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DETERMINATION OF TRAJECTORIES FOR A GLIDING PARACHUTE SYSTEM

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The problem of the automatic guidance of a parachute subject to a constant wind is considered. A trajectory is to be determined, through variations of the bank angle, to land on an intended target and to approach the target in the upwind direction. Optimal control, involving a minimum expenditure of control energy, requires the solution of a nonlinear two-point boundary value problem. However, the structure of the optimal control law is derived in terms of three unknown parameters. Nonlinear algebraic equations for these parameters involving elliptic integrals can be written, but their solution is tedious. A method

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for computing the parameters is developed through the numerical minimization of a terminal error function. This method requires the integration of a system of twelve differential equations at each iteration. A non-optimal guidance scheme is also given, which prescribes a given geometrical path parameterized by three constants as a trajectory.

Determination of Trajectories for

A Gliding Parachute System

by

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April 1975

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FOREWORD

This report was prepared under contract with Brown University in the Division of Engineering and Lefschetz Center for Dynamical Systems. The work was carried out under Exploratory Development, Project 1F262203AH86, Control of Gliding Parachute Systems, for the U.S. Army Natick Research and Development Command, Natick, Massachusetts. Mr. Arthur L. Murphy, Jr., of the Engineering Science Division, Aero-Mechanical Engineering Laboratory, was the Project Engineer for this effort.

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I. INTRODUCTION

The problem of determining trajectories for the automatic guidance of a gliding parachute system is considered. Uses for such guidance techniques include delivery of cargo in military operations. Basically, the objectives are to control the motion of the parachute, subject to wind drift, so that it lands as close as possible to the intended target and that its ground speed on landing is minimized to protect the cargo from damage. The latter condition requires that the parachute head into the wind at touchdown. The trajectory of the parachute is controlled by variations of the bank angle, which can be accomplished by a servo motor that pulls on the shroud lines.

An analysis of the optimal control problem is presented in which a measure of the expenditure of control energy is to be minimized (see [5]* for somewhat different formulations of the optimal control problem). The optimal control, as a function of the state vector of the parachute, is discovered to be parametrized by three constants; methods for determining these constants are discussed. Also, a non-optimal guidance scheme is given which has simple computational requirements.

*[5] A. E. Pearson, "Optimal Control of a Gliding Parachute System", TR-73-30-AD, U.S. Army Natick Labs., August 1972.

II. DYNAMICS OF PARACHUTE MOTION

The forces acting on the parachute consist of the resultant aerodynamic force $\underset{\sim}{F}$, and the weight $M\underset{\sim}{g}$. The bank angle σ is the angle between the vector $\underset{\sim}{F}$ and the vertical (see Fig. 1). The parachute is considered as a point particle acted upon by these two forces, as shown in Fig. 2 below.

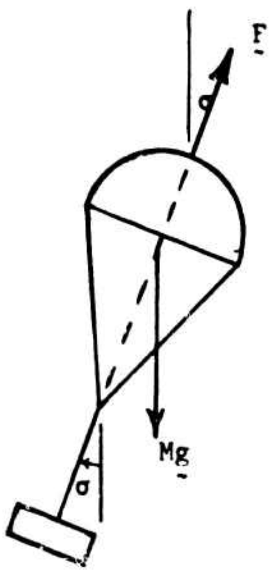


Figure 1

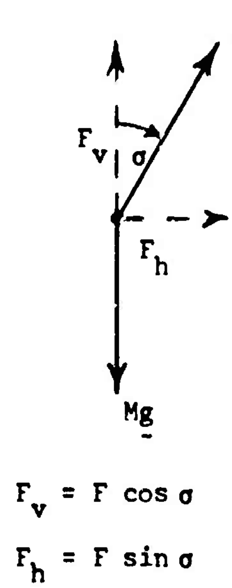


Figure 2

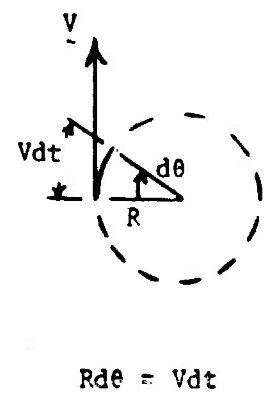


Figure 3

It is assumed that the terminal descent speed has already been reached, so that the vertical velocity U is constant. Hence

$$F \cos \sigma = Mg \; . \tag{1}$$

It is also assumed that the horizontal airspeed of the parachute, V, is constant. If the instantaneous radius of curvature of the horizontal flight path is denoted by R, then

$$F \sin \sigma = \frac{MV^2}{R} \; . \tag{2}$$

Dividing (2) by (1), the bank angle and radius of turn are related by

$$\tan \sigma = \frac{V^2}{gR} . \tag{3}$$

The above assumptions of steady aerodynamic flight require that the bank angle be restricted to angles less than thirty degrees, approximately.

The horizontal flight path during an infinitesimal time interval dt is shown in Fig. 3. If θ is the angle that the horizontal velocity vector $\underset{\sim}{V}$ makes with a fixed direction, the infinitesimal arc traversed has length V dt and subtends an angle $d\theta$. So $V\,dt = R d\theta$, and

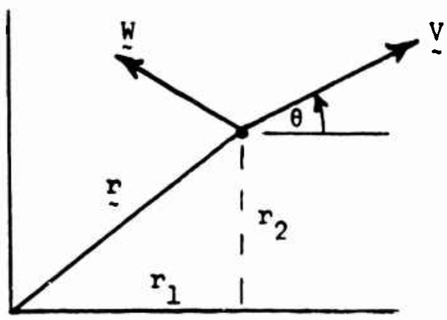
$$R = V/\dot{\theta} \tag{4}$$

which yields the equation

$$\dot{\theta} = \frac{g}{V} \tan \sigma \tag{5}$$

as the relationship between θ and σ.

If the parachute is launched from an initial height h at time $t = 0$, the touchdown occurs at time $T = h/U$. At some time t_o, intermediate between launch time and touchdown, assume that a constant horizontal wind $\underset{\sim}{W} = (W_1, W_2)$ is present. Consider just the horizontal motion of the parachute in a coordinate system whose origin is the target, as shown in Fig. 4. Letting (r_1, r_2) be the coordinates of the position vector in the horizontal plane, the dynamics of the parachute motion are described by the equations



$$\begin{aligned} \dot{r}_1 &= V \cos \theta + W_1 \\ \dot{r}_2 &= V \sin \theta + W_2, \quad t_o \le t \le T \\ \dot{\theta} &= \frac{g}{V} \tan \sigma \end{aligned} \tag{6}$$

Figure 4

Define the normalized coordinates

$$x_1 = \frac{r_1 + (T-t)W_1}{V(T-t_o)} , \quad x_2 = \frac{r_2 + (T-t)W_2}{V(T-t_o)} , \quad x_3 = \theta \tag{7}$$

and the normalized control variable

$$u = \frac{(T-t_o)g}{V} \tan \sigma \tag{8}$$

and the normalized time

$$\eta = \frac{t-t_o}{T-t_o} . \tag{9}$$

The equations of motion of the system in non-dimensional form become

$$\begin{aligned} \frac{dx_1}{d\eta} &= \cos x_3 \\ \frac{dx_2}{d\eta} &= \sin x_3 , \qquad 0 \leq \eta \leq 1 \\ \frac{dx_3}{d\eta} &= u \end{aligned} \tag{10}$$

For the remainder of this report, the variable t will be used to denote normalized time instead of the variable η, and $(\cdot)$ will indicate differentiation with respect to t.

There are two desired conditions on the motion of the parachute at touchdown; that it land on target and that it approach the target in the upwind direction in order to minimize the horizontal speed relative to the ground on impact. Since the orientation of the coordinate system is arbitrary, assume for convenience that the wind is out of the east ($W_1 = -W, W_2 = 0$). Then the landing constraints are that $r_1 = r_2 = 0$ and $\theta = 2n\pi$ (n is any integer) at touchdown, or

$$x_1(1) = x_2(1) = 0 \quad \text{and} \quad x_3(1) = 2n\pi \quad . \tag{11}$$

The initial conditions for Equations (10) are assumed to be known from measurement. Thus, the basic control problem is to find a function $u(t)$ such that the trajectory from (10) satisfies the given initial conditions and desired terminal conditions (11). Optimal control would do this in a manner which minimizes some measure of control effort.

Some general observations may be made for any control $u(t)$ (optimal or otherwise):

1. The speed of the parachute in the (normalized) x_1-x_2 plane is unity, since an element of arc length is $ds = (dx_1^2 + dx_2^2)^{1/2}$ and the speed is

$$\frac{ds}{dt} = \left[\left(\frac{dx_1}{dt}\right)^2 + \left(\frac{dx_2}{dt}\right)^2\right]^{1/2} = (\cos^2 x_3 + \sin^2 x_3)^{1/2} = 1 \; . \tag{12}$$

   Thus, a trajectory from $t = 0$ to $t = 1$ must have unit length and it is necessary that the initial conditions $x_1(0)$ and $x_2(0)$ be inside the unit circle in order to satisfy the terminal conditions.

2. The control $-u(t)$ results in a trajectory that is the mirror image in the x_1 axis of the trajectory given by $u(t)$. This follows if the substitutions $y_1 = x_1$, $y_2 = -x_2$, $y_3 = -x_3$ are made in (10), corresponding to which the equations of motion become

$$\begin{aligned} \dot{y}_1 &= \cos y_3 \\ \dot{y}_2 &= \sin y_3 \\ \dot{y}_3 &= -u \end{aligned} \tag{13}$$

   Therefore, it is sufficient to study initial conditions in the upper half x_1-x_2 plane only.

III. OPTIMAL CONTROL

The dynamical equations of motion of the system in state space notation are

$$\dot{x}_1 = \cos x_3$$
$$\dot{x}_2 = \sin x_3 \qquad t \in [0,1] \tag{14}$$
$$\dot{x}_3 = u$$

The specific optimal control problem to be considered is to minimize the "control energy" expended

$$E = \int_0^1 |u(t)|^2 dt \tag{15}$$

subject to the terminal constraints

$$x_1(1) = x_2(1) = 0$$
$$x_3(1) = 2n\pi . \tag{16}$$

Using optimal control theory to find the optimum u(t) (see, for example, [1]*), form the Hamiltonian

$$H(x,p,u) = u^2 + p_1 \cos x_3 + p_2 \sin x_3 + p_3 u . \tag{17}$$

The necessary condition for an extremum is

$$\frac{\partial H}{\partial u} = 2u + p_3 = 0 \tag{18}$$

together with the system equations

$$\dot{x}_1 = \cos x_3$$
$$\dot{x}_2 = \sin x_3 \tag{19}$$
$$\dot{x}_3 = u$$

*[1] A. E. Bryson, Jr. and Y. C. Ho, "Applied Optimal Control", Ginn and Co., Waltham, MA., 1969.

and the adjoint equations

$$\dot{p}_1 = -\frac{\partial H}{\partial x_1} = 0$$
$$\dot{p}_2 = -\frac{\partial H}{\partial x_2} = 0 \qquad (20)$$
$$\dot{p}_3 = -\frac{\partial H}{\partial x_3} = p_1 \sin x_3 - p_2 \cos x_3 \;.$$

The boundary conditions are

$$x_i(0) \text{ given} \;, \quad x_i(1) = 0 \;, \quad i = 1,2,3 \;. \qquad (21)$$

This is a system of six differential equations and one algebraic equation for the seven variables $u(t)$, $x_i(t)$, $p_i(t)$, $i = 1,2,3$. Since the boundary conditions are split between $t = 0$ and $t = 1$, this is a two point boundary value problem, which presents considerable computational difficulties for its solution.

Using an algorithm based on dynamic programming [4]*, a region in the x_1-x_2 plane has been mapped out for which the optimal controls and trajectories have been computed regardless of the initial condition $x_3(0)$. This region is shown in Fig. 5. Outside this region, the convergence of the algorithm depended upon $x_3(0)$ and was frequently very slow. See [6]** for a complete discussion.

The solution to the two point boundary value would yield the optimal control as a function of time (i.e., open loop), but the form of the optimal control law in terms of state feedback can be deduced as follows.

*[4] K. Martensson, "A Constraining Hyperplane Technique for State Variable Constrained Optimal Control Problems", ASME Transactions, Journal of Dynamic Systems, Measurement and Control, Vol. 95, No. 4, December 1973.

**[6] K. C. Wei and A. E. Pearson, "Numerical Solution to the Optimal Controls of a Gliding Parachute System", TR-75-017-AMEL U.S. Army Natick Development Center, October 1974.

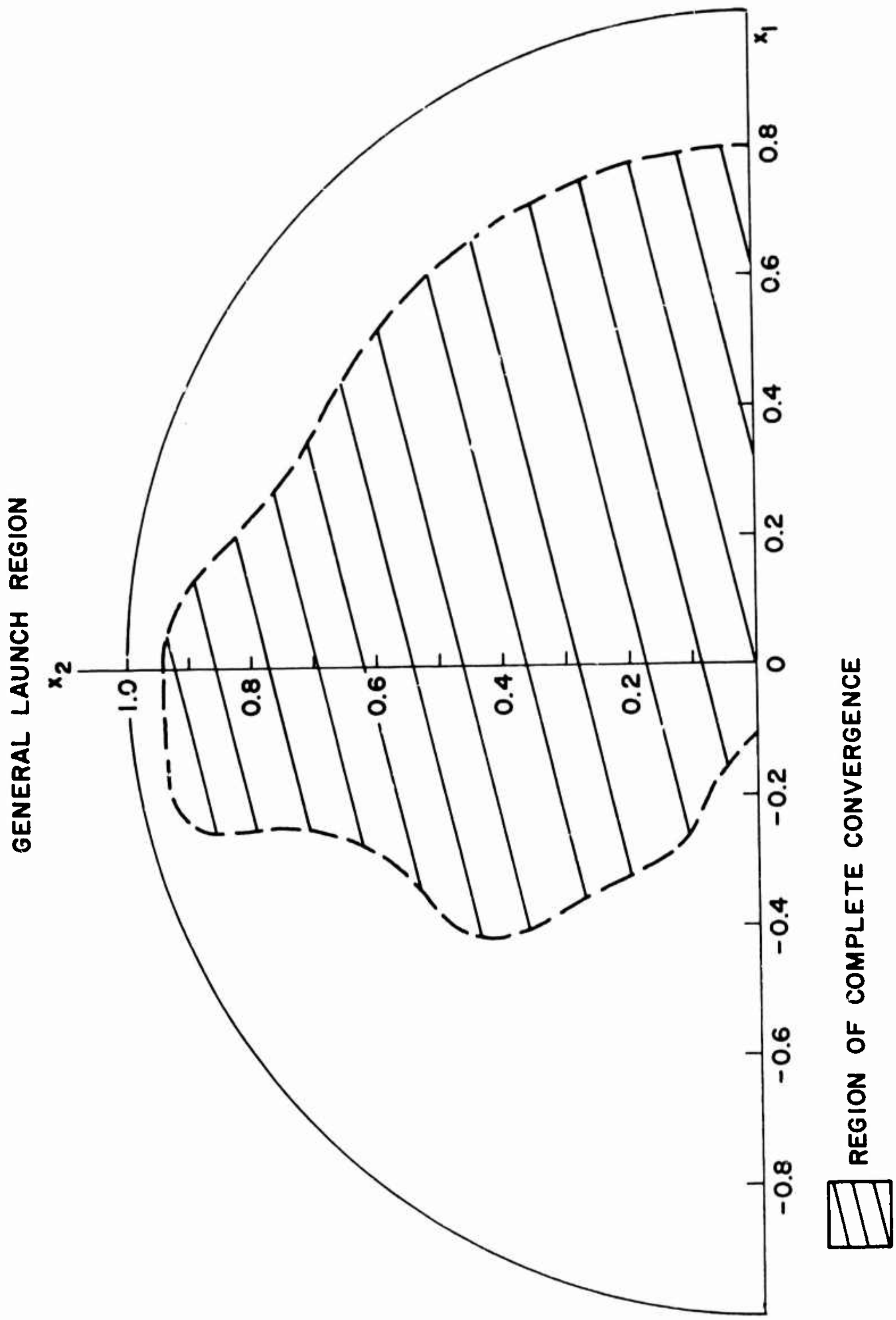

GENERAL LAUNCH REGION
x_2
x_1
1.0
0.8
0.6
0.4
0.2
0
-0.8
-0.6
-0.4
-0.2
0.2
0.4
0.6
0.8
REGION OF COMPLETE CONVERGENCE

Figure 5

From the adjoint equations (20),

$$\begin{aligned} p_1 &= \text{constant} \triangleq -2\alpha_2 \\ p_2 &= \text{constant} \triangleq 2\alpha_1 \\ \dot{p}_3 &= p_1 \sin x_3 - p_2 \cos x_3 = -2(\alpha_1 \cos x_3 + \alpha_2 \sin x_3) \ . \end{aligned} \tag{22}$$

But from the system equations (19),

$$\begin{aligned} \dot{x}_1 &= \cos x_3 \\ \dot{x}_2 &= \sin x_3 \end{aligned} \tag{23}$$

and from the necessary condition (18) for an extremum of the Hamiltonian, it follows from time differentiation that

$$\dot{p}_3 = -2\dot{u} \ . \tag{24}$$

Hence

$$\begin{aligned} &\dot{u} = \alpha_1 \dot{x}_1 + \alpha_2 \dot{x}_2 \\ \text{or} \quad & \\ &u = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 \quad (\alpha_3 \text{ is a constant}) \end{aligned} \tag{25}$$

which is in the form of a state feedback control law.

