THE APPLICATION OF THE FINITE ELEMENT TECHNIQUE TO POTENTIAL FLOW PROBLEMS:

PART I

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NOMENCLATURE

$\frac{d}{dt}$	=	denotes material differentiation
D	=	the domain under consideration
g	=	prescribed boundary condition of ϕ
$\hat{n}$	=	unit normal vector
n_x, n_y	=	direction cosines of $\hat{n}$
$\bar{q}$	=	fluid velocity
$\hat{s}$	=	unit tangential vector
u,v	=	components of q in x,y directions
x_j, y_j	=	Dirichlet integral of the function ϕ
$X(\phi)$	=	Dirichlet integral of the function ϕ
Γ	=	boundary of D
Γ_1	=	part of Γ, on which ϕ is prescribed
Γ_2	=	remaining part of Γ, on which $\frac{d\phi}{dn} = 0$ is prescribed
Δ	=	area of an arbitrary triangle
Δ_5	=	area of triangle number 5
ε,h	=	an arbitrary constant and function respectively, used in the variational procedure
$\bar{\zeta}$	=	vorticity vector
ρ	=	density
ϕ	=	velocity potential
ψ	=	stream function
∇	=	"Del" or gradient operator

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ABSTRACT

In this report, the finite element method is applied to field problems governed by Laplace's equation, and in particular, to potential flow in fluid mechanics. The conditions under which the variational method may be used are examined for Dirichlet, Neumann and mixed boundary conditions, and for both singly- and multiply-connected regions. The discretisation of the field, using finite elements of triangular form is developed, and the resulting equations are solved. A computer program based on this analysis has been developed and is fully described in a subsequent report. This program will solve a two-dimensional potential field for simple or mixed boundary conditions and for singly- or multiply-connected regions. It may be used for multiple-body flow fields, such as aerofoil cascades, with boundary constraints such as the Kutta condition.

INTRODUCTION

Analytical closed-form solutions to field problems involving Laplace's equation are presented in standard texts (such as Ref. 1,2), for simple geometries and boundary conditions. For more complex shapes and boundary conditions such as are found in the aerodynamics of internal and external flows, analytic methods which generally yield a numerical solution have been derived. A summary of the more important of these is given in Ref. 3. For more difficult cases, a numerical method is usually resorted to.

A widely-used numerical method, used for example in the solution of the harmonic and bi-harmonic equations, subject to certain boundary conditions, is the finite difference method. A drawback to this method is that if the boundaries are irregular, in the sense that the points on the boundary do not coincide with the prescribed network, the method becomes more difficult to apply. This difficulty can be overcome with the finite element method, which was recently developed for structural and solid mechanics (Ref. 4). Irregardless of the shape of the boundary the elements may be fitted to this shape without complicating the method any further. Although this technique has been used extensively in structural problems, it has not been widely adopted in fluid mechanics.

The application of the finite element method to the solution of field problems, involving Poison's and Laplace's equations, was proposed by Zienkiewicz and Cheung in 1965, (Ref. 5), though few applications of this method have been made since.

In this report, the solution of Laplace's equation, under various boundary conditions, is investigated. The procedure used is essentially that proposed by Zienkiewicz and Cheung. Recently a similar method has been developed independently by Silvester and Jaeger (Ref. 6) for the solution of Helmholtz's equation. The original contribution of the present work consists of (a) the detailed development of the method for the mixed boundary conditions often encountered in potential flow, such as the Kutta condition (b) the derivation of a method for obtaining the stream function ψ separately from the velocity potential ϕ (c) the verification of the finite element application to multiply-connected regions (d) the development of a general-purpose computer program to economically solve two-dimensional Laplace field problems.

This report is primarily devoted to analysis. A subsequent report (Ref. 7) considers the computation. The scope is restricted to two-dimensional fields. The extension of the method to three-dimensions is currently being investigated.

2. BASIC EQUATIONS

The quasi-static flow of an incompressible, inviscid, fluid is considered. It is furthermore assumed that the flow is two-dimensional.

Since the fluid is incompressible

$$\frac{d\rho}{dt} = 0. \qquad (2.1)$$

Hence it follows from the equation of continuity,

$$\frac{d\rho}{dt} + \rho \nabla \cdot \overline{q} = 0, \tag{2.2}$$

that

$$\nabla \cdot \overline{q} = 0. \tag{2.3}$$

In cartesian coordinates, this gives for two-dimensional flow

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{2.4}$$

The fluid is assumed to be inviscid, and hence the motion is irrotational, consequently

$$\overline{q} = -\nabla \phi. \tag{2.5}$$

Substitution of equation (2.5) into equation (2.3) results in

$$\nabla^2\phi = 0, \tag{2.6}$$

which in cartesian coordinates becomes

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0, \tag{2.7}$$

which is Laplace's equation in two dimensions.

