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WSEG REPORT 250

QUANTITY-QUALITY TRADEOFFS

January 1975

Including

IDA STUDY S-443

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21 FEB 1975

MEMORANDUM FOR DIRECTOR OF DEFENSE RESEARCH AND ENGINEERING

SUBJECT: Quantity-Quality Tradeoffs

1. The study contained in this report, WSEG 250, was prepared for DDR&E (P&A) in response to WSEG Task Order 216, dated 14 November 1973.

2. The objective of this report is to develop the theory in the tradeoff between quantity and quality. The study represents an initial effort to examine several concepts which might be usefully applied with some additional investigation. It was recognized at the outset that the scope of the study and the complexity of the problem almost precluded definitive results. Four different areas which showed promise were investigated: (a) general purpose fighter/bomber aircraft designed for the multiple roles of intercept, airbase attack and close air support; (b) medium tanks; (c) mathematical attrition processes in multiple engagements and (d) historical records of air to air combat from World War II. All four investigations produced results which could prove useful in more detailed decision-oriented analyses.

3. The first analysis investigated aircraft quantity-quality tradeoffs at the theater level. The measure of effectiveness (MOE) was FEBA movement as a function of the ratio of combat firepower, of which air firepower (delivered CAS sorties) was only a portion of the total. The results of this analysis tend to support the idea that increasing quantity is a better investment than increasing quality in the range of forces and effectiveness parameters considered. In some cases the MOE was quite insensitive to changes in the aircraft quality parameters. However, the MOE was always sensitive to changes in quantity. The analysis did not determine the cost relationship between the individual quality parameters and quantity. Therefore, the results are not conclusive for decision making.

4. In the second analysis (tanks) the primary MOE was loss-rate difference as determined from multiple computer runs of a tank exchange model, simulating tank engagements of 1 to 5 tanks on defense and 1 to 10 tanks in the assult. Here again, there is so little parametric cost data on tanks that the final link between the quantity, quality, effectiveness and cost could not be made.

5. The last two analyses were primarily theoretical and may prove useful in further investigations of the quantity-quality tradeoffs.

M. H. Sappington

M. H. SAPPINGTON
Rear Admiral, USN
Acting Director

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STUDY S-443

QUANTITY-QUALITY TRADEOFFS

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January 1975

This report has been prepared by the Program Analysis Division of the Institute for Defense Analyses in response to the Weapons Systems Evaluation Group Task Order DAHC15 73 C 0200 T-216, dated 14 November 1973.

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SUMMARY

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1. INTRODUCTION

The purpose of this study is to develop methodology for trading off quantity and quality of weapons systems. The methodology that has been developed deals with determining effectiveness as a function of quantity and quality of weapons systems. To make decisions on optimal mixes, costs are needed as a function of the same variables. The study does not treat costs, but concentrates on the effectiveness aspects of quantity-quality trade-offs.

Developing effectiveness functions is usually thought of as more difficult than developing cost functions, because peacetime cost-accounting data are routinely maintained and peacetime costs of new systems can be predicted, at least in principle, as functions of physical and operational variables of existing systems, while wartime effectiveness depends upon both friendly and enemy physical and operational variables, and useful wartime data are not generally available--even for existing systems. This study treats mostly modeling aspects of determining effectiveness functions, although some empirical combat data are analyzed in Chapter 4.

In the four chapters of this study, methodology is the key aspect of the work. The results of the exploration are (1) to illustrate analytical approaches to trading off quantity and quality and (2) to show how areas of sensitivity (where quantity-quality trade-offs are most important) can be identified by the methodology. Example data are used in simulation models of aircraft (in Chapter 1) and tanks (in Chapter 2). Theoretical work is emphasized in Chapter 3. Some methodological developments for processing combat data are presented in Chapter 4.

Chapter 1 considers general-purpose aircraft at the theater level. A theater-level model is employed that simulates ground and air warfare and that includes optimal allocation of general-purpose aircraft to missions. Trade-offs between quantity of aircraft and quality of aircraft measured by firepower per combat air support (CAS) sortie are analyzed. Various effectiveness levels for both sides in the intercept (INT) and airbase attack (ABA) missions are considered. If the "killing-of-aircraft" parameters associated with INT and ABA are low, there is a large effect on the ground-air war (measured in terms of FEBA advance) from CAS sorties and sensitive trade-offs between quantity and quality. When one or more of the killing-of-aircraft parameters are high, the trade-offs are not sensitive. For most of the force levels and effectiveness parameters considered, variations in quantity of aircraft have a greater effect than variations in quality as measured by CAS firepower.

Chapter 2 treats tanks at the engagement level. Engagements of up to 10 offensive tanks against 5 defensive tanks are explored. A Monte Carlo simulation of tank engagements is used to perform the analysis. A statistical methodology is developed that permits (1) selection of the quality parameters to which the model is most sensitive and (2) estimation of effectiveness according to various measures as a function of quantity and quality. Results are given for selected cases.

Chapter 3 explores mathematical attrition processes, mostly from a theoretical point of view. Quantity-quality trade-offs with Lanchester square and Lanchester linear mathematical structures have been thought to be important for some time. However, it is quite difficult to apply the theory. This chapter identifies combat situations having physical properties leading to various mathematical forms, including Lanchester square, Lanchester linear, multiple-engagement binomial, single-engagement binomial, and barrier penetration. This is a step towards applying these methods in quantity-quality trade-offs; for once a mathematical form is justified to be appropriate and parameters are available, quantity-quality trade-offs can be made. The work presented here does not make the trade-offs, but it is believed to be a contribution to the eventual application of mathematical structures to weapons-systems decisions.

Chapter 4 develops some new methodologies for relating physical characteristics to performance parameters used in simulations. A demonstrative empirical investigation based on World War II fighter aircraft is explored. Data on outcomes of engagements and on physical characteristics of the aircraft involved are developed for U.S. and German aircraft. Probability of kill is estimated as a function of wing loading, power loading, maximum speed, maximum ceiling, number of machine guns, and number of cannon. Several other aspects of methodology for using historical data and subjective data in weapons-systems design are discussed in the chapter.

2. DISCUSSION AND SUMMARY

A. GENERAL-PURPOSE AIRCRAFT: THEATER LEVEL

1. Methodology

Trade-offs between quantity of general-purpose aircraft and quality of general-purpose aircraft as measured by fire-power per CAS sortie are investigated in Chapter 1. The OPTSA model (Reference [1]) is used for the investigation. It simulates a two-sided conventional war with ground forces and tactical air forces, adding the air-delivered firepower from CAS missions to the ground firepower and forming the force ratio

$$\frac{\text{Blue ground firepower plus Blue air firepower}}{\text{Red ground firepower plus Red air firepower}},$$

which is used to calculate movement of the forward edge of the battle area (FEBA) and casualties to the ground forces of both sides. Aircraft of both sides are assigned to CAS, INT, and ABA missions to optimize the position of the FEBA at a particular day of the war.

The reasons for using a game-theoretic model such as OPTSA are that war outcomes are sensitive to assignments of aircraft to missions and that optimal assignments of aircraft to missions are sensitive to effectiveness parameters. Thus, a model with built-in mission assignments can yield very different war outcomes from those that occur with two-sided optimal, or mutually enforceable, strategies. To deny either side his enforceable strategy is to underestimate his capabilities. The OPTSA model substantially reduces the concern about mission allocation. It incorporates FEBA movement computations, casualty-estimation methods, and air-attrition equations similar to those used in other commonly applied theater-level models. Thus, it is believed to be an appropriate vehicle for studying quantity-quality trade-offs.

2. Data

Data described in Chapter 1 are hypothetical, but they bear some relationship to usual planning data. Blue and Red divisions number 24 and 80, respectively. Blue aircraft quantity is varied; Red aircraft quantity is fixed at 2,000. Firepower per Blue CAS sortie is varied; firepower per Red CAS sortie is fixed at 1/100 of firepower per Red division (higher than most planning numbers, though within the range).

A 10-day war is considered. Without CAS, the position of the FEBA would be -100 at D+10, based on the function used for FEBA movement as a function of force ratio.

3. Application of Methodology

Figures S-1 and S-2 give isovalue contours for FEBA position at D+10 as a function of number of Blue aircraft and firepower per Blue CAS sortie, for low and high killing-of-aircraft parameters, respectively. Killing-of-aircraft parameters reflect probability of kill, given detection, in the INT and ABA missions, by both Blue and Red.

In Figure S-1, starting from the upper left circle at 1,000, 0.10 (where FEBA position is -82), note that adding 1,000 aircraft leads to -19 and adding 0.04 units of fire-power per sortie leads to -72. However, starting from the lower right circle at 2,000, 0.02 (where FEBA position is at -117), note that adding 1,000 aircraft leads to -97, while adding 0.04 units of firepower per sortie leads to -49. The conclusion is that additional quantity yields more at 1,000, 0.10 and additional quality yields more at 2,000, 0.06. That

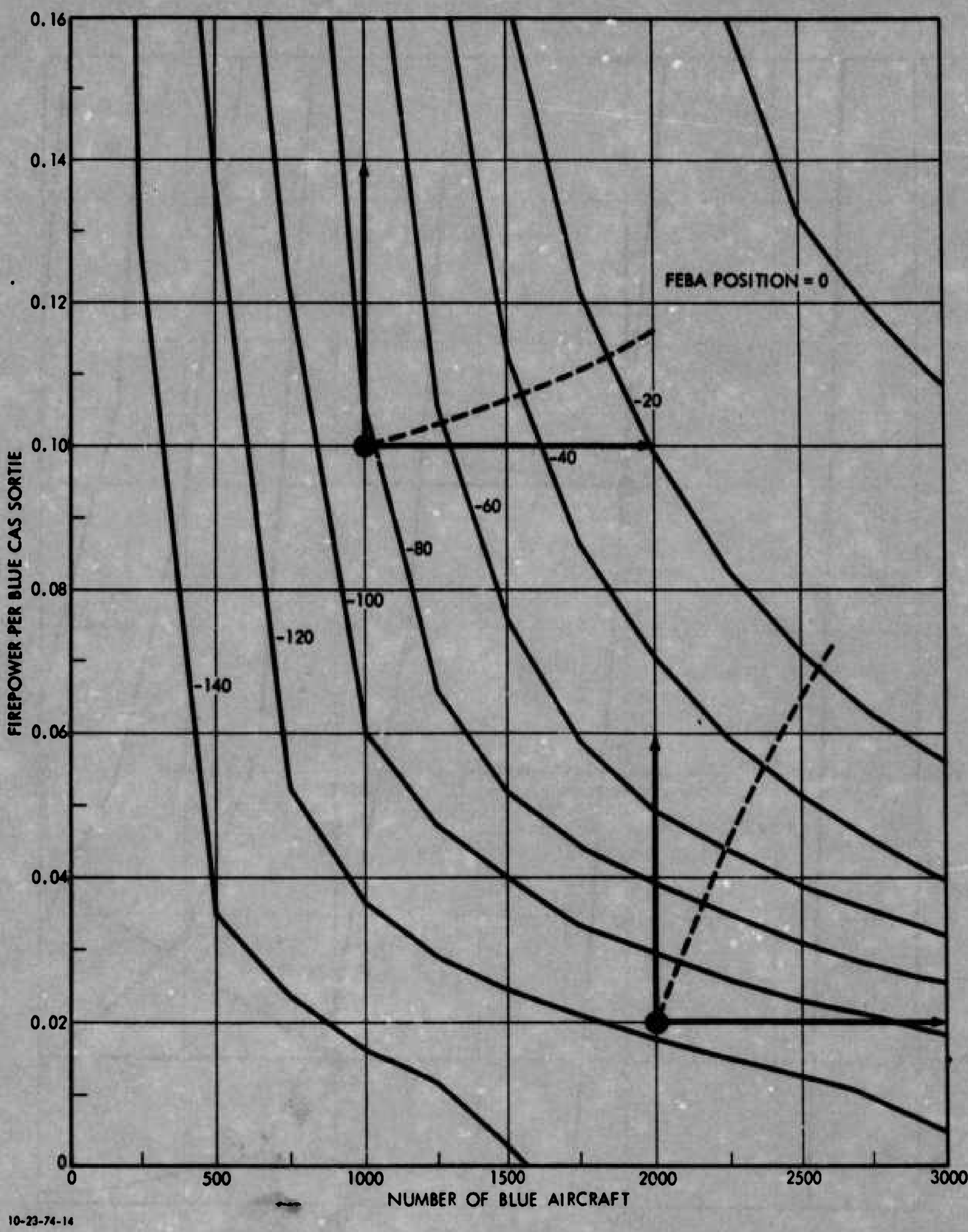

Figure S-1. FEBA POSITION AT D+10 AS A FUNCTION OF BLUE QUANTITY AND QUALITY, WITH LOW KILLING-OF-AIRCRAFT PARAMETERS

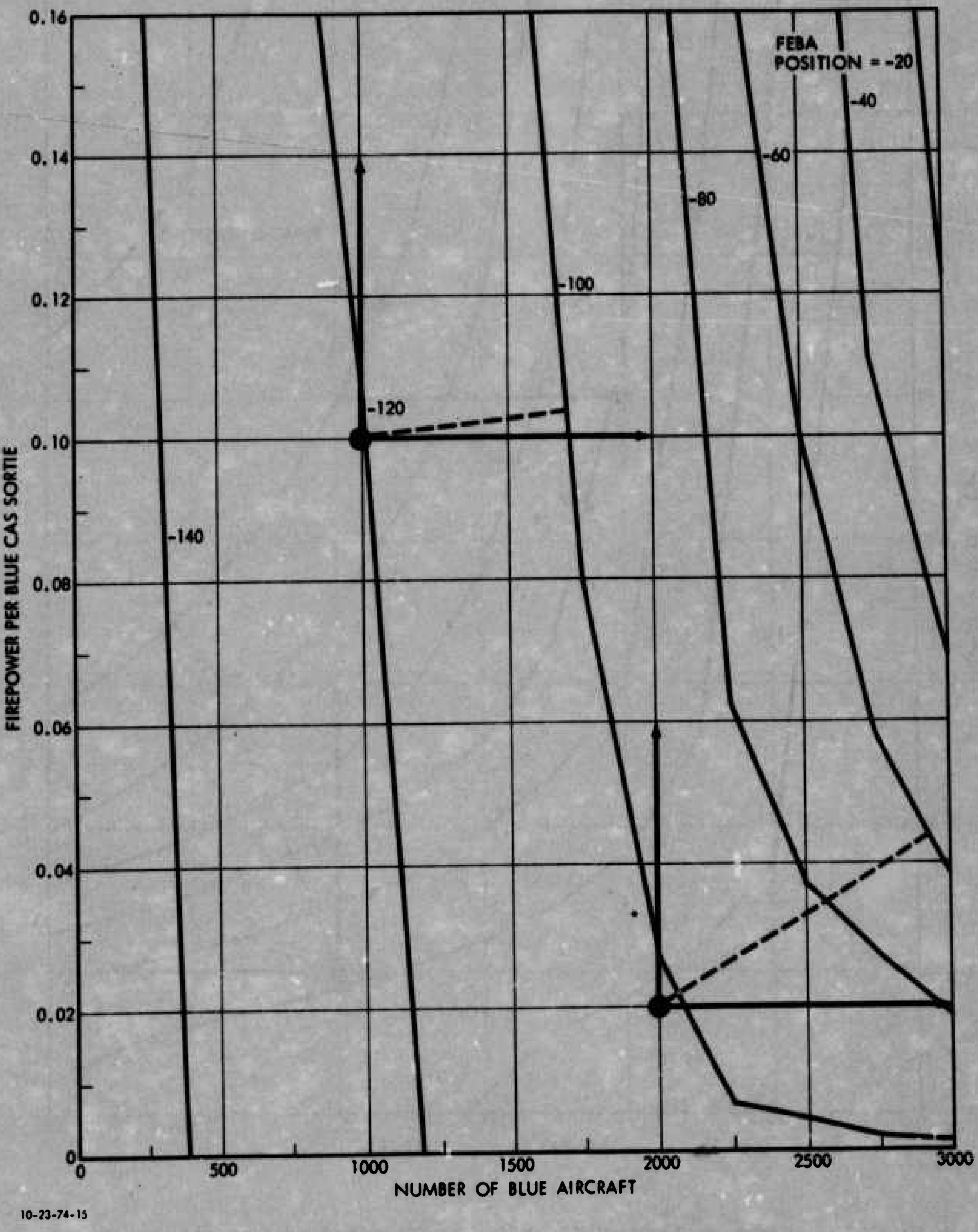

Figure S-2. FEBA POSITION AT D+10 AS A FUNCTION OF BLUE QUANTITY AND QUALITY, WITH HIGH KILLING-OF-AIRCRAFT PARAMETERS is, if quantity is low and quality is high, adding quantity is preferred; but if quantity is high and quality is low, adding quality is preferred. If aircraft are survivable, these changes make a significant difference in war outcomes.

