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ALGEBRAIC FOUNDATIONS OF STABILITY THEORY: A COMPUTERIZED LINEAR ALGEBRA BIBLIOGRAPHY

CUMULATIVE REPORT

on

GRANT AF-AFOSR 72-2164

October 1, 1971 - September 30, 1976

Marvin Marcus

Henryk Minc

Robert C. Thompson

SUMMARY

The toal of 118 papers, monographs and books completed during the five year period under review by the principal investigators Marvin Marcus, Henryk Minc, R. C. Thompson span the entire field of linear and multilinear algebra and cognate areas. The work contains significant new results in: numerical linear algebra; stability theory; theory of inequalities; eigenvalue localization theory; invariant theory; generalized matrix functions; combinatorial analysis; tensor algebra.

It is generally recognized in the mathematical community that the group working in the Algebra Institute at Santa Barbara under the sponsorship of the AFOSR comprise a leading center for modern research in the above fields. This is evidenced in a number of ways: one of the two international journals, LINEAR AND MULTILINEAR ALGEBRA, is edited by Marvin Marcus and R. C. Thompson; NSF and other funding agencies send many proposals to be reviewed to members of this group; a number of graduate students have completed Ph.d thesis work in the Institute; both younger and senior mathematicians, working in the field, have asked to visit the Institute on a temporary basis to work with members of the group.

Beyond the research itself, a major new information resource has been created in the Algebra Institute, the Linear Algebra Bibliography (LAB). This is a computerized interactive data retrieval system currently available on-line. which contains summaries of the entire literature in pure and applied linear algebra. essentially from the beginnings of the subject (about 100 years ago). It is anticipated that the LAB will be made current by Oct. 1, 1977 so that it can be maintained and augmented at modest cost thereafter.

The research activity described in this report certainly could not have been carried out without the sponsorship of the AFOSR.

KEYWORDS: matrix, eigenvalues. linear algebra. stability. tensor, inequalities, numerical analysis, localization, determinant, permanent, quadratic forms, numerical range, field of values.

Introduction

This report is organized in four sections, labelled A, B, C, D.

Section A is a list of publications generated by the three principal investigators during the 1971-76 period covered by this Grant. This list comprises a total of 118 items.

Section B is a brief description of the research level journal founded during the term of the grant by Marcus and Thompson.

Section C is a list of the graduate students trained by the principal investigators during the term of the grant.

Section D describes in narrative form, some of the results obtained by the principal investigators. Owing to the very substantial number of publications, a detailed listing of the new results obtained in the individual papers is not possible. Instead, a description in broad outline is given of some of the major trends, together with discussions of selected items from several of the papers.

Publications of M. Marcus. During the five year period under review, some 38 research papers written by M. Marcus evolved as follows. Four further research papers were written by graduate students under the direction of M. Marcus. There are also seven ' ›oks.

I. Papers comprising research completed just prior to the 1971-76 period but published during this period:

1. Partitioned hermitian matrices (with W. Watkins), Duke Math. J. 38, (1971), 237-249.

2. On the degree of the minimal polynomial of a commutator operator (with M. S. Ali), Pacific J. Math. 37 (1971), 561-565.

3. An extension of the Minkowski Determinant Theorem (with W. R. Gordon), Proc. Edinburgh Math. Soc. 17 (1971), 321-324.

4. Linear transformations on matrices. J. Res. Math. Sci. Sect. Nat. Bur. Stds., 75B (1971), 107-113.

II. Papers comprising research completed and published during the 1971-76 period:

5. Antiderivations on the exterior and symmetric algebras (with R. Merris), Linear Algebra and Appl., 5 (1972). 13-18.

6. A dimension inequality for multilinear functions, Article in Inequalities-III, Proceedings of the Third Symposium on Inequalities Held at the University of California. Los Angeles, September 1-9, 1969, Academic Press, New York, 1972, 217-224.

7. An analysis of equality in certain matrix inequalities II. (with W. R. Gordon), SIAM J. Numer. Anal. 9 (1972), 130-136.

8. An inequality for Schur functions (with H. Minc), Linear Algebra and Appl. 5 (1972), 19-28.

9. Minimal polynomials of additive commutators and Jordan products (with M. S. Ali), J. Algebra 22 (1972), 12-33.

10. Groups of linear operators defined by group characters (with J. Holmes), Trans. Amer. Math. Soc., 172 (1972), 177-194.

11. On projections in the symmetric power space (with W. R. Gordon), Monatsh. Math. 76 (1972), 130-134.

12. Rational tensor representations of Hom(V,V) and an extension of an inequality of I. Schur (with W. R. Gordon), Canad. J. Math., 24 (1972), 686-695.

13. A note on the multiplicative property of the Smith normal form (with Ernest E. Underwood). J. of Research, 76B, Nos. 3 and 4, July - December 1972, 205-206.

14. Trace inequalities, Proceedings of the 5th Gatlinburg symposium on Numerical Algebra, U. S. Atomic Energy Commission, June 1972, 14.1-14.6.

15. A relation between permanental and determinantal adjoints (with R. Merris), J. Austral. Math. Soç. 15 (1973), 270-271.

16. Images of bilinear symmetric and skew-symmetric functions (with M. Shafqat Ali), Illinois J. Math. 17 (1973), 505-5..2.

17. Derivations, Plücker relations, and the numerical range, Indiana Univ. Math. J. 22, 12 (1973), 1137-1149.

18. Review of "Multiparameter Eigenvalue Problems: Matrices and Compact Operators," F. V. Atkinson, Academic Press, New York, 1972. SIAM Review 15, (1973), 678-679.

19. Annihilating polynomials for similarity and commutator mappings (with W. Watkins), Linear and Multilinear Algebra, 2, 1 (1974), 81-84.

20. Elementary divisors of derivations (with W. Watkins), Linear and Multilinear Algebra, 2, 1 (1974), 65-80.

21. On the degree of the minimal polynomial of the Lyapunov operator (with M. Shafqat Ali), Monatsh. Math., 78 (1974), 229-236.

22. Partial derivations and B. Iversen's linear determinants (with E. Wilson), Aequationes Mathematicae, 10, 2/3 (1974), 123-127.