From equation (2.4) it is clear that with the substitutions,

$$u = -\frac{\partial\psi}{\partial y},$$

and

$$v = \frac{\partial\psi}{\partial x}, \tag{2.8}$$

the equation of continuity is satisfied, and hence that a function ψ exists and is related to velocity through equations (2.8).

The motion is assumed irrotational, hence the vorticity vector,

$$\bar{\zeta} = \nabla \times \bar{q}, \tag{2.9}$$

vanishes, and consequently

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0. \tag{2.10}$$

Substitution of equations (2.8) into equation (2.10) results in

$$\nabla^2\psi = 0, \tag{2.11}$$

which in cartesian coordinates becomes

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0. \tag{2.12}$$

From

$$\bar{q} = u\,\hat{i} + v\,\hat{j}, \tag{2.13}$$

and equation (2.5), there follows that

$$u = -\frac{\partial \phi}{\partial x},$$

and

$$v = -\frac{\partial \phi}{\partial y}. \tag{2.14}$$

Comparing equations (2.14) with equations (2.8), results in

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y},$$

and

$$\frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}, \tag{2.15}$$

which may be recognized as the Cauchy-Riemann conditions.

The formulation of the flow problem may either be in terms of the velocity potential, ϕ, that is, subject to

$$\nabla^2\phi = 0, \tag{2.16}$$

where

$$u = - \frac{\partial \phi}{\partial x}, \text{ and } v = - \frac{\partial \phi}{\partial y}, \tag{2.17}$$

or in terms of the stream function, ψ, subject to

$$\nabla^2 \psi = 0, \tag{2.18}$$

where

$$u = - \frac{\partial \psi}{\partial y}, \text{ and } v = \frac{\partial \psi}{\partial x} . \tag{2.19}$$

3. BOUNDARY CONDITIONS

When a fluid is in contact with a solid wall, then the fluid and the wall must have the same velocity normal to the wall, if the fluid does not penetrate the wall. Consequently, if the walls are at rest, then the component of the velocity normal to the wall must be zero. Hence,

$$\bar{q} \cdot \hat{n} = 0. \tag{3.1}$$

Substitution of equation (2.5) into equation (3.1) results in

$$\frac{\partial \phi}{\partial x} n_x + \frac{\partial \phi}{\partial y} n_y = 0. \tag{3.2}$$

Similarly in terms of the stream function, ψ, there is obtained from equations (2.8), (2.13), and (3.1)

$$\frac{\partial \psi}{\partial y} n_x - \frac{\partial \psi}{\partial x} n_y = 0. \tag{3.3}$$

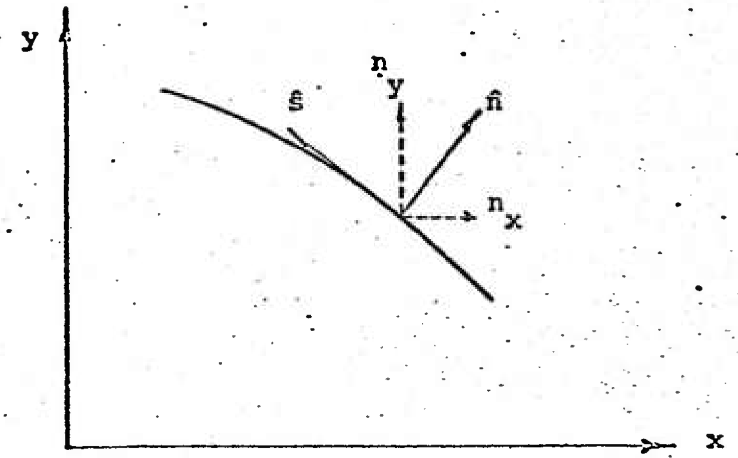

Fig. 3.1 Direction Cosines of n̂

From Fig. 3.1, it is clear that

$$n_x = \frac{\partial x}{\partial n} = \frac{\partial y}{\partial s} ,$$

and (3.4)

$$n_y = \frac{\partial y}{\partial n} = - \frac{\partial x}{\partial s} .$$

Consequently equations (3.2) and (3.3) reduce to

$$\frac{d\phi}{dn} = \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial n} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial n} = 0, \tag{3.5}$$

and

$$\frac{d\psi}{ds} = \frac{\partial \psi}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial \psi}{\partial y}\frac{\partial y}{\partial s} = 0, \tag{3.6}$$

as equivalent statements of the boundary condition (3.1).
Furthermore, it is noted that by expanding both sides of the following, through the chain rule, and substituting equations (2.15) and (3.4), that,

$$\frac{d\phi}{dn} = \frac{d\psi}{ds} , \tag{3.7}$$

and

$$\frac{d\psi}{dn} = - \frac{d\phi}{ds} \tag{3.8}$$

4. STATEMENT OF PROBLEM

The flow problem being considered reduces therefore to the solution of

$$\nabla^2\phi = 0, \tag{4.1}$$

subject to

$$\frac{d\phi}{dn} = 0 \tag{4.2}$$

along solid walls.