A different expression for this feedback law can be obtained by noting that the Hamiltonian does not explicitly depend upon time. Therefore, the Hamiltonian is a constant of the motion, i.e., the value of H is constant on the optimal trajectory (see [1], p. 49*). On the optimal trajectory then,

$$p_3 = -2u$$

and

$$\begin{aligned} H &= u^2 + p_1 \cos x_3 + p_2 \sin x_3 + p_3 u \\ &= p_1 \cos x_3 + p_2 \sin x_3 - u^2 \\ &= 2\alpha_1 \sin x_3 - 2\alpha_2 \cos x_3 - u^2 . \\ &\triangleq -C \ . \end{aligned} \tag{26}$$

*[1] A. E. Bryson, Jr. and Y. C. Ho, "Applied Optimal Control", Ginn and Co., Waltham, MA., 1969.

Thus

$$u^2 = 2\alpha_1 \sin x_3 - 2\alpha_2 \cos x_3 + C$$

or

$$u = \pm(2\alpha_1 \sin x_3 - 2\alpha_2 \cos x_3 + C)^{1/2} \tag{27}$$

where the sign is not known from the above information.

Comparing the two expressions (25) and (27) at the terminal time $t = 1$, it is necessary that

$$[\alpha_1 x_1(1) + \alpha_2 x_2(1) + \alpha_3]^2 = 2\alpha_1 \sin x_3(1) - 2\alpha_2 \cos x_3(1) + C$$

or

$$\alpha_3{}^2 = -2\alpha_2 + C \quad . \tag{28}$$

This means that the optimal control satisfies

$$u = \pm\sqrt{2\alpha_1 \sin x_3 - 2\alpha_2 \cos x_3 + 2\alpha_2 + \alpha_3{}^2} \quad . \tag{29}$$

Comparing (25) and (27) at the initial time $t = 0$,

$$[\alpha_1 x_1(0)+\alpha_2 x_2(0)+\alpha_3]^2 = 2\alpha_1 \sin x_3(0)-2\alpha_2 \cos x_3(0)+2\alpha_2+\alpha_3{}^2, \quad \text{or} \tag{30}$$
$$2\alpha_1 \sin x_3(0)+2\alpha_2[1-\cos x_3(0)] - [\alpha_1 x_1(0)+\alpha_2 x_2(0)]^2 - 2\alpha_3[\alpha_1 x_1(0)+\alpha_2 x_2(0)] = 0 \ .$$

If $\alpha_1 x_1(0) + \alpha_2 x_2(0) \neq 0$, Eq. (30) can be solved for α_3 in terms of α_1 and α_2 yielding

$$\alpha_3 = \frac{\alpha_1 \sin x_3(0)+\alpha_2[1-\cos x_3(0)] - \frac{1}{2}[\alpha_1 x_1(0)+\alpha_2 x_2(0)]^2}{\alpha_1 x_1(0) + \alpha_2 x_2(0)} \tag{31}$$

Thus, one equation has been obtained which provides a relationship among the three parameters α_1, α_2, α_3 corresponding to the optimal control.

IV. DETERMINATION OF OPTIMAL CONTROL PARAMETERS FROM ALGEBRAIC EQUATIONS

In this section an attempt is made to proceed further analytically with finding the values of the parameters in the optimal control law from algebraic equations.

Using an unsubscripted notation (x,y,θ) instead of (x_1,x_2,x_3), the optimal trajectory is described by the equations

$$\begin{aligned} \dot{x} &= \cos\theta & x(0) &= x_o \\ \dot{y} &= \sin\theta & y(0) &= y_o \\ \dot{\theta} &= u = \alpha_1 x + \alpha_2 y + \alpha_3 & \theta(0) &= \theta_o \end{aligned} \tag{32}$$

It remains to choose α_1, α_2, α_3 such that $x(1) = y(1) = 0$, $\theta(1) = 2n\pi$. By differentiating the last equation once, the variable $\theta(t)$ can be decoupled from $x(t)$ and $y(t)$ resulting in the equation

$$\ddot{\theta} = \dot{u} = \alpha_1\dot{x} + \alpha_2\dot{y}$$

or

$$\ddot{\theta} = \alpha_1 \cos\theta + \alpha_2 \sin\theta \tag{33}$$

for $\theta(t)$. Let

$$\begin{aligned} \alpha_1 &= \alpha\cos\beta & &\left.\begin{aligned} x &= \rho\cos\phi \\ y &= \rho\sin\phi \end{aligned}\right\} \text{(polar coordinates)} \\ \alpha_2 &= \alpha\sin\beta & & \\ \alpha_3 &= \gamma & &\theta = \psi + \beta + \frac{\pi}{2} \qquad (\alpha \geq 0,\ \rho \geq 0) \end{aligned} \tag{34}$$

Then

$$u = \alpha\rho\cos(\phi-\beta) + \gamma \tag{35}$$

and the relation for γ in terms of α,β from (31) becomes

$$\gamma = \frac{\sin\beta - \sin(\beta-\theta_o) - \frac{1}{2}\alpha\rho_o^2\cos^2(\phi_o-\beta)}{\rho_o\cos(\phi_o-\beta)} \quad . \tag{36}$$

In terms of ψ instead of θ, Eq. (33) is reduced to

$$\ddot{\psi} = -\alpha\sin\psi \qquad \begin{aligned} \psi(0) &= \theta_o - (\tfrac{\pi}{2} + \beta) \\ \psi(1) &= -(\tfrac{\pi}{2} + \beta) \end{aligned} \tag{37}$$

which is the well known general equation of motion of a simple pendulum, that is, the equation of motion without the usual linear approximation. For a pendulum, α is replaced by g/ℓ, where ℓ is the length of the pendulum.

The solution to Eq. (37) would yield $\psi(t;\alpha,\beta)$. To determine α and β, and hence γ through (36), it is necessary to apply the terminal conditions $x(1) = y(1) = 0$. Now

$$\begin{aligned} x(1) - x(0) &= \int_o^1 \cos\theta(t)dt \\ y(1) - y(0) &= \int_o^1 \sin\theta(t)dt \ . \end{aligned} \tag{38}$$

Applying the terminal conditions and substituting for $\theta(t)$ in terms of $\psi(t;\alpha,\beta)$ results in

$$\begin{aligned} \int_o^1 \cos(\psi + \beta + \tfrac{\pi}{2})dt &= -x_o = -\rho_o\cos\phi_o \\ \int_o^1 \sin(\psi + \beta + \tfrac{\pi}{2})dt &= -y_o = -\rho_o\sin\phi_o \quad . \end{aligned} \tag{39}$$

Expanding the trigonometric functions in the integrands will yield the equations

$$\cos\beta \int_0^1 \sin\psi(t)dt + \sin\beta \int_0^1 \cos\psi(t)dt = \rho_o\cos\phi_o$$

$$\cos\beta \int_0^1 \cos\psi(t)dt - \sin\beta \int_0^1 \sin\psi(t)dt = -\rho_o\sin\phi_o$$

or

$$\int_0^1 \cos\psi(t;\alpha,\beta)dt = \rho_o\sin(\beta-\phi_o)$$
$$\int_0^1 \sin\psi(t;\alpha,\beta)dt = \rho_o\cos(\beta-\phi_o) \quad . \tag{40}$$

These two equations would determine α and β.

In order to solve the pendulum equation, multiply both sides of (37) by $\dot{\psi}$:

$$\dot{\psi}\,\ddot{\psi} = -\alpha\dot{\psi} \sin \psi$$

or

$$\frac{1}{2}\frac{d}{dt}(\dot{\psi}^2) = \alpha \frac{d}{dt}(\cos \psi) \quad . \tag{41}$$

Therefore,

$$\frac{1}{2}\dot{\psi}^2 - \alpha \cos \psi = \text{constant} \triangleq E_o \tag{42}$$

which represents the statement of conservation of energy for the pendulum. Evaluating the left hand side at $t = 1$, and noting that $\dot{\psi}(t) = \dot{\theta}(t) = u(t)$, we find that

$$\begin{aligned} E_o &= \frac{1}{2}[\dot{\psi}(1)]^2 - \alpha \cos \psi(1) \\ &= \frac{1}{2}[u(1)]^2 - \alpha \cos [\theta(1) - \beta - \frac{\pi}{2}] \\ &= \frac{1}{2}\gamma^2 + \alpha \sin \beta \quad . \end{aligned} \tag{43}$$

Note that the constant value of the Hamiltonian for the optimization problem is $H = -C$, and

$$C = \alpha_3^{\,2} + 2\alpha_2 = \gamma^2 + 2\alpha\sin\beta = 2E_o \quad . \tag{44}$$

This relationship is not surprising, since the kinetic energy of the pendulum is a homogeneous quadratic function of the generalized velocity and the potential energy is independent of the generalized velocity. This implies that the Hamiltonian of the pendulum is, in fact, the total energy of the pendulum, which is characterized by E_o (see [2]*, p. 54).

Integrating both sides of (42) from $t = 0$ to $t = 1$ yields an expression for the control energy E in terms of the control law parameters:

$$\begin{aligned} E &= \int_o^1 u^2 dt = \int_o^1 \dot{\psi}^2 dt = 2\int_o^1 [E_o + \alpha\cos\psi(t)]dt \\ &= 2[E_o + \alpha\int_o^1 \cos\psi(t)dt] = 2[\tfrac{1}{2}\gamma^2 + \alpha\sin\beta + \alpha\rho_o\sin(\beta-\phi_o)] \\ &= \gamma^2 + 2\alpha[\sin\beta + \rho_o\sin(\beta-\phi_o)] \qquad (45) \\ &= \alpha_3^{\,2} + 2\alpha_2 + 2(\alpha_2 x_o - \alpha_1 y_o) \quad . \qquad (46) \end{aligned}$$

The solution to the pendulum equation can be written in terms of elliptic integrals. However, it is necessary to consider two separate cases. In the first case, the values of α,β,γ are such that $E_o(\alpha,\beta,\gamma) > \alpha$. This case corresponds to the pendulum having sufficient energy to execute complete revolutions about its axis, resulting in a non-oscillatory motion. In the second case, $E_o(\alpha,\beta,\gamma) \leq \alpha$, and the pendulum executes oscillations about the position $\psi = 0$.

*[2] H. Goldstein, "Classical Mechanics", Addison-Wesley Pub. Co., Inc., Reading, MA., 1950.

Case I: $E_o > \alpha$

From Eq. (42),

$$\frac{d\psi}{dt} = \pm\sqrt{2(E_o + \alpha\cos\psi)} \quad . \tag{47}$$

In this case $\frac{d\psi}{dt}$ always has the same sign and we can use either

$$\dot{\psi}(0) = u(0) = \alpha\rho_o\cos(\beta-\phi_o) + \gamma \tag{48}$$

or

$$\dot{\psi}(1) = u(1) = \gamma \tag{49}$$

to determine this sign. Using the latter for convenience, we can write

$$\frac{d\psi}{dt} = \mathrm{sgn}\gamma\sqrt{2(E_o + \alpha\cos\psi)} \qquad \psi(0) \triangleq \psi_o = \theta_o - (\frac{\pi}{2} + \beta) \tag{50}$$

where, from (43),

$$E_o = \frac{1}{2}\gamma^2 + \alpha\sin\beta \quad . \tag{51}$$

(γ cannot equal zero, otherwise $E_o = \alpha\sin\beta \leq \alpha$. Hence $\mathrm{sgn}\gamma = \pm 1$).

Thus

$$\mathrm{sgn}\gamma\int_o^t dt' = \int_{\psi_o}^{\psi} \frac{d\psi'}{\sqrt{2(E_o + \alpha\cos\psi')}} \qquad \psi_o = \theta_o - (\frac{\pi}{2} + \beta) \ . \tag{52}$$

Using the identity $\cos\psi' = 1-2\sin^2\frac{\psi'}{2}$ and letting $\xi = \frac{\psi'}{2}$ leads to

$$(\operatorname{sgn}\gamma)t = \int_{\psi_o/2}^{\psi/2} \frac{2d\xi}{\sqrt{2(E_o + \alpha - 2\alpha\sin^2\xi)}}$$

$$= \frac{k}{\sqrt{\alpha}} \int_{\psi_o/2}^{\psi/2} \frac{d\xi}{\sqrt{1 - k^2\sin^2\xi}}, \quad \text{where} \quad k \triangleq \sqrt{\frac{2\alpha}{E_o + \alpha}} < 1$$

$$= \frac{k}{\sqrt{\alpha}} [F(\frac{\psi}{2}, k) - F(\frac{\psi_o}{2}, k)] \tag{53}$$

where

$$F(\psi,k) = \int_o^{\psi} \frac{d\xi}{\sqrt{1 - k^2\sin^2\xi}}$$

is the elliptic integral of the first kind. This relationship implicitly determines $\psi(t;\alpha,\beta,\gamma)$. We also know that

$$\gamma(\alpha,\beta) = \frac{\sin\beta - \sin(\beta - \theta_o)}{\rho_o\cos(\phi_o - \beta)} - \frac{1}{2}\alpha\rho_o\cos(\phi_o - \beta) \tag{54}$$

$$\psi(1) \triangleq \psi_1 = -(\frac{\pi}{2} + \beta) \tag{55}$$

$$\int_o^1 \cos\psi(t)dt = \rho_o\sin(\beta - \phi_o) \tag{56}$$

and applying these relationships will determine α,β,γ. To express the last relation (56) in terms of elliptic integrals note that $dt = d\psi/\frac{d\psi}{dt}$ so that

$$\int_0^1 \cos\psi(t)dt = \frac{1}{\operatorname{sgn}\gamma} \int_{\psi_o}^{\psi_1} \frac{\cos\psi\, d\psi}{\sqrt{2(E_o + \alpha\cos\psi)}}$$

$$= \frac{1}{\operatorname{sgn}\gamma} \int_{\psi_o/2}^{\psi_1/2} \frac{(1 - 2\sin^2\xi)2d\xi}{\sqrt{2(E_o + \alpha - 2\alpha\sin^2\xi)}}, \quad \xi = \psi/2$$

$$= \frac{1}{\operatorname{sgn}\gamma} \frac{k}{\sqrt{\alpha}} \int_{\psi_o/2}^{\psi_1/2} \frac{1 - 2\sin^2\xi}{\sqrt{1 - k^2\sin^2\xi}}\, d\xi$$

$$= \frac{1}{\operatorname{sgn}\gamma} \frac{k}{\sqrt{\alpha}} [(1 - \frac{2}{k^2}) \int_{\psi_o/2}^{\psi_1/2} \frac{d\xi}{\sqrt{1 - k^2\sin^2\xi}} + \frac{2}{k^2} \int_{\psi_o/2}^{\psi_1/2} \sqrt{1 - k^2\sin^2\xi}\, d\xi]$$

$$= \frac{1}{\operatorname{sgn}\gamma} \frac{k}{\sqrt{\alpha}} [(1 - \frac{2}{k^2})\{F(\frac{\psi_1}{2},k) - F(\frac{\psi_o}{2},k)\} +$$

$$\frac{2}{k^2}\{E(\frac{\psi_1}{2},k) - E(\frac{\psi_o}{2},k)\}] \tag{57}$$

where $E(\psi,k) = \int_0^\psi \sqrt{1 - k^2\sin^2\xi}\, d\xi$ is the elliptic integral of the second kind.