23. Normal and Hermitian tensor products (with Dean W. Hoover), Linear Algebra and its Applications, 9 (1974), 1-8.

24. Elementary Divisors of Tensor Products (with H. Robinson), Communications of the ACM, 18, 1 (January 1975), 36-39.

25. A note on the Hodge star operator (with H. Robinson), Linear Algebra and its Applications, 10 (1975), 85-87.

26. Rearrangement and extremal results for hermitian matrices, Linear Algebra and its Applications, 11 (1975), 95-104.

27. Adjoints and the numerical range, Linear and Multilinear Algebra, 3 (1975), 81-89.

28. Congruence maps and derivations (with M. Shafqat Ali), Linear and Multilinear Algebra, 3 (1975), 115-133.

29. Bilinear functionals on the Grassmannian Manifold (with H. Robinson), Linear and Multilinear Algebra, 3 (1975), 215-225.

30. On two theorems of Frobenius (with H. Minc), Pac. J. Math., 60 (1975), 149-151.

31. On exterior powers of endomorphisms (with H. Robinson), Linear Algebra and its Applications, 14 (1976), 219-225.

32. Converses of the Fischer inequality and an inequality of A. Ostrowski, Linear and Multilinear Algebra, 4 (1976), 137-148.

III. Papers comprising research completed during the five year period, accepted by journals and in the process of being published:

33. The range multiplicity of an Hermitian matrix, (with M. Ishaq),

Linear and Multilinear Algebra.

34. The numerical radius of exterior powers, (with P. Andresen), Linear Algebra and its Applications.

35. Pencils of real symmetric matrices and the numerical range, Aequationes Mathematicae.

36. Isometries of matrix algebras, (with R. Grone), Journal of Algebra.

37. Constrained extrema of bilinear functionals, (with P. Andresen), Monatschefte fur Mathematik.

38. Weyl's inequality and quadratic forms on the Grassmannian, (with P. Andresen), Pacific Journal of Mathematics.

IV. Books

39. College Trigonometry (with H. Minc), Houghton-Mifflin, Boston, 1971.

40. Integrated Analytic Geometry and Algebra with Circular Functions (with H. Minc), General Learning Corporation, 1973.

41. Finite Dimensional Multilinear Algebra, Part I, Marcel Dekker, Inc., New York, 1973, 1-292.

42. Algebra, University of California, Santa Barbara, 1974.

43. Linear Algebra, University of California, Santa Barbara, 1974.

44. Finite Dimensional Multilinear Algebra, Part II, Marcel Dekker, Inc., New York, 1975.

45. Modern Algebra, Marcel Dekker, Inc., New York.

V. Papers written by graduate students under the direction of M. Marcus.

46. R. Grone, A note on the dimension of an orbital subspace, Linear Algebra and its Applications, to appear.

47. R. Grone, Decomposable tensors as a quadratic variety, Proceedings of the American Mathematical Society, to appear.

48. R. Grone, Off-diagonal elements of normal matrices, Journal of Research of the National Bureau of Standards, submitted.

49. H. Robinson, Quadratic forms on symmetry classes of tensors, Linear and Multilinear Algebra, to appear.

Publications of H. Minc. During the five year period under review, some 21 research papers written by H. Minc evolved as follows. There are also two books.

I. Papers comprising research completed just prior to the 1971-76 period but published during this period:

1. Rearrangements, Trans. Amer. Math. Soc. 159 (1971), 497-504.

II. Papers comprising research completed and published during the 1971-76 period:

2. Nearly decomposable matrices, Linear Algebra and Appl. 5 (1972), 181-187.

3. An inequality for Schur functions (with M. Marcus), Linear Algebra and Appl. 5 (1972), 19-28.

4. On permanents of circulants, Pacific J. Math. 42 (1972), 477-484.

5. Eigenvalues of matrices with prescribed entries (with D. London), Proc. Amer. Math. Soc. 34 (1972), 8-14.

6. Diagonals of nonnegative matrices (with P. Erdös), Linear and Multilinear Algebra 1 (1973), 89-95.

7. (0,1)-matrices with minimal permanents, Israel J. Math. 15 (1973), 27-30.

8. A remark on a theorem of M. Hall, Canad. Math. Bull. 17 (1974), 547-548.

9. Irreducible matrices, Linear and Multilinear Algebra 1 (1974), 337-342.

10. The structure of irreducible matrices, Linear and Multilinear Algebra 2 (1974), 85-90.

11. An unresolved conjecture on permanents of (0,1)-matrices, Linear and Multilinear Algebra 2 (1974), 97-113.

12. Linear transformations on nonnegative matrices, Linear Algebra and Appl., 9 (1974), 149-153.

13. Six letters from Alexander C. Aitken (edited by H. Minc), Linear and Multilinear Algebra 2 (1974), 1-12.

14. On two theorem of Frobenius (with M. Marcus), Pacific J. Math. 60 (1975), 149-151.

15. Spectra of irreducible matrices, Proc. Edinburgh Math. Soc. 19 (1975), 231-236.

16. Doubly stochastic matrices with minimal permanents, Pacific J. Math. 58 (1975), 155-157.

17. Subpermanents of doubly stochastic matrices, Linear and Multilinear Algebra 3 (1975), 91-94.

18. Which nonnegative matrices are self inverse? (with F. Harary), Math. Mag. 49 (1976), 91-92.

19. The invariance of elementary symmetric functions, Linear and Multilinear Algebra 4 (1976), 209-215.

III. Papers comprising research completed during the five year period, accepted by journals, and in the process of being published:

20. Linear transformations on matrices: rank 1 preservers and determinant preservers, Linear and Multilinear Algebra (to appear).

21. Evaluation of permanents (submitted for publication).

IV. Books

22. Permanents, Encyclopedia of Mathematics and its Applications, Addison-Wesley (in preparation).

23. Lecture Notes: Nonnegative matrices, Technion-Israel Institute of Technology, Haifa, 1974.

List of publications

Publications of R. C. Thompson. During the five year period under review, some 41 research papers written by R. C. Thompson evolved as follows. Some of these papers carry graduate students as coauthors. Three further research papers were written by a graduate student under the direction of R. C. Thompson. There is also one book, and one additional publication, a dedication to O. Taussky-Todd.

I. Papers comprising research completed just prior to the 1971-76 period but published during this period:

1. Principal submatrices VIII: Principal sections of a pair of forms, Rocky Mountain J. Math. 2 (1972), 97-110.

2. Principal submatrices IX: Interlacing inequalities for singular values of submatrices, Linear Algebra and Appl., 5 (1972), 1-12.