Or in terms of the stream function, ψ, the solution of

$$\nabla^2\psi = 0, \tag{4.3}$$

subject to

$$\frac{d\psi}{ds} = 0 \tag{4.4}$$

along solid walls.

5. THE VARIATIONAL BASIS OF THE METHOD

5.1 The Dirichlet Principle

The technique which is developed in this report is founded on the Dirichlet Principle (Ref. 8). This theorem, which may be stated as follows, converts a Laplace field problem into a problem of the calculus of variations.

"Consider a domain D with a function g defined on its boundary Γ:

For each sufficiently differentiable function u defined in D + Γ, which assumes the prescribed values on the boundary, define the number

$$X(u) = \iint_D \tfrac{1}{2}(u_x^2 + u_y^2)\,dxdy \qquad (5.1)$$

where $u_x = \frac{\partial u}{\partial x}$, and $u_y = \frac{\partial u}{\partial y}$. This quantity, the Dirichlet integral of the function u, attains a lower bound which is non-negative. Let v be the function for which this lower bound occurs, (i.e., for which X (u) is minimized). Then it can be shown that v satisfies Laplace's equation throughout D.

Moreover, it can be shown that there is only one function for which X (u) is a minimum i.e., that v is unique."

For our purposes, this theorem will be restated as follows;

"The solution to

$$\nabla^2 u = 0 \quad \text{over, } D, \qquad (5.2)$$

and

$$u = g \quad \text{over, } \Gamma,$$

is that function which minimizes the Dirichlet integral over D."

This variational principle may be applied directly to the flow problem, if it is formulated in terms of the stream function, ψ, or in terms of the potential function, φ, provided that ψ or φ is known at the boundaries. This is not always the case. For example, in the flow past an object, the value of ψ is determined up to an arbitrary constant at the surface, hence prescribing the exact boundary condition cannot be accomplished. However, from the section dealing with boundary conditions, it is clear that

$$\frac{d\phi}{dn} = 0 \qquad (5.3)$$

along the surface of the object. It would seem reasonable, therefore, to first formulate the problem in terms of the potential function, ϕ.

5.2 The Variational Solution

Although the problem can be solved directly by application of the Dirichlet Principle, it is more instructive to derive the variational solution from first principles as in the following. The constraints or the boundary conditions are then clearly obtained.

Consider the Dirichlet Integral (5.1). Since it is the sum of squares, it must always be non-negative. Thus, in general, it will have a minimum (positive) value. Let the function u for which this minimum is obtained be denoted by ϕ.

Consider now any function h(x,y) and any arbitrary constant ε. Formulating the function

$$u = \phi + \varepsilon h, \tag{5.4}$$

and substituting into equation (5.1) gives

$$X(\varepsilon) = \iint_D \tfrac{1}{2}\{(\phi_x + \varepsilon h_x)^2 + (\phi_y + \varepsilon h_y)^2\}\, dxdy \tag{5.5}$$

As ε varies in equation (5.5), so X takes on various corresponding values. The necessary condition for X to have a stationary value with respect to ε is

$$\frac{\partial X}{\partial \varepsilon} = 0 . \tag{5.6}$$

Evaluating equation (5.6) and then substituting in the condition $\varepsilon = 0$

into equations (5.4) and (5.6), gives respectively

$$u = \phi \,, \tag{5.7}$$

and

$$\left(\frac{\partial X}{\partial \varepsilon}\right)_{\varepsilon=0} = \iint_D (\phi_x h_x + \phi_y h_y)\, dxdy = 0 \,. \tag{5.8}$$

But, from the definition of ϕ, the condition that $\varepsilon=0$, is the condition that also minimizes X in equation (5.5). In this case, then, equation (5.8) which previously was the condition for a stationary value of X at $\varepsilon=0$, is now the condition for a stationary and minimum value of X.

In summary, if ϕ is the function minimizing the Dirichlet Integral then ϕ is the solution of

$$\iint_D (\phi_x h_x + \phi_y h_y)\, dxdy = 0 \,, \tag{5.9}$$

where h(x,y) is an arbitrary function. The application of this result to the problem at hand, incorporating the boundary conditions, now follows.

Using Green's identity :

$$\iint_D (\phi_x h_x + \phi_y h_y) dxdy \equiv - \iint_D h\nabla^2\phi dxdy + \int_\Gamma h \frac{d\phi}{dn}\, ds \,, \tag{5.10}$$

there obtains as an alternate statement of equation (5.9)

$$\iint_D h\nabla^2\phi dxdy - \int_\Gamma h \frac{d\phi}{dn}\, ds = 0 \,. \tag{5.11}$$

Since h is arbitrary, it may be chosen to be zero on Γ but otherwise arbitrary in D. This choice leads to

$$\int_{\Gamma} h \frac{d\phi}{dn} ds = 0 , \tag{5.12}$$

and

$$\iint_{D} h \nabla^2\phi dxdy = 0 , \tag{5.13}$$

hence

$$\nabla^2\phi = 0 \text{ over } D \tag{5.14}$$

If, however, h is chosen non-zero it follows from equation (5.12) that $\frac{d\phi}{dn} = 0$ on Γ. Alternatively, if $\frac{d\phi}{dn}$ is prescribed as being zero on the boundary, then equations (5.12), (5.13) and (5.14) hold with h still being arbitrary on the boundary.