Another way of thinking about marginal changes using these isovalue contours is to consider the gradient at any point--namely, the line perpendicular to the tangent of the isovalue contour at the point. Gradients are approximated in Figure S-1 as dotted lines. The best combined quantity-quality marginal changes from an effectiveness point of view are along the gradients. Starting from 1,000, 0.10 relatively more quantity is preferred, while starting from 2,000, 0.02 relatively more quality is preferred.

Now consider Figure S-2, which has high killing-of-aircraft parameters. First, the situation is much worse for Blue, with the 1,000, 0.10 case having FEBA position -120 as opposed to -82. Starting at 1,000, 0.10 (with FEBA position -120), adding 1,000 aircraft results in FEBA position -92; adding 0.04 units of firepower per sortie leads to -117. Changes in quantity and quality have less effect than in Figure S-1, and changes in quality have much less effect. Starting at 2,000, 0.02 (with FEBA position -102), adding 1,000 aircraft leads to -79; adding 0.04 units of firepower per sortie leads to -95. That is, if aircraft are not very survivable, neither improvements in quantity nor quality make a significant difference.

Chapter 1 discusses in detail the factors that drive the overall results. Quantity and quality changes in general-purpose aircraft--and related issues (air defense and aircraft shelters)--are explored in some detail. In general, the findings are that Blue aircraft can significantly improve the ground war results for Blue when killing-of-aircraft parameters are low. Red aircraft can somewhat improve the ground war results for Red when killing-of-aircraft parameters are high.

B. TANKS: ENGAGEMENT LEVEL

1. Methodology

The Tank Exchange Model (Reference [2]) is used for the investigation of tanks described in Chapter 2. The model simulates an engagement for up to 10 attack tanks. The simulation is Monte Carlo and can accommodate a great deal of detail about effectiveness parameters, tactics, terrain, timing, and so on.

The methodology consists basically of estimation of mathematical functions, fit to many replications of the Tank Exchange Model, which can be used to compute engagement outcomes as a function of quantity and selected quality measures on both sides. The selection of the quality measures that affect engagement outcomes is based on results obtained in the simulations. By using the estimated functions, quantity-quality trade-offs can be constructed.

2. Data

The data input set used for this study is a modified version of the test case presented in Reference [2]. Inputs can be classified into the following categories: general control information and starting conditions, tactical doctrine, firing doctrine, desired movement changes and speeds, mechanical failure and mechanical repair parameters, terrain descriptions, and detection parameters. The major change made in this study is that quantity and selected quality measures (performance parameters) are permitted to vary within a range, rather than being fixed.

3. Application of Methodology

Five measures of effectiveness are studied. Let A and D denote numbers of attackers and defenders; and let $\dot{A}$ and $\dot{D}$

denote numbers of attackers lost and defenders lost. The measures of effectiveness are (where E(·) denotes expected value):

(1) $E(\dot{A}/\dot{D})$
(2) $E(\dot{A})/E(\dot{D})$
(3) $E(\dot{A})/A \;/\; E(\dot{D})/D$
(4) $E(\dot{A}-\dot{D})$
(5) $E(\dot{A})/A - E(\dot{D})/D$.

There is an extensive discussion of the proper measure of effectiveness in Chapter 2. Measures (3) and (5) are justified as most appropriate for quantity-quality trade-offs.

Table S-1 displays the effectiveness parameters initially investigated and gives their coefficients for equations of the form

$$\dot{A} = B_0 + B_1X_1 + \cdots + B_{30}X_{30}$$
$$\dot{D} = B_0 + B_1X_1 + \cdots + B_{30}X_{30} ,$$

where $X_1, \ldots, X_{30}$ are the variables and $B_0, B_1, \ldots, B_{30}$ are the coefficients. Only the coefficients found to be statistically significant are displayed. This investigation led to the choice of several significant quality measures for study: vulnerability, error, and lethality (derived from the Probability Attack or Defense Classified "Kill Given Hit" noted in Table S-1). These quality parameters are defined in detail in Chapter 2.

Figures S-3 and S-4 give results for the loss-rate-difference measure of effectiveness, LRD = $E(\dot{R})/R - E(\dot{B})/B$, as a function of number of Blue tanks and Blue lethality, error and vulnerability, for Blue on the defense against 10 Red attackers and 6 Red attackers, respectively. When LRD = 0, the forces can be said to be in a parity situation. For instance, if Red has 2,000 attackers and Blue has 1,000 defenders in the

Table S-1. EFFECTIVENESS PARAMETERS INITIALLY INVESTIGATED (BLUE ON DEFENSE)

Variable Investigated	Coefficients Found Statistically Significant	
	Attack Losses	Defense Losses
Number of Attack Tanks	0.26	0.28
Number of Defense Tanks	0.73	0.31
Attack Vulnerability Measure	1.50	-0.84
Defense Vulnerability Measure	-0.38	0.28
Attack Error Measure	0.88	-0.44
Defense Error Measure	-1.26	0.61
Attack Tank Height	1.25	-- [a]
Defense Tank Height	--	--
Attack Load-Time Measure	0.38	--
Defense Load-Time Measure	-0.41	--
Attack Aim-Time Measure	--	--
Defense Aim-Time Measure	--	0.27
Probability Attack Classified $K \mid H$ [b]	0.92	-0.53
Probability Defense Classified $K \mid H$ [b]	--	--
Attack Probability of Misfire	--	--
Defense Probability of Misfire	--	--
Attack Time to Clear Misfire	--	--
Defense Time to Clear Misfire	--	--
Acceleration	--	--
Deceleration	-0.39	--
Attack Maximum Detection Range	-0.47	--
Defense Maximum Detection Range	--	--
Attack Detection-Range Multiplier	--	--
Defense Detection-Range Multiplier	0.35	-0.24
Attack Maximum Detection-Range Weapon Sign	--	0.22
Defense Maximum Detection-Range Weapon Sign	--	--
Attack Probability Defense-Weapon Signature Detected	-1.29	0.68
Defense Probability Defense-Weapon Signature Detected	--	--
Attack Probability Target Detected, given Weapon Signature Detected	-1.16	0.83
Defense Probability Target Detected, given Weapon Signature Detected	--	-0.23
Constant Found in Regression Equation	0.31	1.81
Coefficient of Determination (R^2)	0.67	0.79
Residual MSS (Mean Sum-of-Squares Error)/ Pure Error SS (Sum of Squares)	4.48 [c]	3.70 [c]

a. Everywhere it is not shown, the calculated t statistic < 2.
b. Killed, given a hit.
c. Has an F distribution.

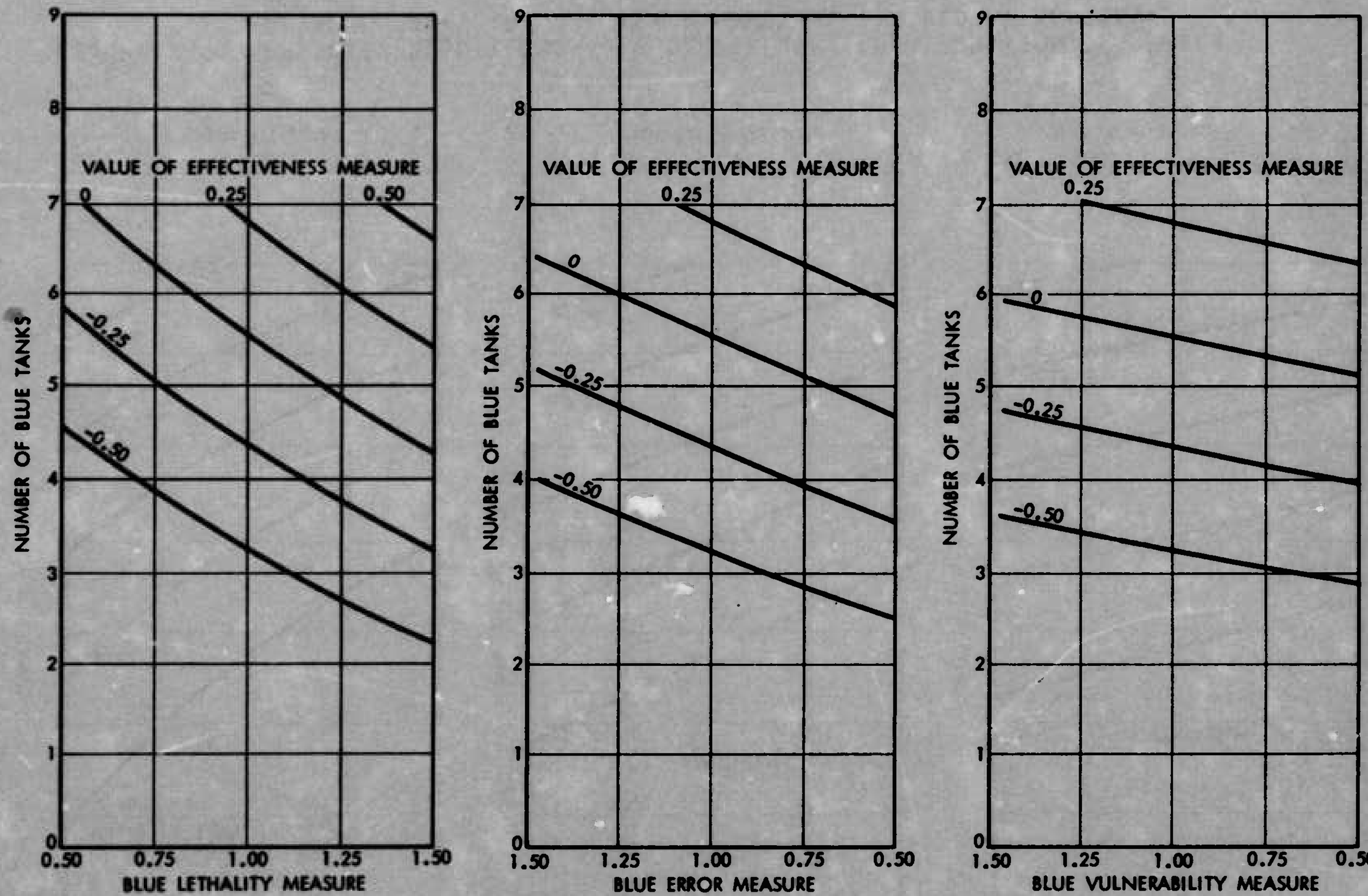

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Figure S-3. EFFECTIVENESS (LOSS-RATE DIFFERENCE) AS A FUNCTION OF NUMBER OF BLUE TANKS AND QUALITY PARAMETERS - 10 RED ATTACK TANKS

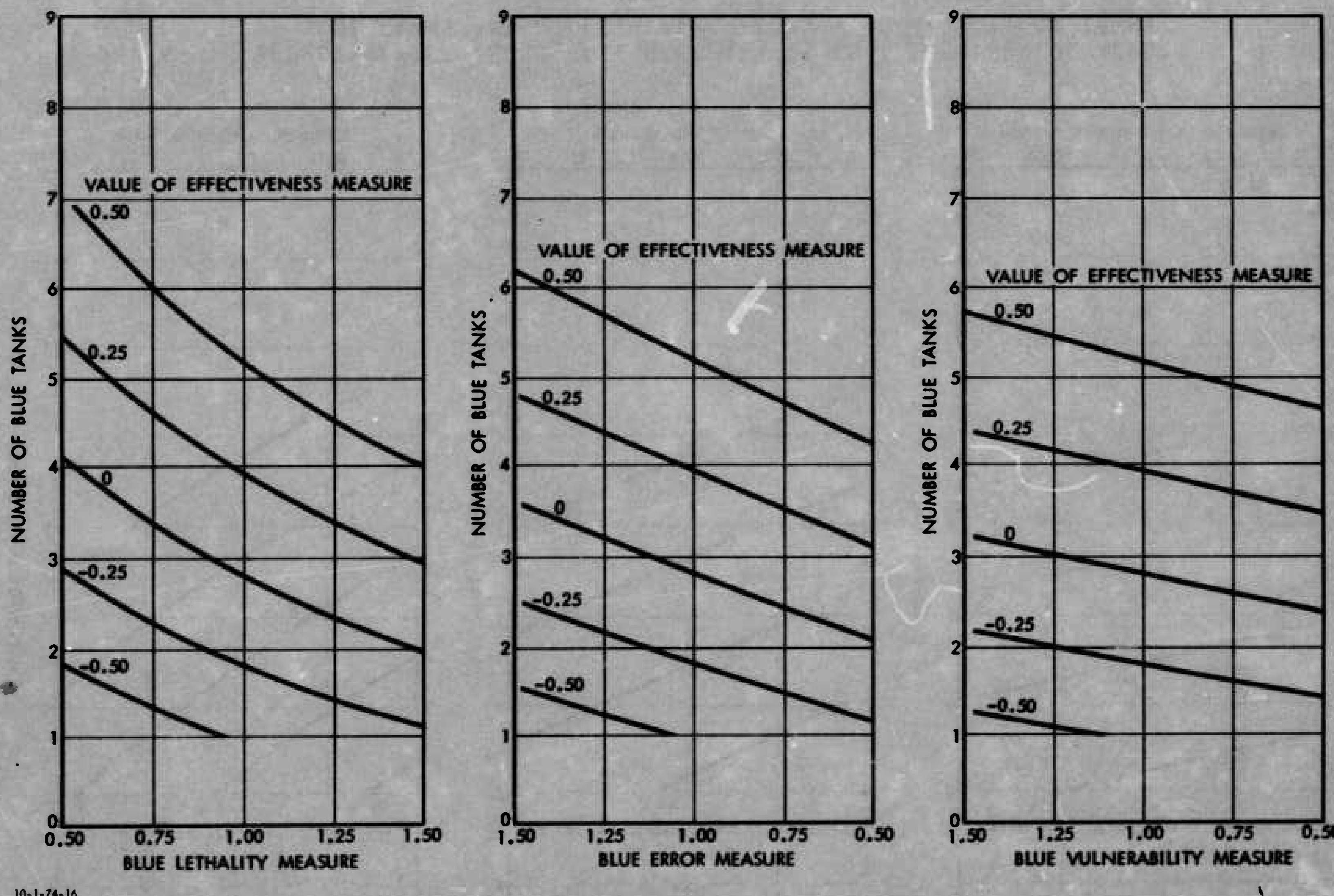

Figure S-4. EFFECTIVENESS (LOSS-RATE DIFFERENCE) AS A FUNCTION OF NUMBER OF BLUE TANKS AND QUALITY PARAMETERS - 6 RED ATTACK TANKS theater, and the losses per period are 200 and 100, respectively, LRD = 200/2,000 - 100/1,000 = 0, and the forces will always be in the same ratio. In Chapter 2, the various measures of effectiveness are discussed in some detail.

Considering the left-most graph in Figure S-3, note that, for LRD = 0, 7 tanks with lethality 0.57 are equal to 5 tanks with lethality 1.20. The other quality measures shown in Figure S-3 have less steep slopes, and the quantity-quality trade-off is not so pronounced. The definitions of lethality, error, and vulnerability control how difficult the quality would be to change; and the physical changes required depend on the physical nature of Red (as well as Blue) tanks.

Comparing Figures S-3 and S-4 (in particular the left-most graphs), note that to preserve LRD = 0, for Blue lethality 1.00, Blue must have 5.6 tanks against 10 attackers but 2.8 tanks against 6 attackers. Thus, the size of engagement to be expected must be an input to the quantity-quality trade-off determination. If most engagements are to involve 10 attacking tanks, Figure S-3 should be used. If most engagements are to involve 6 tanks, Figure S-4 should be used. If some mixture will occur, the figures should be weighted. Also, the loop should be closed such that Blue has a sufficient number of tanks in the theater. (For instance, if Blue decides that the expected engagement involves 8 Red attack tanks and Blue decides on 4 defense tanks per engagement with a certain lethality, then if Red has 2,000 tanks Blue should have 1,000 tanks to preserve LRD = 0.)

In the above discussion, Blue is always on defense. Results are given at the end of Chapter 2 for Blue on attack. A decision process for theater-level planning would require that the percent of time Blue is on the attack be considered. The quantity-quality trade-offs at engagement level need to be aggregated for theater-level planning. Thus, this analysis should be considered to be a component of a broader planning process.

C. MATHEMATICAL MODELS OF ATTRITION PROCESSES

Chapter 3 treats combat processes and mathematical models of attrition.

A classification scheme is constructed that contains qualitative characteristics, quantifiable characteristics, and important factors that are possibly neither quantifiable nor easy to deal with qualitatively.

Three principal categories concerning the nature of the interaction are identified, as follows:

(1) Each side searching to find the other.

(2) One side maintaining a barrier through which the other side attempts to penetrate.

(3) One side attempting to destroy passive targets on the other.

For each of the above categories, there is a recommended attrition process. For the first category, the Lanchester-square and Lanchester-linear processes are considered to be appropriate, in that the mathematical assumptions of these processes match the physical nature of the combat situation. A generalized Lanchester model containing the processes is presented in Chapter 3. For the second category, a new barrier-penetration model is considered to be appropriate and is presented in Chapter 3. For the third category, recently developed single-engagement binomial processes and multiple-engagement binomial processes are considered to be appropriate and are reviewed in Chapter 3.