3. On the eigenvalues of sums of Hermitian matrices (with Linda Freede), Linear Algebra and Appl. 4 (1971), 369-376.

4. On the eigenvalues of sums of Hermitian matrices. III (with Linda Freede), J. Res. Nat. Bureau of Standards, 75 B (1971), 115-120.

II. Papers comprising research completed and published during the 1971-76 period:

5. The singular values of matrix products (with S. Therianos), Scripta Math. 29 (1973), 99-110.

6. The singular values of matrix products. II, Scripta Math. 29 (1973), 111-114.

7. On the real and absolute singular values of a matrix product, Linear Algebra and Appl. 4 (1971), 243-254.

8. On the singular values of matrix products. III (with S. Therianos), Scripta Math. 29 (1973), 115-123.

9. The eigenvalues and singular values of matrix sums and products. VII (with S. Therianos), Canad. Math. Bull. 16 (1973), 561-569.

10. Skew circulant quadratic forms (with Dennis Garbanati), J. Number Theory 4 (1972), 557-572.

11. The eigenvalues and singular values of matrix sums and products. VIII (with S. Therianos), Aequationes Math. 7 (1972), 219-242.

12. Classes of unimodular abelian group matrices (with D. Garbanati), Pacific J. Math. 43 (1972), 633-646.

13. Inequalities connecting the eigenvalues of Hermitian matrices with the eigenvalues of complementary principal submatrices (with S. Therianos), Bull. Austral. Math. Society 6 (1972), 117-132.

14. Inertial properties of eigenvalues, J. Math. Anal. Appl. 41 (1973), 192-198.

15. The eigenvalues of complementary principal submatrices of a positive definite matrix (with S. Therianos), Canad. J. Math. 24 (1972), 658-667.

16. On a construction of B. P. Zwahlen (with S. Therianos), Linear and Multilinear Algebra, 2 (1974), 309-325.

17. On the eigenvalues of a product of unitary matrices. I, Linear and Multilinear Algebra, 2 (1974), 13-25.

18. The eigenvalues of a partitioned hermitian matrix involving a parameter, Linear Algebra and Appl., 9 (1974), 243-260.

19. Dissipative matrices and related results, Linear Algebra and Appl., 11 (1975), 155-169.

20. Singular value inequalities for matrix sums and minors, Linear Algebra and Appl., 11 (1975), 251-269.

21. The characteristic polynomial of a principal subpencil of a Hermitian matrix pencil, Linear Algebra and Appl., 14, 1976, 135-177.

22. Dissipative matrices and the matrix $A^{-1}A^{*}$, Houston J. of Math., vol 1, number 1, 1975, 137-148.

23. The bilinear field of values, Monats. Math., 81, 1976, 153-167.

24. The behavior of eigenvalues and singular values under perturbations of restricted rank, Linear Algebra and Appl., 13, 1976, 69-78.

25. Extreme determinants on the convex hull of matrices with prescribed singular values, Linear and Multilinear Algebra, vol. 3, 1975, 15-17.

26. The intersection of the convex hulls of proper and improper real matrices with prescribed singular values, Linear and Multilinear Algebra, vol. 3, 1975, 155-160.

27. The singular value range of a matrix sum or product, Bulletin of the Institute of Mathematics, Academia Sinica, vol. 3, 1975, 275-281.

28. Sections of matrix pairs: The second interlacing principle (with Z. Gordon), Houston Journal of Mathematics, 2, 1976, 427-437.

29. On Iohvidov's proofs of the Fischer-Frobenius theorem. Journal of Research of the National Bureau of Standards, 80B2, 1976, 269-272.

30. A matrix inequality, Commentationes Mathematicae Universitatis Carolinae, 17, 1976, 393-397.

III. Papers comprising research completed during the five year period, accepted by journals, and in the process of being published:

31. Singular values, diagonal elements, and convexity, SIAM Journal on Applied Mathematics, in press, 27 printed pages.

32. Convex and concave functions of singular values of matrix sums, Accepted, Pacific Journal of Mathematics, 9 typed pages.

33. Inertial properties of eigenvalues II, J. Math. Anal. Appl., accepted, 9 typed pages.

34. Matrix type metric inequalities, Linear and Multilinear Algebra, awaiting publication, 29 typed pages.

35. The convex hull of signed rearrangements with prescribed sign parity, Linear and Multilinear Algebra, awaiting publication, 11 typed pages.

IV. Papers in manuscript stage, incorporating results obtained during the five year period, but not yet submitted to journals:

36. The discovery of an inequality in linear algebra. To be submitted. 46 typed pages.

37. Invariants, signatures, and inertial signatures in hermitian matrix pencils, 33 typed pages.

38. Pencils of complex and real symmetric and skew matrices, 46 typed pages, about to be submitted.

V. Papers written just after the five year period, incorporating results evolving from the research completed during the five years (all still in manuscript stage):

39. Principal minors of complex symmetric and skew matrices.

40. Interlacing inequalities for invariant factors, 14 typed pages.

41. On the change in similarity invariants under a rank 1 perturbation.

VI. Papers written by graduate students under the direction of R. C. Thompson.

42. D. Garbanati, Classes of nonsingular abelian group matrices over fields, J. Algebra, 27, (1973), 422-435.

43. D. Garbanati, Abelian group matrices over the p-adic and rational integers, J. Number Theory,

44. D. Garbanati, Classes of circulants over p-adic and rational integers, Pacific J. Math.,

VII. Books

45. Elementary modern algebra, R. C. Thompson, Scott Foresman, 1974, 454 pages.

VIII. Additional publications

46. Dedication, by M. Marcus and R. C. Thompson, Linear and Multilinear Algebra, vol. 3, p. 1, 1975.

B. Journal

The research journal LINEAR AND MULTILINEAR ALGEBRA, published by Gordon and Breach, was founded by R. C. Thompson and M. Marcus. Publishing quarterly, this journal has achieved recognition as the leading research journal in its area.

C. Graduate students trained

The following Ph.D. students completed their degree during the five year term of this grant under the direction of R. C. Thompson:

1. D. Garbanati, 1972 (presently on the faculty of the University of Maryland).
2. S. Therianos, 1974 (presently at Chapman College, in the U. S. Navy program of shipboard construction).
3. Z. Gordon, 1971 (presently teaching in Victoria, Canada).
4. (in progress) S. Johnsen.