If on part of the boundary, Γ_1, ϕ is prescribed as $\phi=g$, and $\frac{d\phi}{dn} \neq 0$, and on the remaining part, Γ_2, ϕ is not prescribed but $\frac{d\phi}{dn} = 0$, then for $\Gamma = \Gamma_1 + \Gamma_2$, there obtains the results (5.12), (5.13) and (5.14) as before, by choosing $h = 0$ on Γ_1. The form of equation (5.11) is then

$$\iint_{D} h \nabla^2\phi dxdy - \int_{\Gamma_2} h \frac{d\phi}{dn} ds = 0 , \tag{5.15}$$

from which (5.12), (5.13) and (5.14) follow.

6. THE FINITE ELEMENT RELATIONS

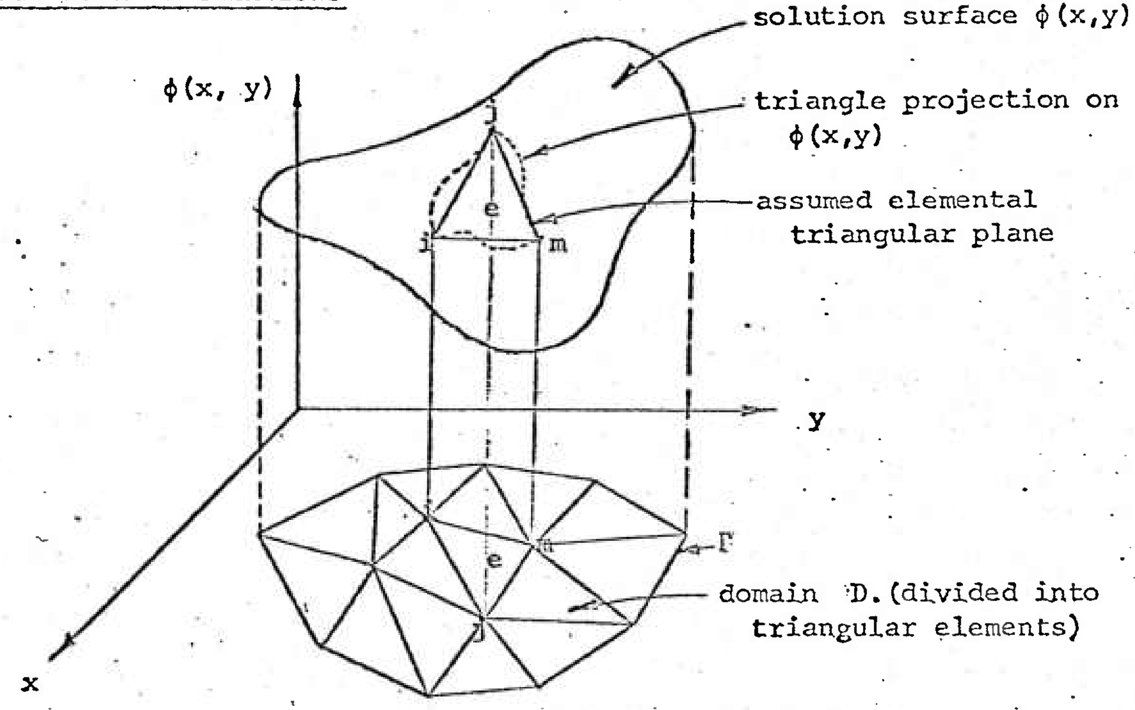

Fig. 6.1 Domain D and Its Solution Surface $\phi(x, y)$

The solution to the problem being considered is that of finding the function $\phi = \phi(x,y)$ satisfying Laplace's equation throughout the domain D, where ϕ is prescribed on Γ_1 and $\frac{d\phi}{dn} = 0$ on Γ_2.

The function $\phi = \phi(x,y)$ represents a smooth surface. As an approximation, this surface is assumed to be made up of elemental triangular planes as shown in Fig. 6.1. Let there be n elements and m nodal points. At the boundaries, it is assumed that each nodal point is a *pair-point* consisting of one point on Γ and one point in D immediately adjacent. Since no discontinuities are assumed in D + Γ, the ϕ value is the same for each point of a pair.

Consider a typical triangular element e as in Fig. 6.2. The nodal points are numbered i, j and m in an anti-clockwise manner.