Applying the three relations (54), (55), (56), and using Eqs. (53) and (57), the algebraic equations that determine the parameters α,β,γ are:

$$\frac{1}{\operatorname{sgn}\gamma} \frac{k}{\sqrt{\alpha}} [F(\frac{\psi_1}{2},k) - F(\frac{\psi_o}{2},k)] = 1$$

$$(1 - \frac{2}{k^2}) + \frac{1}{\operatorname{sgn}\gamma} \frac{2}{k\sqrt{\alpha}} [E(\frac{\psi_1}{2},k) - E(\frac{\psi_o}{2},k)] = \rho_o \sin(\beta - \phi_o) \tag{58}$$

$$\gamma = \frac{\sin\beta - \sin(\beta - \theta_o)}{\rho_o\cos(\phi_o - \beta)} - \frac{1}{2}\alpha\rho_o\cos(\phi_o - \beta)$$

where

$$\psi_o = -(\frac{\pi}{2} + \beta) + \theta_o)$$

$$\psi_1 = -(\frac{\pi}{2} + \beta)$$

$$E_o = \frac{1}{2}\gamma^2 + \alpha \sin \beta$$

$$k = \sqrt{\frac{2\alpha}{\alpha + E_o}}$$

<u>Case II: $E_o \leq \alpha$</u>

Again,

$$\frac{d\psi}{dt} = \pm \sqrt{2(E_o + \alpha \cos \psi)} \quad . \tag{59}$$

However, in this case the sign of $\frac{d\psi}{dt}$ changes each time the motion of the pendulum reaches its amplitude. The amplitude, A, is reached when $\frac{d\psi}{dt} = 0$, therefore

$$E_o + \alpha \cos A = 0 \tag{60}$$

and

$$A = \cos^{-1}(-\frac{E_o}{\alpha}) \quad . \tag{61}$$

The period of oscillation, τ, can be calculated through

$$\frac{\tau}{4} = \int_o^A \frac{d\psi}{\sqrt{2(E_o + \alpha \cos \psi)}} = \int_o^A \frac{d\psi}{\sqrt{2\alpha(\cos \psi - \cos A)}}$$

$$= \int_o^A \frac{d\psi}{\sqrt{4\alpha(\sin^2 \frac{A}{2} - \sin^2 \frac{\psi}{2})}} \quad . \tag{62}$$

Let $\kappa = \sin \frac{A}{2}$ and make the transformation (since $\kappa \leq 1$)

$$\sin \frac{\psi}{2} = \kappa \sin \xi \ , \tag{63}$$

Then

$$\frac{1}{2} \cos \frac{\psi}{2} \, d\psi = \kappa \cos \xi \, d\xi$$

and it can easily be established that

$$\frac{d\psi}{\sqrt{\kappa^2 - \sin^2 \frac{\psi}{2}}} = \frac{2 \, d\xi}{\sqrt{1 - \kappa^2 \sin^2 \xi}} \tag{64}$$

so

$$\tau = \frac{2}{\sqrt{\alpha}} \int_0^A \frac{d\psi}{\sqrt{\kappa^2 - \sin^2 \frac{\psi}{2}}} = \frac{4}{\sqrt{\alpha}} \int_0^{\pi/2} \frac{d\xi}{\sqrt{1 - \kappa^2 \sin^2 \xi}} = \frac{4}{\sqrt{\alpha}} K(\kappa) \tag{65}$$

where $K(\kappa) = F(\frac{\pi}{2},\kappa)$ is the complete elliptic integral of the first kind, and where

$$\kappa^2 = \sin^2 \frac{A}{2} = \frac{1 - \cos A}{2} = \frac{1 + E_o/\alpha}{2} = \frac{\alpha + E_o}{2\alpha} \ . \tag{66}$$

Thus, κ is the reciprocal of k from Case I.

Since the period is now known in terms of (α,β,γ), the times when $\frac{d\psi}{dt}$ changes sign can be determined, and it is conceivable that expressions can be formed to which the terminal conditions at $t = 1$ can be applied.

It should be remembered that it is not known in advance which case applies, since the triple (α,β,γ) is precisely what is to be determined. Therefore, it is necessary that the solution of both cases be found, and solutions not compatible with the condition of that case be eliminated. In view of the fact that elliptic functions are involved, and in view of the complexity that would be necessary to express the relations needed to complete Case II, it is felt that the results of this section are not practical for numerical computations, although they are of theoretical interest. A more practical method for computational purposes is developed in the next section.

V. PARAMETER OPTIMIZATION THROUGH MINIMIZATION OF TERMINAL ERROR

Using the form of the optimal control law

$$u = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 \tag{67}$$

the optimal trajectories are governed by the equations

$$\begin{aligned} \dot{x}_1 &= \cos x_3 \\ \dot{x}_2 &= \sin x_3 \qquad\qquad \underset{\sim}{x}(o) = \underset{\sim}{x}_o, \quad t\epsilon[o,1] \\ \dot{x}_3 &= \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 \end{aligned} \tag{68}$$

where the vector $\underset{\sim}{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ is to be chosen to satisfy $\underset{\sim}{x}(1) = \underset{\sim}{0}$. Let the solution to (68) for a given $\underset{\sim}{\alpha}$ be denoted by $\underset{\sim}{x}(t;\underset{\sim}{\alpha})$, and define the terminal error to be

$$F(\underset{\sim}{\alpha}) = x_1^2(1;\underset{\sim}{\alpha}) + x_2^2(1;\underset{\sim}{\alpha}) + \sin^2 \frac{x_3(1;\underset{\sim}{\alpha})}{2} \tag{69}$$

The terminal error serves as a penalty function which penalizes undesirable terminal states by the extent to which they deviate from the desired terminal state $\underset{\sim}{x}(1) = \underset{\sim}{0}$. Since x_3 is physically an angle, a trigonometric function of x_3 which measures its deviation from any multiple of 2π was used, rather than $x_3^2(1;\alpha)$. In this way, the resulting computations will be the same if an arbitrary multiple of 2π is added to x_3. Of course, $F(\underset{\sim}{\alpha}) = 0$ only if $x_1(1;\underset{\sim}{\alpha}) = x_2(1;\underset{\sim}{\alpha}) = 0$ and $x_3(1;\underset{\sim}{\alpha}) = 2n\pi$.

It is desired to minimize $F(\underset{\sim}{\alpha})$ numerically, where the functional relationship between F and $\underset{\sim}{\alpha}$ is defined through the solution to (68) at t = 1. In view of the already established relation among the optimal values of the components of $\underset{\sim}{\alpha}$ given by equation (31), it is natural to attempt to eliminate α_3 in favor of α_1 and α_2, thereby reducing the dimensionality of the parameter search by one. However, if $\alpha_1 x_1(0) + \alpha_2 x_2(0) = 0$, Eq. (31) cannot be solved for α_3 in terms of α_1 and α_2. Instead, the following relationship holds from (30):

$$\alpha_1 \sin x_3(o) + \alpha_2[1 - \cos x_3(0)] = 0. \tag{70}$$

This could be used to eliminate either α_1 or α_2. If $\sin x_3(0) = 0$ or $\cos x_3(0) = 1$ or both, the consideration of further sub-cases is necessary. Of course, the vector $\underset{\sim}{\alpha}$ is not known in advance and hence it is impossible to know beforehand which case applies. In a numerical method, difficulties would occur not only when the solution lies on the line $\alpha_1 x_1(0) + \alpha_2 x_2(0) = 0$, but also anytime a particular iteration produced values near this line of singularity. Therefore, reduction of the search to two parameters does not prove advantageous.

A popular method for the minimization of a function is Newton's method, where the value of $\underset{\sim}{\alpha}$ at the (n+1) iteration is computed according to

$$\underset{\sim}{\alpha}(n+1) = \underset{\sim}{\alpha}(n) - H^{-1}[\underset{\sim}{\alpha}(n)]\nabla F[\underset{\sim}{\alpha}(n)] \tag{71}$$

and where $H[\underset{\sim}{\alpha}(n)]$ is the Hessian matrix of $F(\underset{\sim}{\alpha})$ whose elements are the second partial derivatives of F, i.e., $H_{ij} = \partial^2 F/\partial\alpha_i \partial\alpha_j$. Newton's method, however, requires the evaluation of the function $F(\underset{\sim}{\alpha})$, its gradient vector and its Hessian matrix, all of whose values must be calculated by solving three differential equations per component. Each iteration of Newton's method would therefore require the solution of 30 differential equations (note that the Hessian matrix is symmetric).

A quasi-second order technique, such as a so-called variable metric method, is one which attempts to preserve the desirable convergence properties of Newton's method without the actual evaluation of second order partial derivatives. In such a method the iterations are computed according to

$$\underset{\sim}{\alpha}(n+1) = \underset{\sim}{\alpha}(n) - \lambda_n H_n \nabla F[\underset{\sim}{\alpha}(n)] \tag{72}$$

where H_n is not the Hessian matrix, but a symmetric positive definite matrix computed internally through a recursion formula. In general, H_{n+1} is a function of H_n, $\underset{\sim}{\alpha}(n) - \underset{\sim}{\alpha}(n-1)$, and $\nabla F[\underset{\sim}{\alpha}(n)] - \nabla F[\underset{\sim}{\alpha}(n-1)]$. The scalar λ_n is chosen to minimize $F(\underset{\sim}{\alpha}(n) - \lambda_n H_n \nabla F[\underset{\sim}{\alpha}(n)])$. Further details of variable metric methods and other minimization techniques can be found in [3]*.

The gradient of $F(\underset{\sim}{\alpha})$ is calculated as

$$\frac{\partial F}{\partial \alpha_i} = 2x_1(1;\underset{\sim}{\alpha}) \frac{\partial x_1}{\partial \alpha_i}(1;\underset{\sim}{\alpha}) + 2x_2(1;\underset{\sim}{\alpha}) \frac{\partial x_2}{\partial \alpha_i}(1;\underset{\sim}{\alpha}) + \frac{1}{2}\sin x_3\,(1;\underset{\sim}{\alpha}) \frac{\partial x_3}{\partial \alpha_i}(1;\underset{\sim}{\alpha})$$

or

$$\nabla F(\underset{\sim}{\alpha}) = \left[2x_1 \;\; 2x_2 \;\; \frac{1}{2}\sin x_3\right] \begin{bmatrix} \frac{\partial x_1}{\partial \alpha_1} & \frac{\partial x_1}{\partial \alpha_2} & \frac{\partial x_1}{\partial \alpha_3} \\ \frac{\partial x_2}{\partial \alpha_1} & \frac{\partial x_2}{\partial \alpha_2} & \frac{\partial x_2}{\partial \alpha_3} \\ \frac{\partial x_3}{\partial \alpha_1} & \frac{\partial x_3}{\partial \alpha_2} & \frac{\partial x_3}{\partial \alpha_3} \end{bmatrix} \Bigg|_{t=1} . \tag{73}$$

Define $g_{ij}(t;\underset{\sim}{\alpha}) = \frac{\partial x_i}{\partial \alpha_j}(t;\underset{\sim}{\alpha})$. To develop equations for $g_{ij}(t;\underset{\sim}{\alpha})$ it is necessary to differentiate (15) with respect to α_j and interchange this operation with the differentiation with respect to time. This leads to

*[3] S. L. S. Jacoby, J. S. Kawalik, and J. T. Pizzo, "Iterative Methods for Nonlinear Optimization Problems", Prentice-Hall, Inc., Englewood Cliffs, NJ, 1972.

$$\frac{d}{dt}\frac{\partial x_1}{\partial \alpha_j} = -\sin x_3 \frac{\partial x_3}{\partial \alpha_j} \qquad \left.\frac{\partial x_i}{\partial \alpha_j}\right|_{t=0} = 0,\ i = 1,2,3$$

$$\frac{d}{dt}\frac{\partial x_2}{\partial \alpha_j} = \cos x_3 \frac{\partial x_3}{\partial \alpha_j} \tag{74}$$

$$\frac{d}{dt}\frac{\partial x_3}{\partial \alpha_j} = \alpha_1 \frac{\partial x_1}{\partial \alpha_j} + \alpha_2 \frac{\partial x_2}{\partial \alpha_j} + \begin{cases} x_j, & j = 1,2 \\ 1, & j = 3 \end{cases}$$

or

$$\dot{g}_{1j} = -g_{3j} \sin x_3 \qquad g_{ij}(0) = 0,\ i = 1,2,3$$

$$\dot{g}_{2j} = g_{3j} \cos x_3 \tag{75}$$

$$\dot{g}_{3j} = \alpha_1 g_{1j} + \alpha_2 g_{2j} + \begin{cases} x_j, & j = 1,2 \\ 1, & j = 3 \end{cases}$$

Together with (68), these form a system of 12 differential equations whose solution at $t = 1$ is then used to compute $F(\underset{\sim}{\alpha})$ and $\nabla F(\underset{\sim}{\alpha})$:

$$\begin{aligned} \dot{x}_1 &= \cos x_3 \\ \dot{x}_2 &= \sin x_3 \qquad\qquad x(0) = x_o \\ \dot{x}_3 &= \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 \\[1ex] \dot{g}_{11} &= -g_{31} \sin x_3 \\ \dot{g}_{21} &= g_{31} \cos x_3 \\ \dot{g}_{31} &= \alpha_1 g_{11} + \alpha_2 g_{21} + x_1 \\[1ex] \dot{g}_{12} &= -g_{32} \sin x_3 \\ \dot{g}_{22} &= g_{32} \cos x_3 \qquad\qquad g_{ij}(0) = 0 \\ \dot{g}_{32} &= \alpha_1 g_{12} + \alpha_2 g_{22} + x_2 \end{aligned} \tag{76}$$

$$\dot{g}_{13} = - g_{33} \sin x_3$$

$$\dot{g}_{23} = g_{33} \cos x_3$$

$$\dot{g}_{33} = \alpha_1 g_{13} + \alpha_2 g_{23} + 1$$

Numerical Results:

A FORTRAN computer program PAROM was written which finds the minimum of $F(\alpha)$, starting with an initial guess $\alpha(0)$, using a variable metric method known as the Davidon-Fletcher-Powell algorithm. At each iteration Eqs. (76) are integrated via a Runge-Kutta routine. The subroutines used are found in the IBM Scientific Subroutine Package (they are FMFP for the Davidon-Fletcher-Powell method and RKGS for the Runge-Kutta integration). After the optimal value of $\underset{\sim}{\alpha}$ is computed, the program TRAJ is used to compute the resulting trajectory.

The minimization of $F(\underset{\sim}{\alpha})$ was considered to converge when $F(\underset{\sim}{\alpha}) < 0.0200$. For some initial conditions the process did not converge, even for a large number of iterations. The results of the parameter optimization are shown in Table 1. The values of $\underset{\sim}{\alpha}$ and $F(\underset{\sim}{\alpha})$ shown are those obtained after the number of iterations carried out. The speed of convergence varied widely with the initial condition $\underset{\sim}{x}(0)$. The initial guess $\underset{\sim}{\alpha}(0)$ was chosen as $\underset{\sim}{0}$, since it is not obvious as to how $\underset{\sim}{\alpha}$ depends upon $\underset{\sim}{x}(0)$. If some information concerning this dependence were discovered, the speed of convergence could be greatly improved, since the optimal value of $\underset{\sim}{\alpha}$ is often quite different from $\underset{\sim}{0}$.

Following Table 1 are Figs. 6-12, which consist of a graph showing the location of the selected initial conditions with respect to the unit circle, followed by plots of the optimal controls and trajectories for these initial conditions. The left side of each of these figures, i.e., Figs. 7-a to 12-a, show the trajectories for the function $F(\underset{\sim}{\alpha})$ defined by Eq. (69). It is evident from these plots that the error in the angle x_3 was weighted too heavily in relation to the terminal error in x_1 and x_2. Therefore, these runs were re-computed using a ten-to-one weighting factor in favor of x_1 and x_2, i.e., minimizing the function

$$F(\underset{\sim}{\alpha}) = 10[x_1^2(1;\underset{\sim}{\alpha}) + x_2^2(1;\underset{\sim}{\alpha})] + \sin^2 \frac{x_3(1;\underset{\sim}{\alpha})}{2} .$$

The corresponding trajectories for the minimization of this function are shown in Figs. 7-b to 12-b. A more acceptable terminal accuracy is evident in these plots.

The cost of these computations is approximately 10¢ per iteration.

Table 1

Optimum Values of $\underline{\alpha}$

$x_1(0)$	$x_2(0)$	$x_3(0)$	no. iter. carried out	no. iter. to convg.	α_1	α_2	α_3	$F(\underline{\alpha})$	H	E
0	.8	0°	85	50	21.029	-25.704	9.864	0.0008	-63.05	28.53
		90°	135	-	*	*	*	0.1874	*	*
		180°	72	51	8.449	0.038	4.692	0.0088	-22.23	10.29
		-90°	47	27	6.683	-11.001	5.964	0.0043	-21.41	9.79
.4	.4	0°	105	99	34.045	-33.851	-12.987	0.0162	-98.94	56.78
		90°	153	-	52.940	-60.875	-13.218	0.0331	-162.80	100.05
		180°	45	30	-10.510	-3.748	4.297	0.0003	-9.47	14.75
		-90°	215	-	3.984	-47.794	8.210	0.0542	-94.72	34.80
-4	.4	0°	98	-	-26.323	-31.920	-2.152	0.0476	44.56	18.63
		90°	122	-	36.151	-1.361	3.082	0.0483	-69.85	30.57
		180°	56	32	-4.032	9.503	0.787	0.0046	-19.44	14.69
		-90°	84	-	-11.382	-12.310	0.058	0.0077	22.67	4.50
.8	.2	0°	200	-	15.977	-63.935	-16.147	0.0940	-133.01	64.52
		90°	160	-	-5.401	-26.220	16.377	0.0960	-57.21	29.01
		180°	70	-	-8.073	-21.682	9.439	0.0298	-45.20	20.65
		-90°	150	-	35.381	-40.362	-26.339	0.0702	-108.07	43.16
-.8	.2	0°	35	-	3.250	-0.294	1.079	0.0286	-1.91	0.84
		90°	30	20	16.140	6.037	3.802	0.0009	-30.17	13.73
		180°	85	71	-11.883	33.072	-4.465	0.0135	-69.71	27.77
		-90°	20	15	-6.475	3.367	-1.169	0.0033	-8.99	5.85
0	.2	45°	95	23	24.632	-8.111	-8.957	0.0003	-65.61	55.98
		135°	275	275	50.166	-31.587	-5.962	0.0184	-124.52	99.74
		-45°	200	-	*	*	*	0.4361	*	*
		-135°	55	48	-36.528	-12.387	-4.801	0.0021	-18.83	32.70

* The value of $F(\underline{\alpha})$ is too large to consider the results obtained as meaningful.