The following M.A. students completed theses during the five year term of the grant under the direction of R. C. Thompson:

1. R. Gorby, 1973 (presently employed in missile research, at Pt. Magu, Ca. military facility).
2. R. Rubin, 1974 (presently employed in computer development at Burroughs Corp.).
3. L. Gin, 1974 (presently employed at General Electric Tempo research facility).
4. L. Zilz, 1974 (employment not known).

The following Ph.D. student completed his degree during the term of the grant under the direction of Henryk Minc:

1. S. Therianos, 1974 (presently at Chapman College, in the U. S. Navy program of shipboard construction). (This student was directed jointly by R. C. Thompson and H. Minc.)

The following mathematicians have completed their Ph.D. work during the five year term of the grant under the direction of M. Marcus:

1. Dr. Elizabeth Wilson, Mathematician, Naval Labs. Pt. Mugu, Ca.
2. Dr. James Holmes, Asst. Professor, Westmont College, Santa Barbara, Ca.

3. Dr. Herbert Robinson, NRC Post-Doctoral Fellow, National Bureau of Standards, Washington, D. C.

4. Dr. Patricia Andresen, Assistant Professor, University of Alaska, Fairbanks, Alaska.

5. Dr. Robert Grone.

6. I. Filippenko - in progress.

D. Description of Research

MARVIN MARCUS

The most important item to appear during the five year span covered by this report is the two volume book FDMA, vols. 1 and 7, published by M. Dekker, 1972, 1974. This book contains a good deal of research on multilinear algebra and group representation completed under various Air Force grants by Marcus.

There are too many other items produced during the period of this report to be able to quote them all here. Some broad categories, however, are: (1) the numerical range, (2) stability theory and the Lyapunov map, (3) properties of tensor products of linear transformations, (4) inequalities, (5) norms.

1. The numerical range. Originally called the field of values, but now called the numerical range, this subject has seen explosive research activity in recent years in the hands of very many mathematicians. This is because of its importance in infinite dimensional situations, and also because of its growing importance in linear numerical algebra, particularly in questions arising from numerical schemes for solving partial differential equations. Many new results have been obtained by Marcus and various student collaborators. Among others, these include most recently an efficient computer algorithm for calculating various generalized numerical ranges. For most of the generalized numerical ranges, this is the first efficient computer program available for their calculation. Apart from this, some specific items are the following.

Beginning with the paper by W. A. Beck and C. R. Putnam in 1956, the problem of finding conditions under which an operator is similar to its adjoint has received considerable attention over the last twenty years. These investigations and a number of similar questions usually involve hypotheses on the occurrence of the number 0 in the numerical range. Paper (27)* introduced a new concept called the range multiplicity of an arbitrary complex number z which essentially measures the number of times z can occur in the numerical range. By this is meant the largest integer k for which k orthonormal vectors exist so that the value of the quadratic form on each of these vectors is z. The paper (27) develops a normal form in terms of the range multiplicity of 0 and numerous results are obtained that improve earlier work of S. K. Berberian (1962), P. A. Fillmore (1969), C. A. McCarthy (1964), I. H. Sheth (1966), Olga Taussky-Todd (1970) and (1972), and J. P. Williams (1969). This research was described in an invited paper by Marcus at the Special Session on Matrix Theory, AMS meeting number 719, November 23, 1974, Houston, Texas.

In item 33 the concept of the range multiplicity of a complex number is further investigated, and it is shown that the range multiplicity of 0 for the unitary part in the polar factorization of A is at most twice the range multiplicity of 0 associated with A.

Closely related to the results of the last paragraph is an investigation of various types of bilinear numerical ranges. Roughly speaking, a bilinear numerical range analyzes the properties of nonprincipal submatrices parallel to the way in which the usual numerical range analyzes properties of principal submatrices. There were very few prior publications on bilinear numerical ranges, two by Mirsky and an interesting one by Thompson. A wealth of results have been obtained, the most surprising being the unexpected discovery that the numerical range properties of a nonprincipal submatrix depends on the location of the submatrix within the full matrix, particularly depending on the extent of its overlap with the principal diagonal.

Item 37 made considerable progress towards an analysis of some of these questions. Essentially what is done in this paper is to examine

*The references on pp. 14-23 are to the papers listed on pp. 3-5.

the relationship between off-diagonal subdeterminants of an Hermitian matrix and the eigenvalues of that matrix. For example, it is quite simple to see that if λ_1 and λ_n are respectively the largest and smallest eigenvalues of an Hermitian matrix A then every off-diagonal element lies in a circle centered at the origin of radius $1/2(\lambda_1 - \lambda_n)$. This simple theorem is extended to off-diagonal subdeterminants and the totally unexpected phenomenon just mentioned is seen to occur. Namely, that for an Hermitian matrix whose eigenvalues are given, the possible values that a non-principal subdeterminant can assume depend crucially on the position of the subdeterminant in the matrix. More precisely, the possible values depend on the intersection multiplicity of the row and column numbers that determine the subdeterminant. Such results are useful because they permit one to estimate the "spread" $\lambda_1 - \lambda_n$, and similar expressions involving differences of products of eigenvalues, by examining suitably situated non-principal subdeterminants.

In manuscript 29 earlier results of I. C. Gohberg and M. G. Krein (Introduction to the Theory of Linear Nonselfadjoint Operators, Trans. of Math. Monographs, Vol. 18, Amer. Math. Soc. (1969)) and of M. Goldberg, G. Zwas and E. Tadmor (Linear Algebra and its Appl., 8 (1974), 427-428); (Linear and Multilinear Algebra, 2 (1975), 317-326) were examined. The new results obtained were again concerned with the sizes of principal subdeterminants of an arbitrary matrix and here the relationship of the absolute value of these subdeterminants to the eigenvalues and singular values of the matrix was further investigated.

Paper (17) dealt with the general numerical range, investigating its convexity. These results showed among other things, that if A is an $m \times m$ matrix then the set of numbers det (UAU*) obtained as U runs over all $n \times m$ matrices satisfying U*U = identity is not in general convex. (A first result of this type had been obtained by Thompson some years ago.)