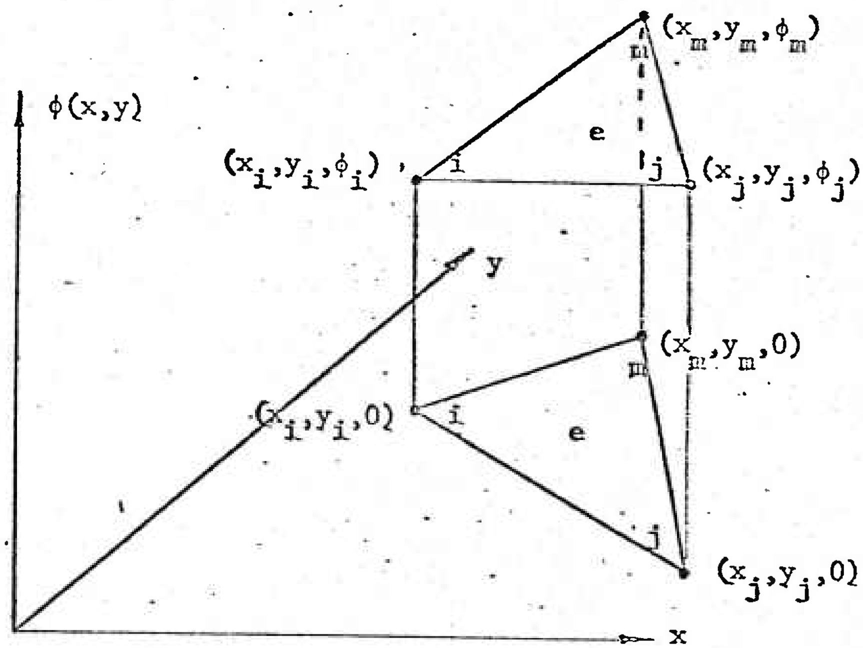

Fig. 6.2 A Typical Triangular Element e.

Under the previous assumption, the equation of the function ϕ becomes for the element e the equation of a plane, that is

$$\phi = \alpha_1 + \alpha_2 x + \alpha_3 y , \tag{6.1}$$

where α_1, α_2 and α_3 are constants to be determined by the nodal values of ϕ, and hence

$$\alpha_k = f_k \; (\phi_i, \; \phi_j, \; \phi_m), \text{ where } k = 1,2, \text{ or } 3 . \tag{6.2}$$

From equation (6.1),

$$\phi_i = \alpha_1 + \alpha_2 x_i + \alpha_3 y_i ,$$

$$\phi_j = \alpha_1 + \alpha_2 x_j + \alpha_3 y_j , \qquad (6.3)$$

$$\phi_m = \alpha_1 + \alpha_2 x_m + \alpha_3 y_m .$$

This system of equations (6.3) yields a unique solution for the coefficients α_1, α_2, and α_3, provided that the coefficient matrix

$$2\Delta = \begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_m & y_m \end{vmatrix} \neq 0. \qquad (6.4)$$

It can simply be shown by trigonometry that twice the area Δ of the projected triangle is equal to the above determinant, as indicated. Since the area of the triangle is always finite, $\Delta \neq 0$, and equation (6.4) is proved. With the introduction of the notation

$$a_i = x_j y_m - x_m y_j$$

$$b_i = y_j - y_m \qquad (6.5)$$

$$c_i = x_m - x_j$$

and solving for the coefficients α_1, α_2, and α_3, there obtains

$$\phi = \frac{1}{2\Delta}\{(a_i + b_i x + c_i y)\phi_i + (a_j + b_j x + c_j y)\phi_j + (a_m + b_m x + b_m y)\phi_m\} \tag{6.6}$$

where b_j, c_j, b_m, and c_m are obtained by cyclic permutation of the indices in relations (6.5). Equation (6.6) may be written as the matrix equation

$$\phi = \frac{1}{2\Delta}\{A\Phi + B\Phi x + C\Phi y\} , \tag{6.7}$$

where the matrices A, B, C, Φ are given by

$$\begin{aligned} A &= (a_i \;\; a_j \;\; a_m) \\ B &= (b_i \;\; b_j \;\; b_m) \\ C &= (c_i \;\; c_j \;\; c_m) \end{aligned} \tag{6.8}$$

and

$$\Phi = \begin{bmatrix} \phi_i \\ \phi_j \\ \phi_m \end{bmatrix} \tag{6.9}$$

It was shown in Section 5.2 that the solution to the problem is obtained by finding the function ϕ which satisfies equation (5.9) or, equivalently, minimizes the integral

$$X(\phi) = \iint_D \tfrac{1}{2}(\phi_x^2 + \phi_y^2)\, dxdy . \tag{6.10}$$

This relation may be rewritten as the sum over all elements, that is

$$X(\phi) = \sum_{i=1}^{n} \iint_{D_i} \tfrac{1}{2}(\phi_x^2 + \phi_y^2)\, dxdy \tag{6.11}$$

From equations(6.6) and (6.11), it is clear that

$$X(\phi) = F(\phi_1, \phi_2, \phi_3, \ldots\ldots, \phi_m) \quad (6.12)$$

The necessary condition for this to attain a minimum is shown in Ref. 9 to result in the set of m equations,

$$\frac{\partial X}{\partial \phi_i} = 0, \text{ where } i = 1, \ldots.m \quad (6.13)$$

(N.B. It is conventional to denote the triangle vertices by i,j,m. This m is an identifier and should not be confused with the m of equations (6.12) and (6.13).