H = Value of Hamiltonian

E = Value of Control Input Energy

LOCATION OF SELECTED INITIAL CONDITIONS WITHIN THE UNIT CIRCLE

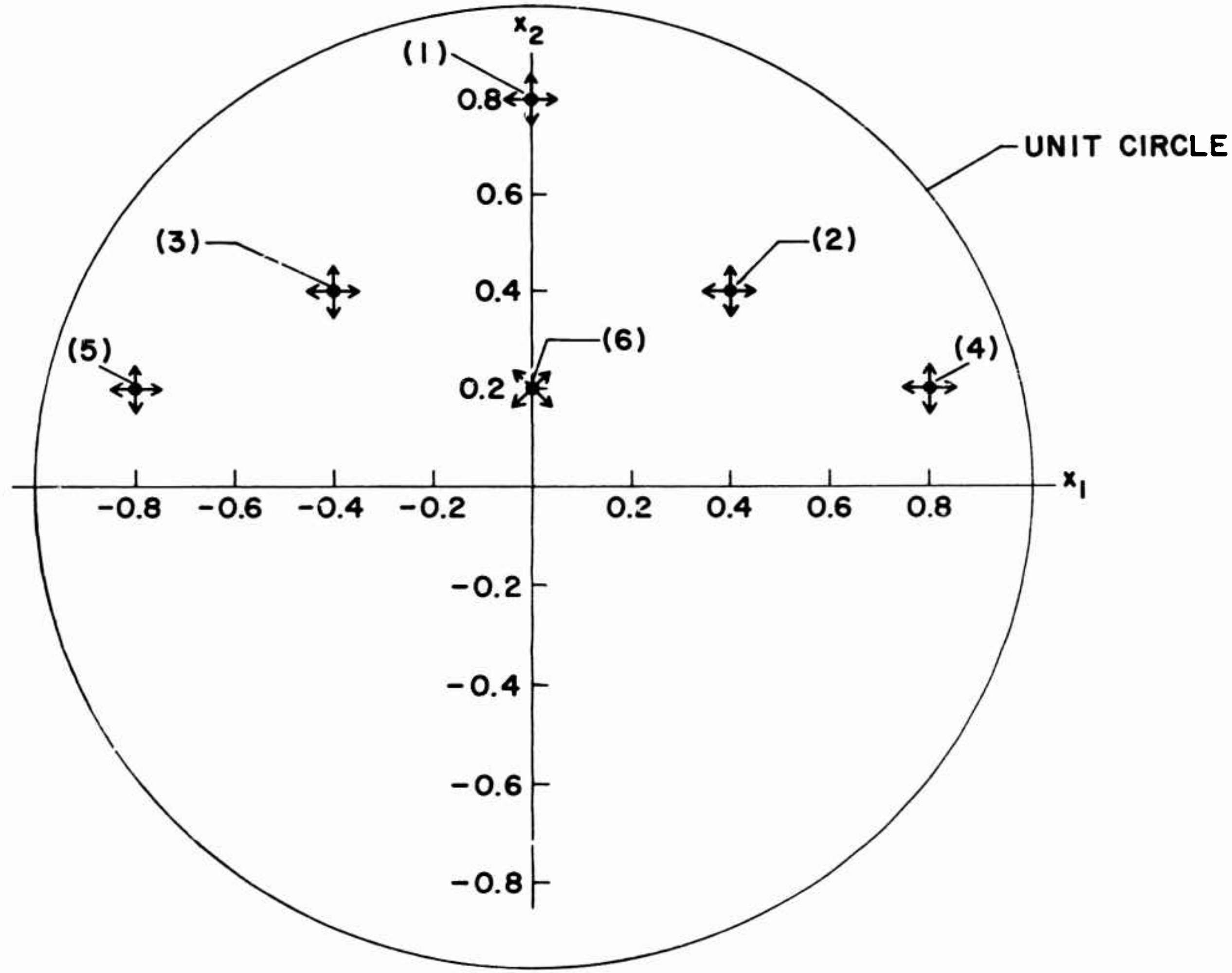
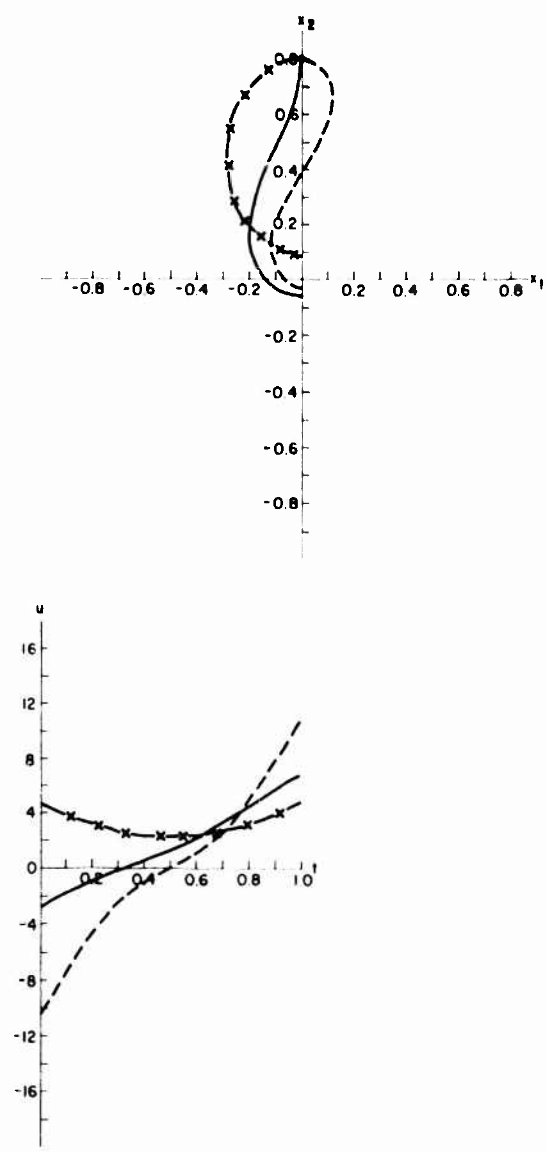


Figure 6

TRAJECTORIES AND CONTROL FOR POINT I

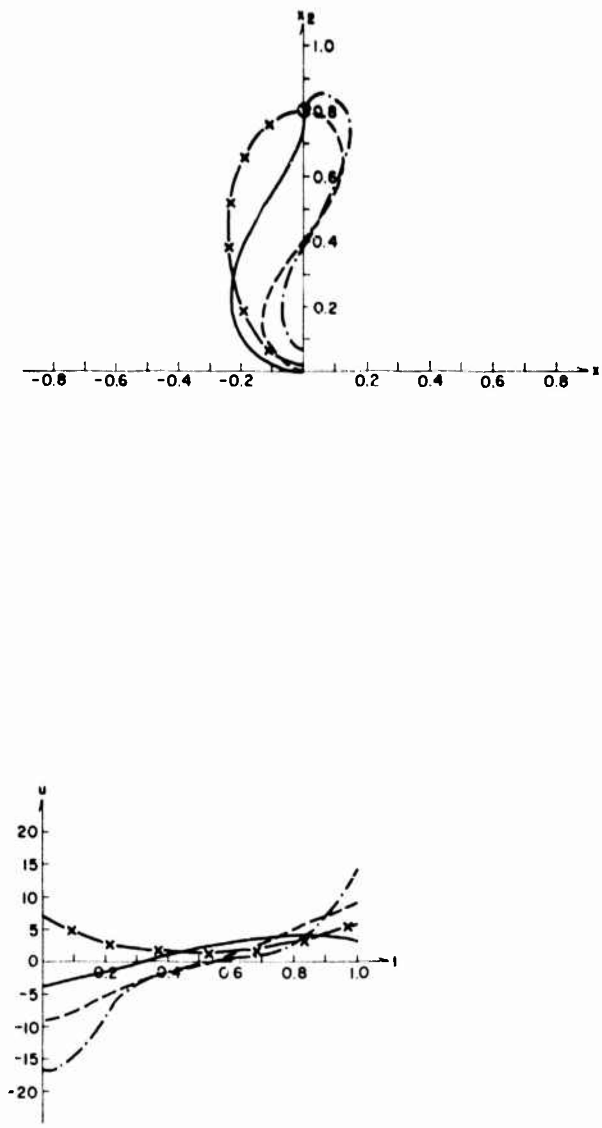
$$\text{POINT I} \triangleq \left[\begin{array}{ll} X_1(0) = 0 & \\ X_2(0) = 0.8 & \\ X_3(0) = \left\}\begin{array}{l} -90^\circ \\ 0^\circ \\ 90^\circ \\ 180^\circ \end{array}\right. & \end{array}\right.$$

- -90° ——————
- 0° — — — —
- 90° —— · ——
- 180° ——x——



ONE TO ONE WEIGHTING

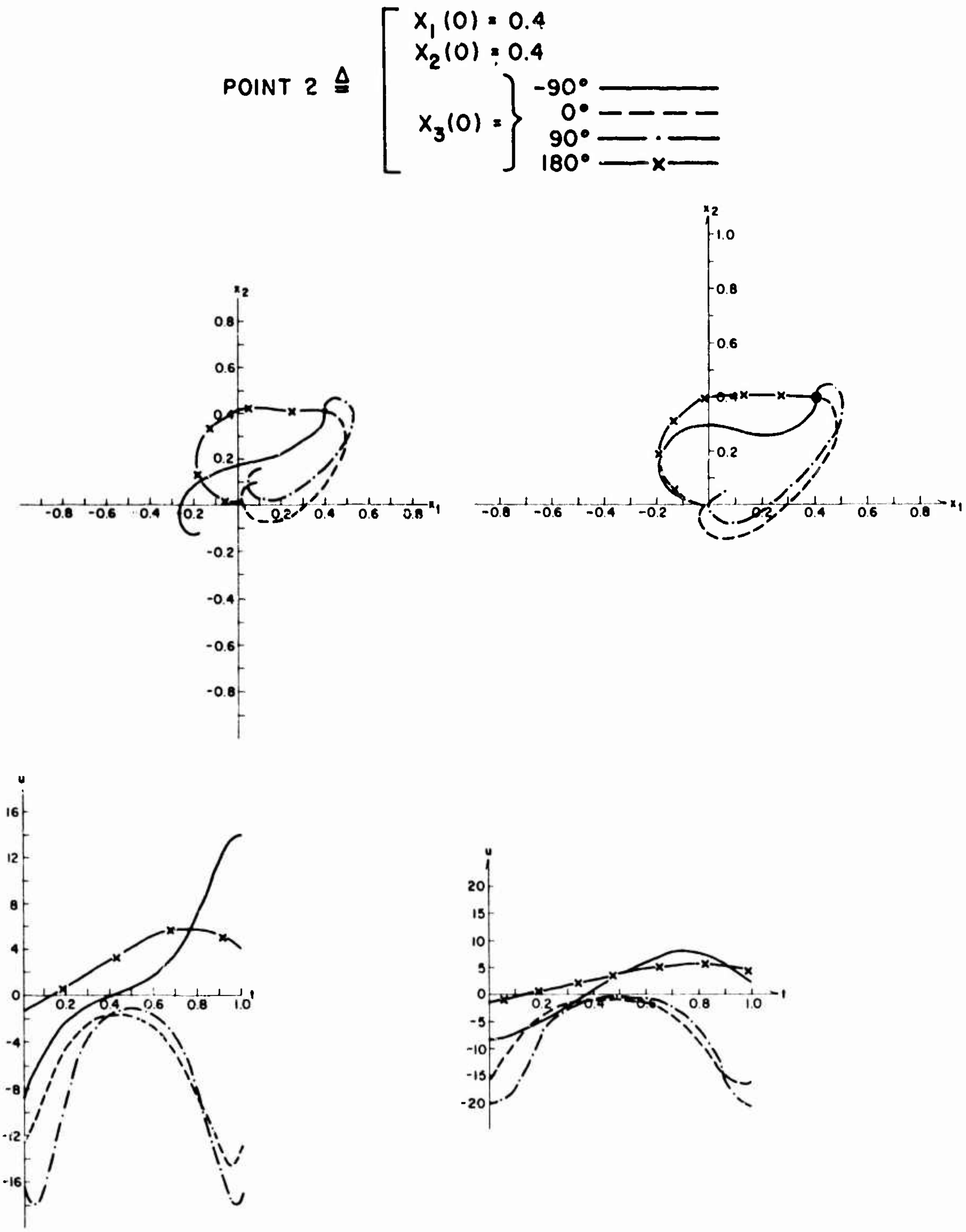
Figure 7a



TEN TO ONE WEIGHTING

Figure 7b

TRAJECTORIES AND CONTROL FOR POINT 2



ONE TO ONE WEIGHTING

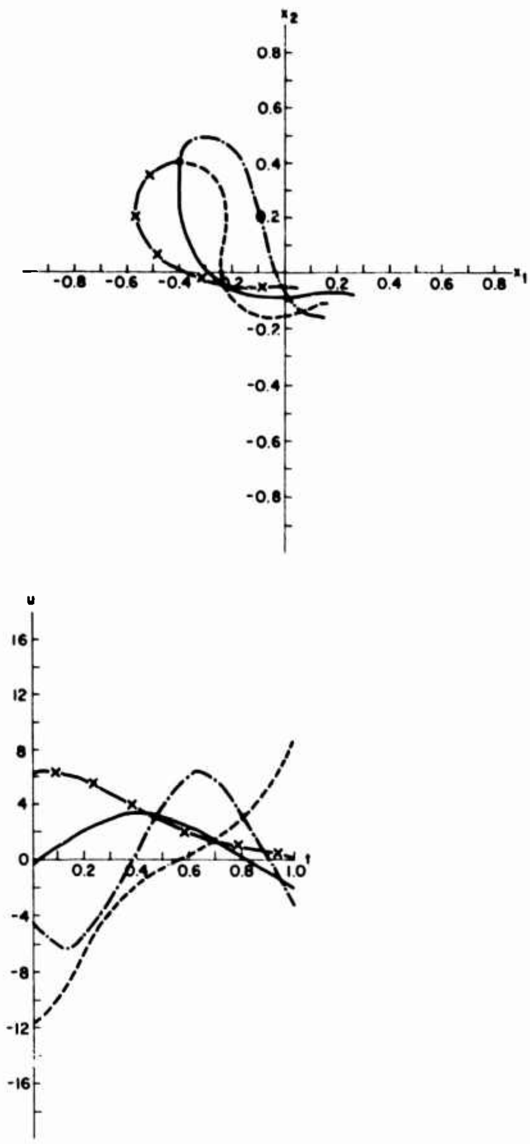
Figure 8a

TEN TO ONE WEIGHTING

Figure 8b

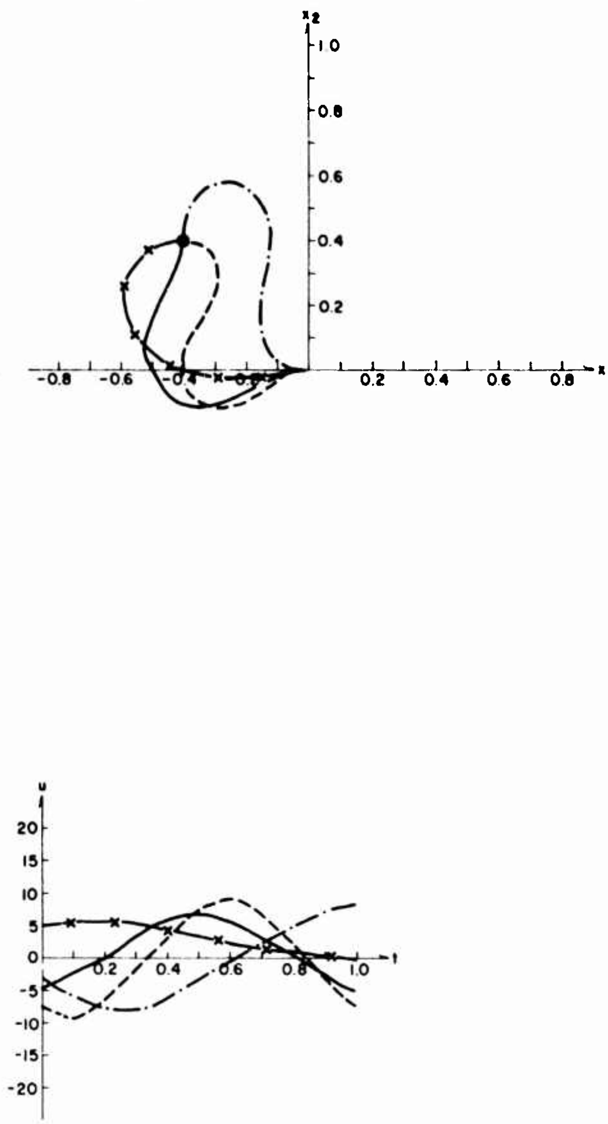
TRAJECTORIES AND CONTROL FOR POINT 3

POINT 3 $\triangleq$ $\left[\begin{array}{l} X_1(0) = -0.4 \\ X_2(0) = 0.4 \\ X_3(0) = \left\{\begin{array}{l} -90° \\ 0° \\ 90° \\ 180° \end{array}\right. \end{array}\right.$