Item 38 contains more results on the set of values taken on by the conjugate bilinear functional (Ax,y) as x and y range over all unit vectors with prescribed inner product. By analyzing the same problem for the induced functional on the Grassmannian, further results on non-principal subdeterminants are also obtained. The following are typical of the contents of the paper:

Let

$$W^q(A) = \{(Ax,y) : \|x\| = \|y\| = 1, \ |(x,y)| = q\},$$

We proved the following results:

THEOREM. *Let* $A \in M_n(C)$. *Then* $W^q(A)$ *is an annulus centered at the origin.*

THEOREM. *Let* $A \in M_n(C)$ *be hermitian with eigenvalues* $\lambda_1 \geq \cdots \geq \lambda_n$. *Then*:

(i) *If* A *is indefinite,* $W^q(A)$ *is a closed disc centered at the origin with radius*

$$r = \max\{[(1+q)\lambda_1 - (1-q)\lambda_n]/2, \ -[(1+q)\lambda_n - (1-q)\lambda_1]/2\};$$

(ii) If $A > 0$, $0 \leq q < 1$, *and*

$$\frac{q}{\sqrt{1-q^2}} \leq \frac{\lambda_1 - \lambda_n}{2\sqrt{\lambda_1\lambda_n}},$$

then $W^q(A)$ *is a closed disc centered at the origin with radius*

$$r = \left(\frac{1+q}{2}\right)\lambda_1 - \left(\frac{1-q}{2}\right)\lambda_n;$$

(iii) _If_ $A > 0$, $0 \leq q < 1$, _and_

$$\frac{q}{\sqrt{1-q^2}} > \frac{\lambda_1 \lambda_n}{2\sqrt{\lambda_1 \lambda_n}} ;$$

then $W^q(A)$ _is an annulus with inner radius_

$$r_I = \left(\frac{1+q}{2}\right)\lambda_n - \left(\frac{1-q}{2}\right)\lambda_1 ,$$

and outer radius

$$r_0 = \left(\frac{1+q}{2}\right)\lambda_1 - \left(\frac{1-q}{2}\right)\lambda_n .$$

Item 34 is concerned with the largest absolute value taken on by an m-square principal subdeterminant in any unitary transform of an n-square complex matrix A. For $m = 1$ this maximum coincides with the numerical radius of A. The results obtained constitute generalizations of the Gohberg-Krein analysis of the case of equality in Weyl's inequalities relating eigenvalues and singular values. A typical result is the following:

Let A _be an_ m-_square complex matrix with eigenvalues_ $\lambda_1 \cdots \lambda_n$ _arranged so that_ $|\lambda_1| \geq \cdots \geq |\lambda_n|$. _The largest absolute value taken on by an_ m-_square principal subdeterminant in any unitary transform of_ A _is at least_ $|\lambda_1 \cdots \lambda_m|$, $m = 1, \ldots, n$. _This largest absolute value is equal to_ $|\lambda_1 \cdots \lambda_m|$ _for_ $m = 1, \ldots, n$ _iff_ A _is normal_.

Item 35 extends the Milnor-Calabi result on simultaneous diagonalization of a pencil of real symmetric matrices to pencils in which no 2×2 principal subdeterminant vanishes in any real congruence transform. The proof depends on applying multilinear algebra methods to a canonical form of R. C. Thompson for simultaneous congruence reduction. The major result here is:

THEOREM. _Let_ A _and_ B _be_ n-_square real symmetric matrices_. _Assume_ *that*

$$(\det X^{\mathsf T} A X)^2 + (\det X^{\mathsf T} B X)^2 > 0$$

for all $n \times 2$ *matrices* X *of rank* 2. *If* $n \geq 3$, *then* A *and* B *are simultaneously congruent over* $\mathbb{R}$ *to diagonal matrices as follows:*

$$P^{\mathsf T} A P = \operatorname{diag}(\alpha_1, \ldots, \alpha_p, \overbrace{0, \ldots, 0}^{q}, \overbrace{\pm 1, \ldots, \pm 1}^{k})$$

$$P^{\mathsf T} B P = \operatorname{diag}(\overbrace{\pm 1, \ldots, \pm 1}^{p}, \overbrace{\pm 1, \ldots, \pm 1}^{q}, \overbrace{0, \ldots, 0}^{k}) ,$$

p, q, k *are non-negative integers satisfying* $p + q + k = n$, *and* $\alpha_1 \cdots \alpha_p \neq 0$. *If* $n = 2$, *then either* A *and* B *are simultaneously congruent to diagonal matrices or they are simultaneously congruent to a pair of* 2×2 *matrices as follows:*

$$P^{\mathsf T} A P = \epsilon \begin{bmatrix} 0 & \alpha \\ \alpha & 0 \end{bmatrix} ,$$

$$P^{\mathsf T} B P = \epsilon \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} , \quad \epsilon = \pm 1$$

or

$$P^{\mathsf T} A P = \epsilon \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} ,$$

$$P^{\mathsf T} B P = \epsilon \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} , \quad \epsilon = \pm 1 .$$

2. Stability and the Lyapunov map. The papers (19), (20), and (21) constitute contributions to the study of the elementary divisor structure of the Lyapunov map

$$X \to AX - XA^{\mathsf T} .$$

This map, of course, is of crucial importance in analyzing the behavior of an autonomous system of ordinary differential equations of the type appearing in the study of feedback control of linear systems. The general result is that a fully complete analysis has been obtained of the elementary divisor structures of the Lyapunov (and some other) linear maps. For example, Paper (24) is concerned with the determination of the elementary divisors of a tensor product of linear maps. This, of course, is a standard theorem which dates back to the thirties and is due to Roth. However, we have found a new and simple argument which is purely combinatorial in nature. It is also interesting that the method contains some interesting applications to combinatorial matrix theory. To illustrate:

Suppose that the first and last rows of a $(p+1) \times k$ matrix are prescribed as consisting of strictly increasing sequences of integers chosen from $1, \ldots, q$, $q \geq k \geq 1$, $p \geq 1$. In how many ways can the $p-1$ intermediate rows be selected from the integers $1, \ldots, q$ so that they satisfy: (i) the integers in each row strictly increase from left to right; (ii) the integers down every column are nondecreasing and no adjacent integers in a column differ by more than 1? For example, in how many ways can the 7×3 matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 4 & 5 & 6 \end{bmatrix} \qquad (1)$$

be so completed? The answer is det M = 980 where

$$M = \begin{bmatrix} \binom{6}{3} & \binom{6}{4} & \binom{6}{5} \\ \binom{6}{2} & \binom{6}{3} & \binom{6}{4} \\ \binom{6}{1} & \binom{6}{2} & \binom{6}{3} \end{bmatrix} .$$

Thus there are 980 ways of filling in the matrix (1) using only the integers 1, ..., 6 so that each row strictly increases, the integers down a column do not decrease and adjacent integers in a column never differ by more than 1. Of course, there are 6^{15} ways of completing the matrix altogether using the integers 1, ..., 6 so that the probability of finding one in a random search is about 2.08×10^{-9} This work is closely related to earlier work of W. Givens [Argonne Nat. Lab. Rep. (ANL-6456), 1961), J. W. Neuberger [Israel J. Math. 6 (1968), 121-132] and H. Neudecker [SIAM J. Appl. Math. 17. 3 (1969), 603-60[illegible]].