If, however, some of the ϕ's of equation (6.12) are prescribed, in value, then these ϕ_i's are absorbed into F, since they are numbers. For example, suppose ϕ_K, and ϕ_L are prescribed values, relation (6.12) then becomes

$$X(\phi) = F(\phi_1, \phi_2 \ldots. \phi_{K-1}, \phi_{K+1} \ldots \phi_{L-1}, \phi_{L+1} \ldots. \phi_m) , \quad (6.14)$$

and hence this results in the set of equations (6.13) becoming the set of m-2 equations

$$\frac{\partial X}{\partial \phi_i} = 0, \text{ where } i = 1, \ldots.m \quad i \neq K \quad i \neq L \quad (6.15)$$

In order to utilize equation (6.13) the quantities $\frac{\partial X}{\partial \phi_i}$ need to be calculated. From equation (6.11),

$$X = \sum_{i=1}^{n} X^i \quad (6.16)$$

where X^i denotes the Dirichlet integral over the i"th element.

Differentiating equation (6.16) yields

$$\frac{\partial X}{\partial \phi_i} = \sum_{j=1}^{n} \frac{\partial X^j}{\partial \phi_i} . \qquad (6.17)$$

The physical meaning of equation (6.17) is that $\frac{\partial X}{\partial \phi_i}$ is (the limit of) the ratio of the change in X to the change in ϕ at a particular nodal point i. One can think of the ϕ surface over D as being a multi-faceted tent of some stretchable material. Changing ϕ at some nodal point i, i.e., varying ϕ_i, corresponds to lifting the 'tent corner' at i vertically up or down. This alters the tent surface *only* over those elements which have a vertex at i, and alters the contribution from these elements to the total X. The contribution to the total X from other elements *has not changed*. Thus $\frac{\partial X}{\partial \phi_i}$ only needs to be calculated over those elements which have a vertex at i, since the contribution to $\frac{\partial X}{\partial \phi_i}$ from other elements is zero. Thus in equation (6.17) the summation over the n elements of the surface, need only be taken over *the elements which have a vertex at the nodal point i*. This conclusion can also be shown by simple mathematical reasoning. Consequently, the quantity of interest is the sum of the contributions of the elements around the nodal point i to the relation $\frac{\partial X}{\partial \phi_i}$, and hence the basic relation needed is $\frac{\partial X^e}{\partial \phi_i}$, where e is one of the elements which contribute to the nodal point i. Applying equation (6.11) to the element e and differentiating w.r.t. ϕ_i yields

$$\frac{\partial X^e}{\partial \phi_i} = \iint_{D^e} [\phi_x (\phi_x)_{\phi_i} + \phi_y (\phi_y)_{\phi_i}] dxdy , \qquad (6.18)$$

where

$$(\phi_x)_{\phi_i} = \frac{\partial}{\partial \phi_i} \left(\frac{\partial \phi}{\partial x}\right), \text{ and } (\phi_y)_{\phi_i} = \frac{\partial}{\partial \phi_i} \left(\frac{\partial \phi}{\partial y}\right) \tag{6.19}$$

But differentiating relation (6.7) w.r.t. x and y gives

$$\phi_x = \frac{1}{2\Delta} B\phi = \frac{1}{2\Delta} (b_i \quad b_j \quad b_m) \begin{bmatrix} \phi_i \\ \phi_j \\ \phi_m \end{bmatrix}$$

and (6.20)

$$\phi_y = \frac{1}{2\Delta} C\phi = \frac{1}{2\Delta} (c_i \quad c_j \quad c_m) \begin{bmatrix} \phi_i \\ \phi_j \\ \phi_m \end{bmatrix}$$

respectively.

Differentiating equations (6.20) w.r.t. to ϕ_i then gives

$$(\phi_x)_{\phi_i} = \frac{1}{2\Delta} b_i$$

and (6.21)

$$(\phi_y)_{\phi_i} = \frac{1}{2\Delta} c_i$$

respectively.

