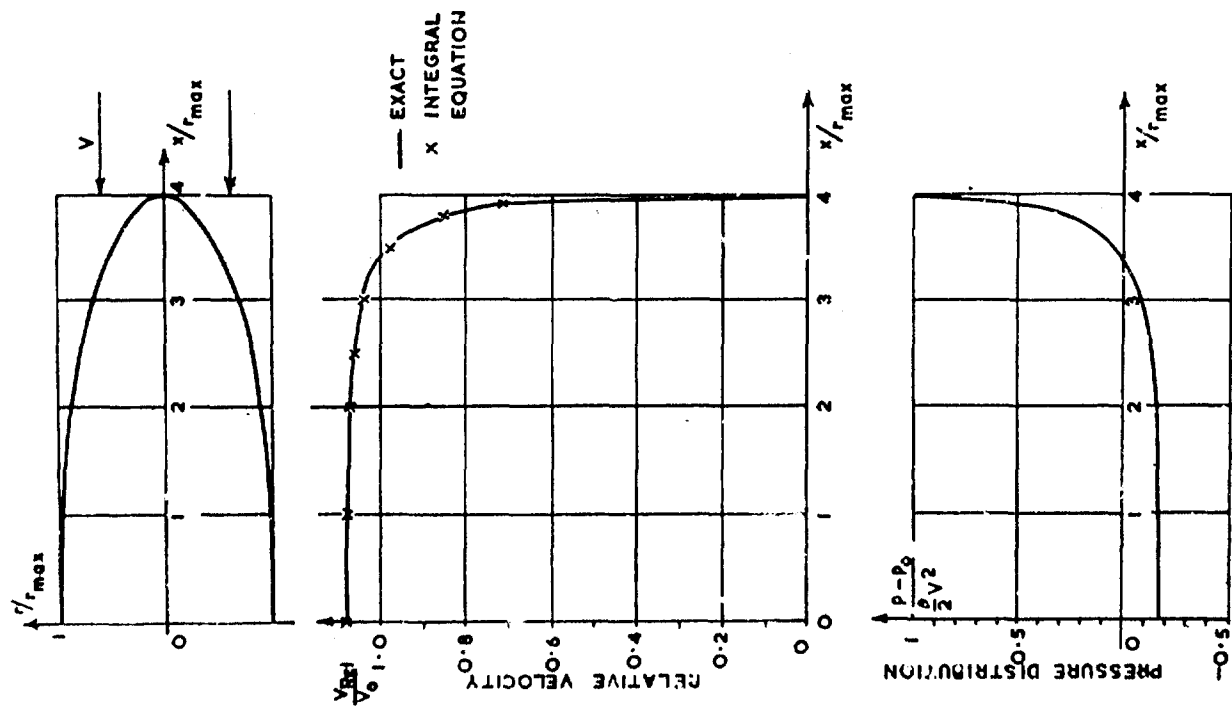


Fig. 4. Ellipsoid 1 : 4.—Longitudinal translation.