The above paper was invited to appear in the Communications of the Association for Computing Machinery in a special issue honoring A. S. Householder. The reason that it was submitted there is that some of the results are pertinent to recent work on tensor products, successive approximation, Kronecker products, and matrix equation systems.

An additional paper (31) is concerned with traces of exterior powers of the sum of two endomorphisms on a projective module. One of the consequences of this theorem has to do with the polynomial det $(\lambda A + \mu X)$. Here one is interested in the coefficients in the polynomial expansion of this function in λ and μ. As a side result, this theorem is used to obtain some rather curious combinatorial applications.

3. Properties of tensor products of linear transformations. This is a continuing investigation which has been in progress for many years. Here are some of the new results.

Item 25 contains a very short proof that the Hodge star operator on the Grassmann algebra is independent of the orthonormal basis used to define it. The standard proof depends on showing that the operator can be decomposed into a composition of three maps, each of which is independent of the basis.

In item 31 two problems are considered. First, it is shown that if the exterior power of a matrix is diagonalizable then the reduction to diagonal form can be accomplished by a similarity which itself is an exterior power of a nonsingular matrix. Further results along this line are obtained which generalize earlier work of S. Belcerzyk, Colloquium Mathematicum, XXIII (1971), 203-211. An interesting purely matrix theoretic result which is used in this investigation seems to be useful in other contexts:

Assume that the field R contains the eigenvalues of matrices $A, B \in M_n(R)$, $AB = 0$, and A has precisely p nonzero eigenvalues (including multiplicities). Then A and B are simultaneously similar to matrices of the form

$$\begin{bmatrix} 0 & * & \\ & \ddots & * \\ * & 0 & \\ & 0 & A' \end{bmatrix}, \quad \begin{bmatrix} B' & * \\ & \\ 0 & 0 \end{bmatrix}$$

respectively, where $A' \in GL_p(R)$ is upper triangular and $B' \in M_{n-p}(R)$ is upper triangular.

In paper 23 the problem of determining under what circumstances a Kronecker product of matrices is normal was considered. This paper was suggested by a recent paper of Y. Kuo [Linear Alg. Appl. 7, (1973), 63-70]. To be precise the results state:

If $A_1, \ldots, A_p$ are square complex matrices and $A = A_1 \otimes \cdots \otimes A_p \neq 0$ is normal then A_i is normal, $i = 1, \ldots, p$. Similar results are available when A is hermitian or skew-hermitian. If $T : V \to V$ is linear and $K(T)$ is the induced linear map on a general symmetry class of tensors then normal, Hermitian, and skew-hermitian properties of $K(T)$ imply similar properties for T.

In item 36 all isometries of the algebra of n-square complex matrices which preserve the following norm are determined: the norm is the sum of the first k singular values. For $k = 1$ and $k = n$ the result specializes to earlier work of K. Morita and B. Russo respectively.

Research was also initiated on the very difficult and deep work being done by H. Busemann and his students on irreducible length in the Grassmann space. This concept, of course, arises in other symmetry classes of tensors, e.g., the contravariant tensor space. The general question, which seems enormously difficult, is to determine how "long" an irreducible tensor in a symmetry class can be. By the length of a tensor is simply meant the number of decomposable elements in the shortest representation of the tensor as a sum of decomposable elements.

4. Matrix inequalities. One typical paper, 32, contains an extension of an inequality of A. Ostrowski and exhibits the relationship of the inequality to various converses of the classical generalization by E. Fischer of the Hadamard determinant theorem. The following is a typical result:

THEOREM. Assume that $1 \leq k \leq m \leq n$ and that H is the m-square

Gram matrix based on the linearly independent vectors $a_1, \ldots, a_m$ *in* C^n. *Let* $X^1, \ldots, X^{m-k}$ *be nonzero orthogonal vectors in* C^n *for which*

$$(a_i, X^j) = 0, \quad i = 1, \ldots, k, \quad j = 1, \ldots, m-k,$$

and

$$(a_{k+i}, X^j) = r_{ij}, \quad i,j = 1, \ldots, m-k.$$

Then

$$\frac{\det H}{\det H_{11}} \geq \frac{|\det \rho|^2}{\prod_{j=1}^{m-k} \|X^j\|^2}$$

where H_{11} *is the* k-*square leading principal submatrix of* H *and* ρ *is the* $(m-k)$-*square matrix* $[r_{ij}]$.

D. Description of Research

HENRYK MINC

The research conducted during the period of the grant can be roughly divided into three areas: (i) Nonnegative matrices; (ii) Permanents of (0,1) matrices and of doubly stochastic matrices; computation of permanents; (iii) Linear transformation on matrices.

Nonnegative matrices are of importance in many branches of pure and applied mathematics. Minc's research concerns both general nonnegative matrices (items #1, 5, 8, 9, 11, 13, 14, 15, 18)* and special types of nonnegative matrices: doubly stochastic matrices and (0,1)-matrices.

In papers #8 and 9 a study is made of the structure of imprimitive matrices. In 1912 Frobenius showed that if A is an irreducible matrix with index of imprimitivity h, then there exists a permutation matrix P such that PAP^T is in a superdiagonal block form with h zero square blocks along the main diagonal. In papers #8 and 9 a converse result is obtained: a matrix in a superdiagonal block form without zero rows or columns is irreducible if and only if the product of the superdiagonal (nonzero) blocks is irreducible. These papers contain also other related results.