Hence equation (6.18) reduces to

$$\frac{\partial x^e}{\partial \phi_i} = \iint_{D^e} \left[\frac{1}{4\Delta^2} b_i (b_i, b_j, b_m) \begin{bmatrix} \phi_i \\ \phi_j \\ \phi_m \end{bmatrix} + \frac{1}{4\Delta^2} c_i (c_i, c_j, c_m) \begin{bmatrix} \phi_i \\ \phi_j \\ \phi_m \end{bmatrix}\right] dxdy \tag{6.22}$$

The integrand in the above expression is a constant, and since

$$\iint_{D^e} dxdy = \Delta \ , \tag{6.23}$$

there obtains

$$\frac{\partial x^e}{\partial \phi_i} = \frac{1}{4\Delta} [b_i(b_i,b_j,b_m) + c_i(c_i,c_j,c_m)] \begin{bmatrix} \phi_i \\ \phi_j \\ \phi_m \end{bmatrix} . \tag{6.24-a}$$

Similarly it can be shown that

$$\frac{\partial x^e}{\partial \phi_i} = \frac{1}{4\Delta} [b_j(b_i,b_j,b_m) + c_j(c_i,c_j,c_m)] \begin{bmatrix} \phi_i \\ \phi_j \\ \phi_m \end{bmatrix} , \tag{6.24-b}$$

and

$$\frac{\partial x^e}{\partial \phi_m} = \frac{1}{4\Delta} [b_m(b_i,b_j,b_m) + c_m(c_i,c_j,c_m)] \begin{bmatrix} \phi_i \\ \phi_j \\ \phi_m \end{bmatrix} . \tag{6.24-c}$$

Equations (6.24) can be summarized in matrix form as

$$[\frac{\partial x}{\partial \phi}]^e = \begin{bmatrix} \frac{\partial x^e}{\partial \phi_i} \\ \frac{\partial x^e}{\partial \phi_j} \\ \frac{\partial x^e}{\partial \phi_m} \end{bmatrix} \tag{6.25}$$

Substitution of equation (6.24-a) into equation (6.17) and using the minimization condition for X yields

$$\frac{\partial X}{\partial \phi_i} = \sum_{k=1}^{n} \frac{1}{4\Delta} [b_i(b_i,b_j,b_m) + c_i(c_i,c_j,c_m)] \begin{bmatrix} \phi_i \\ \phi_j \\ \phi_m \end{bmatrix} = 0 , \qquad (6.26)$$

where the quantity in the summation is calculated over those elements, which contribute to the nodal point i. If, in fact, element p does not contribute, then the RHS of equation (6.26) does not contain a submatrix with an i suffix for $k = p$, and therefore, the contribution from element p is zero. From equation (6.13), the condition for minimization of X is that the RHS of equation (6.26) be zero, as indicated.

If certain boundary conditions are imposed, for example

$$\phi_i = g_i, \text{ where } i = 1, \ldots\ldots, p \text{ where } p < m \qquad (6.27)$$

then from the equation corresponding to (6.15) there obtains the set of $m - p$ equations

$$\frac{\partial X}{\partial \phi_i} = 0, \text{ where } i = p + 1, \ldots\ldots\ldots, m \qquad (6.28)$$

Together with the further p equations

$$\phi_i = g_i, \text{ where } i = 1, \ldots\ldots\ldots, p \qquad (6.29)$$

these form a set of m equations.

The method of solution, is from this point on, more simply explained in terms of an example rather than in general formulae. The procedure outlined in the following is that used in the computer program later described.

The set of nodal points over the domain D are in an actual problem assigned identification numbers in some convenient sequence. Consider, for the purpose of our example, that the i'th nodal point is the point denoted as 5 and that it is surrounded by the elements denoted by e = 12, 13, 14 with their other nodal points identified as shown in Fig. 6.3 below.

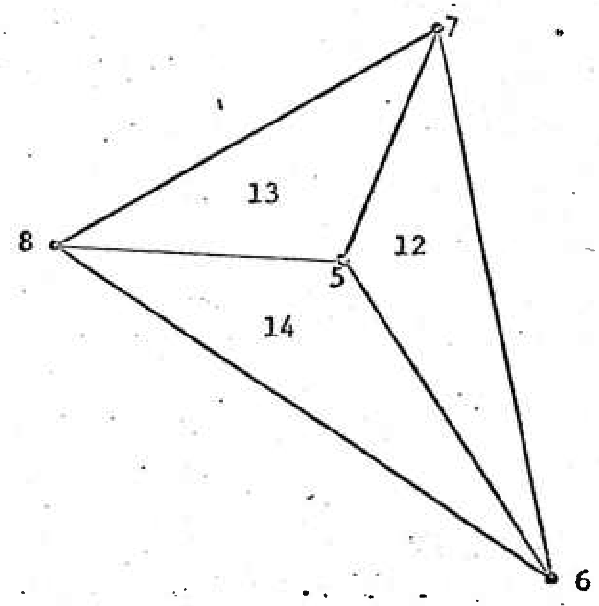

Fig. 6.3. A Nodal Point With Its Surrounding Elements

There are assumed to be no prescribed values of ϕ in the region under consideration. Equation (6.26) is now used to calculate $\frac{\partial x}{\partial \phi_i}$ for the domain D, bearing in mind that in the RHS of equation (6.26) only the elements around i need to be considered. For our case, these elements are 12, 13, and 14.