D. PHYSICAL CHARACTERISTICS AND WEAPON-SYSTEM EFFECTIVENESS PARAMETERS

Chapter 4 presents two methodological approaches to quantifying effectiveness parameters (quality) as a function of physical characteristics. The first approach can use either data from history or outputs of simulation models. The second approach uses data from subjective preferences of experts. The first approach is applied to historical data from World War II (in Chapter 4).

Trading off quantity and quality requires a method for estimating effectiveness of various mixes of quantity and quality and a method for estimating costs of various mixes of quantity and quality. Usually, a model is used to estimate effectiveness, and it requires as input effectiveness parameters such as probability of kill, given detection, of a particular target by a particular shooter. Cost relationships, however, are often stated in terms of overt physical characteristics. Thus it is difficult to link the effectiveness parameter (or quality measure) with the cost of achieving the physical characteristics necessary to yield the effectiveness.

Chapter 4 explores methods aimed at estimating weapon-system effectiveness as a function of physical characteristics.

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[1] Bracken, J. *Two Optimal Sortie Allocation Models.* 2 vols. IDA Paper P-992, December 1973.

[2] Graves, J. W. *Tank Exchange Model.* 2 vols. IDA Paper P-916, November 1973.

[3] Karr, A. F. *On a Class of Binomial Attrition Processes.* IDA Paper P-1031, June 1974.

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Chapter 1

QUANTITY-QUALITY TRADE-OFFS OF GENERAL-PURPOSE AIRCRAFT: THEATER-LEVEL ANALYSIS

Eleanor L. Schwartz

A. OVERVIEW

This chapter presents an analysis of effectiveness of general-purpose aircraft at the theater level.

The principal focus is on the trade-off between quantity of aircraft and quality of aircraft as measured by firepower units per combat air support (CAS) sortie. However, quality of aircraft as measured by probability of kill given detection in intercept (INT) and airbase attack (ABA) is also treated, as are aircraft shelters and air defenses.

The model used in this analysis is the OPTSA model of conventional theater-level ground and air warfare (References [5] and [10]). The principal feature of this model is that it computes two-sided allocations of aircraft to missions in such a way that the strategies are mutually enforceable. Neither side can do better against the other side's enforceable strategy.

Having the allocations of aircraft to missions computed internally removes a major item of difficulty from the analysis, for it is well-known that the mutually enforceable strategies change as quantity and quality of aircraft change. Having the model compute the allocations leaves the analyst free to concentrate on other issues.

The OPTSA model computes FEBA movement and casualties to ground forces as a function of

$$\frac{\text{Blue ground firepower plus Blue air firepower}}{\text{Red ground firepower plus Red air firepower}},$$

in the same spirit as the ATLAS (Reference [11]) and IDAGAM I

(References [1,2,3,9]) models of ground and air warfare. It computes losses to aircraft using the same type of attrition functions as the TAC CONTENDER (Reference [15]) model of air warfare and the IDAGAM I model.

The measure of effectiveness (MOE) used in this analysis is position of the FEBA at D+10. Other MOEs are available (e.g., Blue minus Red firepower delivered by CAS sorties). FEBA is used in order to isolate the areas of quantity and quality in which tactical aircraft made a big difference in the ground-air war. Dealing directly with this MOE created efficiencies in determining which regions of the many parameters are most interesting from this point of view.

The chapter is organized as follows: First, in Section B, the OPTSA model is discussed. The game structure, assessment procedure, and possible limitations are reviewed. This section includes a presentation of new explorations of the density of allocation set and the density of the number of decision periods. Section C presents the use of the model to explore quantity-quality trade-offs, starting with the data set and covering the various combinations of parameters analyzed.

B. STRUCTURE OF THE OPTSA MODEL

1. Game Structure

OPTSA determines strategies for allocating Blue and Red general-purpose aircraft to three missions: CAS, INT, and ABA. The basic scenario is a multiperiod war--whose length (in days) is an input--that is modeled as a multistage adaptive game. On a number of specified "decision days" (the first day of the war is always a decision day), both Blue and Red pick an allocation of aircraft to the three missions from an input list of allowable allocations (i.e., "pure strategies"). Each allocation in the list specifies the *proportion* of the general-purpose aircraft inventory to be assigned to each mission. The particular allocation picked is used for all days in that period, until the next decision day. Given the length of the war (in days), the decision days, and a list of allowable allocations of aircraft, OPTSA finds two-sided optimal adaptive strategies for each side, for picking an allocation on each decision day. That is, for each side on each decision day, a set of probabilities with which to choose each allowable allocation is found. The strategy can depend on allocations made in previous periods--hence the term "adaptive." At present, OPTSA can process a war of up to 90 days, with two or three decision days. It would be desirable to have more decisions. However, this is not computationally feasible. Models exist that claim to find approximately optimal solutions to games in which an allocation decision is made every day (Reference [15]), but counterexamples have been developed that discredit the claims of optimality (References [4], [6], [7], and [8]). OPTSA solves *exactly* the game in which allocation decisions are made on two or three specified days. As will be discussed further (below), outcome of the war seems to be fairly insensitive to changes in the decision days.

The multistage game is solved by a backwards induction process, where the payoff for a (one-stage matrix) game at stage k is the game value of a stage k+1 game, corresponding to the expected war outcome in the situation when both sides, given a particular state at stage k, play optimally from then on. (At present, OPTSA can process a two- or three-stage game.) Improvements (described in Reference [14]) that substantially reduce the running time have been made to the game-solving procedure. The major aspect of these improvements is that, to solve a game, it is usually necessary to calculate only a small portion of the payoff entries. Since most of OPTSA's computer time is spent computing payoff entries, the fewer payoff entries that need to be computed, the shorter the running time. Running times for most of the cases explored here varied from 3 to 10 seconds per game (for a two-period, 10-day war).

2. Assessment Procedure

a. General

Payoffs of games at stage k are game values of games at stage k+1. The final payoffs are actual war outcomes. The particular final-stage game is determined by the Blue and Red allocations played during all but the last period.

b. Inputs Needed

OPTSA determines optimal strategies for aircraft allocation, *given* certain input parameters. The input parameters (explained in detail in Reference [5]) are as follows:

- Number of days in war.
- Decision days for allocations.
- Number of ground divisions added each day (including day 1) by type of division. Up to three kinds of ground divisions can be played on each side.
- Number of general-purpose aircraft added each day.
- Number of special-purpose aircraft, if any, added each day, by type. There are three types: special-purpose CAS, special-purpose ABA, and special-purpose INT.
- Number of aircraft shelters.
- Firepower per CAS sortie, for general-purpose and special-purpose CAS planes.
- Effectiveness parameters (detection and kill probabilities) of planes on the INT mission (the general-purpose planes and special-purpose interceptors).
- Effectiveness parameters for aircraft on CAS and ABA to kill interceptors.
- Effectiveness parameters for ABA missions (against sheltered and nonsheltered aircraft) for general-purpose and special-purpose ABA planes.
- Function for daily FEBA advance as a function of force ratio.
- Functions for daily Blue and Red division destruction, as a function of force ratio.
- Allowable allocations of general-purpose aircraft.

c. <u>Computation of War Outcome</u>

Given the input data and an allocation of general-purpose aircraft to missions by each side for each period, the assessment routine fights a war in the following manner (the procedure similar to that of many combat-simulation models):[1] On each day for each side, a starting ground-division inventory (by type) and a starting aircraft inventory (by type) are found by summing the previous day's starting inventory, minus losses, plus today's additions. Using the allocation appropriate for the period, the general-purpose aircraft are assigned to the three air missions and the special-purpose aircraft are assigned to their appropriate missions, producing a total number of aircraft that will fly each mission that day. Then, on each side of the battlefield, there is air-to-air combat between one side's interceptors and the other side's attackers (planes on CAS and ABA). The remaining planes are considered vulnerable to ABA. All attrition is computed by the exponential equation

$$\dot{T} = T\left(1-e^{-(1-e^{-dT})k\frac{S}{T}}\right),$$

where

$\dot{T}$ = targets killed;

T = targets;

S = shooters;

d = probability that a particular shooter detects a particular target; and

k = probability of kill given detection.

Red planes assigned to ABA oppose the total stock of Blue planes not killed by interceptors. Blue planes are sheltered

[1]See Reference [5] for a more detailed description of the assessment procedure.

if there are aircraft shelters available. The number of Blue planes killed is computed by two exponential attrition equations, one each for sheltered and nonsheltered aircraft. However, the Blue aircraft inventory is not reduced until the following day. Similarly, Blue planes assigned to ABA oppose the total stock of Red planes, and losses to Red are computed.

CAS-assigned planes that were not killed by interceptors deliver firepower. The total air firepower delivered (that day) for each side is found by multiplying the number of surviving planes on CAS by an input firepower value. Similarly, a ground-firepower score is found by multiplying the number of ground divisions by the firepower per ground division. The force ratio

$$x = \frac{\text{Blue ground firepower + Blue air firepower}}{\text{Red ground firepower + Red air firepower}}$$

is formed, and the daily FEBA advance is computed as a function of x, using an input function. The FEBA position at the end of the day is the previous day's FEBA position plus the daily FEBA advance. Cumulative totals of the ground and air firepower for each side are also kept. Ground division casualties to each side are computed by interpolation as a function of the force ratio x. If desired, all ground casualties can be assumed to be replaced.

This procedure is repeated each day. At the end of the last day, the appropriate MOE (e.g., FEBA position) becomes a payoff entry in a final-stage game, as explained earlier.

We can see that there are two opposing factors affecting the desirable allocation of aircraft. If no aircraft are assigned to CAS missions, then the total contribution to the force ratio must come from ground firepower. However, if all aircraft are assigned to CAS, there will be no resources on INT or ABA to stop Red from delivering CAS. OPTSA attempts to find the optimal trade-off between the missions.

3. Possible Limitations of the Model

Internal to the model, two factors that possibly could affect the results of the OPTSA model have already been alluded to. We wish now to discuss them in more detail. The first factor is the fineness of the strategy space (i.e., the number and detail of the list of allowable allocations of aircraft). Does the set of allocations either of half the aircraft to one mission, half to another, or of all aircraft to one mission give a reasonably wide range of choices? Or should we be able to allocate aircraft by, say, multiples of one-tenth? The second factor is the number and spacing of the periods--or allocation decision days. Are two or three decision periods really enough in a long war, or can one side gain a significantly greater advantage than the other if both sides are able to choose more frequently? Also, given two or three decision periods, how does the choice of decision days affect the result? Is the outcome when the first decision period is long and the second short substantially different from the situation in which the reverse is true?

As has been stated already, these problems in some form are common to all aircraft-allocation optimization models. The reason for limiting the options is due to the extremely long computer time that would be necessary to solve a game in which either an extremely fine allocation set or many decision periods are used. (Reference [14] shows methods for estimating running times as a function of both the allocation-set size and the number of decision periods.) The compromises made by OPTSA, however, do not weaken the validity of the model's results--so far as we can determine.

a. Fineness of Allocation Set

The use of the set of six allocations consisting of all the ways of devoting all the general-purpose aircraft to one of the three missions or half the aircraft to one mission and half to another has resulted in reasonable model running-times; and it does seem to give a reasonably wide range of choices. In the work described in the rest of this paper, a seventh choice (devoting one third of the aircraft to each of the three missions) is available, but it is very seldom optimal. However, other strategies--perhaps those that allow some unequal split of aircraft between missions (e.g., 2/3 to CAS, 1/3 to ABA)--might often be optimal over certain ranges of input values. Giving both sides a spectrum of ways of dividing aircraft between two missions might conceivably yield a large improvement for one side when that side finds two missions profitable to some degree.

This is a serious problem, and more attention should be given it. However, explorations that have been made so far (giving each side up to 20 allowable allocations) indicate that changes in game value (using the FEBA measure of effectiveness) resulting from using enlarged allocation sets are often zero--that is, the original strategy of the six remains optimal. Even when a new strategy becomes optimal, we have not yet encountered game-value changes of more than about five percent. Using the air-firepower-difference measure of effectiveness (which might tend to be more sensitive than FEBA position), Reference [12] provides evidence that the game value when an exceedingly fine choice of allocation is available to each side is only two percent different from the game value when the set of six allocations is used.

b. Decision Periods

We exhibit in Table 1-1 an instance of striking insensitivity of the game value to reasonable changes in the decision days in a 10-day war, for both two- and three-period cases. Of course, the decision days must be the same for Blue and for Red. The data for these cases are the basic data set described in the next part of this chapter. Two cases were explored: Case A, with all kill parameters at 0.1; and Case B, with all kill parameters of interceptors and planes on ABA at 0.9. Because of the nature of the model's game-solving procedure, running times vary widely with the spacing of the decision days; but the game-value differences are always under five percent. Examination of the optimal strategies, however, reveals that Red can do a little better in a three-period than in a two-period war, finding it advantageous to use slightly different first- and second-period allocations. This change in Red's strategy suggests that, in a war longer than 10 days, the number and spacing of decision days might make a larger difference in game value.

C. USE OF OPTSA TO EXPLORE AIRCRAFT QUANTITY-QUALITY AND OTHER RELATIONSHIPS

We now proceed to examine the results of running the OPTSA model with different variations in the data, which will give insights into the way various factors affect the outcome of a battle. A basic data set (described below) is used for most of the work. First, trade-offs between aircraft quantity and CAS firepower-value are considered, for various ranges of the effectiveness parameters (detection and kill) of airplanes on ABA and INT missions. Isovalue curves for quantity versus CAS firepower are developed to illustrate further the nature of the trade-off. Later, we explore the effects of air defenses

Table 1-1. EFFECT (ON GAME VALUE) OF CHANGING THE DECISION DAYS IN A 10-DAY WAR

Number of Periods	Decision Days in a 10-Day War*	Game Value: Low Kill-Parameters	Game Value: High Kill-Parameters
2	1 7 -	-78.7	-117.9
2	1 6 -	-78.7	-117.7
2	1 5 -	-78.6	-117.1
3	1 5 8	-81.8	-118.4
3	1 4 8	-81.3	-118.6
3	1 3 8	-80.8	-118.9
3	1 5 7	-81.3	-118.2
3	1 4 7	-81.7	-118.5
3	1 3 7	-81.9	-119.1
3	1 5 6	-80.2	-117.7
3	1 4 6	-81.0	-118.0
3	1 3 6	-81.3	-118.8

*A decision day is the first day of a period.

(SAMs and AAA) and aircraft shelters on the outcome, although not strictly from the point of view of trading off aircraft quantity versus quality. Finally, we examine the results of the model with a data set thought to be somewhat more realistic than the basic data set.

1. Basic Data Set and Parameters Treated

A two-period, 10-day war, with allocation decisions made on days 1 and 6 is the case examined. This case has a sufficient number of days and periods for all the effects in OPTSA to show up (in a one-day war many parameters are simply irrelevant), yet is short enough for games to be solved quickly. The set of seven strategies available for each side in each period is as follows:

Strategy	Proportion of Aircraft to--		
	CAS	ABA	INT
1	1.00	0.00	0.00
2	0.50	0.50	0.00
3	0.00	1.00	0.00
4	0.33	0.34	0.33
5	0.50	0.00	0.50
6	0.00	0.50	0.50
7	0.00	0.00	1.00

The payoff MOE is FEBA position at the end of the war.

The pattern of exploring changes in parameters is generally the same as in the case of the one-day war [13]. The main consideration is how CAS firepower per general-purpose aircraft and number of general-purpose aircraft (no special-purpose aircraft were used) affect the game value for various values of the INT and ABA effectiveness-parameters. Ground firepower is also a factor.

Following is a summary of the ranges of parameters examined. (Reference [5] provides a full description of the meaning of these parameters.)

Ground Firepower. There is one type of ground division on each side. Blue has 24 divisions at 10 firepower units each, or 240 firepower units. Blue's number of divisions is later raised to 36. Red has 80 divisions at 6 firepower units each (or 480 firepower units). No divisions are added after the first day, but all ground casualties are replaced.

Number of Aircraft. Only general-purpose aircraft are used. For the first series of runs, the number of Red aircraft is held constant at 2,000 the first day with 100 (5 percent) entering on each subsequent day. Results are examined for 1,000, 2,000, 3,000, and 5,000 Blue aircraft the first day. In each case, 5 percent of the first-day number of aircraft enters on each subsequent day. In the second series of runs, the effects of varying the number of Red aircraft are considered.

Firepower per Aircraft. For the first series of runs, the Red firepower per aircraft is held constant at 0.06 firepower units. That is, one successful CAS sortie delivers 0.01 of the firepower of the ground division. Similarly, the base-case Blue firepower per aircraft is 0.1. Effects are examined for Blue firepower values of 0.06 (the same as for Red), 0.1, and 0.14. These values are felt to represent fairly reasonable lower and upper limits on the firepower value. In the second series of runs, both the Blue and Red firepower values vary down to 0.01.

Detection Parameters. In the first series of runs, detection parameters (which are allowed to vary between 0 and 1) are *all* set to 1. Perfect detection is assumed, and analysis is concentrated on the effect of the kill parameters. In the second series of runs, detection parameters of 0.0001, 0.001, and 0.01 are considered. These form a realistic framework for the true value.