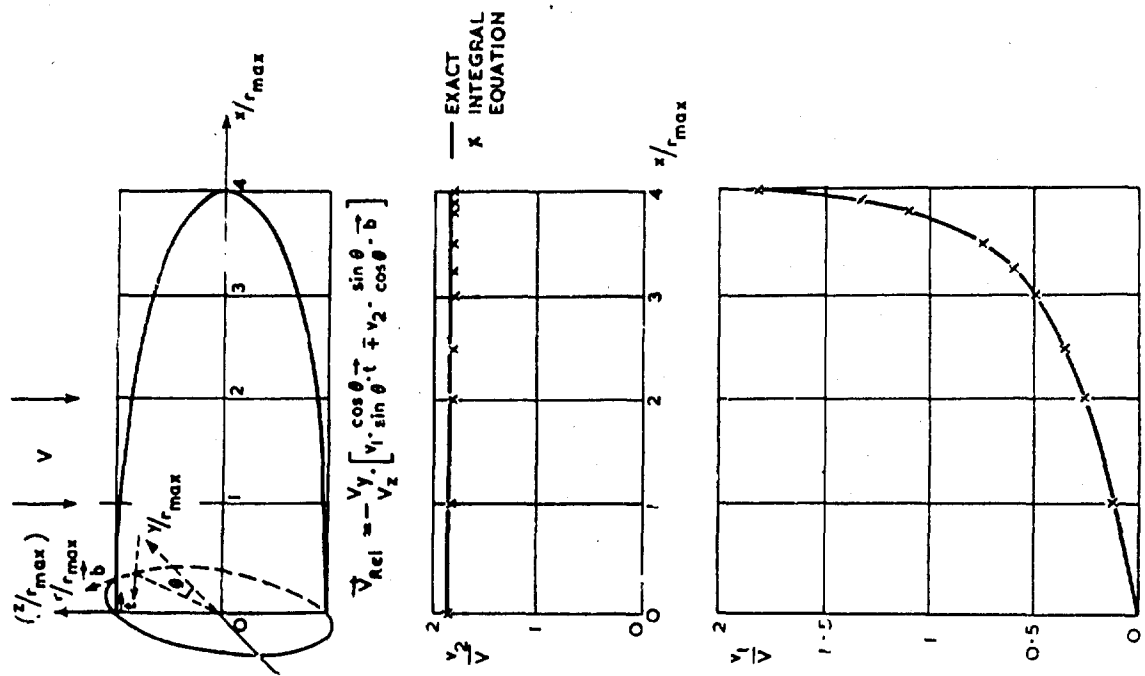
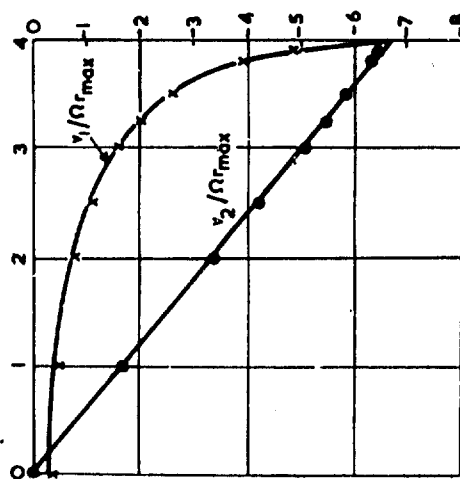
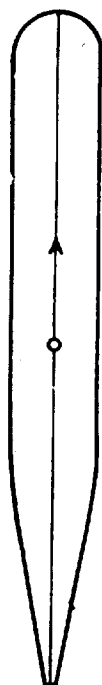


Fig. 5. Ellipsoid 1 : 4.—Lateral translation.



$\frac{x}{r_{\max}}$	$\frac{v_1}{\Omega r_{\max}}$	EXACT	$\frac{v_2}{\Omega r_{\max}}$	EXACT
4	-4.83	(-6.75)	-6.46	(-6.75)
3.9	-3.94	(-4.92)	-6.35	(-6.58)
3.8	-2.56	(-3.97)	-5.90	(-6.42)
3.5	-1.97	(-2.57)	-5.45	(-5.91)
3.25	-1.57	(-1.98)	-5.02	(-5.49)
3	-1.07	(-1.57)	-4.20	(-5.07)
2.5	-0.76	(-1.07)	-3.36	(-4.22)
2	-0.43	(-0.76)	-1.66	(-3.38)
1	-0.34	(-0.41)	0	(-1.69)
0		(-0.31)		(0)

FIG. 6. Ellipsoid 1: 4.—Relative velocity in the case of rotation about the axis of y or z.