The problem whether a given n complex numbers are the spectrum of a nonnegative matrix is of crucial importance. Unfortunately, it is still unsolved. In paper #14 a result is obtained which somewhat simplifies the problem by relating the eigenvalues of an $n \times n$ irreducible matrix of index h to the eigenvalues of an m-square primitive matrix, where $m \leq n/h$. In paper #11 it is essentially shown that spectral problems involving nonnegative matrices cannot be further simplified by applying linear transformations that map nonnegative matrices into nonnegative matrices and hold their spectra fixed.

*The references on pp. 24-27 are to the papers listed on pp. 6-7.

In paper #13 simple proofs are given for two classical results of Frobenius. Papers #1 and 5 deal with combinatorial problems involving non-negative matrices.

Permanents of (0,1)-matrices have been studied extensively. They are of interest in combinatorics since the permanent of a (0,1)-matrix is equal to the number of systems of distinct representatives of the corresponding configuration of subsets of a finite set. In paper #1 a lower bound for the permanent of a square (0,1)-matrix is given, and a necessary and sufficient condition for equality is obtained. In paper #7 it is shown that the hypotheses for a well-known lower bound for permanents of a (0,1)-matrix, due to M. Hall, can be relaxed. In paper #10 a partial affirmative answer is obtained to a conjecture proposed by Minc in 1963 (that research was supported by a previous AFOSR grant). The conjectured upper bound for permanents of (0,1)-matrices has been now proved by a Russian mathematician L. M. Bregman.

Papers #3, 16 and 17 are related to the famed unresolved conjecture of van der Waerden on the permanent function of doubly stochastic matrices. Paper #3 gives an explicit formula for the permanent of an $n \times n$ circulant $\alpha I + \beta P + \gamma P^2$. The case when the circulant is doubly stochastic is also studied. In paper #16 a short proof is given of a result of D. London on permanental minors of a doubly stochastic matrix with a minimum permanent. In paper #17 it is shown that if in any doubly stochastic matrix all subpermanents of order r, $1 \leq r \leq n-2$, are equal then all the entries of the matrix are equal.

Permanents of $m \times n$ matrices with the entries in any field are usually computed by a method of Ryser (1963) based on the principle of inclusion and exclusion. It is not generally known that in 1812 Binet obtained evaluation formulas for permanents using the same principle. Binet proved his formulas for $m = 2$ and 3, and gave a formula for the permanent of a $4 \times n$ matrix. A method of proof was suggested by Joachimstal (1856). In paper #21 a general formula for computing the permanent of an $m \times n$ matrix is given and proved.

A problem of considerable interest and importance in linear algebra is the determination of the form of linear transformations on $m \times n$ matrices that preserve certain prescribed properties of the matrices. Frobenius (1897) found the form of linear transformations on $n \times n$ matrices that keep the determinant fixed. Marcus and Purves (1959) determined all linear transformations T on $n \times n$ matrices that hold the r^{th} elementary symmetric function fixed, $4 \leq r \leq n - 1$. Beasley (1970) extended the result to the case $r = 3$. In paper #11 it is shown that if T holds both the determinant and the trace fixed then T is a similarity transformation, possibly coupled with transpositions. In paper #19 similar results were established for linear transformations preserving either the trace or the second elementary symmetric function of the eigenvalues <u>and</u> one of the following invariants or properties: the determinant, the permanent, an elementary symmetric function of singular values squared, the property of being unitary, or the property of being of rank 1.

The set of linear transformations preserving rank 1 was determined by Marcus and Moyls (1959) using methods of multilinear algebra. Apart from its intrinsic interest this result has been used extensively in proofs of other results on linear transformations holding some invariants fixed. In paper #20 the result of Marcus and Moyls is re-proved by elementary methods and it is used to prove the result of Frobenius on determinant preservers.

Item #22 is a comprehensive monograph on permanents containing a complete bibliography on the subject (of over 275 papers and books). The work is scheduled to be completed in 1977.

The remaining three papers, #2, 4 and 8 deal with general matrix functions, eigenvalues of matrices with prescribed entries, and with involutory non-negative matrices, respectively.

D. Description of Research

R. C. THOMPSON

The principal thrust of Thompson's research during the five years 1971-76 has been the investigation of the eigenvalues and singular values of matrix sums, products, and minors. Besides this, there was published research on the numerical range, on matrix norms, on pencils of matrices, and on certain number theoretical questions. Many results have been obtained but the following exhibits the broad features of the research.*

(a) Research on eigenvalues of matrix sums, products, and minors. The objective of this research was twofold: First, to take the already existing literature (which was extremely disorganized and contained many results which either were not as sharp as possible or which were organized in a way that concealed the basic principles involved) and do a complete investigation to obtain the sharpest possible results, and to identify the basic principles. Second, having exposed the basic principles and techniques, to build on this foundation and construct a complete theory. The progress to date is this: much of the first part of this program is complete, and substantial progress towards the second is in hand. Some general principles obtained are:

(i) Uniform methods have been developed for studying the eigenvalues of matrix sums, of matrix products, and of matrix minors, with the consequence that any theorem proved for eigenvalues of sums will also apply in slightly different form to the eigenvalues of products or minors.

(ii) A family of inequalities known as the Amir-Moez inequalities, which were believed for 15 years to be the basic results governing the eigenvalues of sums and products, have been shown to be worthless in

*References on pp. 28-34 are to the papers listed on pp. 8-10.

the sense of being much too complicated and not sharp: a much simpler and uniformly sharper result was found.

(iii) A basic device, (known as the Courant-Fischer max-min principle) for establishing results about eigenvalues. which has been used and still continues to be used by nearly every mathematician involved in eigenvalue estimation, has been shown to be of much less value than previously believed, and in particular has been shown to conceal the underlying geometry.

(iv) The basic geometrical tool appropriate to the estimation of eigenvalues of sums, products, and minors has been identified. It is the Schubert calculus of algebraic geometry, in the form in which it studies the intersection properties of three nested collections of subspaces of a given vector space (in the language of algebraic geometry: intersection properties of three "flags"). The full description of the intersection properties of three flags is a long outstanding open question in algebraic geometry, involving some very advanced concepts, e.g., cohomology, differential geometry. However, much is known.