Kill Parameters for the ABA and INT Missions. All kill parameters are constrained to vary between 0 and 1. Therefore, it is felt that if results are examined for a low and high value of one or more of these parameters, a framework can be established for how the game value changes. Regions of sensitivity to these parameters can then be determined; and results can be found for intermediate values of the appropriate parameter, to identify the region of greatest sensitivity. Here, results are examined when the four kill parameters for the Blue and Red INT and ABA missions are either 0.1 or 0.9--but varied independently of each other. That is, there are 2^4 cases, each involving a different subset of the four parameters being set to 0.9 and the rest set to 0.1.

The kill parameter for CAS or ABA planes against enemy interceptors is kept constant at 0.1. In the second series of runs, it is set to half the kill parameter of interceptors. There are no aircraft shelters; the ABA kill-parameter is against unsheltered aircraft. Later, the effect of shelters will be considered.

Payoff Measure. The MOE is FEBA position at the end of the war (day 10). The FEBA position is the algebraic sum of the FEBA movements for each day. The FEBA movement each day is a function of the force ratio

$$x = \frac{\text{Blue ground firepower + Firepower from successful Blue CAS that day}}{\text{Red ground firepower + Firepower from successful Red CAS that day}}$$

with the property that $F(\frac{1}{x}) = -F(x)$. Linear interpolation was used between the following input set of breakpoints and values:

Force ratio x	1	1.5	2	3	5	10
FEBA advance F(x)(km)	0	2	10	20	40	60

With no air firepower at all, the force ratio $\frac{240}{480} = \frac{1}{2}$ results in a FEBA movement of -10 km per day or -100 km at the end of the war. This is a benchmark value around which the result of the game will vary.

As stated previously, the war has 10 days and two decision periods, decisions being made at the beginning of days 1 and 6. Each side uses the set of seven strategies previously described.

Table 1-2 compactly displays the basic data set, and Figure 1-1 shows the graph of FEBA advance as a function of force ratio.

We note here that we have not examined the effects of two important factors: special-purpose aircraft and division destruction. Special-purpose aircraft are never a hindrance to the side that has them. The question is, Under what circumstances (and by how much) will they affect the game value? The use of division destruction (instead of replacing all casualties) might in some cases make the ground firepower a much less dominate force in determining the general outcome of the war. Regions of sensitivity should be found for both these factors.

2. Results of Varying Quantity, CAS Parameter, and ABA and INT Kill-Parameters for High Detection-Parameters (First Series of Runs)

Tables 1-3 and 1-4 show the results of the first series of runs--the number of Blue ground divisions being 24 and 36, respectively.

Based on the results of the one-day wars [13], several hypotheses were made for the 10-day, two-period war. Because of the ground firepower asymmetry, it was thought that Red would quickly send more of his planes to INT and ABA as his effectiveness parameters for these missions increased. Therefore, although Blue could gain something by increasing his

Table 1-2. BASIC DATA SET FOR THE OPTSA RUNS IN A 10-DAY, TWO-PERIOD WAR

Variable	Value of Variable for--	
	Blue	Red
Number of Ground Divisions[a]	24 (later, 36)	80
Firepower per Ground Division	10.0	6.0
Number of Aircraft[b] (general-purpose only)	1,000; 2,000; 3,000; 5,000	2,000
Firepower per CAS Sortie	0.06; 0.10; 0.14	0.06
Number of Shelters	0.0	0.0
Detection Parameters	1.0	1.0
Kill Parameters:		
Attackers kill Interceptors	0.1 (in some runs, 0.05 or 0.45)	0.1 (in some runs, 0.05 or 0.45)
Intercept	0.1 or 0.9	0.1 or 0.9
ABA (nonsheltered)	0.1 or 0.9	0.1 or 0.9

Ten-day war, two periods, aircraft allocations made on days 1 and 6.
Pay-off MOE is FEBA position at end of war, computer as shown in Figure .
SAM parameters are added in some runs.

Strategy space on each side:

Pure Strategy	Proportion of Aircraft to-- CAS	ABA	INT
1	1.00	0.00	0.00
2	0.50	0.50	0.00
3	0.00	1.00	0.00
4	0.33	0.34	0.33
5	0.50	0.00	0.50
6	0.00	0.50	0.50
7	0.00	0.00	1.00

[a] All ground casualties are replaced.

[b] This is the number the first day. Five percent of this number of aircraft enters each subsequent day in addition. No special-purpose aircraft are played.

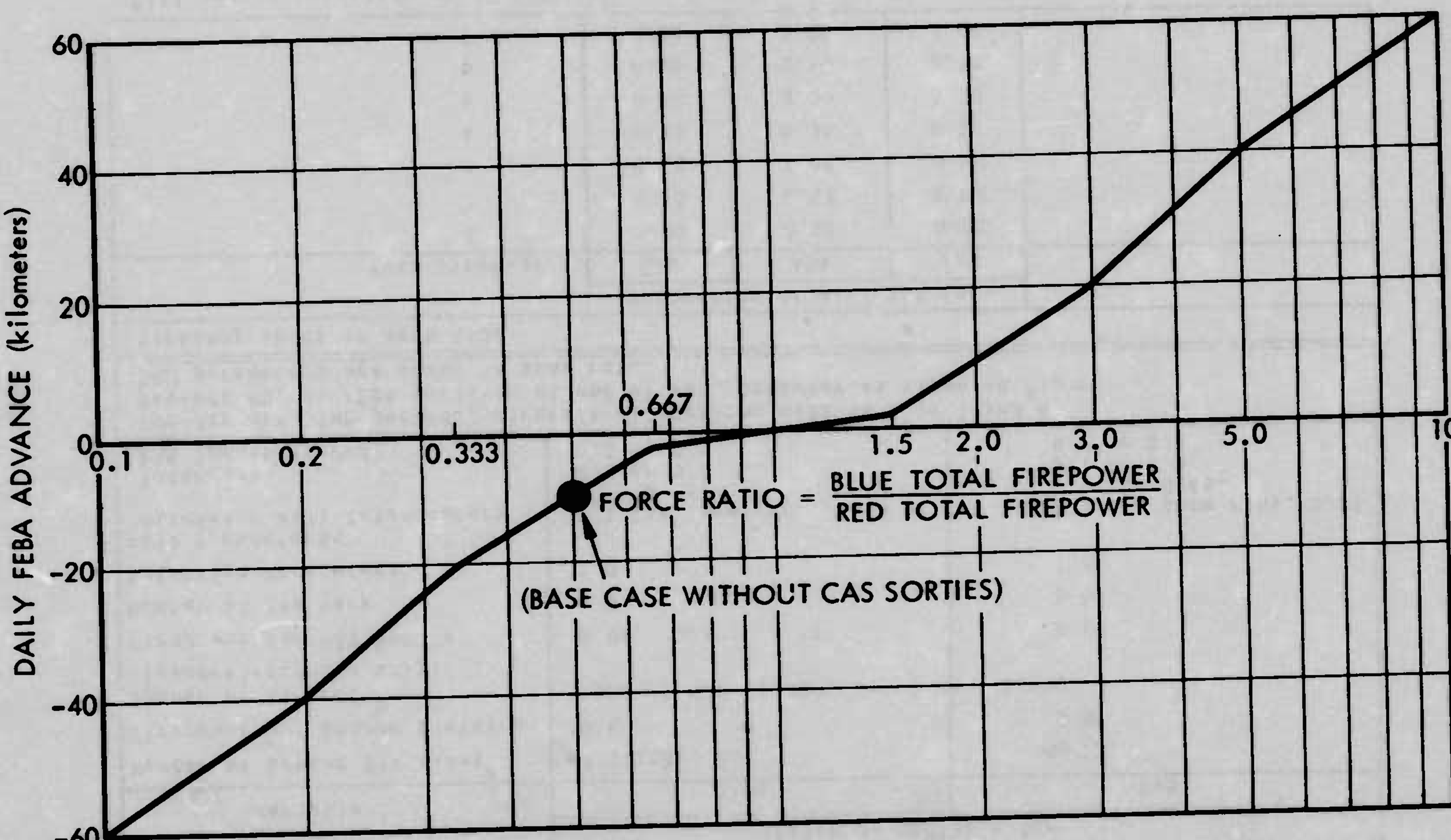

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Figure 1-1. OPTSA DAILY FEBA ADVANCE AS A FUNCTION OF FORCE RATIO THAT DAY

Table 1-3. FEBA MOVEMENT (km) AS A FUNCTION OF QUANTITY, CAS PARAMETER, AND ABA AND INT KILL-PARAMETERS

[Blue ground firepower = 240]

Red ABA Parameter = 0.1

Blue INT Parameter	Firepower per Blue Aircraft	Blue ABA Parameter = 0.1: 1,000	2,000	3,000	4,000	5,000	Blue ABA Parameter = 0.9: 1,000	2,000	3,000	4,000	5,000
		Red INT Parameter = 0.1									
0.1	0.06	-100	-49	-16	--	-1	-76	-35	-15	--	0
	0.10	-79	-19	-2	--	15	-60	-14	-1	--	33
	0.14	-68	-7	7	--	60	-42	-3	11	--	70
0.9	0.06	-97	-42	-14	--	0	-76	-34	-14	--	0
	0.10	-77	-16	-1	--	24	-60	-14	-1	--	33
	0.14	-67	-6	8	--	66	-42	-3	11	--	70
		Red INT Parameter = 0.9									
0.1	0.06	-129	-106	-85	--	-42	-126	-101	-60	--	-8
	0.10	-126	-99	-80	--	-22	-125	-94	-56	--	-4
	0.14	-124	-96	-78	--	-13	-123	-92	-52	--	27
0.9	0.06	-106	-96	-84	--	-42	-106	-96	-60	--	-8
	0.10	-104	-91	-80	--	-22	-104	-91	-55	--	-4
	0.14	-103	-88	-78	--	-13	-103	-88	-52	--	27

Red ABA Parameter = 0.9

Blue INT Parameter	Firepower per Blue Aircraft	Blue ABA Parameter = 0.1: 1,000	2,000	3,000	4,000	5,000	Blue ABA Parameter = 0.9: 1,000	2,000	3,000	4,000	5,000
		Red INT Parameter = 0.1									
0.1	0.06	-133	-113	-95	--	-71	-121	-96	-55	--	-7
	0.10	-124	-99	-81	--	-57	-119	-87	-35	--	4
	0.14	-117	-89	-71	--	-41	-115	-82	-22	--	23
0.9	0.06	-120	-74	-27	--	-5	-120	-67	-25	--	-3
	0.10	-118	-66	-13	--	14	-118	-59	-11	--	18
	0.14	-115	-62	-7	--	50	-115	-55	-4	--	54
		Red INT Parameter = 0.9									
0.1	0.06	-135	-117	-99	--	-75	-130	-109	-60	--	-8
	0.10	-129	-107	-94	--	-67	-128	-104	-56	--	4
	0.14	-126	-103	-92	--	-55	-126	-102	-52	--	23
0.9	0.06	-120	-96	-84	--	-53	-120	-96	-60	--	-8
	0.10	-118	-91	-80	--	-37	-118	-91	-55	--	4
	0.14	-115	-88	-78	--	-26	-115	-88	-52	--	24

Note: Minus sign indicates Blue withdrawal.

Table 1-4. FEBA MOVEMENT (km) AS A FUNCTION OF QUANTITY, CAS PARAMETER, AND ABA AND INT KILL-PARAMETERS

[Blue ground firepower = 360]

Red ABA Parameter = 0.1

Blue INT Parameter	Firepower per Blue Aircraft	Blue ABA Parameter = 0.1					Blue ABA Parameter = 0.9				
		Number of Blue Aircraft									
		1,000	2,000	3,000	4,000	5,000	1,000	2,000	3,000	4,000	5,000
		Red INT Parameter = 0.1									
0.1	0.06	-25	-9	-3	--	6	-13	-4	1	--	12
	0.10	-19	-2	6	--	35	-9	2	11	--	57
	0.14	-14	4	16	--	89	-7	8	35	--	91
0.9	0.06	-16	-7	-1	--	10	-13	-4	1	--	12
	0.10	-13	-1	8	--	49	-9	2	11	--	57
	0.14	-10	4	20	--	89	-7	8	35	--	91
		Red INT Parameter = 0.9									
0.1	0.06	-31	-21	-14	--	-5	-27	-16	-8	--	4
	0.10	-27	-18	-12	--	0	-24	-15	-6	--	21
	0.14	-26	-16	-10	--	10	-24	-14	-4	--	51
0.9	0.06	-16	-13	-11	--	-5	-16	-13	-8	--	4
	0.10	-15	-13	-10	--	0	-15	-13	-6	--	21
	0.14	-15	-12	-9	--	10	-15	-12	-4	--	51

Red ABA Parameter = 0.9

Blue INT Parameter	Firepower per Blue Aircraft	Blue ABA Parameter = 0.1					Blue ABA Parameter = 0.9				
		Number of Blue Aircraft									
		1,000	2,000	3,000	4,000	5,000	1,000	2,000	3,000	4,000	5,000
		Red INT Parameter = 0.1									
0.1	0.06	-38	-30	-22	--	-11	-22	-13	-8	--	5
	0.10	-35	-26	-17	--	-2	-20	-13	-5	--	19
	0.14	-32	-21	-11	--	13	-19	-12	-2	--	48
0.9	0.06	-22	-11	-4	--	7	-22	-9	-3	--	8
	0.10	-20	-9	1	--	39	20	-7	3	--	43
	0.14	-18	-7	6	--	68	-18	-5	10	--	72
		Red INT Parameter = 0.9									
0.1	0.06	-38	-30	-22	--	-12	-31	-17	-8	--	4
	0.10	-35	-26	-17	--	-4	-29	-16	-6	--	19
	0.14	-32	-21	-13	--	4	-27	-16	-4	--	48
0.9	0.06	-22	-13	-11	--	-5	-22	-13	-8	--	4
	0.10	-20	-13	-10	--	0	-20	-13	-6	--	19
	0.14	-18	-12	-9	--	10	-18	-12	-4	--	49

Note: Minus sign indicates Blue withdrawal.

firepower per plane, this gain would be very sensitive to the Red ABA and INT parameters and would not be significant if these parameters were large. An increase in the number of Blue aircraft would produce less sensitive results, and an increase in Blue ground firepower would put Blue in a better position yet. Furthermore, because Blue has fewer aircraft, a change in Blue ABA or INT parameters results in little improvement; however, as the number of Blue planes increased, this difference would become greater.

The first series of runs, in general, confirms these hypotheses. Let us start with the base case, in which Blue has 1,000 aircraft (each with a firepower value of 0.1) and 24 ground divisions, and examine the result of varying the Blue and Red ABA and INT kill-parameters. Each of the four parameters is either high (0.9) or low (0.1). We summarize the results in Table 1-5.

Several things can be deduced from these data. First, there is only one really large jump. Four of the values are around -60 or -70; all the rest are below -100. The game value is higher (better for Blue) when *both* the Red INT and Red ABA kill-parameters are low. Given this, raising the Blue ABA parameter makes a substantial improvement for Blue, from -78 to -60. Raising the Blue INT parameter yields a change in strategy but no real change in game value. However, if either the Red INT or Red ABA parameter increases, the game value decreases rapidly and, moreover, becomes insensitive to changes in the Blue ABA parameter.

The other changes are of a lesser order of magnitude. When the Red INT or Red ABA parameters are high, the Blue INT parameter makes some difference. Oddly enough, this is especially true when the Red INT parameter is high. Examining the optimal strategies in this case is informative. First, even when Blue has a high ABA parameter, Blue will not fly ABA often.

Table 1-5. SUMMARY OF FIRST SERIES OF RUNS IN A 10-DAY, TWO-PERIOD WAR

Red Parameters		Blue Parameters		Game Value (FEBA Movement --km)
ABA	INT	ABA	INT	
0.1	0.1	0.1	0.1	-79
0.1	0.1	0.1	0.9	-77
0.1	0.1	0.9	0.1	-60
0.1	0.1	0.9	0.9	-60
0.1	0.9	0.1	0.1	-126
0.1	0.9	0.1	0.9	-104
0.1	0.9	0.9	0.1	-125
0.1	0.9	0.9	0.9	-104
0.9	0.1	0.1	0.1	-124
0.9	0.1	0.1	0.9	-118
0.9	0.1	0.9	0.1	-119
0.9	0.1	0.9	0.9	-118
0.9	0.9	0.1	0.1	-129
0.9	0.9	0.1	0.9	-118
0.9	0.9	0.9	0.1	-128
0.9	0.9	0.9	0.9	-118

As Blue's INT parameter increases, Blue goes to an all-INT strategy in the first period. Meanwhile, Red has been playing ABA if his ABA parameter is high; INT, otherwise. In this latter case, both Blue and Red devote all their planes to INT; that is, neither side flies attack missions, but they both keep all their aircraft hovering over their own battlefields. This seems very surprising, but seems more plausible if we remember that if any attack mission is highly vulnerable to INT it will be unlikely to be flown. Therefore, by devoting all planes to INT, Blue can prevent Red from flying attack missions. And if Red flies ABA, Blue can intercept a lot of it.