In evaluating the RHS quantity in the summation for each triangle, the suffices i, j, m must be attached to each triangle in turn, keeping i the nodal point 5, and j, m being the other vertices identified anti-clockwise. This procedure, using equation (6.26) in the form

$$\left[\frac{b_ib_i + c_ic_i}{4\Delta}, \frac{b_ib_j + c_ic_j}{4\Delta}, \frac{b_ib_m + c_ic_m}{4\Delta}\right] \begin{bmatrix} \phi_i \\ \phi_j \\ \phi_m \end{bmatrix} = 0 , \qquad (6.30)$$

yields the following equation:

$$\left[\frac{b_5b_5 + c_5c_5}{4\Delta_{12}}, \frac{b_5b_6 + c_5c_6}{4\Delta_{12}}, \frac{b_5b_7 + c_5c_7}{4\Delta_{12}}\right] \begin{bmatrix} \phi_5 \\ \phi_6 \\ \phi_7 \end{bmatrix}$$

$$+$$

$$\left[\frac{b_5b_5 + c_5c_5}{4\Delta_{13}}, \frac{b_5b_7 + c_5c_7}{4\Delta_{13}}, \frac{b_5b_8 + c_5c_8}{4\Delta_{13}}\right] \begin{bmatrix} \phi_5 \\ \phi_7 \\ \phi_8 \end{bmatrix} \qquad (6.31)$$

$$+$$

$$\left[\frac{b_5b_5 + c_5c_5}{4\Delta_{14}}, \frac{b_5b_8 + c_5c_8}{4\Delta_{14}}, \frac{b_5b_6 + c_5c_6}{4\Delta_{14}}\right] \begin{bmatrix} \phi_5 \\ \phi_8 \\ \phi_6 \end{bmatrix} = 0.$$

By expanding equation (6.31), collecting terms in ϕ_5, ϕ_6, ϕ_7, ϕ_8, and reassembling in matrix form, equation (6.31) can be written in the following form

$$[\, d_5^5, d_6^5, d_7^5, d_8^5 \,] \begin{bmatrix} \phi_5 \\ \phi_6 \\ \phi_7 \\ \phi_8 \end{bmatrix} = 0 \,, \tag{6.32}$$

where the superscript on the d's identify them as deriving from the nodal point i = 5, and the subscripts are chosen as corresponding to the subscripts of the ϕ's for convenience.

Let m, as before, be the total number of nodal points including those on the boundaries, and let p be the number of nodal points on the boundary where ϕ is prescribed. By letting i now successively denote all the (m-p) nodal points over the region D that do not have prescribed ϕ values, one obtains (m-p) equations similar to equation (6.32). The boundary conditions yield a further p equations of the type (6.29). There is thus a total of m equations involving (m-p) unknown ϕ's and p known ϕ's. There is thus, in general, sufficient equations to solve for the unknown ϕ's,

Equation (6.32) can be written in the form below

$$[0\ 0\ 0\ 0\ d_5^5\ d_6^5\ d_7^5\ d_8^5\ 0\ 0\ 0\ 0\ 0\ 0 \ldots\ldots\ldots 0] \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \\ \phi_7 \\ \phi_8 \\ \phi_9 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \phi_m \end{bmatrix} = 0 , \qquad (6.33)$$

where each matrix contains m terms.

The other (m-p) equations of the form (6.32) can be similarly written.

The p equations of type (6.29) can be written in the form

$$[0\ 0\ 1\ 0\ 0\ \ldots.0\ 0\ 0\ \ldots.\ 0\ 0] \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \phi_m \end{bmatrix} = g_3 , \qquad (6.34)$$

where each matrix contains m terms and the only non-zero term in the row matrix is a 1 in the r'th position if r is the subscript of the g on the RHS.

It may be simply shown that the (m-p) equations of type (6.33) and the p equations of type (6.34) can be assembled into a matrix equation of the following form

$$\begin{bmatrix} \cdot & & & & & & & & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & \\ 0 & 0 & 1 & 00 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & d^5_5 & d^5_6 & d^5_7 & d^5_8 & 0 & \cdots & 0 \\ & & & & & & & & & & \\ & & & & & & & & & & \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \vdots \\ \phi_m \end{bmatrix} = \begin{bmatrix} \cdot \\ \cdot \\ g_3 \\ \cdot \\ 0 \\ \vdots \\ \cdot \end{bmatrix}, \qquad (6.35)$$

where

(a) The row matrix of an equation such as (6.33) goes in the i'th row of the above (m x m) matrix, where i is the nodal point for which equation (6.33) is obtained. A zero goes into the same row of the RHS column matrix of equation (6.35)

(b) The row matrix of an equation such as (6.34) goes in the r'th row of the (m x m) matrix, where the RHS of equation (6.34) is g_r. The value g_r goes into the same row of the RHS column matrix of equation (6.35).

Equation (6.35) can be rewritten in the form

$$H \tilde{\Phi} = G. \qquad (6.36)$$

There is thus obtained a matrix equation of type (6.36), whose solution yields the desired values of the unknown ϕ's.

Equation (6.36) can be solved by inversion or iteration procedures as described in the succeeding report (Ref. 7).

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