ONE TO ONE WEIGHTING

Figure 9a

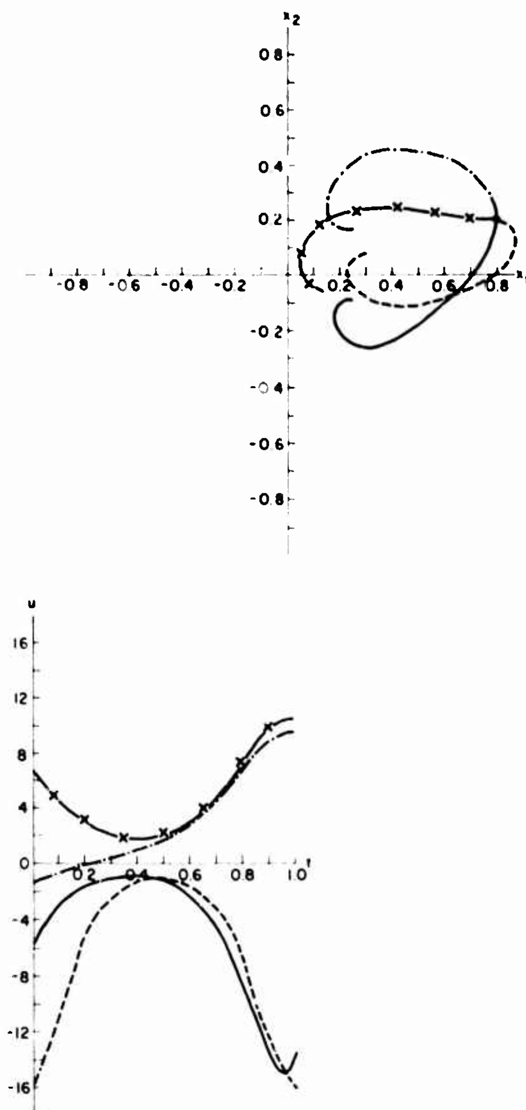


TEN TO ONE WEIGHTING

Figure 9b

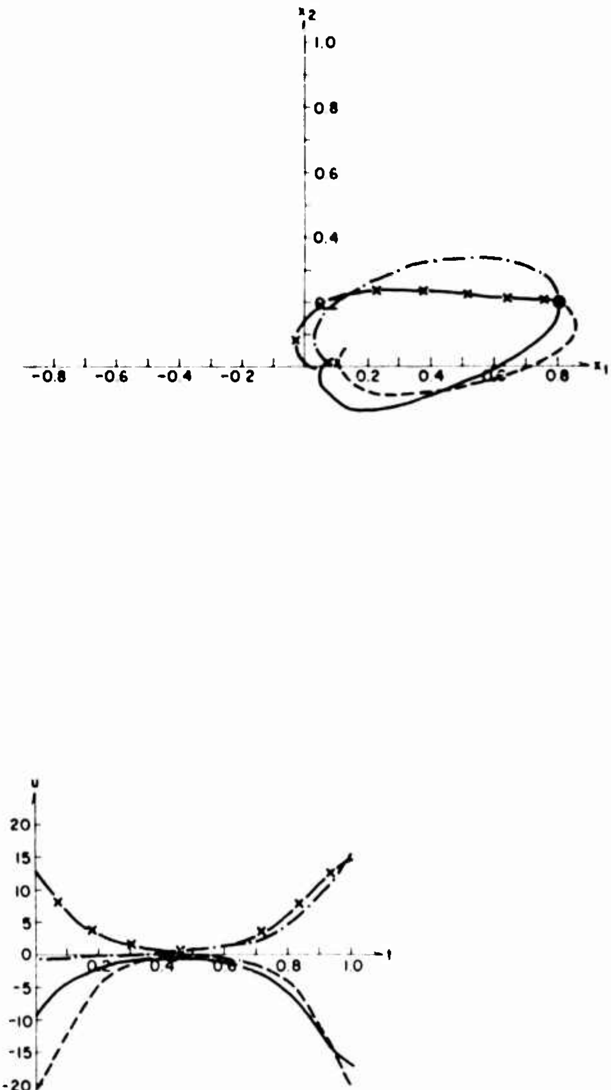
TRAJECTORIES AND CONTROL FOR POINT 4

POINT 4 $\triangleq \left[\begin{array}{l} X_1(0) = 0.8 \\ X_2(0) = 0.2 \\ X_3(0) = \left\} \begin{array}{l} -90° \text{ ———} \\ 0° \text{ – – –} \\ 90° \text{ —·—} \\ 180° \text{ —x—} \end{array} \right. \end{array} \right.$



ONE TO ONE WEIGHTING

Figure 10a

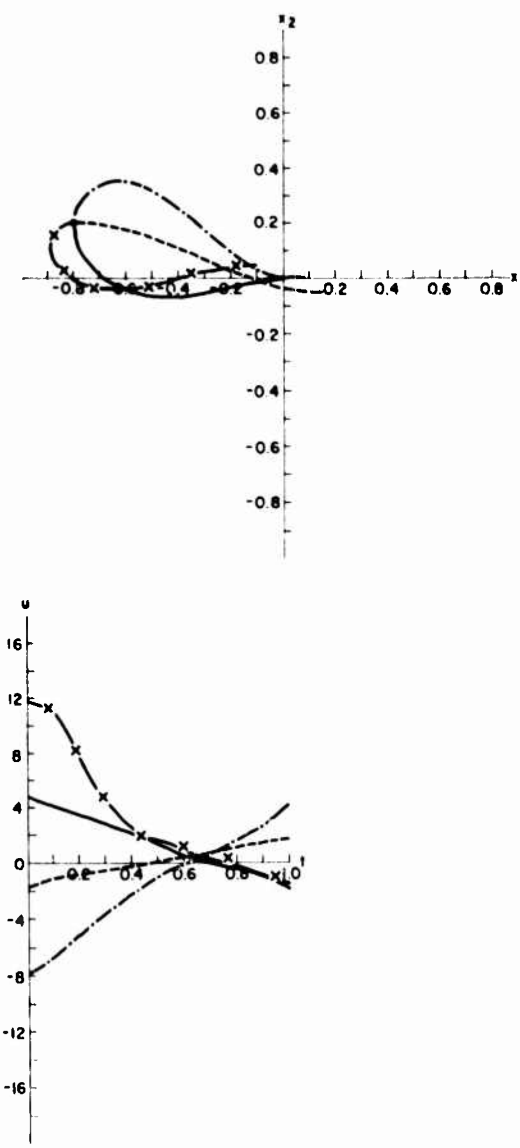


TEN TO ONE WEIGHTING

Figure 10b

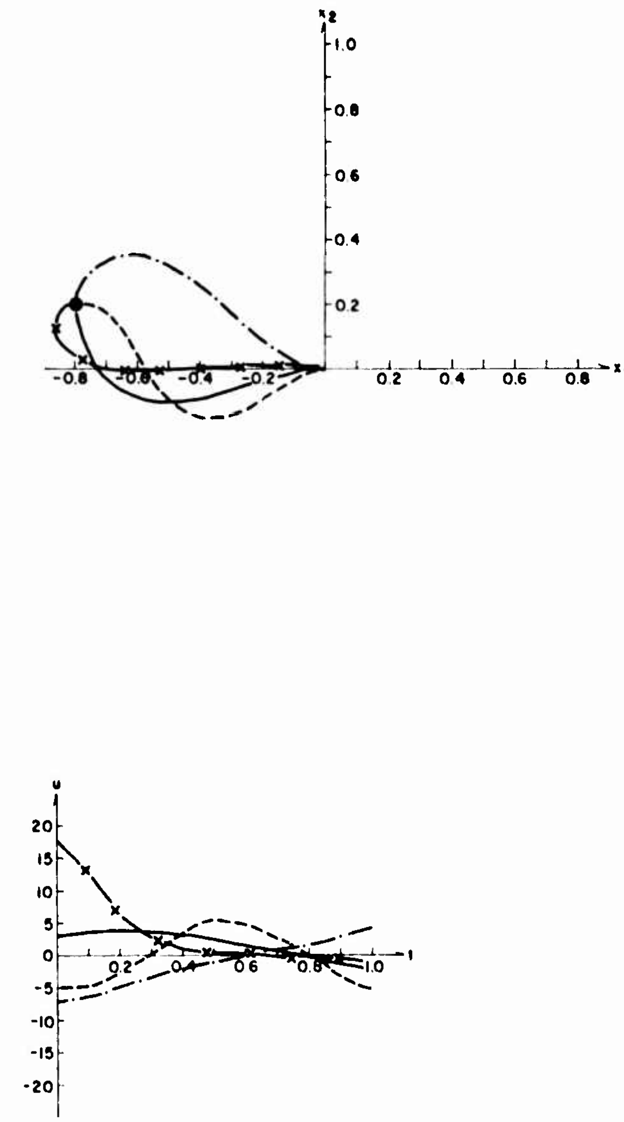
TRAJECTORIES AND CONTROL FOR POINT 5

POINT 5 $\left[\begin{array}{l} X_1(0) = -0.8 \\ X_2(0) = 0.2 \\ X_3(0) = \left\} \begin{array}{l} -90° \\ 0° \\ 90° \\ 180° \end{array}\right. \end{array}\right.$



ONE TO ONE WEIGHTING

Figure 11a

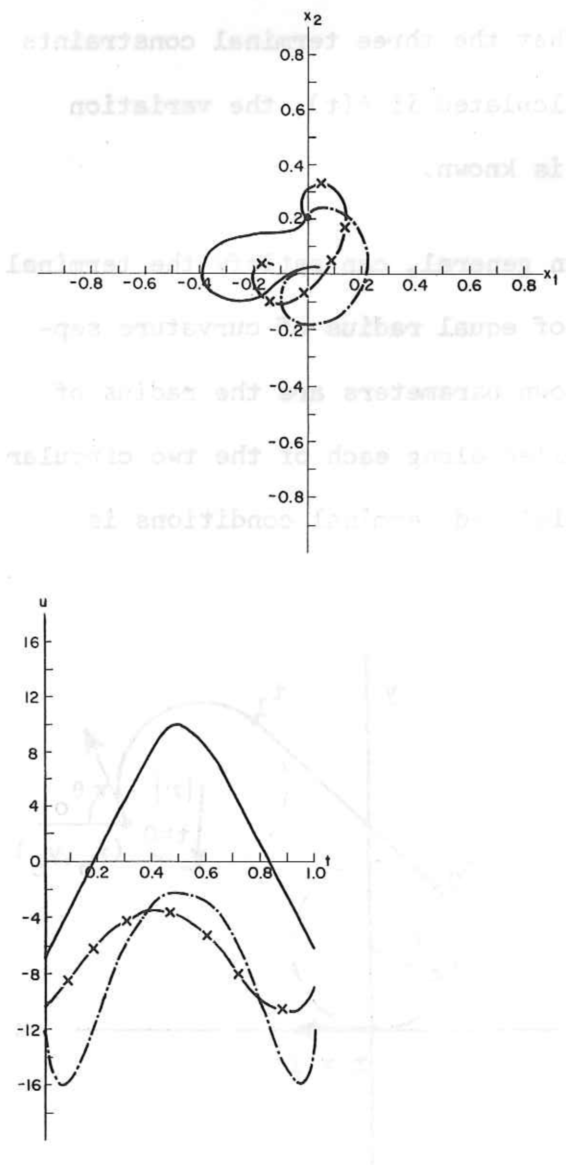


TEN TO ONE WEIGHTING

Figure 11b

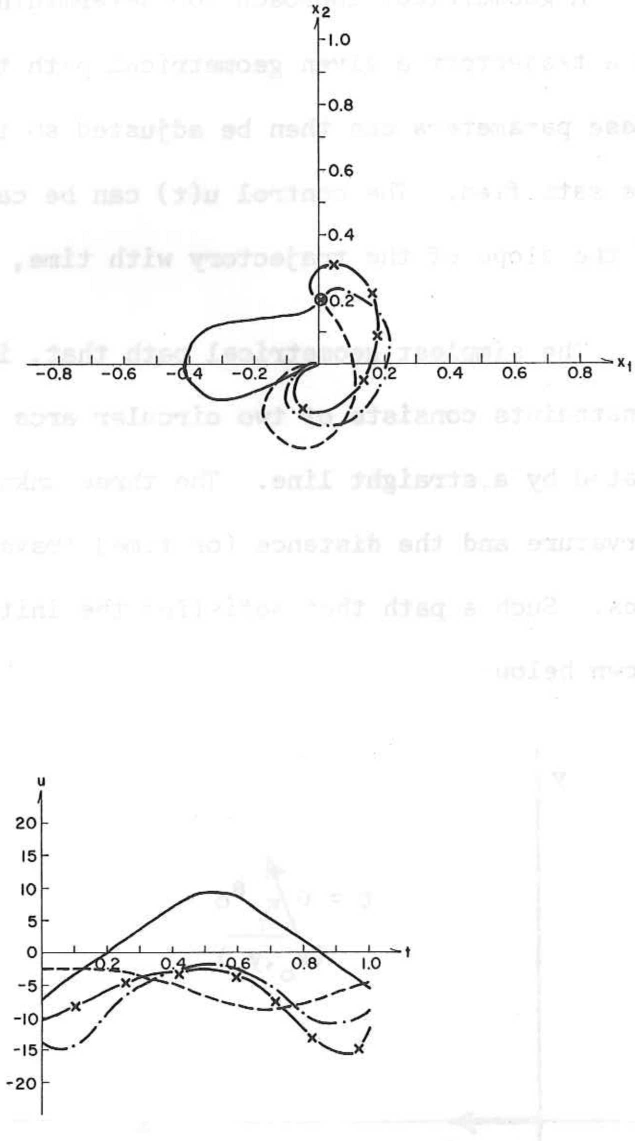
TRAJECTORIES AND CONTROL FOR POINT 6

$$\text{POINT 6} \triangleq \left[\begin{array}{l} X_1(0) = 0.0 \\ X_2(0) = 0.2 \\ X_3(0) = \left\{\begin{array}{l} -135° \\ -45° \\ 45° \\ 135° \end{array}\right. \end{array}\right.$$



ONE TO ONE WEIGHTING

Figure 12a



TEN TO ONE WEIGHTING

Figure 12b

VI. A GEOMETRICAL APPROACH TO TRAJECTORY DETERMINATION

$$\dot{x} = \cos\theta \qquad x(0) = x_o \qquad y(0) = y_o \qquad \theta(0) = \theta_o$$

$$\dot{y} = \sin\theta \qquad \text{Determine } u(t) \text{ to satisfy} \tag{77}$$

$$\dot{\theta} = u \qquad x(1) = 0 \qquad y(1) = 0 \qquad \theta(1) = 2n\pi$$

A geometrical approach for determining trajectories is to prescribe as a trajectory a given geometrical path that has three unknown parameters. These parameters can then be adjusted so that the three terminal constraints are satisfied. The control u(t) can be calculated if θ(t), the variation of the slope of the trajectory with time, is known.

The simplest geometrical path that, in general, can satisfy the terminal constraints consists of two circular arcs of equal radius of curvature separated by a straight line. The three unknown parameters are the radius of curvature and the distance (or time) travelled along each of the two circular arcs. Such a path that satisfies the initial and terminal conditions is shown below:

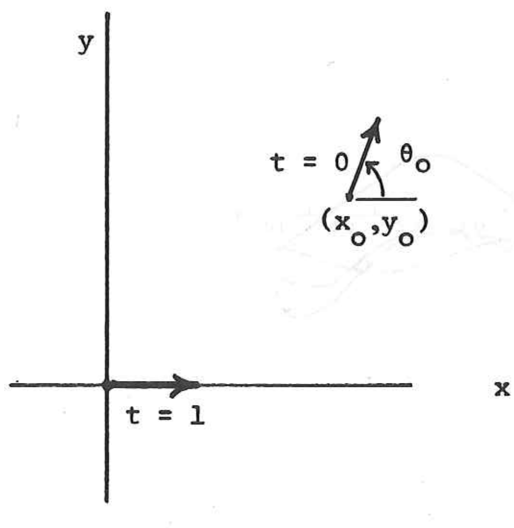


Figure 13

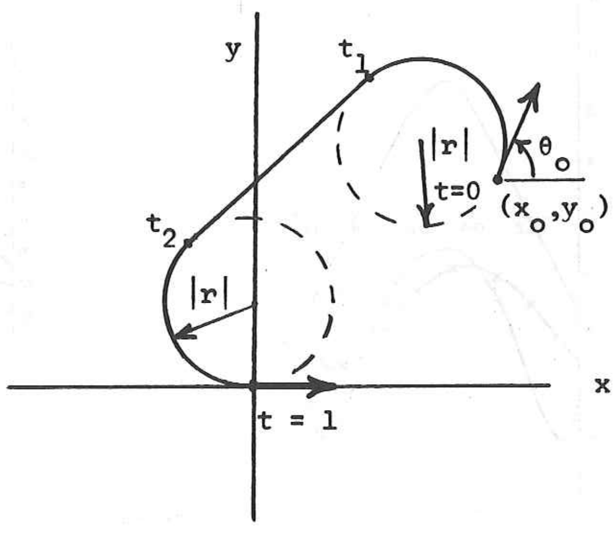


Figure 14

On a straight line path, $\theta(t)$ is constant and hence $u(t) = 0$. On a circular arc whose radius of curvature is r, we have $u(t) = 1/r$, where

if $r > 0$, the arc is traversed counter-clockwise (since $\dot{\theta} > 0$)

if $r < 0$, the arc is traversed clockwise (since $\dot{\theta} < 0$).