When the Blue firepower per plane and number of planes vary, many things happen. Let us first consider varying the Blue firepower per plane, as it makes quite a bit less difference. First, if Blue is in a region where it is optimal to fly all ABA or INT in one or more periods, firepower values on CAS are not that important--which is true in the cases mentioned above, when the Red INT and Blue INT parameters are both high. No CAS is flown in the first period and rarely flown in the second. Varying the Blue firepower per plane from 0.06 to 0.14 results in the game value's changing from -120 to -115 for the high-ABA case and -105 to -103 for the low Red ABA case. This change occurs probably because, with higher firepower per plane, Blue has to forgo more CAS and, hence, is more reluctant to devote all planes to INT. Red, not having this problem, continues to fly all INT and to kill the Blue CAS planes, preventing the delivery of *more* Blue firepower than previously. When Blue's INT and ABA parameters were low, the changes were a bit larger, but not really significant. Even with low Blue INT, low Blue ABA, low Red INT, but high Red ABA (where all CAS was the optimal Blue strategy), the game value goes only from -133 to -117, as Red plays all ABA in the first period, which undercuts Blue's gain.

The region where an increase in Blue firepower value makes the most difference is the region where Blue has done best from the start (i.e., where Red has low ABA and INT parameters). Here CAS remains the optimal Blue strategy always in the second period and in the first period when Blue's ABA parameter is low. In this case, varying the Blue firepower value from 0.06 to 0.14 changes the game value from -100 to -68. With high Blue ABA, Blue flies mainly ABA the first period and all CAS the second. The game value goes from -76 to -42. We can conclude, therefore, that an increase in Blue firepower per plane can increase the game value, but the amount of increase falls off rapidly as Red's INT or ABA parameters rise.

Changing the number of Blue planes can have a far more dramatic effect on the game value. Some examples from our base case (where the Blue firepower per plane remains 0.1) are shown below.

Red Parameters		Blue Parameters		Number of Blue Planes			
ABA	INT	ABA	INT	1,000	2,000	3,000	5,000
0.1	0.1	0.1	0.1	-79	-19	-2	15
0.1	0.9	0.1	0.1	-126	-99	-80	-22
0.1	0.9	0.1	0.9	-104	-91	-80	-22
0.9	0.1	0.1	0.9	-118	-66	-13	14
0.9	0.1	0.9	0.1	-128	-104	-56	4

As the examples show, as the number of Blue aircraft rises the game value rises in different ways, depending on the INT and ABA situations. Some general observations can be made first. As the number of Blue planes rises but the other variables are unchanged, the Blue firepower score makes more of a difference than previously, although the increase in game value due to higher firepower is still markedly reduced by higher Red ABA or INT parameters. Additional aircraft make a difference to Blue *sooner* if the Red INT or ABA parameters are low.

Changing the number of Blue aircraft affects some regions of sensitivity quite a bit--but others not at all. In particular, the Blue ABA parameter makes an increasing amount of difference in the result as the number of Blue planes gets larger, even when Red parameters are also high. The increased number of Blue planes makes it possible to go from a CAS-INT strategy to one that uses more ABA. The sensitivity to the Blue INT parameter sometimes decreases with increasing Blue planes (e.g., when the Red INT parameter is 0.9 and the Red ABA parameter is 0.1), but sometimes increases (e.g., when the Red INT parameter is 0.1 and the Red ABA parameter is 0.9). This effect should be explored more thoroughly.

Figures 1-2 and 1-3 show the graph of the game value versus number of Blue aircraft when the number of Red aircraft stays at 2,000. The basic data set is used, with 24 Blue ground divisions. In Figure 1-2, all kill-parameters are 0.1; in Figure 1-3, they are all 0.9 except for attackers killing interceptors, which remain 0.1. The S-shaped nature for the curves is evident.

The effect of increasing the ground firepower was also considered. Table 1-4 shows the results when all the situations in Table 1-3 were run--changing the number of planes, firepower per plane, and kill parameters in exactly the same way (but with 360 ground firepower units delivered by Blue each day, instead of 240). Comparison of Tables 1-4 and 1-3 shows that the results not only increase dramatically for Blue in all cases, but the sensitivity of game value to increase in the Red INT and ABA parameters is markedly reduced. The differences from increasing the firepower and number of airplanes in general become smaller, because the force ratio of Blue to Red firepower delivered each day becomes closer to unity--due to the much larger term (360) in the numerator. Therefore, the ratio is in a section of the FEBA advance function F, with gentler slope; and the FEBA position (each payoff and, hence, the game value) is smoothed out.

An increase of 50 percent in the current Blue ground firepower would be expected to yield a larger increase in game value than a 50-percent increase in the (base case) number of Blue aircraft or firepower per aircraft, because the former increase would contribute 120 additional firepower units to the Blue firepower delivered daily, while the latter would contribute only 50 (i.e., 0.1 x 500 or 0.05 x 1,000). Work should be done comparing the results of adding the same number of firepower units by increasing (1) ground firepower, (2) the number of aircraft, (3) the firepower per aircraft. Our guess is that the first method will be much more robust (i.e., much less sensitive to changes in Red air parameters). From a real-world standpoint,

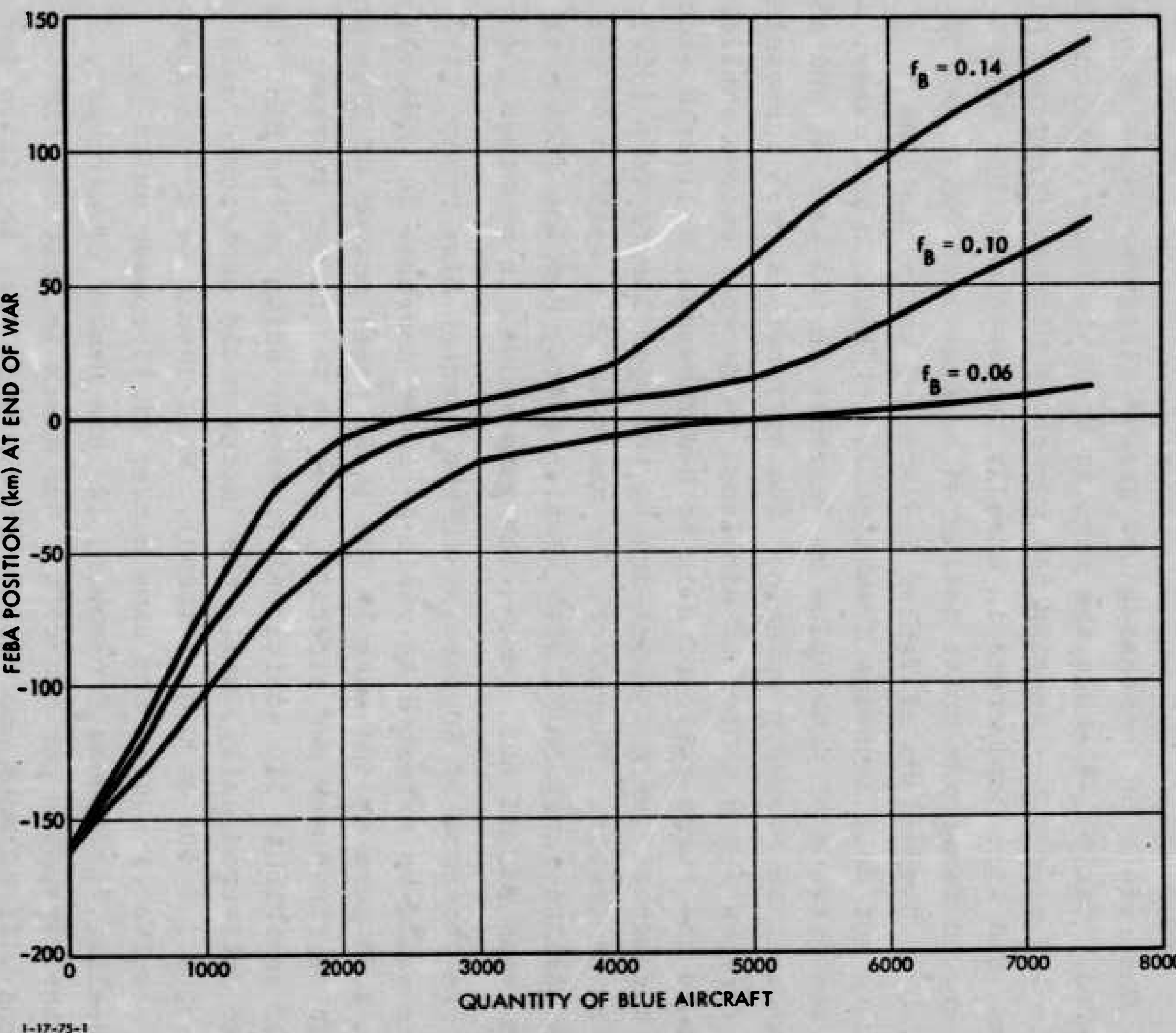

Figure 1-2. FEBA MOVEMENT AS A FUNCTION OF QUANTITY OF BLUE AIRCRAFT (LOW KILL-PARAMETER)

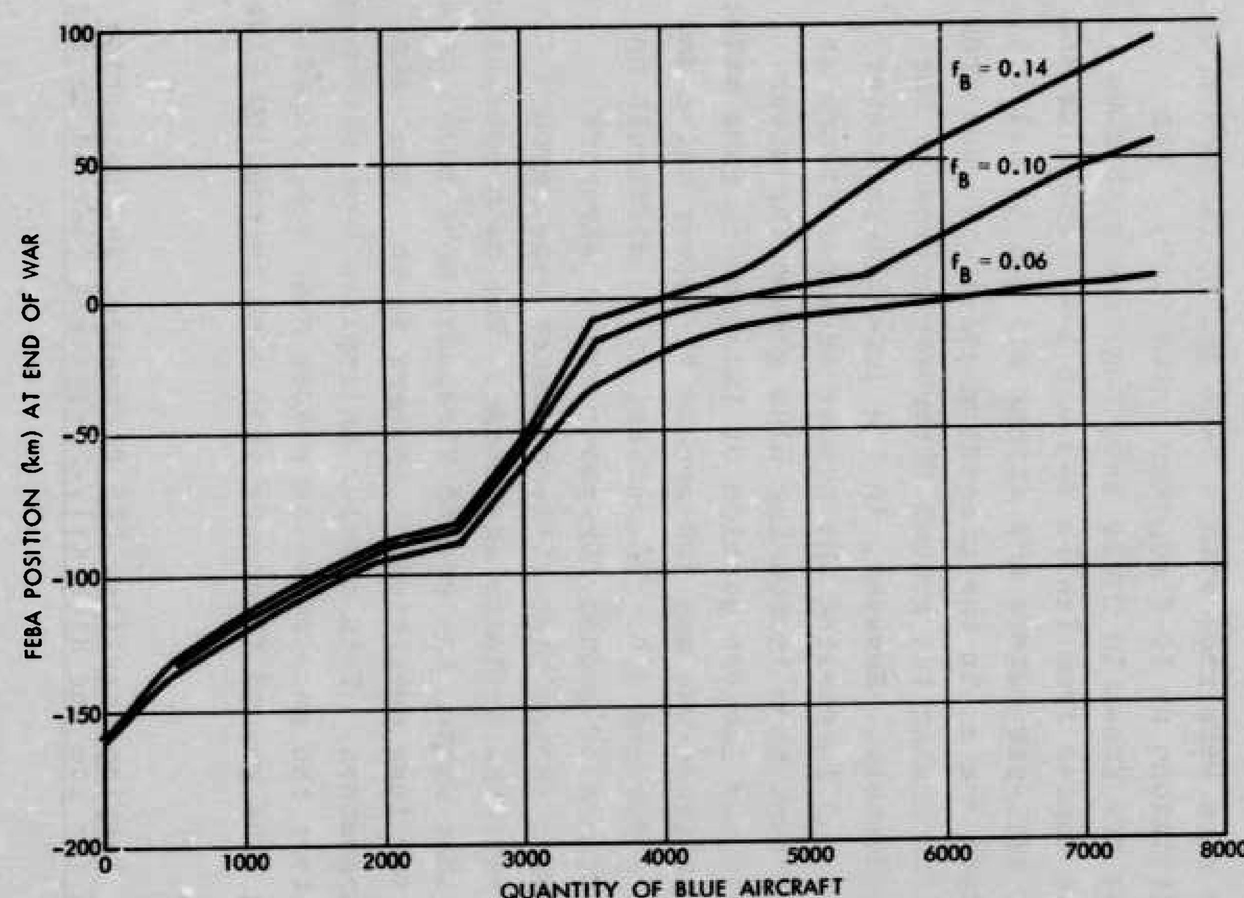

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Figure 1-3. FEBA MOVEMENT AS A FUNCTION OF QUANTITY OF BLUE AIRCRAFT (HIGH KILL-PARAMETER) however, it might not be feasible to increase the number of Blue divisions.

To shed a little more light on the effects of Blue ground firepower, some runs were made with a Blue ground firepower of 300 units (30 divisions at 10 firepower units each). The results of these are shown in Table 1-6, together with some appropriate data copied from Tables 1-3 and 1-4. The Blue and Red ABA and INT kill-parameters are either all 0.1 or all 0.9, and the other data are as in the preceding cases. We see from Table 1-6 that increasing the ground firepower increases the game value considerably. However, in the low-kill-parameter case, the effects of increasing the number of aircraft are as (or more) significant than increasing Blue ground firepower. And in both cases the largest portion of increase in game value sometimes comes between 240 and 300 ground firepower units and sometimes between 300 and 360. Therefore, it is difficult to trade off the effects of ground firepower versus aircraft quantity. However, in the high-kill-parameter case, ground firepower seems to have the definite advantage. The main conclusion we can draw is that making the ground firepower more equal makes the game value far less sensitive to changes in the (Red) ABA and INT kill-parameters. This conclusion implies that with high Red kill-parameters the game value increases much more rapidly with increasing Blue ground firepower than with increasing Blue air firepower.

3. Results of Varying Quantity, CAS Parameter, and Detection Parameters for Low and High Kill-Parameters (Second Series of Runs)

Table 1-7 shows another series of runs, using slightly different data, that were made to explore the effects of changing various other parameters, particularly the detection parameters in the INT and ABA missions, which in previous runs were set very high. The term e^{-dR} in the attrition equation

Table 1-6. EFFECT (ON THE GAME VALUE) OF INCREASING BLUE GROUND FIREPOWER

Blue Ground Fire-power	Red and Blue Kill Param-eters	Blue Firepower per Aircraft	Number of Blue Aircraft				
			1,000	2,000	3,000	4,000	5,000
240	0.1	0.06	-100	-49	-16	--	-1
		0.10	-79	-19	-2	--	15
		0.14	-68	-7	7	--	51
	0.9	0.06	-120	-96	-60	--	-8
		0.10	-118	-91	-55	--	4
		0.14	-115	-88	-52	--	24
300	0.1	0.06	-55	-21	-8	--	3
		0.10	-45	-8	3	--	22
		0.14	-37	-1	12	--	75
	0.9	0.06	-63	-34	-24	--	-1
		0.10	-60	-33	-20	--	9
		0.14	-58	-32	-17	--	37
360	0.1	0.06	-25	-9	-3	--	-6
		0.10	-19	-2	6	--	35
		0.14	-14	4	16	--	89
	0.9	0.06	-22	-13	-8	--	4
		0.10	-20	-13	-6	--	19
		0.14	-18	-12	-4	--	48

goes to zero for d much above 0.01. However, real-world values of d are smaller than this. It was desired to bu... framework for examining sensitivity to the detection parameters. The input data are described below.

As before, the war has 10 days and two periods, allocation choices being made on days 1 and 6. The set of seven pure strategies (previously described) was used. The Blue and Red ground firepower are 240 firepower units (24 divisions at 10 units each) and 480 firepower units (80 divisions at 6 units each), as before. All ground casualties are replaced. The number of Blue aircraft is 1,000, 2,000, or 3,000; the firepower per Blue aircraft is 0.01, 0.055, or 0.10. The Red air resources are also allowed to vary; the Red firepower per aircraft is 0.01 or 0.055 and the number of Red aircraft is 1,000, 2,000, or 3,000. This results in (3 x 3 x 2 x 3 =) 54 Blue and Red aircraft situations. For each set of 54 situations, a run was made as follows. The kill parameters for Red *and* Blue on ABA *and* INT are either *all* 0.1 or *all* 0.9. Attackers kill interceptors with a parameter half that of the interceptor parameter (i.e., 0.05 or 0.45). There are no special-purpose aircraft or shelters. For each of these two kill-parameter situations, the detection parameters for *all* missions are 0.0001, 0.001, or 0.01. Therefore, six arrays (with 54 game values per array) were constructed, as in Table 1-7. The payoff MOE is FEBA position at the end of the war, which is computed as described earlier.

We are interested mainly in the results of changing the detection parameters. Since we make this change for both sides simultaneously, it is not immediately evident which side will benefit by the increased parameters. We can conjecture, however, that it is the side that is more likely to use the non-CAS missions.