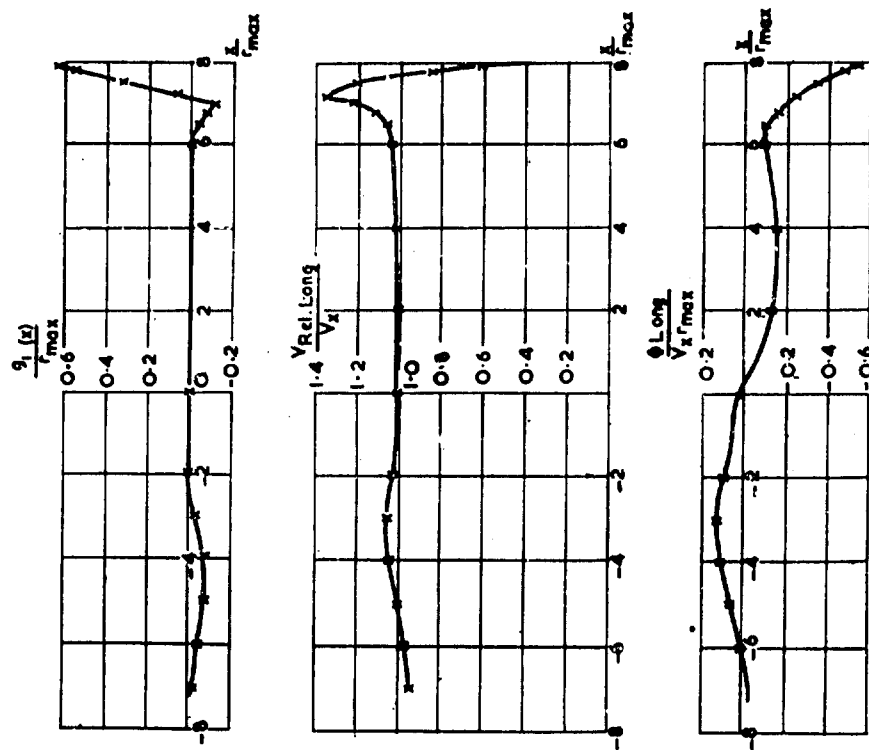


FIG. 7. Torpedo in longitudinal translation.—Source distribution, relative velocity, and potential.