To see this, note that if $u(t) = 1/r$, then $\theta(t) = 1/r\, t + \theta_o$.

$$\dot{x} = \cos\left(\frac{1}{r} t + \theta_o\right) \qquad \dot{y} = \sin\left(\frac{1}{r} t + \theta_o\right)$$

$$x(t) = r \sin\left(\frac{1}{r} t + \theta_o\right) + a \qquad y(t) = -r \cos\left(\frac{1}{r} t + \theta_o\right) + b$$

Thus $(x-a)^2 + (y-b)^2 = r^2$, which is the equation of a circle of radius r.

In this connection, a control law is considered of the form

$$u(t) = \begin{cases} \frac{1}{r}, & 0 \leq t < t_1 \\ 0 & t_1 \leq t \leq t_2 \\ \pm \frac{1}{r}, & t_2 < t \leq 1 \end{cases} \tag{78}$$

i.e., a "bang-zero-bang" control law, where r can be either positive or negative. Then

$$\theta(t) = \begin{cases} \frac{1}{r} t + \theta_o, & 0 \leq t < t_1 \\ \frac{1}{r} t_1 + \theta_o, & t_1 \leq t \leq t_2 \\ \pm \frac{1}{r}(t \pm t_1 - t_2) + \theta_o, & t_2 < t \leq 1 . \end{cases} \tag{79}$$

To satisfy the terminal constraint $\theta(1) = 2n\pi$ it is necessary that

$$\pm \frac{1}{r}(1 \pm t_1 - t_2) + \theta_o = 2n\pi . \tag{80}$$

To simplify the following calculation let $z = x + jy$. Then $\dot{z} = \dot{x} + j\dot{y} = \cos\theta + j \sin\theta = e^{j\theta}$. Integrating from $t = 0$ to $t = 1$ results in

$$z(1) - z(0) = \int_o^1 e^{j\theta(t)}dt$$

$$= \int_o^{t_1} e^{j(\frac{1}{r}t+\theta_o)}dt + \int_{t_1}^{t_2} e^{j(\frac{1}{r}t_1+\theta_o)}dt + \int_{t_2}^{1} e^{j(\pm\frac{1}{r}[t\pm t_1-t_2]+\theta_o)}dt$$

$$= \left[\frac{r}{j} + (t_2-t_1) \mp \frac{r}{j}\right] e^{j(\frac{1}{r}t_1+\theta_o)} - \frac{r}{j}\left[e^{j\theta_o} \mp 1\right] \qquad (81)$$

since, from (80), $e^{j(\pm\frac{1}{r}[1\pm t_1-t_2]+\theta_o)} = e^{j2n\pi} = 1$.

Applying the terminal constraint $z(1) = 0$ and multiplying through by j yields

$$jx_o - y_o + [r \mp r + j(t_2-t_1)]e^{j(\frac{1}{r}t_1+\theta_o)} - r(e^{j\theta_o} \mp 1) = 0 \qquad (82)$$

or, explicitly setting both real and imaginary parts to zero, we have

$$-y_o + (r\mp r)\cos(\frac{1}{r}t_1+\theta_o) - (t_2-t_1)\sin(\frac{1}{r}t_1+\theta_o) - r(\cos\theta_o \mp 1) = 0 \qquad (83)$$

$$x_o + (r\mp r)\sin(\frac{1}{r}t_1+\theta_o) + (t_2-t_1)\cos(\frac{1}{r}t_1+\theta_o) - r\sin\theta_o = 0 \quad . \qquad (84)$$

The upper solution (the one corresponding to the + sign in (78)) represents trajectories in which the circular arcs are both traversed in the same sense, while the lower solution represents trajectories in which they are traversed in opposite senses.

A simplification results when the upper solution is used, as the second term in Eqs. (83) and (84) drops out. This simplification is essential for decoupling r from t_1 and t_2.

Therefore, considering only the upper solution, Eqs. (80), (83), (84) form a system of 3 equations for r, t_1, t_2, which can be written as

$$
\begin{aligned}
\frac{1}{r}[1 - (t_2 - t_1)] &= 2n\pi - \theta_o \\
r \sin \theta_o - x_o &= (t_2 - t_1)\cos(\frac{1}{r} t_1 + \theta_o) \\
r(1 - \cos \theta_o) - y_o &= (t_2 - t_1)\sin(\frac{1}{r} t_1 + \theta_o) \quad .
\end{aligned} \tag{85}
$$

A physically acceptable trajectory is any solution to (85) in which

$$
\begin{aligned}
&\text{(i)} \quad t_1 \geq 0 \\
&\text{(ii)} \quad t_2 \leq 1 \\
&\text{(iii)} \quad t_2 - t_1 \geq 0 \\
&\text{(iv)} \quad r \neq 0
\end{aligned} \tag{86}
$$

The radius of curvature, r, can be decoupled as follows: Define the net angle traversed as $\psi_n \triangleq 2n\pi - \theta_o$ and the angle of inclination of the straight line segment as $\theta^* = \frac{1}{r} t_1 + \theta_o$. From the first member of Eqs. (85),

$$
1 - r\psi_n = t_2 - t_1 \quad . \tag{87}
$$

Squaring the second and third members of (85) and summing yields

$$
[r \sin \theta_o - x_o]^2 + [r(1 - \cos \theta_o) - y_o]^2 = (t_2 - t_1)^2 = (1 - r\psi_n)^2
$$

or

$$
[\psi_n^2 - 2(1 - \cos\theta_o)]r^2 - 2[\psi_n - y_o(1 - \cos\theta_o) - x_o\sin\theta_o]r + (1 - x_o^2 - y_o^2) = 0. \tag{88}
$$

In polar coordinates, $x_o = \rho_o \cos \phi_o$ and $y_o = \rho_o \sin \phi_o$.

Defining

$$
\begin{aligned}
a &\triangleq \psi_n^2 - 2(1 - \cos \theta_o) \\
b &\triangleq \psi_n - y_o(1 - \cos\theta_o) - x_o \sin \theta_o = \psi_n + \rho_o[\sin (\phi_o - \theta_o) - \sin \phi_o] \\
c &\triangleq 1 - x_o^2 - y_o^2 = 1 - \rho_o^2
\end{aligned} \tag{89}
$$

then

$$ar^2 - 2br + c = 0 \tag{90}$$

is a quadratic equation which r must satisfy. The coefficients a, b, c are determined from the initial conditions and the specification of the integer n in $\psi_n = 2n\pi - \theta_o$.

Some observations concerning the coefficients may be noted. First, for initial conditions within the unit circle, $c > 0$. Second, "a" is a positive definite function of ψ_n, as shown in Fig. 15.

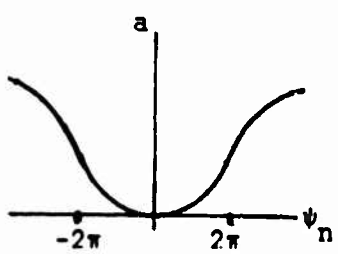


Figure 15

$$a = \psi_n^2 - 2(1-\cos\theta_o) = \psi_n^2 - 2(1-\cos\psi_n) \tag{91}$$

$$\frac{da}{d\psi_n} = 2(\psi_n - \sin\psi_n) \tag{92}$$

$$\frac{d^2a}{d\psi_n^2} = 2(1 - \cos\psi_n) \geq 0 \ . \tag{93}$$

The only critical point is at $\psi_n = 0$ and it is a minimum. There is a point of inflection when ψ_n is a multiple of 2π. Hence, "a" must have the shape of a positive definite function of ψ_n.

The solution to (90) is

$$r = \frac{b \pm \sqrt{b^2 - ac}}{a} \quad . \tag{94}$$

As long as $\psi_n \neq 0$, then $a \neq 0$.

If

$-\sqrt{ac} < b < \sqrt{ac}$, then two complex roots occur

$b \geq \sqrt{ac}$, then two positive roots occur

$b \leq -\sqrt{ac}$, then two negative roots occur.

Of course, complex roots have no physical significance. Note that a given value of n determines two values of r. From (85) and (87), θ^* must satisfy

$$\cos\theta^* = \frac{r\sin\theta_o - \rho_o\cos\phi_o}{1 - r\psi_n}$$
$$\sin\theta^* = \frac{r(1-\cos\theta_o) - \rho_o\sin\phi_o}{1 - r\psi_n} \tag{95}$$

The value of θ^* is unique to within arbitrary multiples of 2π. The times t_1 and t_2 are then determined from r and θ^* as

$$t_1 = r(\theta^* - \theta_o) \tag{96}$$

$$t_2 = 1 - r(2n\pi - \theta^*) \ . \tag{97}$$

Carrying out this procedure for all values of n will yield all possible solutions to (85). However, only those solutions which also satisfy conditions (86) are acceptable. A specific computational algorithm which enumerates all acceptable trajectories is developed.

The energy expended by a trajectory is given by

$$E = \int_o^1 u^2\,dt = \frac{t_1}{r^2} + \frac{1-t_2}{r^2} = \frac{1}{r^2}[1-(t_2-t_1)]$$
$$= \frac{1}{r^2}(r\psi_n) = \frac{\psi_n}{r} \quad . \tag{98}$$

Of all acceptable trajectories, the one(s) with the smallest value of E can be chosen.

Bounds on the energy can be expressed by noting that i, ii, iii of (86) imply that $0 \leq t_2 - t_1 \leq 1$, or $0 \leq r\psi_n \leq 1$, from (87). Hence

$$\psi_n^2 \leq E \leq \frac{1}{r^2} \tag{99}$$

<u>Computational Algorithm</u>

Restrict θ_o to the interval $-\pi < \theta_o \leq \pi$. There is no loss in generality by doing this since n is variable. Assume $\rho_o < 1$. Then there cannot be any "straight line" trajectories, i.e., trajectories for which $\psi_n \triangleq 2n\pi - \theta_o$ is zero, since such a trajectory must start from $\rho_o = 1$ in order to have unit length (there is only one solution for an initial condition on the boundary of the unit circle, which occurs when $\rho_o = 1$, $\phi_o = \pi$, $\theta_o = 0$).

1. Choose a value of n and compute $\psi_n = 2n\pi - \theta_o$. It is not necessary to consider a value of n for which $\psi_n = 0$. Thus, $a > 0$.

2. Compute

$$\begin{aligned} a &= \psi_n^2 - 2(1 - \cos\theta_o) && (a > 0) \\ b &= \psi_n + \rho_o\left[\sin(\phi_o - \theta_o) - \sin\phi_o\right] && \\ c &= 1 - \rho_o^2 && (c > 0) \end{aligned} \tag{100}$$

and $r = \dfrac{b \pm \sqrt{b^2 - ac}}{a}$.

If $b^2 - ac < 0$, eliminate that value of r. Since $ac > 0$, $r \neq 0$ and condition (iv) of (86) is satisfied. From (87), we have $1 - r\psi_n = t_2 - t_1$ and hence a physically acceptable trajectory must have $0 \leq r\psi_n \leq 1$. If this is not satisfied, eliminate that value of r. Hence condition (iii) of (86) is now satisfied.

3. If $r\psi_n = 1$, then we have a "circular" trajectory, i.e., a trajectory which consists solely of a circular arc. For such a trajectory, $u = \frac{1}{r} = \psi_n$, $0 \leq t \leq 1$, and $E = \psi_n/r = \psi_n^2$ (the values of θ^*, t_1 and t_2 are unimportant). If $r\psi_n \neq 1$, solve

$$\cos\theta^* = \frac{r\sin\theta_o - \rho_o\cos\phi_o}{1 - r\psi_n} \triangleq f_1 \tag{101}$$

$$\sin\theta^* = \frac{r(1-\cos\theta_o) - \rho_o\sin\phi_o}{1 - r\psi_n} \triangleq f_2$$

simultaneously for θ^*. There will be a solution for θ^* since (90) guarantees that $f_1^2 + f_2^2 = 1$. The value of θ^* is unique to within arbitrary multiples of 2π and can be expressed as

$$\theta^* = \begin{cases} (\operatorname{sgn} f_2)\cos^{-1} f_1, & f_1 \neq -1 \\ \pi, & f_1 = -1 \end{cases} \tag{102}$$

where the value of the function $\cos^{-1} f_1$ is taken in the principal value interval $[0,\pi]$. Hence, we have $-\pi < \theta^* \leq \pi$.

4. Consider the range of possible values for θ^* by adjusting θ^* via

$$\theta_k^* = \theta^* + 2\pi k\ , \quad k \text{ an integer} \tag{103}$$

and compute the switching times

$$t_1^{(k)} = r(\theta_k^* - \theta_o), \quad t_2^{(k)} = 1 - r(2n\pi - \theta_k^*) \tag{104}$$

Check that $t_1^{(k)} \geq 0$ and $t_2^{(k)} \leq 1$ for each k. Because of this restriction, it is necessary to examine k only in the interval $[0,n]$ if $r > 0$ or $[n,0]$ if $r < 0$. Now conditions (i) and (ii) of (86) are satisfied, and solutions to (85) have been constructed which satisfy all conditions of (86).

5. Choose another value for n and repeat steps 1 through 5.

6. Compute $E = \psi_n/r$ for all acceptable trajectories and select the one(s) with the smallest value of E.

Numerical Results

A FORTRAN computer program named GEO was written which enumerates all physically acceptable trajectories using the above algorithm.

It is not apparent as to what range of values for n should be examined. The numerical results given for the program GEO used the interval [-2,2] as the range of values for n. From the computational experience it seems that this range is sufficient, since all minimum energy trajectories corresponded to n in the interval [-1,1].

The minimum energy trajectories, computed for the same set of initial conditions as in Table 1, are given in Table 2. Plots of the trajectories in the x-y plane and the control variables are depicted in Figs. 16-21. For some initial conditions, there were two or more trajectories that yielded the same minimum energy; only one of these is shown in Table 2. It is interesting to compare the data in the energy column for Table 1 and Table 2. For most initial conditions, as expected, parameter optimization resulted in trajectories having smaller control energies than the geometrical method. However, there were two cases in which the reverse was true. This may have occurred because parameter optimization led to a local minimum of $F(\underset{\sim}{\alpha})$, or because the numerical procedure was not carried out to a sufficient number of iterations.

The largest absolute value of the control on the time interval $0 \leq t \leq 1$ is perhaps a more meaningful index of performance than the control energy. Except for the initial condition $\rho_o = .2$, $\phi_o = 90°$, $\theta_o = 135°$, the trajectory with the minimum value of E also had the minimum value of $|u|$. In this case, there was a trajectory beside the one given in Table 2 with a slightly higher value of E (E = 113.14), but a slightly lower value of $|u|$ (u = 11.08).

```
      DIMENSION R(2),E(2),G(2),F(2),TSTAR(2),TSTARK(2,20),T1(2,20),T2(2,
     C20),U(2)
      WRITE(6,1)
      READ(5,2) RHO,PHI,THETA
      TWOPI=2.*ARCOS(-1.0)
      WRITE(6,3)
      N1=5
      DO 19 NN=1,N1
      N=NN-(1+N1)/2
      FN=FLOAT(N)
      PSIN=TWOPI*FN-THETA
      IF(PSIN.EQ.0.0) GO TO 19
      IF(SIN(THETA)-PSIN*RHO*COS(PHI).NE.0.0) GO TO 7
      IF(1.0-COS(THETA)-PSIN*RHO*SIN(PHI).NE.0.0) GO TO 7
      IF(IABS(N).GE.2) GO TO 19
      R(1)=1.0/PSIN
      U(1)=PSIN
      E(1)=PSIN**2
      WRITE(6,25) N,PSIN,R(1),U(1),E(1)
   25 FORMAT(1H ,5X,I2,5X,2(F9.4,5X),17X,'T1=T2',20X,2(F9.4,5X))
      GO TO 19
    7 A=PSIN**2-2.0*(1.0-COS(THETA))
      B=PSIN+RHO*(SIN(PHI-THETA)-SIN(PHI))
      C=1.0-RHO**2
      D=B**2-A*C
      IF(D.LT.0.0) GO TO 19
      R(1)=(B+SQRT(D))/A
      R(2)=(B-SQRT(D))/A
      IF(D.NE.0.0) GO TO 15
      J=1
      GO TO 8
   15 J1=0
      J2=0
      IF(R(1)*PSIN.LE.0.0) GO TO 5
      IF(R(1)*PSIN.GE.1.0) GO TO 5
      J1=1
    5 IF(R(2)*PSIN.LE.0.0) GO TO 6
      IF(R(2)*PSIN.GE.1.0) GO TO 6
      J2=1
    6 J=J1+J2
      IF(J.EQ.0) GO TO 19
      IF(J.EQ.2) GO TO 8
      IF(J1.EQ.1) GO TO 8
      R(1)=R(2)
    8 DO 9 I=1,J
      U(I)=1./R(I)
      E(I)=PSIN/R(I)
      F(I)=(R(I)*SIN(THETA)-RHO*COS(PHI))/(1.-R(I)*PSIN)
      G(I)=(R(I)*(1.-COS(THETA))-RHO*SIN(PHI))/(1.-R(I)*PSIN)
      IF(F(I).EQ.-1.0) GO TO 10
      TSTAR(I)=SIGN(1.0,G(I))*ARCOS(F(I))
      GO TO 11
   10 TSTAR(I)=ARCOS(F(I))
   11 K1=IABS(N)+1
      DO 12 KK=1,K1
      IF(R(I)) 16,16,18
   16 K=KK-1+N
      GO TO 20
   18 K=KK-1
   20 TSTARK(I,K)=TSTAR(I)+TWOPI*FLOAT(K)
      T1(I,K)=R(I)*(TSTARK(I,K)-THETA)
      T2(I,K)=1.0-R(I)*(TWOPI*FN-TSTARK(I,K))
      IF(T1(I,K).LT.0.0) GO TO 12
      IF(T2(I,K).GT.1.0) GO TO 12
   17 WRITE(6,4) N,PSIN,R(I),T1(I,K),T2(I,K),TSTARK(I,K),U(I),E(I)
   12 CONTINUE
    9 CONTINUE
   19 CONTINUE
      STOP
    1 FORMAT(1H0,2X,'RHO(0)',4X,'PHI(0)',3X,'THETA(0)')
    2 FORMAT(3F10.5)
    3 FORMAT(1H0,6X,'N',9X,'PSI N',11X,'R',13X,'T1',12X,'T2',6X, 'THETA
     CSTAR K',7X,'U',13X,'E',/)
    4 FORMAT(1H ,5X,I2,5X,7(F9.4,5X))
      END
```