Table 1-7. EFFECT (ON FEBA POSITION AT D+10) OF VARYING THE DETECTION PARAMETERS

Firepower per Blue Aircraft	Number of Blue Aircraft	Detection Parameter for Blue and Red: 0.0001						0.001						0.01					
		Firepower per Red Aircraft																	
		0.010	0.055	0.010	0.055	0.010	0.055	0.010	0.055	0.010	0.055	0.010	0.055	0.010	0.055	0.010	0.055	0.010	0.055
		Number of Red Aircraft																	
		1,000	1,000	2,000	2,000	3,000	3,000	1,000	1,000	2,000	2,000	3,000	3,000	1,000	1,000	2,000	2,000	3,000	3,000
Kill Parameter = 0.1																			
0.010	1,000	-92	-117	-100	-144	-105	-170	-92	-117	-100	-142	-105	-169	-93	-114	-100	-141	-106	-168
	2,000	-78	-107	-85	-132	-93	-158	-78	-106	-86	-130	-93	-155	-79	-98	-87	-128	-94	-154
	3,000	-65	-96	-72	-122	-79	-146	-65	-94	-73	-119	-80	-142	-65	-85	-73	-115	-81	-141
0.055	1,000	-36	-65	-43	-100	-49	-122	-46	-65	-61	-100	-73	-122	-57	-68	-80	-100	-91	-130
	2,000	-13	-19	-14	-43	-15	-71	-15	-20	-22	-46	-35	-72	-16	-21	-29	-50	-49	-75
	3,000	-5	-10	-6	-16	-7	-27	-6	-10	-10	-17	-14	-40	-7	-10	-10	-18	-17	-42
0.100	1,000	-14	-23	-16	-51	-17	-81	-19	-28	-36	-59	-52	-86	-32	-36	-66	-79	-83	-113
	2,000	-1	-5	-2	-11	-3	-16	-4	-6	-9	-13	-16	-39	-5	-6	-12	-17	-35	-49
	3,000	9	4	7	-1	5	-5	6	4	2	-1	-2	-7	6	4	2	-1	-4	-7
Kill Parameter = 0.9																			
0.010	1,000	-92	-115	-100	-137	-105	-158	-99	-103	-103	-119	-107	-141	-100	-101	-104	-123	-108	-146
	2,000	-81	-104	-90	-123	-96	-140	-90	-92	-98	-103	-103	-118	-87	-89	-99	-101	-103	-121
	3,000	-70	-92	-81	-109	-89	-126	-78	-80	-90	-94	-98	-104	-76	-77	-88	-91	-99	-101
0.055	1,000	-50	-65	-65	-100	-76	-123	-87	-93	-98	-116	-102	-139	-96	-98	-101	-121	-105	-143
	2,000	-19	-27	-36	-59	-49	-83	-51	-51	-86	-91	-95	-113	-43	-43	-93	-98	-98	-117
	3,000	-11	-12	-21	-31	-35	-58	-22	-22	-54	-54	-85	-90	-19	-19	-47	-47	-90	-97
0.100	1,000	-23	-32	-42	-64	-56	-91	-78	-82	-95	-111	-99	-135	-93	-95	-99	-117	-104	-140
	2,000	-8	-8	-16	-24	-31	-54	-26	-26	-78	-82	-91	-106	-23	-23	-88	-92	-96	-113
	3,000	1	1	-6	-8	-15	-20	-7	-7	-32	-32	-76	-82	-5	-5	-29	-29	-85	-91

Examining Table 1-7 yields some interesting results. First, as before, increasing the Blue firepower per plane increases the game value significantly. However, except when the detection parameters are 0.0001, this increase is considerably undercut when the Red kill parameters are high (even though the Blue kill parameters are too). Therefore, the improvement gained through increased Blue firepower per aircraft is sensitive to the Red kill-parameter values. Exactly the same argument is true for Red: an increase in Red firepower per plane lowers the game value. However, when the kill parameters (for Red and Blue) are high and the detection probability is not 0.0001, the improvement for Red is much less. In fact, sometimes an increase of five times the Red firepower per aircraft resulted in *no* change in the game value, as Red was in a region where the optimal strategy involved Red's playing only ABA and INT. However, when the detection parameters were 0.0001, these missions evidently resulted in too few planes being detected to be profitable, even though the kill parameters were high. The Red CAS firepower then made more difference.

The case in which all kill parameters (Red and Blue, ABA and INT) were high usually resulted in air improvement for Red (sometimes large, sometimes small). The improvement for Red was greatest when Blue and Red both had a large number of planes but the Blue firepower score was large (0.10). As the Red parameters increase, Red's strategy shifts from all CAS in the second period to ABA and INT, while Blue's strategy remains all CAS. Raising all the kill parameters improved the game value for *Blue* when the Blue firepower score was low (0.01) and Red's high (0.055), independent of the number of aircraft. This is a situation familar from the one-day war: if the disparity in firepower scores per aircraft is large, as an effectiveness parameter (ABA or INT kill) increases for the side with the smaller firepower score, that side quickly shifts from CAS to the other missions.

Unsurprisingly, increasing the number of aircraft on a side helps that side a lot, even when the kill parameters are high or the opposing side has many aircraft--as can be observed in Table 1-7. We note again that the improvement from an aircraft increase is far less sensitive to changes in the opposing side's effectiveness parameters than an increase in firepower per plane.

We now examine the effect of detection parameters. The value 0.0001 is a realistic lower bound on the values. A detection parameter value of 0.01 is high enough to result often in the exponential term's going to zero. Between these extremes, there is sometimes a lot of change--more often, little. Sometimes the change is not monotone (i.e., from 0.0001 to 0.001 the situation improves for Blue, but from 0.001 to 0.01 the situation improves for Red, or vice versa).

Table 1-8 shows the cases in which the difference in game value was greatest. We note that when increasing the detection parameters on both sides leads to a very high change in game value, the improvement is always for Red. Furthermore, when the kill parameters are 0.9, the major part of the improvement occurs between 0.0001 and 0.001, while the increase is more evenly spread when the kill parameters are 0.1. These factors can be explained (in part) as follows: Because of the disparity in ground firepower, Red is always more ready than Blue to fly the non-CAS missions if the effectiveness parameters are reasonably high. Hence, a change in detection parameter--*other things being equal*--is more likely to benefit Red, and the *greater* changes are more likely to come from Red's changing from CAS to the other missions. (The first-period strategy is rarely all CAS; the second-period strategy often is. In general, the second-period strategy changes more markedly away from CAS as the relevant parameter rises.)

Table 1-8. CASES IN WHICH DETECTION PARAMETERS HAVE A LARGE EFFECT ON THE GAME VALUE

Number of Blue Planes	Blue FP/Plane	Number of Red Planes	Red FP/Plane	Blue and Red Kill Parameters	Game Value When Blue and Red Detection Parameters Are--		
					0.0001	0.001	0.01
2,000	0.100	3,000	0.010	0.1	-3	-16	-35
2,000	0.100	3,000	0.055	0.1	-16	-39	-49
1,000	0.100	3,000	0.010	0.1	-17	-51	-83
1,000	0.100	2,000	0.010	0.1	-16	-36	-66
2,000	0.055	3,000	0.010	0.1	-15	-35	-49
1,000	0.055	1,000	0.010	0.9	-50	-87	-96
2,000	0.055	2,000	0.010	0.9	-36	-86	-92
2,000	0.055	3,000	0.010	0.9	-49	-95	-98
3,000	0.055	3,000	0.010	0.9	-35	-85	-90
1,000	0.100	1,000	0.010	0.9	-23	-78	-93
1,000	0.100	2,000	0.010	0.9	-42	-95	-99
2,000	0.100	2,000	0.010	0.9	-16	-78	-88
2,000	0.100	2,000	0.055	0.9	-24	-82	-92
3,000	0.100	3,000	0.010	0.9	-15	-76	-85
3,000	0.100	3,000	0.055	0.9	-20	-82	-91

These great increases come when the number of Red planes is greater but the Blue firepower per plane is larger. As has been noted before, this is a common situation, in which Blue is very vulnerable to increased Red air effectiveness. What is true for a change in Red kill-parameters holds true also for a change in Red detection-parameters, though less markedly. The corresponding effect for Blue showed up in the kill parameters but not in the detection parameters because of the ground-firepower asymmetry.

4. Isovalue Curves for the Quantity-Quality Trade-Off

We now describe two charts that were prepared to shed more light on the effects on the outcome of OPTSA from varying the quantity and quality of aircraft. More specifically, it was desired to clarify the difference in increase in the game value from increasing the number of Blue aircraft versus increasing the firepower value per Blue plane (i.e., ordnance delivered by a successful Blue CAS sortie). For the data set described below, the model was run for several different values of quantity and firepower value of Blue aircraft. From the results, isovalue contours were constructed to show the trade-off between quantity and quality of aircraft.

The data set is almost identical to the basic data set used for the work described in the previous subsection. The one difference is the ability of planes assigned to CAS and ABA to kill enemy interceptors, which was previously 0.1 but is now half the kill parameter of the enemy interceptors. Two tables (Tables 1-9 and 1-10) were constructed. In Table 1-9, the kill parameters for Red and Blue on INT and ABA missions are 0.1. Planes on CAS and ABA have a kill parameter of 0.05 against enemy interceptors. In Table 1-10, the kill parameters are 0.9 and 0.45, respectively, for both Red and Blue. Because all detection parameters are set at 1.0, the exponential term

e^{-dR} or e^{-dB} in the attrition equations becomes zero. The payoff MOE, as before, is FEBA position (in kilometers) at the end of the war.

The Blue firepower per aircraft varies from 0.00 (an absolute lower limit) to 0.16 (a reasonable upper limit) by increments of 0.01. The number of Blue aircraft on the first day varies from zero to 3,000 by increments of 250. Five percent of the first-day number is added each subsequent day.

Table 1-9 shows the game values for all the combinations of Blue aircraft quantity and firepower value when the kill parameters are low (0.1 and 0.05, as described above); Table 1-10 shows the game values with high kill-parameters (0.9 and 0.45). From these tables, isovalue contours are constructed for the game values 0, -20, -40, -60, -80, -100, -120, and -140. That is, points where the outcome is one of those values are connected to form a line. All the points (i.e., combinations of Blue aircraft quantity and firepower value) on a line yield the same game value when the model is run with that point's data as input--that game value being one of the values above. Figures 1-4 and 1-5 show piecewise linear approximations to the isovalue contours resulting from Tables 1-9 and 1-10, respectively.

The tables and figures speak for themselves, but we mention some salient features here. Comparing Tables 1-9 and 1-10, we see that the value in Table 1-10 is usually much better for Red. The exception is when Blue planes each have a low firepower-score. In this case, Blue is giving up so little firepower by not playing CAS that they will attack Red planes--with good results. However, as the Blue firepower-score increases, Blue is more likely to play CAS and Red moves heavily to the non-CAS missions. This relative decrease in game value from Table 1-9 to Table 1-10 generally increases with increasing Blue firepower-value, for a given number of Blue airplanes.

Table 1-9. GAME VALUE AS A FUNCTION OF BLUE QUANTITY AND QUALITY WITH LOW AIR-EFFECTIVENESS PARAMETERS

Firepower per Blue Aircraft	Number of Blue Aircraft												
	0	250	500	750	1,000	1,250	1,500	1,750	2,000	2,250	2,500	2,750	3,000
0.16	-161	-138	-116	-94	-69	-44	-22	-9	-3	1	5	8	12
0.15	-161	-139	-118	-95	-70	-47	-24	-10	-5	-1	3	6	10
0.14	-161	-139	-119	-97	-72	-49	-27	-12	-7	-3	1	4	7
0.13	-161	-140	-121	-99	-73	-52	-32	-16	-8	-5	-1	2	5
0.12	-161	-140	-123	-101	-76	-55	-36	-21	-10	-6	-3	0	3
0.11	-161	-141	-124	-103	-79	-58	-41	-26	-14	-9	-5	-2	1
0.10	-161	-141	-126	-105	-82	-62	-47	-32	-19	-11	-7	-4	-2
0.09	-161	-142	-128	-108	-86	-66	-52	-38	-25	-15	-10	-7	-4
0.08	-161	-143	-129	-110	-90	-71	-58	-45	-33	-22	-14	-10	-7
0.07	-161	-143	-131	-113	-94	-77	-63	-52	-40	-30	-21	-14	-11
0.06	-161	-144	-133	-116	-100	-84	-69	-59	-49	-39	-30	-22	-16
0.05	-161	-146	-134	-121	-108	-97	-82	-69	-58	-49	-41	-33	-26
0.04	-161	-149	-137	-127	-117	-108	-100	-88	-77	-67	-57	-48	-40
0.03	-161	-152	-143	-134	-127	-119	-112	-106	-100	-91	-82	-74	-67
0.02	-161	-154	-149	-143	-137	-132	-127	-122	-117	-112	-108	-102	-97
0.01	-161	-157	-153	-149	-145	-142	-138	-134	-131	-128	-124	-121	-118
0.00	-161	-158	-155	-151	-148	-144	-141	-138	-134	-132	-128	-125	-123

Table 1-10. GAME VALUE AS A FUNCTION OF BLUE QUANTITY AND QUALITY WITH HIGH AIR-EFFECTIVENESS PARAMETERS

Firepower per Blue Aircraft	Number of Blue Aircraft												
	0	250	500	750	1,000	1,250	1,500	1,750	2,000	2,250	2,500	2,750	3,000
0.16	-161	-140	-132	-124	-115	-108	-103	-95	-88	-62	-55	-30	-14
0.15	-161	-140	-132	-124	-116	-109	-103	-96	-89	-63	-56	-31	-16
0.14	-161	-141	-133	-125	-117	-110	-104	-96	-90	-64	-56	-33	-18
0.13	-161	-141	-133	-126	-118	-111	-104	-97	-90	-65	-57	-35	-21
0.12	-161	-141	-134	-126	-119	-111	-105	-97	-91	-67	-58	-37	-23
0.11	-161	-141	-134	-127	-119	-112	-105	-98	-92	-69	-59	-40	-26
0.10	-161	-142	-134	-128	-120	-113	-106	-98	-93	-71	-60	-42	-29
0.09	-161	-142	-135	-128	-121	-113	-106	-99	-94	-73	-61	-45	-32
0.08	-161	-142	-135	-129	-121	-114	-107	-100	-95	-76	-63	-49	-36
0.07	-161	-142	-136	-129	-122	-115	-107	-102	-96	-78	-66	-54	-40
0.06	-161	-142	-136	-130	-123	-116	-108	-103	-98	-81	-70	-59	-44
0.05	-161	-143	-136	-130	-123	-116	-109	-104	-99	-84	-74	-65	-50
0.04	-161	-143	-137	-131	-124	-117	-110	-105	-99	-87	-78	-71	-59
0.03	-161	-143	-137	-131	-125	-118	-111	-106	-100	-90	-83	-77	-68
0.02	-161	-143	-137	-131	-125	-118	-111	-107	-100	-95	-90	-86	-79
0.01	-161	-143	-137	-131	-125	-118	-111	-107	-101	-98	-97	-94	-91
0.00	-161	-143	-137	-131	-125	-118	-111	-107	-103	-103	-102	-108	-101