(v) Every result known concerning the eigenvalues of matrix sums, products, minors has been pinpointed as an immediate consequence of the Schubert calculus. A graduate student (S. Johnson) is working under Thompson's direction on the applicability of the Schubert calculus to these linear algebra questions. He has made considerable progress, already has material sufficient for a path-breaking paper, but still is continuing his research. There is reasonable hope that Johnson will be able to prove a conjecture of A. Horn concerning eigenvalues of matrix sums that has been unresolved since its publication in 1962.

(vi) It has been realized that many of the results obtained for eigenvalues of matrix sums, products, minors, can be extended to arbitrary (perhaps semisimple) Lie groups. No work has yet been done to exploit this idea; it is hoped to do some in the next five years. However, it has become clear that the many inequalities obtained by Thompson have a universality extending much beyond the framework within which they were discovered: they can be conjectured to hold in any mathematical situation in which they can be meaningfully stated.

(vii) A result obtained by Thompson just after the conclusion of the five year period (paper #40) has further extended the universality of Thompson's inequalities in a completely unexpected direction, namely to the number theory of matrices with integers as entries. For these matrices certain whole numbers known as "invariant factors" play a central role in describing the properties of the matrix which are important in combinatorial investigations and applications. (Indeed, Dr. Morris Newman, until recently at the National Bureau of Standards, Washington, now at the Santa Barbara Math Department, has repeatedly used invariant factors to resolve questions brought to him in his consulting role at the Bureau by chemists and physicists.) Thompson's newest result shows that certain of his inequalities apply to invariant factors of integral matrices, and it is conjectured that many of his other results apply equally well to invariant factors.

(viii) Two curiosities obtained are the following: There is an inequality for eigenvalues that is generally valid only when the matrices have an odd number of rows and columns. There are inequalities for the eigenvalues of matrix sums that hold for commutative matrices only and do not generally hold for noncommutative matrices. The interlacing inequalities for eigenvalues and the Sylvester Law of Inertia have been shown to both follow from a single eigenvalue inequality.

Suffice it to say that during the five years a broad and sound foundation has been produced for the estimation of eigenvalues of matrix sums, products, minors. So many results have been obtained that there is more than sufficient material to fill a book on this topic. It is, however, premature to write such a book now since new results are being obtained by Thompson and his students so rapidly that the book would soon be obsolete.

References: Papers 1, 2, 3, 4, 5, 6, 7, 8, 9, 13, 14, 15, 16, 17, 18, 19, 20, 28, 33.

(b) Research on singular values. The singular values of a matrix A are the eigenvalues of $(AA^*)^{1/2}$, where A^* is the complex conjugate transpose of A. Though known in this country, they have not exploited here as much as they should be, since they have been shown by Russian mathematicians (particularly by M. G. Krein and collaborators) to be fundamental tools in very many different types of investigations. Part of Thompson's work during the five year period under review has been a systematic development of the properties of singular values, building towards a comprehensive knowledge.

It is worth noting that singular values are finding increasingly many applications in numerical linear algebra. Thus the comprehensive investigations by Thompson of their properties is important from the point of view of applications.

References: Papers 2, 5, 6, 7, 20, 24, 25, 26, 27, 31.

(c) Norms and norm inequalities. Investigations of matrix norms and their properties have been actively pursued by numerous workers in numerical linear algebra, particularly by F. Bauer. A fundamental new turn to this type of study has been given by Thompson in three papers (particularly paper #34). The central discovery here is that of inequalities satisfied by matrix valued norms, i.e., norms in which the norm of a matrix is not a number but another matrix. The discovery is that inequalities for numbers, like the triangle inequality, which, of course, also hold for numerical valued norms, can also hold for matrix valued norms. The precise form (not an obvious one) in which norm inequalities can hold for matrix valued norms was first found in paper #32. A comprehensive investigation is in paper #34.

References: Papers 30, 32, 34.

(d) The numerical range. This is a very active area of investigation, owing in part to its applications in numerical solutions of partial differential equations and in functional analysis. One paper was written by Thompson in this area; it characterizes completely a certain type of numerical range.

References: Paper 23.

(e) Pencils of matrices. Matrix pencils are classical in linear algebra - their study goes back to about 1860. Recently they have become of central importance in control theory, and their renewed study has been advocated by the engineer H. Rosenbock. There are many applications in engineering, particularly in optimal control and stability. Paper #21 initiates a whole new type of investigation pertinent to the study of pencils, namely, the investigation of the relations between the properties of the pencil itself and the properties of the subpencil, particularly with reference to minimal indices, elementary divisors, and inertial signatures. There is no prior published work in this direction, and the paper #21 is just the beginning of a comprehensive investigation. A doctoral student working under Thompson is beginning investigation in this area.

References: Papers 21, 37, 38.

(f) Number theory. Number theory is increasingly important in applied mathematics, e.g., in combinatorics, in algebraic coding theory. Several papers were written on the number theoretical properties of matrices with integers as elements. Roughly speaking, these papers count how many classes of integral matrices there are having certain specified properties. As noted in part (a vii) above, there is prospect of much more work in this direction.

References: Papers 10, 12, 43, 44, 45.

(g) Resolution of a 15-year open question on diagonal elements. A question unresolved since about 1960 is the relationship between the diagonal elements and the singular values of a matrix. A complete solution was found by Thompson in 1974; the comprehensive paper #31 gives this result and many consequences. The form of the result is quite unexpected and surprising, and substantially advances research carried out in the 1950's by many mathematicians on the properties of the diagonal elements of matrices.

References: Paper 31.

(h) Resolution of a 100 year open question on similarity invariants. A question outstanding since the invention of elementary divisor theory in the mid-1800's has been the determination of the relation between the elementary divisors of a matrix and those of a principal submatrix. A complete solution of this question has just been obtained in paper #40: the result is totally unexpected, literally an extraordinary surprise, because it reveals that there is considerable analogy in the behavior of similarity invariants and of singular values. This is a path breaking paper that opens up very many questions for investigation. There are so many results anticipated as a consequence of paper #40 that it is difficult to decide which to investigate first. Paper #40, although written just after the completion of the five year period under review, in fact evolved in the author's thinking from the investigations carried out during the five years, in particular, from paper #21.

(i) Summary. Thompson's research during the five year period (i) solved a number of long unresolved problems, and (ii) built a strong foundation of basic results needed for progress in a number of areas. Many questions are now within reach that were formerly regarded as beyond attack.