Table 2

Minimum Energy Trajectories For Some Initial Conditions As Table 1

ρ_o	ϕ_o	θ_o	n	ψ_n	r	θ_k^*	t_1	t_2	u	E
.8	90°	0°	1	360°	0.032	270°	0.150	0.950	31.42	†197.39
(x_o=0, y_o=.8)		90°	1	270°	0.053	274°	0.171	0.920	18.74	88.31
		180°	1	180°	0.175	270°	0.275	0.725	5.71	17.93
		-90°	1	450°	0.029	268°	0.182	0.953	34.36	269.90
.5657	45°	0°	1	360°	0.069	225°	0.271	0.837	14.47	†90.90
(x_o=.4, y_o=.4)		90°	1	270°	0.132	225°	0.310	0.690	7.59	35.79
		180°	1	180°	0.191	183°	0.009	0.409	5.24	16.46
		-90°	-1	-270°	-0.089	-122°	0.050	0.630	-11.21	52.82
.5657	135°	0°	1	360°	0.069	315°	0.380	0.946	14.47	†90.90
(x_o=-.4, y_o=.4)		90°	1	270°	0.089	328°	0.370	0.950	11.21	52.82
		180°	1	180°	0.191	357°	0.591	0.991	5.24	16.46
		-90°	1	450°	0.067	315°	0.477	0.947	14.83	†116.46
.8246	14.04°	0°	1	360°	0.028	194°	0.094	0.919	35.82	†225.08
(x_o=.8, y_o=.2)		90°	1	270°	0.050	191°	0.088	0.853	20.05	94.49
		180°	1	180°	0.063	185°	0.006	0.809	15.99	50.22
		-90°	-1	-270°	-0.044	-162°	0.055	0.849	-22.95	108.14
.8246	165.96°	0°	1	360°	0.028	346°	0.169	0.993	35.82	†225.08
(x_o=-.8, y_o=.2)		90°	1	270°	0.032	349°	0.145	0.994	31.17	146.90
		180°	1	180°	0.063	355°	0.191	0.994	15.99	50.22
		-90°	1	450°	0.026	347°	0.201	0.994	37.91	†297.78
.2	90°	45°	1	315°	0.148	304°	0.666	0.855	6.77	37.24
(x_o=0, y_o=.2)		135°	-1	-495°	-0.077	-99°	0.314	0.650	-13.01	112.36
		-45°	-1	-315°	-0.135	-68°	0.055	0.313	-7.41	40.73
		-135°	1	495°	0.107	193°	0.612	0.689	9.37	80.91

†Trajectory with minimum energy is not unique.

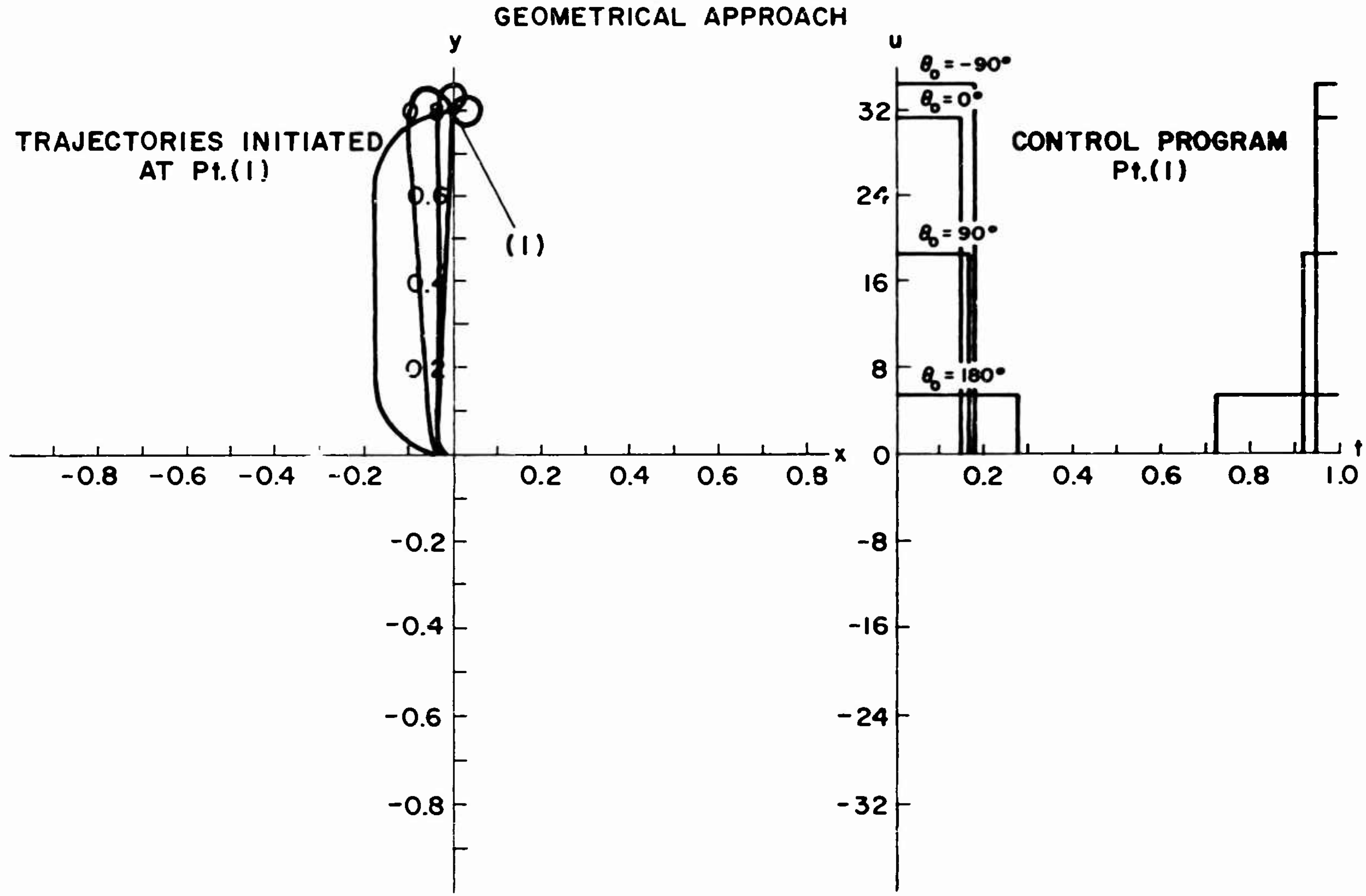

GEOMETRICAL APPROACH
TRAJECTORIES INITIATED
AT Pt.(1)
(1)
y
x
0.6
0.4
0.2
-0.8
-0.6
-0.4
-0.2
0.2
0.4
0.6
0.8
-0.2
-0.4
-0.6
-0.8
CONTROL PROGRAM
Pt.(1)
u
t
$\theta_0 = -90°$
$\theta_0 = 0°$
$\theta_0 = 90°$
$\theta_0 = 180°$
32
24
16
8
0
-8
-16
-24
-32
0.2
0.4
0.6
0.8
1.0

Figure 16

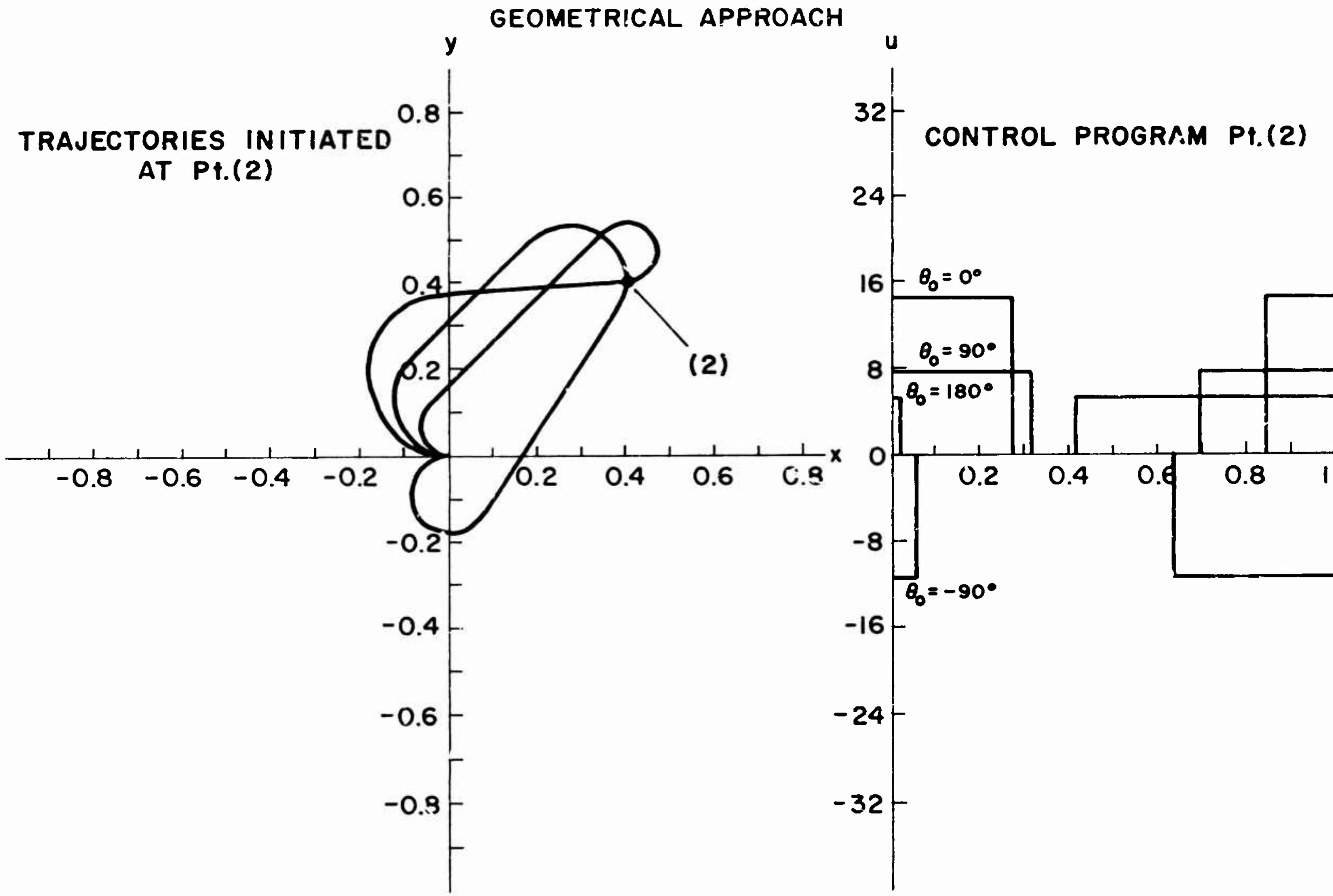

GEOMETRICAL APPROACH
TRAJECTORIES INITIATED AT Pt.(2)
(2)
CONTROL PROGRAM Pt.(2)
$\theta_0 = 0°$
$\theta_0 = 90°$
$\theta_0 = 180°$
$\theta_0 = -90°$
y
x
u
t
0.8
0.6
0.4
0.2
-0.2
-0.4
-0.6
-0.8
32
24
16
8
0
-8
-16
-24
-32
1.0

Figure 17

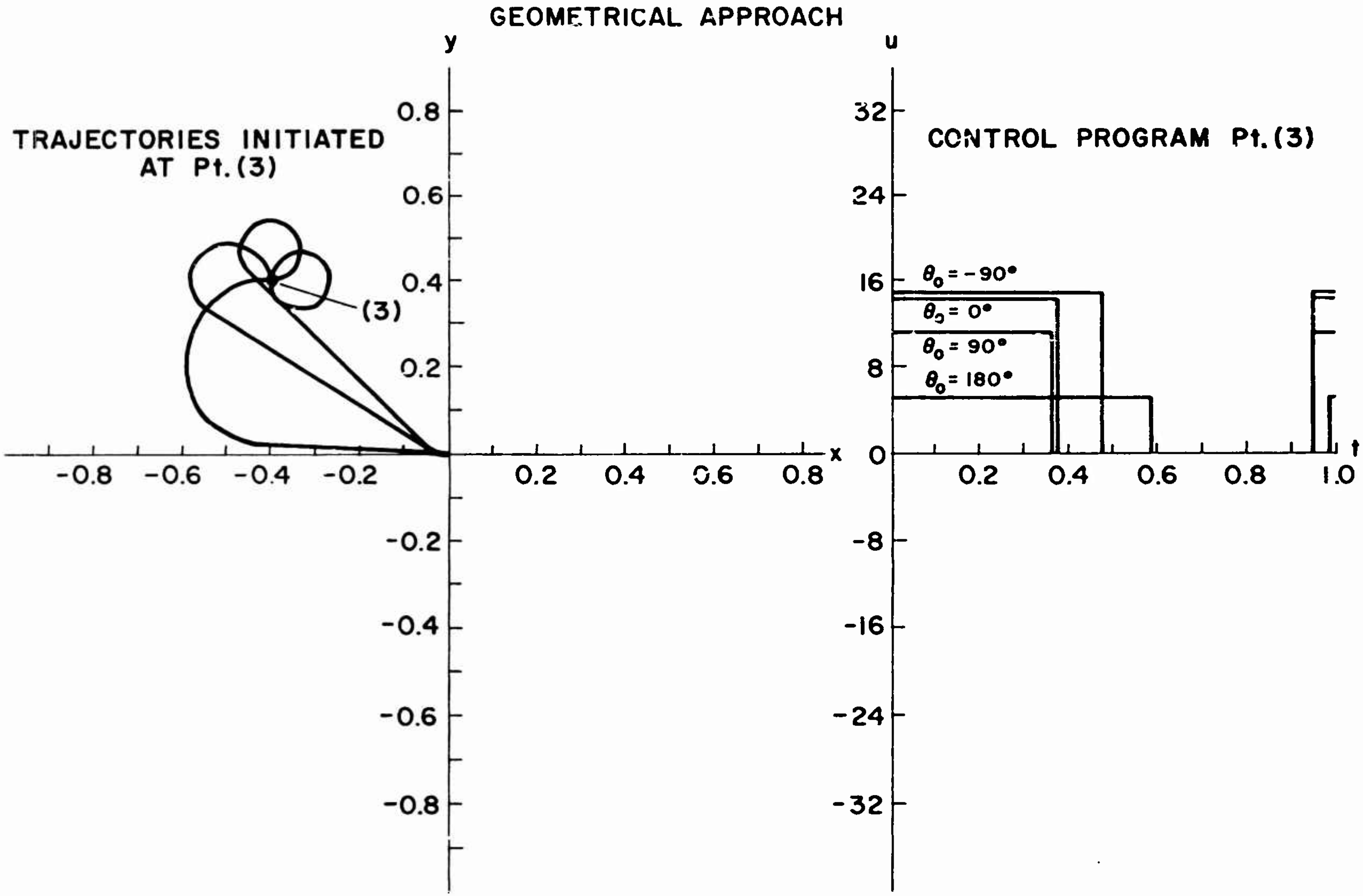

GEOMETRICAL APPROACH
TRAJECTORIES INITIATED AT Pt. (3)
(3)
y
x
0.8
0.6
0.4
0.2
-0.2
-0.4
-0.6
-0.8
CONTROL PROGRAM Pt. (3)
u
t
32
24
16
8
0
-8
-16
-24
-32
1.0
$\theta_0 = -90°$
$\theta_0 = 0°$
$\theta_0 = 90°$
$\theta_0 = 180°$