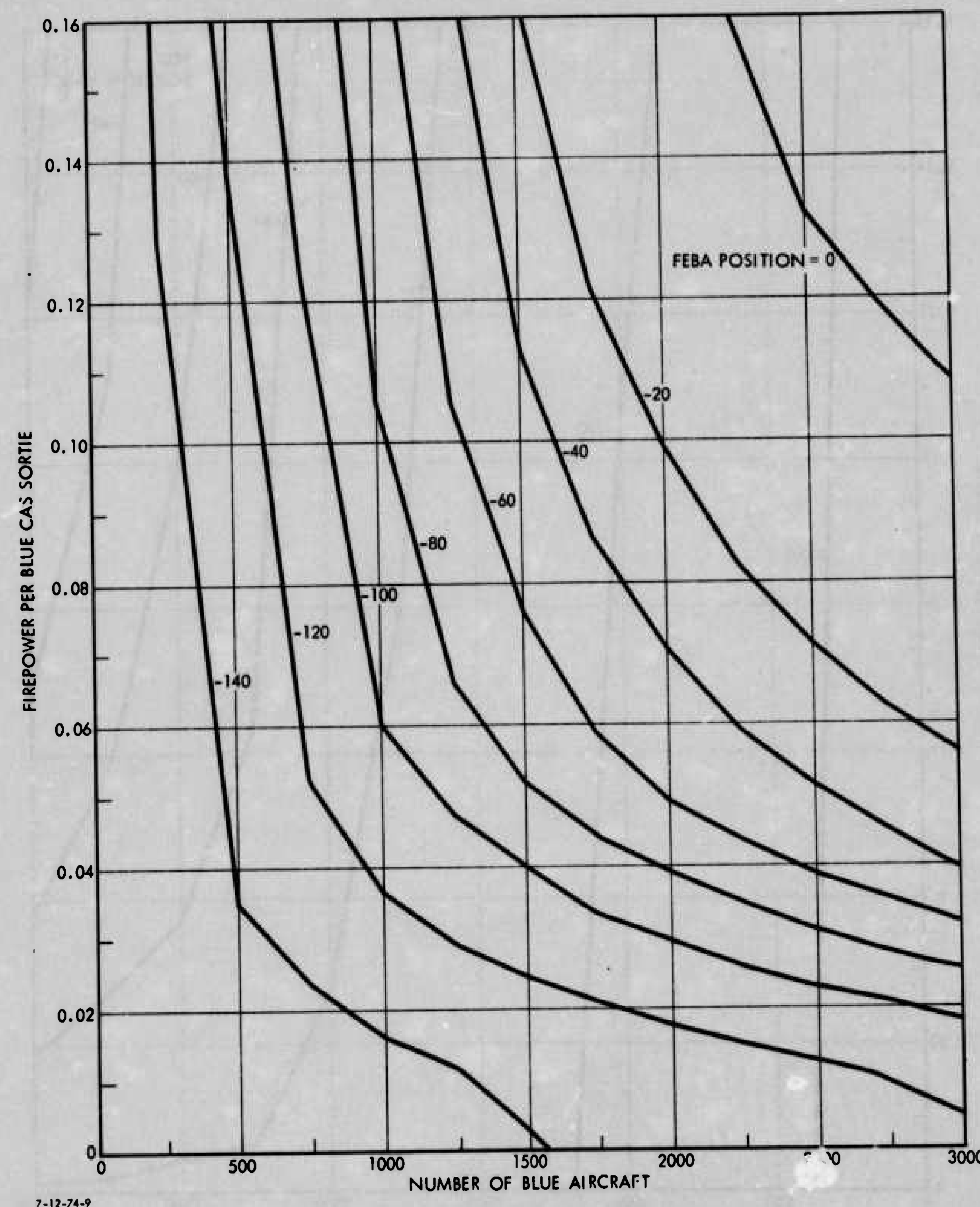

Figure 1-4. FEBA POSITION AS A FUNCTION OF BLUE QUANTITY AND QUALITY, WITH LOW AIR-EFFECTIVENESS PARAMETERS

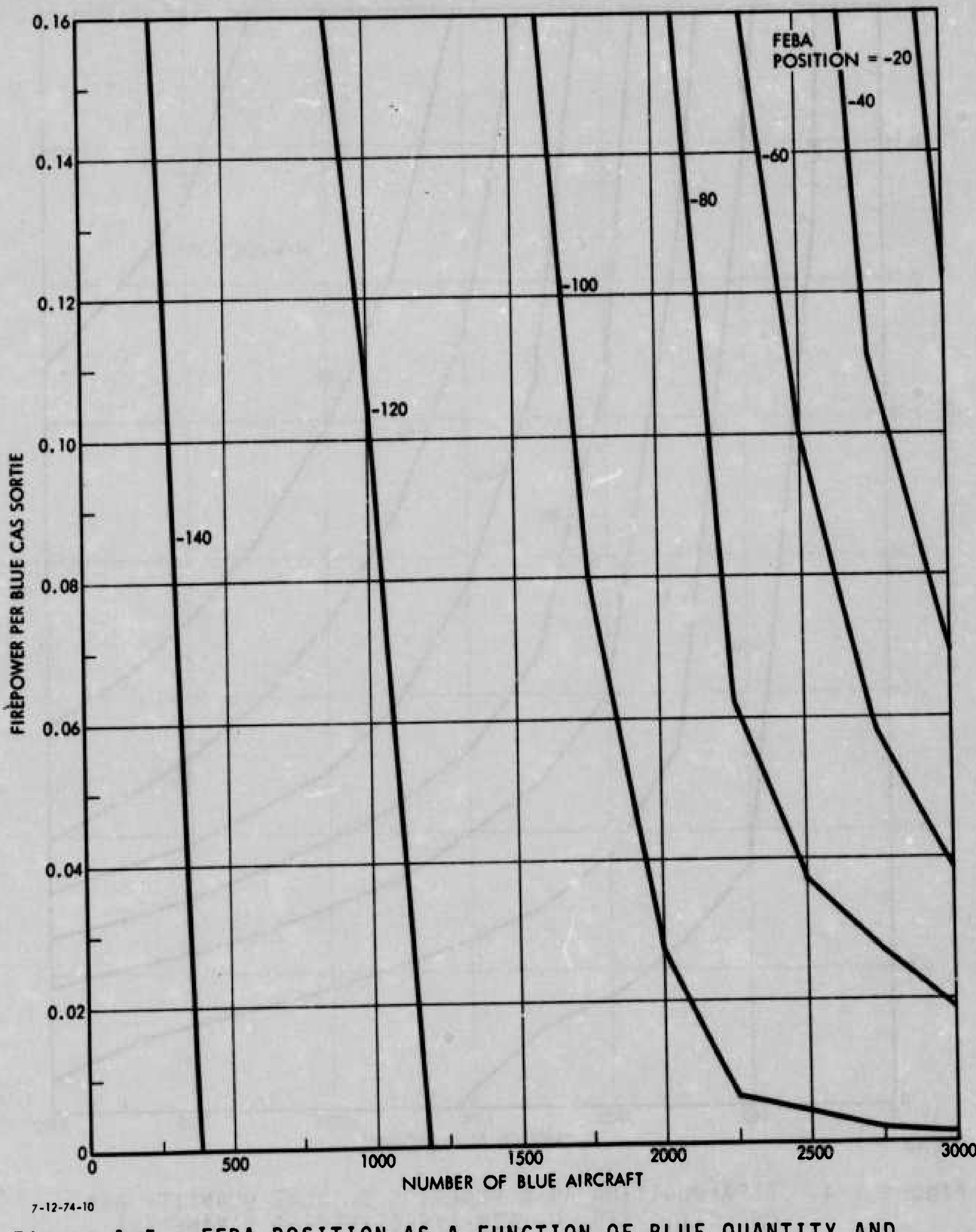

Figure 1-5. FEBA POSITION AS A FUNCTION OF BLUE QUANTITY AND QUALITY, WITH HIGH AIR-EFFECTIVENESS PARAMETERS

Naturally, in each table an increase in Blue firepower-value or number of aircraft results in an increased game value. However, the rates of increase are much larger in Table 1-9, where the kill parameters for ABA and INT are low. (This is borne out by examination of the isovalue contours in Figures 1-4 and 1-5. In Figure 1-5, the high-kill-parameter case, the isovalue contours are quite far apart (much farther apart than in Figure 1-4) and are shifted to the right from Figure 1-4. Therefore, it takes more Blue to get to a higher game value. Also, as we would expect, the curves in Figure 1-5 are far more widely spaced with regard to firepower value, especially in the region where the number of Blue aircraft is high. It is interesting to note, however, that in the lower right corner of Figure 1-4 the curves are almost horizontal and parallel (i.e., an increase in firepower-score yields a large increase in game value). However, with respect to an increase in aircraft quantity only, Figures 1-4 and 1-5 behave quite similarly in this region. We note also that, in the left side of both figures, the curves are very steep--indicating that, for a low number of planes, firepower score makes little difference--which follows from the fact that with so few planes, even if Blue is playing all CAS, the increase in air firepower delivered will not be very large as the firepower value increases.

This paper shows that, for the specific example given, air-effectiveness parameters could make a large difference in the behavior of the game value with respect to quantity and CAS effectiveness of Blue aircraft. More tables like these should be constructed to determine more accurately the behavior of the game value in different situations (especially with regard to varying the disparity in ground firepower).

5. Effects of SAMs on OPTSA Results

We now present some preliminary results that explore the effects of using SAMs in the assessment routine of OPTSA. The assessment routine of the original version of OPTSA [5] did not provide for SAMs to be played at all--an admitted deficiency. At present, extensive revisions of the assessment routine of OPTSA are underway to play SAMs and AAA, as well as adding several other new capabilities. However, since it was desired to have some quick results, a simple but fairly realistic modification was made to play SAMs. This paper presents results of some runs made with this modification. Results of the full revisions of OPTSA will appear later.

To simulate SAMs, two numbers between zero and one are input: the first represents a proportion of Red attack aircraft destroyed by Blue SAMs; the second, a proportion of Blue attack aircraft destroyed by Red SAMs. Let us call these the Blue and Red SAM parameters, respectively. Note that the parameter is *not* an actual number of missiles. The SAM parameters remain constant throughout the game. They enter into the assessment routine thus: on each day after the assignments of aircraft to missions have been made, each side's SAMs fire at the planes of the opposite side assigned to CAS and ABA. The number of attacking planes killed on a side is the SAM parameter of the opposite side times the original number of attacking planes. The relative proportions by mission (CAS and ABA) and aircraft type (general-purpose, special-purpose CAS, and special-purpose ABA) among the survivors remain the same as the original proportions. The attacking planes on each side that survive the SAMs then become vulnerable to interceptors. The assessment then proceeds as in the original version. Aircraft destroyed by SAMs are, of course, added to the total aircraft destroyed for the day. A SAM parameter of zero is equivalent to no SAMs' being played by that side.

The data for particular cases run are shown in Table 1-11. Other than the SAM parameters, the data are the basic data set described earlier.

Table 1-11. DATA FOR THE CASES RUN TO EXPLORE SAM EFFECTS

Variables	Value of Variable			
	Blue		Red	
Number of Ground Divisions	24		80	
Firepower per Ground Division	10		6	
Number of Aircraft[a]	1,000, 2,000, 3,000		2,000	
Firepower per Aircraft	0.06, 0.10, 0.14		0.06	
Number of Shelters	0		0	
Detection Parameters	1.0		1.0	
Kill Parameters	Case A	Case B	Case A	Case B
Attackers Kill Interceptors	0.1	0.1	0.1	0.1
INT	0.1	0.9	0.1	0.9
ABA (nonsheltered)	0.1	0.9	0.1	0.9
SAM Parameters	0.01, 0.10, 0.25		0.01, 0.10, 0.25	

[a]Number of the first day, five percent of which is added each subsequent day (no special-purpose aircraft are played).

Table 1-12 shows the results (game values)[1] for varying SAM parameters for the case where (ABA and INT) kill-parameters are low. Table 1-13 shows the same thing for high kill-parameters. It is expected that a rising SAM parameter will quickly increase the game value for that side.

Though it is difficult to draw general conclusions, a few interesting results can be noted here. First, sensitivity to SAMs is quite low when the kill parameters are high (Case B). In Case A (low kill-parameters), there are many regions of sensitivity. We note that a 0.01 SAM parameter yields an outcome about the same as a zero SAM parameter. As the Red SAM parameter increases, not only does the game value decrease markedly (in Case A), but the decrease is much greater when there are 3,000 Blue aircraft; hence,

[1]The payoff MOE is FEBA position (in kilometers) at the end of the war (Day 10), computed as shown in Figure 1-1 (above).

Table 1-12. FEBA POSITION AS A FUNCTION OF SAM PARAMETERS (CASE A: LOW KILL-PARAMETERS)

Blue SAM Parameter	Red SAM Parameter	Firepower per Blue Aircraft	Number of Blue Aircraft		
			1,000	2,000	3,000
0.00	0.00	0.06 0.10 0.14	-100 -79 -68	-49 -19 -7	-16 -2 7
0.01	0.01	0.06 0.10 0.14	-100 -80 -69	-49 -20 -7	-17 -3 6
0.01	0.10	0.06 0.10 0.14	-118 -99 -91	-81 -56 -40	-52 -23 -7
0.01	0.25	0.06 0.10 0.14	-137 -124 -113	-117 -93 -77	-92 -69 -52
0.10	0.01	0.06 0.10 0.14	-91 -74 -65	-31 -12 -4	-11 1 10
0.10	0.10	0.06 0.10 0.14	-103 -89 -81	-64 -40 -26	-31 -10 -3
0.10	0.25	0.06 0.10 0.14	-117 -108 -99	-97 -77 -66	-77 -50 -31
0.25	0.01	0.06 0.10 0.14	-81 -69 -61	-24 -10 -4	-9 -2 11
0.25	0.10	0.06 0.10 0.14	-90 -79 -74	-55 -36 -26	-25 -9 -2
0.25	0.25	0.06 0.10 0.14	-101 -91 -86	-81 -68 -62	-67 -42 -31

Table 1-13. FEBA POSITION AS A FUNCTION OF SAM PARAMETERS (CASE B: HIGH KILL-PARAMETERS)

Blue SAM Parameter	Red SAM Parameter	Firepower per Blue Aircraft	Number of Blue Aircraft		
			1,000	2,000	3,000
0.00	0.00	0.06	-120	-96	-60
		0.10	-118	-91	-55
		0.14	-115	-88	-52
0.01	0.01	0.06	-119	-96	-62
		0.10	-116	-91	-56
		0.14	-114	-88	-53
0.01	0.10	0.06	-120	-99	-88
		0.10	-117	-95	-83
		0.14	-115	-93	-78
0.01	0.25	0.06	-121	-101	-94
		0.10	-119	-100	-91
		0.14	-118	-98	-89
0.10	0.01	0.06	-108	-95	-62
		0.10	-106	-90	-56
		0.14	-105	-88	-53
0.10	0.10	0.06	-108	-97	-88
		0.10	-107	-93	-83
		0.14	-106	-91	-78
0.10	0.25	0.06	-109	-100	-93
		0.10	-108	-98	-90
		0.14	-108	-96	-88
0.25	0.01	0.06	-101	-92	-62
		0.10	-101	-90	-56
		0.14	-101	-88	-53
0.25	0.10	0.06	-102	-95	-88
		0.10	-101	-91	-83
		0.14	-101	-90	-78
0.25	0.25	0.06	-102	-98	-92
		0.10	-102	-96	-89
		0.14	-101	-94	-88

the improvement in game value from increasing the number of Blue aircraft is considerably undercut by a high Red SAM parameter. The same is true of an increase in Blue firepower per aircraft in Case A. The effect of Red SAMs is similar to that of going from Case A to Case B with no SAMs. If there is either a high Red SAM parameter or a high Red INT parameter, raising the other one does not make much difference in the game value.

The reverse effect, an improvement for Red being undercut by an increased Blue SAM parameter, is not as marked. Equivalently, even in Case A, raising the Blue SAM parameter does not have an extremely great effect--probably because of the ground firepower deficiency of Blue mentioned in Subsection 2 (above). The most dramatic effect is for 1,000 Blue planes at 0.06 firepower value, which yields a change from -100 to -80.

Looking at the optimal strategies that produce these game values sheds some more light on the reasons for the sensitivities and insensitivities encountered in Tables 1-12 and 1-13. While the optimal strategies are often randomized (especially in the second period) and vary with minor changes in the input data, several general, broad patterns of strategy occur, which are summarized in Table 1-14.

Table 1-14. GENERAL PATTERN OF OPTIMAL STRATEGIES FOR CASES WITH SAMs

	Case A (low kill-parameters)	Case B (high kill-parameters)
Blue	Mostly CAS, with various mixtures of ABA and INT.	ABA if Red SAM parameter low and 3,000 Blue aircraft. INT otherwise, often in both periods.
Red	ABA if 1,000 Blue aircraft, low Blue SAM parameter. Much INT, especially if Blue SAM parameter is large.	Partial ABA if 1,000 Blue aircraft and low Blue parameter; CAS second period. INT otherwise, both per periods.

Table 1-14 explains many of the insensitivities. In Case A, when Blue's SAM parameter rises, Red quickly shifts to an all-INT strategy, where they are not vulnerable to SAMs. Red still preserves a lot of their game value, as Red can intercept Blue planes flying CAS or ABA. If Blue tries this tactic as Red's SAM parameter rises, Blue must forgo the CAS firepower that Blue really needs and suffers a loss. If Blue plays as before (CAS), their planes are cut down by Red SAMs. Thus, either way, Blue does a lot worse when the Red SAM parameter is high than Red does when the Blue SAM parameter is high. The driving factor is that, because of the ground firepower disadvantage of Blue (240 ground firepower units versus 480 for Red), Blue *needs* CAS firepower much more than Red; thus, giving both sides equal capability to kill aircraft is more advantageous to Red.

In Case B, with high kill-parameters, even with no SAMs, all INT is often a desirable strategy. There is little CAS played at the outset by either side; thus, the FEBA position is a function mainly of the ground firepower and stays around -100 (the value in the case with no air firepower at all). As SAMs increase, ABA becomes undesirable and a shift to all INT (on both sides) quickly occurs. This strategy change does not, however, affect the game value very much, as CAS is still rarely played.