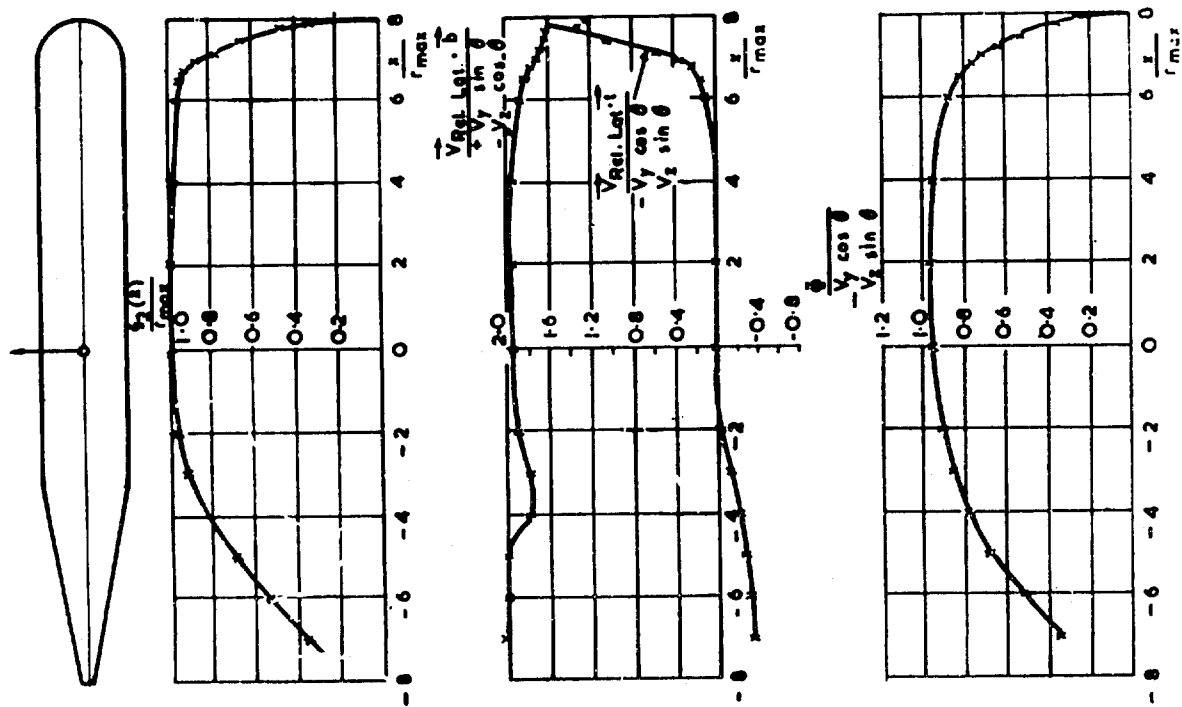


FIG. 8. Torpedo in lateral translation.—Source distribution, relative velocity, and potential.

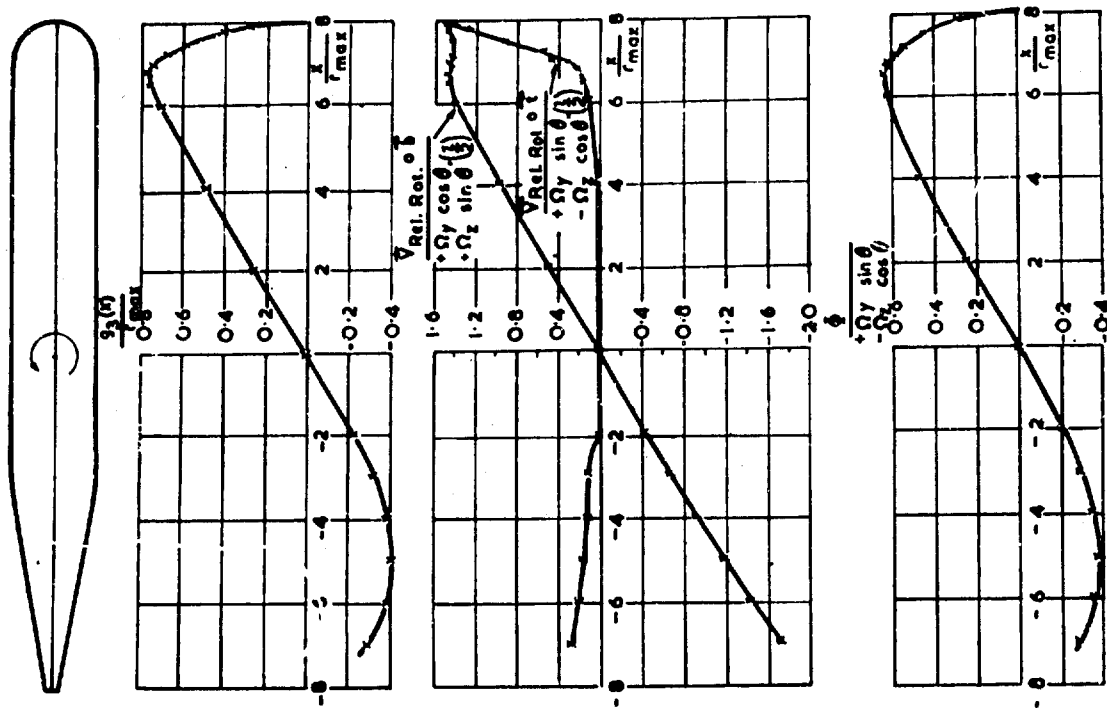


FIG. 9. Torpedo in rotation.—Source distribution, relative velocity and potential.