Figure 18

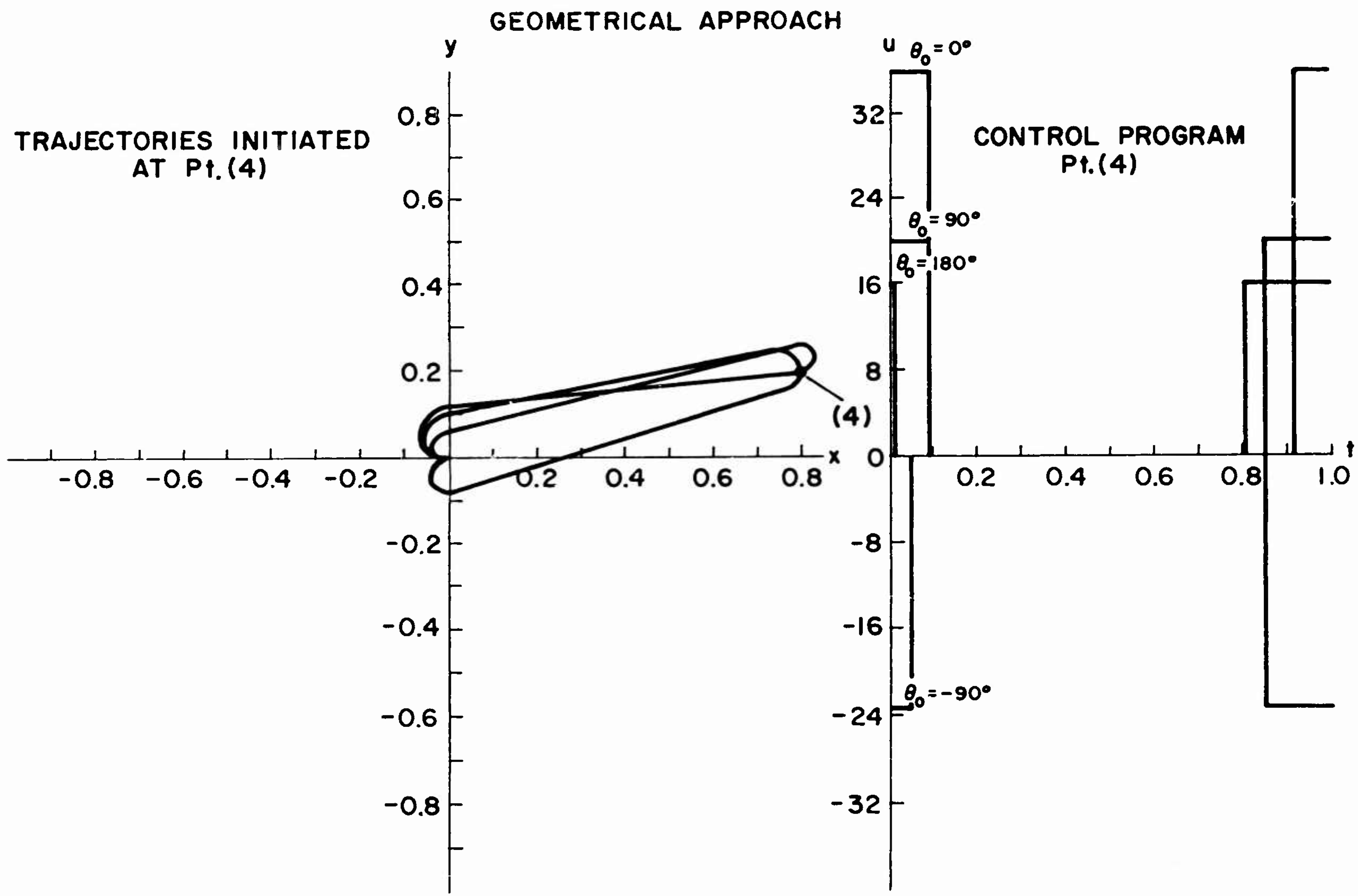

GEOMETRICAL APPROACH
TRAJECTORIES INITIATED
AT Pt.(4)
y
0.8
0.6
0.4
0.2
-0.2
-0.4
-0.6
-0.8
x
-0.8
-0.6
-0.4
-0.2
0.2
0.4
0.6
0.8
(4)
CONTROL PROGRAM
Pt.(4)
u
$\theta_0 = 0°$
$\theta_0 = 90°$
$\theta_0 = 180°$
$\theta_0 = -90°$
32
24
16
8
0
-8
-16
-24
-32
t
0.2
0.4
0.6
0.8
1.0

Figure 19

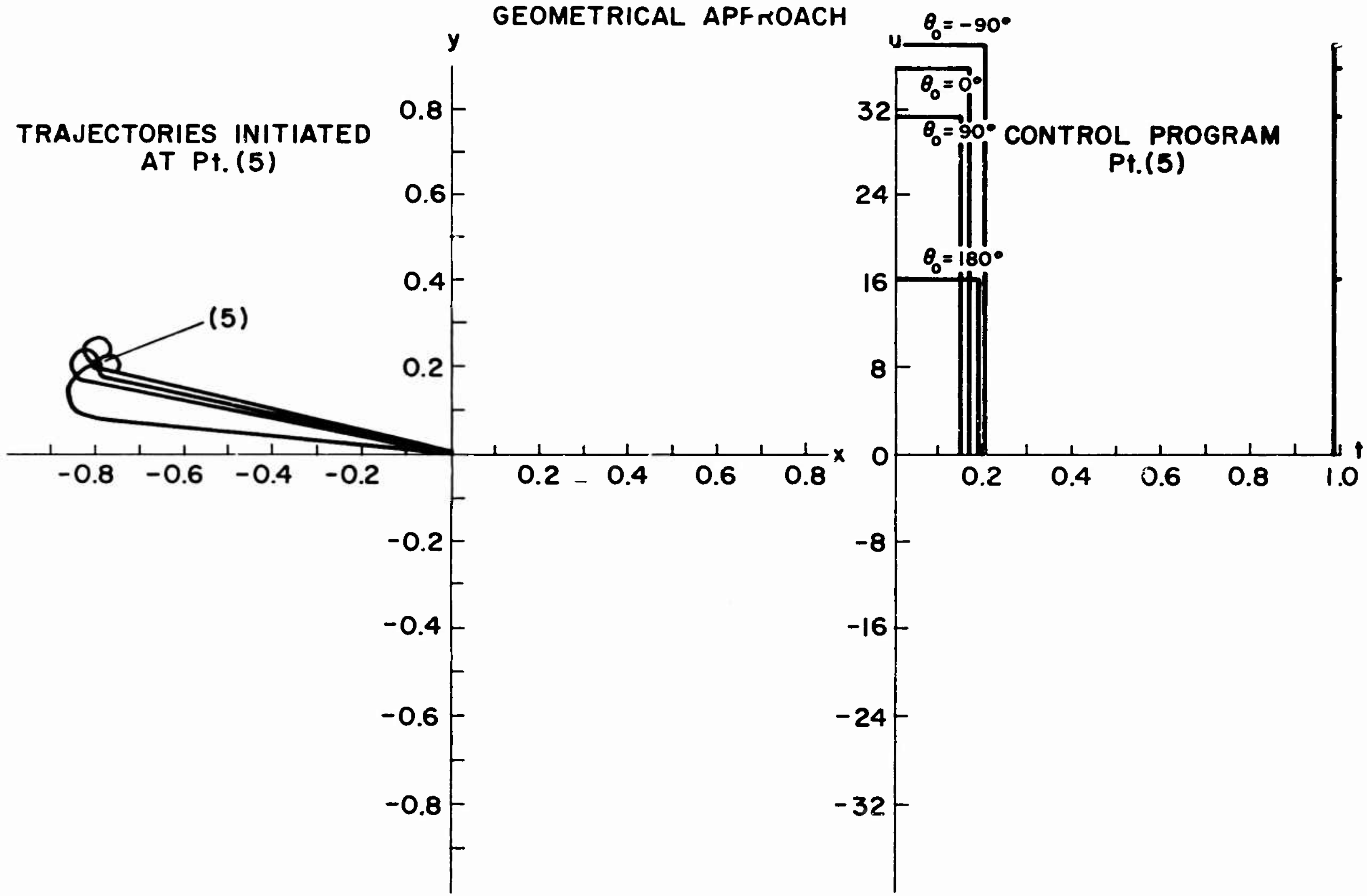

GEOMETRICAL APPROACH
TRAJECTORIES INITIATED
AT Pt.(5)
(5)
y
x
0.8
0.6
0.4
0.2
-0.2
-0.4
-0.6
-0.8
CONTROL PROGRAM
Pt.(5)
u
t
$\theta_o = -90°$
$\theta_o = 0°$
$\theta_o = 90°$
$\theta_o = 180°$
32
24
16
8
0
-8
-16
-24
-32
1.0

Figure 20

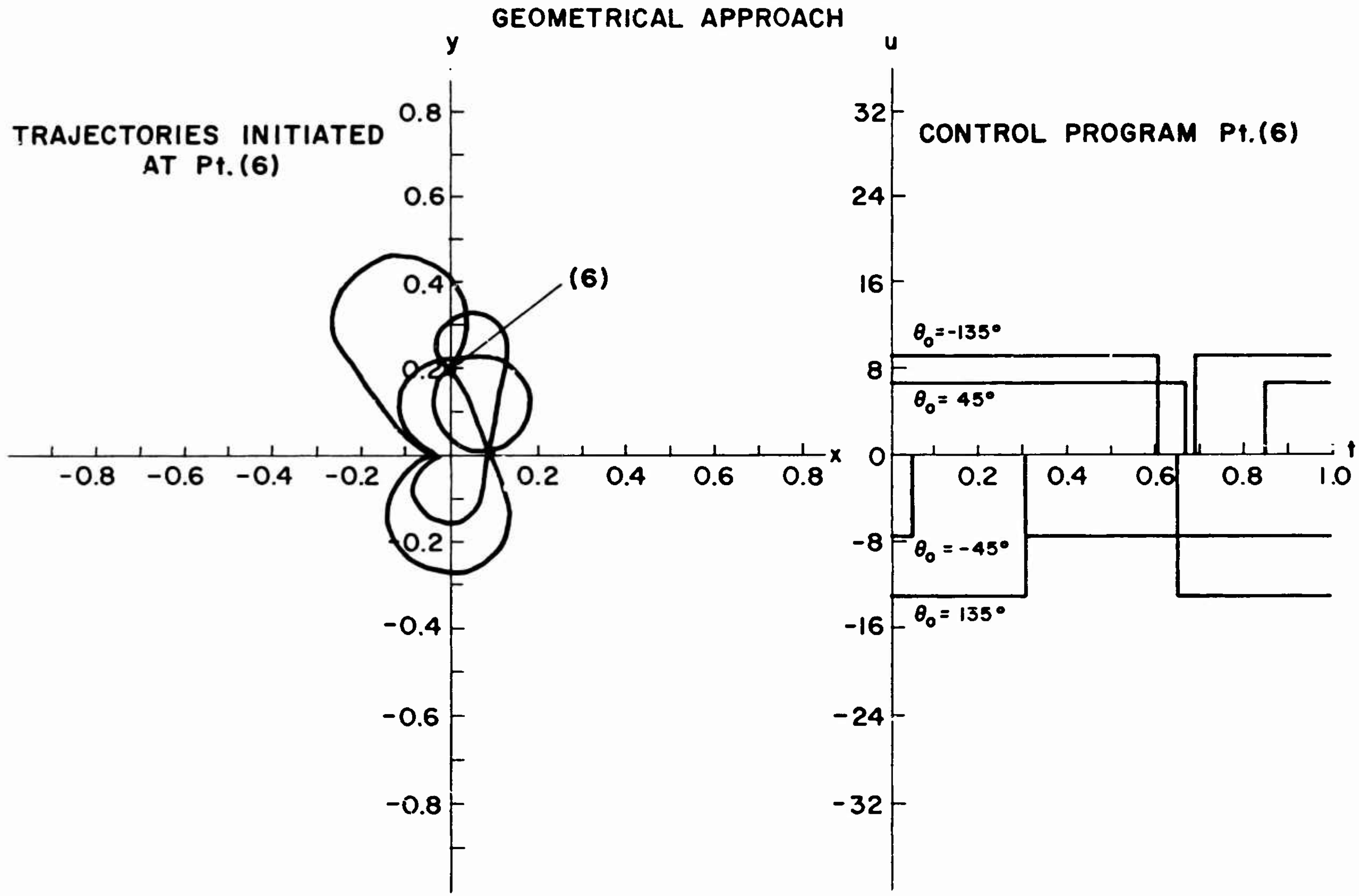

GEOMETRICAL APPROACH
TRAJECTORIES INITIATED AT Pt.(6)
(6)
y
x
0.8
0.6
0.4
0.2
-0.2
-0.4
-0.6
-0.8
CONTROL PROGRAM Pt.(6)
u
t
32
24
16
8
0
-8
-16
-24
-32
0.2
0.4
0.6
0.8
1.0
θ_0=-135°
θ_0= 45°
θ_0 = -45°
θ_0= 135°

Figure 21

VII. SUMMARY AND CONCLUDING REMARKS

The problem of guiding a parachute to a target under the influence of a constant horizontal wind, and having it land against the wind, has been considered. Optimal guidance implies that a measure of the expenditure of control energy is minimized.

Optimal control theory requires the solution of a nonlinear two point boundary value problem, which presents computational difficulties. However, the form of the optimal control in terms of a feedback control law with three unknown parameters is deduced. Nonlinear algebraic equations for these parameters which involve elliptic integrals can be written, but their solution must be broken up into different cases and is tedious.

A method for computing the optimal control law parameters has been developed which consists of parameter optimization through minimization of the terminal error. This method requires the integration of a system of twelve ordinary differential equations at each iteration of the minimization procedure. Computationally, however, the convergence of the method is sometimes slow.

A non-optimal guidance scheme is also developed, in which a given geometrical path involving three unknown parameters is prescribed as a trajectory. The parameters are then computed by applying the terminal constraints. The computational requirements are simple and computing time is negligible.

Both parameter optimization and geometrical trajectory prescription require knowledge of the initial condition vector $\underset{\sim}{x}_o$. In physical terms, this involves the measurement of the horizontal position, velocity, and wind vectors, as well as the altitude and rate of descent, at the time t_o. Parameter optimization results in a control law requiring continuous measurements of the position vector and trajectory direction, whereas the geometrical method yields a completely open-loop control law. In practice, though, the initial time t_o would be updated as new estimates of the wind become available, and the open-loop control becomes, in effect, feedback control with the sequence of initial conditions serving as the current state. The major advantage of parameter optimization is, in general, lower control energies, whereas the advantage of the geometrical method is its computational simplicity.

Nomenclature

A	amplitude of oscillatory motion of pendulum
a,b,c	coefficients of quadratic equation
C	negative of constant value of Hamiltonian
E	control energy
E_o	nondimensional quantity representing total energy of pendulum
$E(\psi,k)$	elliptic integral of the second kind
$\underset{\sim}{F}$	resultant aerodynamic force on parachute
$F(\underset{\sim}{\alpha})$	terminal error function
$F(\psi,k)$	elliptic integral of the first kind
f_1,f_2	constants in solution for θ^*
$G(t;\underset{\sim}{\alpha})$	matrix of partial derivatives of $\underset{\sim}{x}$ with respect to $\underset{\sim}{\alpha}$
$\underset{\sim}{g}$	acceleration due to gravity
H	Hamiltonian
H_n	symmetric, positive definite matrix in variable metric method
$H(\underset{\sim}{\alpha})$	Hessian matrix of $F(\underset{\sim}{\alpha})$
h	initial altitude of parachute
$K(\kappa)$	complete elliptic integral of the first kind
k	modulus of elliptic integral
M	mass of parachute
n	number of multiples of 2π in terminal value of x_3
p_1,p_2,p_3	adjoint variables
R	instantaneous radius of turn of parachute
r	radius of curvature of circular arc
r_1,r_2	coordinates of position of parachute in horizontal plane
T	time at touchdown
t	time, normalized time
t_o	initial time
t_1,t_2	switching times
U	vertical velocity of parachute
u	normalized control variable
V	horizontal velocity of parachute (relative to air)
$\underset{\sim}{W}$	wind vector
$\underset{\sim}{x}$	state vector (x_1,x_2,x_3) of dynamical system

$\left.\begin{matrix} x,y \\ x_1,x_2 \end{matrix}\right\}$	normalized coordinates in horizontal plane
x_3	angle of horizontal velocity vector
z	complex variable denoting $x + jy$
$\underset{\sim}{\alpha}$	vector $(\alpha_1,\alpha_2,\alpha_3)$
α,β,γ	optimal control parameters
$\alpha_1,\alpha_2,\alpha_3$	optimal control parameters
η	normalized time
θ	angle of horizontal velocity vector
θ^*	angle of inclination of straight line
κ	modulus of elliptic integral
λ_n	scaling factor in variable metric method
ξ	integration variable
ρ,ϕ	normalized polar coordinates in horizontal plane
τ	period of oscillatory motion of pendulum
σ	bank angle
ψ	angle of pendulum
ψ_n	net angle traversed for a given n