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ON THE MAXIMUM TRANSFORM AND SEMIGROUPS OF TRANSFORMATIONS

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PREFACE

Part of the Project RAND research program consists of basic supporting studies in mathematics. A problem frequently occurring in applications is that of determining the maximum or minimum value of a function subject to prescribed constraints.

In the present Memorandum the authors show how the mathematical technique of the maximum transform can often be applied effectively to this problem.

SUMMARY

In this Memorandum the authors consider the general problem of determining the maximum value of a function of the form

$$F(x_1,x_2,\ldots,x_N) = \sum_{i=1}^{N} g_i(x_i)$$

over the domain $\sum_{i=1}^{N} x_i = x$, $x_i \geq 0$. They show how the maximum transform, defined by

$$f(y) = M(F) = \max_{x \geq 0} [F(x) - xy],$$

can be applied to problems of this type.

CONTENTS

ON THE MAXIMUM TRANSFORM AND SEMIGROUPS OF TRANSFORMATIONS

1. INTRODUCTION

The problem of determining the maximum of the function

$$(1.1) \qquad F(x_1,x_2,\ldots,x_N) = \sum_{i=1}^{N} g_i(x_i)$$

over the domain D_N defined by $\sum_{i=1}^{N} x_i = x$, $x_i \geq 0$, is one with various ramifications and applications. Analytic solutions and computational algorithms have been obtained in a number of ways; see Karush [7], Bellman [2], Bellman and Karush [3]. We shall now discuss a new way of generating solutions of (1.1). Let $g(x,a)$ be a scalar function of the scalar variable x and the M-dimensional vector a with the group property that

$$(1.2) \qquad \max_{x_1+x_2=x} [g(x_1,a) + g(x_2,b)] = g(x,h(a,b)), \quad x_1,x_2 \geq 0,$$

where $h(a,b)$ is a known function of a and b. It follows inductively that

$$(1.3) \qquad \max_{D_N} \left[\sum_{k=1}^{N} g(x_k,a^{(k)})\right] = g(x,h(a^{(1)},a^{(2)},\ldots,a^{(N)})),$$

where D_N is as above and $h(a^{(1)},a^{(2)},\ldots,a^{(N)})$ is obtained from $h(a,b)$ in a recurrent fashion. The function $g(x,a) = ax^p$, $0 < p \leq 1$, with $a \geq 0$, is a function of the desired type. How can we generate classes of functions with this property, and can we determine all of them?

2. THE MAXIMUM TRANSFORM

In previous papers [3,4,5], we discussed the transform

$$(2.1) \qquad f(y) = M(F) = \max_{x \geq 0} [F(x) - xy],$$

a transform which plays a basic role in the study of convexity; see Fenchel [6] and Beckenbach and Bellman [1]. This transform possesses the important dissolving property

$$(2.2) \qquad M\left[\max_{x_1+x_2=x} [g(x_1,a) + g(x_2,b)]\right] = M(g(x,a)) + M(g(x,b)).$$

Taking advantage of this relation, we can obtain functions satisfying (1.2) by starting with functions $G(x,a)$ satisfying the simpler relation

$$(2.3) \qquad G(x,a) + G(x,b) = G(x,h(a,b)),$$

and inverting:

$$(2.4) \qquad g(x,a) = M^{-1}(G(x,a)) = \min_{y \geq 0} [G(y,a) + xy].$$

3. SOLUTIONS OF THE FUNCTIONAL EQUATION

If a is an M-dimensional vector with components $a_1, a_2, \ldots, a_M$, a very simple class of solutions of (2.3) is given by

$$(3.1) \qquad G(x,a) = (a,G(x)) = \sum_{i=1}^{M} a_i G_i(x),$$

$$h(a,b) = a + b.$$

Under various assumptions of analyticity, it may be shown that aside from inessential changes of variable, $a \to \phi(a)$, these are the only solutions. Rigorous proofs will be given subsequently.

4. PARAMETRIC REPRESENTATION

If it is possible to obtain the minimum value in (2.4) by means of differentiation, we obtain the parametric representation

$$(4.1) \qquad g(x,a) = G(y,a) + xy,$$

$$x = - G_y(y,a).$$

Assuming that $G(x,a)$ is given by (3.1), we face the interesting problem of determining the a_i and $G_i(x)$ so as to fit a given function $g(x)$ in some optimal fashion. Having done this, we can find quick and useful solutions to the original variational problem, (1.1) et seq. The point is that in this way we find exact solutions to approximate problems as opposed to the usual approximate solution to an exact problem.

In subsequent papers, we shall discuss the multidimensional and continuous versions of these problems and techniques.

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