6. Effects of Aircraft Shelters on OPTSA Results

So far, there have been no aircraft shelters in the cases examined. Since aircraft shelters are a very important issue in planning, it is desirable to examine the effect of various numbers of shelters on the outcome of the model. The function of shelters, of course, is to make aircraft less vulnerable to ABA by the enemy. The detection and/or kill parameters input for planes on ABA that kill sheltered aircraft are less than those for unsheltered aircraft. Adding shelters for Blue could therefore influence the outcome of the game in at least two ways: First, if Red had found and still finds it profitable to go on ABA, the lessened vulnerability of Blue aircraft due to sheltering would improve the game value for Blue. Second, if Red had found it desirable to go on ABA with no Blue shelters, the very presence of Blue shelters might cause Red no longer to find ABA desirable and to go on some other mission not so injurious to Blue. Of course, if Red was not finding ABA a desirable mission when there were no Blue shelters, adding Blue shelters would not increase the game value for Blue, although it would not decrease it either. In all these cases, results are dependent on the relative vulnerability of sheltered versus unsheltered aircraft. Several different ranges of vulnerability differences are examined.

a. Treatment of Shelters in the OPTSA Model

The particular way aircraft shelters are simulated in the game can have potentially a large effect on the game value and game strategy. Several reasonable ways to treat shelters exist, and all should be run to determine the differences they make on the outcome. However, in the interest of expediency, one particular method of simulation was chosen. It does not allow shelters to be destroyed but does play the "shell game," in which empty shelters can be hit.

The treatment of shelters is described below for Red planes on ABA attacking Blue airbases. The situation is exactly the same when Blue planes attack Red airbases. On each day, assignments of aircraft to missions are made and then the INT and (if desired) ground-to-air interactions take place, resulting in some attrition to both Red and Blue planes. The remaining Blue planes all become vulnerable to ABA by Red. (If there are no Blue planes left, of course, none are killed.) Blue has a fixed input number of aircraft shelters and shelters all the planes they can. If Blue has many shelters, there might be some unoccupied shelters, indistinguishable to Red from sheltered ones. But if Blue has unsheltered aircraft sitting on the ground, all Blue shelters are filled.

The number of Red planes that will go on ABA is the original number of Red planes devoted to ABA minus attritions from the INT and ground-to-air interactions. There are two types of planes on ABA: Red general-purpose aircraft assigned to ABA and special-purpose Red ABA planes. There are four pairs of detection and kill-effectiveness parameters for Red ABA planes, distinguished by (1) type of Red plane and (2) whether a Red plane is attacking Blue shelters or unsheltered Blue aircraft. According to the relative numbers of the two kinds of Red planes, the parameters are averaged into two d and k pairs--one against Blue shelters (d_s and k_s), the other against unsheltered Blue aircraft (d_{ns} and k_{ns}).

Red comes in to attack the Blue airbase. If there are no unsheltered Blue aircraft, Red devotes all his planes to finding and attacking Blue shelters. To repeat, there is no way for Red to tell whether a shelter is occupied or not. Also, all the shelters themselves are not destroyed. The scenario is that Red can find a shelter with probability d_s and, with probability k_s, can shoot a weapon through the shelter that will destroy a Blue aircraft inside it. The number of shelters shot is given by the exponential equation

$$\dot{S}_B = S_B\left[1 - \exp\left(-\left[1-e^{-d_s S_B}\right]k_s \frac{R}{S_B}\right)\right] ,$$

where S_B is the number of Blue shelters and R the number of Red aircraft on ABA. The number of Blue aircraft killed is $\dot{B} = \dot{S}_B \frac{B}{S_B}$, where B is the total stock of Blue aircraft vulnerable to ABA (i.e., Blue aircraft are uniformly distributed among available shelters).

If there are both sheltered and unsheltered Blue aircraft, Red devotes a part of his force to attacking sheltered aircraft, the rest to attacking unsheltered aircraft. The proportion is chosen to maximize the expected number of Blue aircraft killed. Red general-purpose and special-purpose aircraft are divided in the same proportion. If p is the proportion of Red aircraft devoted to attacking Blue sheltered aircraft and R is the number of Red planes on ABA,

$$\dot{B}_s(p) = B_s\left[1 - \exp\left(-\left[1 - e^{-d_s B_s}\right]k_s \frac{pR}{B_s}\right)\right]$$

(as all Blue shelters are full), and

$$\dot{B}_{ns}(p) = B_{ns}\left[1 - \exp\left(- 1\left[- e^{-d_{ns} B_{ns}}\right]k_{ns} \frac{(1-p)R}{B_{ns}}\right)\right].$$

Red wishes to find $p \varepsilon [0,1]$ maximize $\dot{B}_s + \dot{B}_{ns}$, the total number of Blue planes killed. (All the parameters and aircraft amounts are known to both sides.) The constrained maximizing point of this one-variable, differentiable, unimodal function is easily found by calculus to be

$$p^* = \begin{cases} 0, & \text{if } p' \leq 0 \\ p', & \text{if } p' \varepsilon (0,1) \\ 1, & \text{if } p' \geq 1 \end{cases}$$

where

$$p' = \left[\frac{1}{\left(1-e^{-d_s B_s}\right)k_s \frac{R}{B_s} + \left(1-e^{-d_{ns} B_{ns}}\right)k_{ns} \frac{R}{B_{ns}}}\right]$$

$$\times \left[\left(1-e^{-d_{ns} B_{ns}}\right)k_{ns} \frac{R}{B_{ns}} + \ln \frac{\left(1-e^{-d_s B_s}\right)k_s}{\left(1-e^{-d_{ns} B_{ns}}\right)k_{ns}}\right].$$

The attrition to Blue then is $\dot{B}_s(p^*) + \dot{B}_{ns}(p^*)$. No shelters are destroyed, and the number of Blue aircraft of a certain type killed is in proportion to the number of that type in the original Blue population.

There are several reasonable alternative ways of treating aircraft shelters. It would be interesting to allow shelters to be killed by being bombed on ABA and/or captured by ground troops as the FEBA advances (representing ground capture of actual airfields, rather than the one notionalized airbase). Programming changes are currently underway to handle some of these alternative ways to play shelters.

b. Data, Results, and Discussion

To examine the effects of shelters for both Blue and Red, the model was run for a variety of different combinations of kill parameters against sheltered and unsheltered aircraft. As before, all detection parameters were set to 1.0. In actuality, detection parameters for sheltered aircraft are quite small; however, we have subsumed this in the kill parameter. It was desired to examine extreme cases, as before, to form a framework for interpolation. Table 1-15 shows the parameter combinations run. For each parameter combination, the number of Blue shelters and the number of Red shelters were varied independently from zero to 3,000. The other data were as in the basic data set of Table 1-2 (above). There were 2,000 Red aircraft throughout; for each parameter combination, runs were made for 1,000, 2,000, and 3,000 Blue aircraft.[1] We were most interested in discovering when a side would go to ABA, when the presence of shelters would inhibit another side from going to ABA, and when "shell game" effects would occur as the number of shelters on one side rose.

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[1]As before, an additional five percent of the first-day number of aircraft enter each day, for Blue and for Red.

Table 1-15. KILL-PARAMETER DATA FOR THE TABLES ON SHELTER EFFECTS

Number of Blue Aircraft	Kill Parameters			
	Unsheltered Aircraft		Sheltered Aircraft	
	Blue Kills Red	Red Kills Blue	Blue Kills Red	Red Kills Blue
1,000 2,000 3,000	0.1	0.1	0.01, 0.05	0.01, 0.05
	0.5	0.5	0.01, 0.10	0.01, 0.10
	0.9	0.9	0.01, 0.10, 0.50	0.01, 0.10, 0.50
	0.1	0.9	0.01, 0.05	0.01, 0.10, 0.50
	0.9	0.1	0.01, 0.10, 0.50	0.01, 0.05

All these effects occur, under certain appropriate conditions. We exhibit in Tables 1-16a, b, and c the full results of three of the runs. In Table 1-16a, the kill parameters are low for both sides; in Tables 1-16b and c, they are high. Table 1-16c shows that under certain conditions both Red and Blue shelters can make a substantial difference in the outcome. Table 1-16a shows almost no sensitivity to shelters; in certain regions of Table 1-16b, there is much sensitivity to Blue shelters.

For the runs in general, the most important factor affecting the result is, unsurprisingly, the number of Blue aircraft, which affects the optimal strategies and makes order-of-magnitude changes in the game value. With 1,000 aircraft, Blue practically always devotes all aircraft to CAS, while Red often flies some or all ABA--and all CAS very rarely. Blue sometimes flies some ABA if Red has few shelters and Blue's kill parameter against unsheltered aircraft is 0.9, as in Table 1-16b. Therefore, there are regions where Blue and Red shelters affect the outcome. The most dramatic change occurs when Red's kill parameter against unsheltered aircraft is 0.5. Blue goes from zero to 1,000 shelters. This changes the game value from around -120 to around -70. Adding additional Blue shelters has a "shell game" effect when the

Table 1-16. EFFECTS OF AIRCRAFT SHELTERS ON GAME VALUE (NUMBER OF BLUE AIRCRAFT / BLUE KILL-PARAMETER AGAINST RED SHELTERS / RED KILL-PARAMETER AGAINST BLUE SHELTERS)

a. 1,000 / 0.1 / 0.1						
Kill Parameter Against Shelters		Number of Blue Shelters	Number of Red Shelters			
Blue	Red		0	1,000	2,000	3,000
0.01	0.01	0	-79	-79	-79	-79
		1,000	-78	-78	-78	-78
		2,000	-78	-78	-78	-78
		3,000	-78	-78	-78	-78
0.01	0.05	0	-79	-79	-79	-79
		1,000	-78	-78	-78	-78
		2,000	-78	-78	-78	-78
		3,000	-78	-78	-78	-78
0.05	0.01	0	-79	-79	-79	-79
		1,000	-78	-78	-78	-78
		2,000	-78	-78	-78	-78
		3,000	-78	-78	-78	-78
0.05	0.05	0	-79	-79	-79	-79
		1,000	-78	-78	-78	-78
		2,000	-78	-78	-78	-78
		3,000	-78	-78	-78	-78

(continued on next page)

Table 1-16 (continued)

b. 1,000 / 0.9 / 0.9						
Kill Parameter Against Shelters		Number of Blue Shelters	Number of Red Shelters			
Blue	Red		0	1,000	2,000	3,000
0.01	0.01	0	-119	-122	-124	-124
		1,000	-60	-67	-78	-78
		2,000	-60	-67	-78	-78
		3,000	-60	-67	-78	-78
0.01	0.10	0	-119	-122	-124	-124
		1,000	-60	-67	-78	-78
		2,000	-60	-67	-78	-78
		3,000	-60	-67	-78	-78
0.01	0.50	0	-119	-122	-124	-124
		1,000	-103	-104	-110	-110
		2,000	-80	-89	-94	-94
		3,000	-69	-76	-83	-83
0.10	0.01	0	-119	-122	-124	-124
		1,000	-60	-67	-78	-78
		2,000	-60	-67	-78	-78
		3,000	-60	-67	-78	-78
0.10	0.10	0	-119	-122	-124	-124
		1,000	-60	-67	-78	-78
		2,000	-60	-67	-78	-78
		3,000	-60	-67	-78	-78
0.10	0.50	0	-119	-122	-124	-124
		1,000	-103	-104	-110	-110
		2,000	-80	-89	-94	-94
		3,000	-69	-76	-83	-83
0.50	0.01	0	-119	-122	-124	-124
		1,000	-60	-66	-78	-78
		2,000	-60	-66	-78	-78
		3,000	-60	-66	-78	-78

(continued on next page)

Table 1-16b (concluded)

Kill Parameter Against Shelters		Number of Blue Shelters	Number of Red Shelters			
Blue	Red		0	1,000	2,000	3,000
0.5	0.10	0	-119	-122	-124	-124
		1,000	-60	-66	-78	-78
		2,000	-60	-66	-78	-78
		3,000	-60	-66	-78	-78
0.5	0.5	0	-119	-122	-124	-124
		1,000	-103	-104	-110	-110
		2,000	-80	-89	-94	-94
		3,0C0	-69	-76	-83	-83

(continued on next page)

Table 1-16 (continued)

c. 3,000 / 0.9 / 0.9						
Kill Parameter Against Shelters		Number of Blue Shelters	Number of Red Shelters			
Blue	Red		0	1,000	2,000	3,000
0.01	0.01	0	-35	-74	-81	-81
		1,000	-17	-23	-30	-30
		2,000	-7	-9	-9	-9
		3,000	-1	-2	-2	-2
0.01	0.10	0	-35	-74	-81	-81
		1,000	-17	-24	-37	-37
		2,000	-7	-10	-10	-10
		3,000	-1	-2	-2	-2
0.01	0.50	0	-35	-74	-81	-81
		1,000	-18	-46	-71	-71
		2,000	-10	-21	-54	-54
		3,000	-7	-14	-37	-37
0.10	0.01	0	-35	-74	-81	-81
		1,000	-17	-22	-30	-30
		2,000	-7	-9	-9	-9
		3,000	-1	-2	-2	-2
0.10	0.10	0	-35	-74	-81	-81
		1,000	-17	-23	-37	-37
		2,000	-7	-10	-10	-10
		3,000	-1	-2	-2	-2
0.10	0.50	0	-35	-74	-81	-81
		1,000	-18	-46	-71	-71
		2,000	-10	-21	-54	-54
		3,000	-7	-14	-33	-37
0.5	0.01	0	-35	-65	-76	-79
		1,000	-17	-21	-22	-25
		2,000	-7	-9	-9	-9
		3,000	-1	-2	-2	-2

(concluded on next page)

Table 1-16c (concluded)

Kill Parameter Against Shelters		Number of Blue Shelters	Number of Red Shelters			
Blue	Red		0	1,000	2,000	3,000
0.5	0.10	0	-35	-65	-76	-79
		1,000	-17	-21	-24	-28
		2,000	-7	-9	-10	-10
		3,000	-1	-2	-2	-2
0.5	0.5	0	-35	-65	-76	-79
		1,000	-18	-31	-49	-59
		2,000	-10	-15	-22	-32
		3,000	-7	-10	-14	-17

Red kill-parameter against aircraft is still high enough (0.5) to make ABA profitable. Otherwise, the jump from zero to 1,000 Blue shelters causes Red to change from an ABA to an INT strategy, which remains optimal when further Blue shelters are added. Therefore, the game value remains unchanged. This change in strategy occurs also in Table 1-16a, with low kill-parameters against unsheltered aircraft, although the game value changes very little.

With 2,000 aircraft, Blue chooses mixed strategies that yield mostly CAS; Red, mostly ABA. Blue does some ABA if his kill parameter is high (0.5 or 0.9). If the Red kill-parameter against sheltered aircraft is not high (the breakpoint is somewhere between 0.1 and 0.5) with 2,000 or 3,000 Blue shelters, the game value is often -12, corresponding to an all-CAS strategy by both sides. This is unaffected by the number of Red shelters. If the Red kill-parameter against sheltered aircraft is higher, Red continues to fly ABA as Blue shelters increase; but considerable improvement for Blue (because of the shell game) occurs. As in the case for 1,000 Blue aircraft, the most dramatic change *in general* occurs when the number of Blue shelters goes from zero to 1,000. In some cases, Red shelters have a considerable effect, but usually less than Blue shelters. Table 1-16 illustrates the small difference from a 0.01 versus a 0.1 kill parameter, which occurs in most of the runs. The region of greatest sensitivity with respect to strategy change and game-value change seems to be between a 0.1 and a 0.5 kill parameter.

When Blue has 3,000 aircraft, he flies much more ABA than previously, but only if his kill parameter (averaged for the appropriate number of sheltered versus unsheltered aircraft) is high. Because of the ground-firepower disparity, Blue still prefers to fly mostly CAS. Red usually flies ABA, even with 3,000 Blue shelters; hence, shell-game effects for Blue occur often, as in Table 1-16c. (Red probably flies ABA so much because

Red wants to kill those extra Blue planes and has the ground firepower so that he can afford to do it.) However, there is great insensitivity in game value when Red is ineffective on ABA--which is probably because Blue can deliver enough air firepower to push the force ratio up into the flat region of the FEBA-advance function, where changes in the force ratio result in less change in the game value.

The above discussion shows how important it is to get accurate estimates of the kill (and detection, which was not examined) parameters against both sheltered and unsheltered aircraft. ABA can be a lucrative mission for Red, and results and strategy choices are extremely sensitive to the parameter values. Depending on the parameter values, shelters can make either a huge difference in outcome or no difference at all. Also, alternative ways of modeling the simulation of shelters in combat (e.g., allowing shelters to be destroyed in a number of ways) might affect the game value considerably and should be examined.

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Chapter 2

QUANTITY-QUALITY TRADE-OFFS OF TANKS: ENGAGEMENT-LEVEL ANALYSIS

Morris J. Zusman

The purpose of the research described in this chapter is to develop a generalized methodology that might be applied to make quantity-quality trade-offs and to apply this methodology to the specific case of tanks.

This research uses as its basic working tool an engagement-level model of tank warfare developed at IDA. The model is described in IDA Paper P-916, *Tank Exchange Model*, November 1973 (Reference [3]). It is a highly versatile, limited-engagement Monte Carlo simulation. The model can require up to 17,000 separate pieces of data to simulate a single engagement. All work in this report was done with respect to one basic engagement. Figure 2-1 shows the battlefield that was simulated in all the quantity-quality work of this report. The basic case has up-to-10 tanks attacking up-to-five defense tanks. The reason 10 attack tanks were chosen is that this is the maximum number of attack tanks the model can handle. For cases in which fewer than 10 attack tanks or fewer than five defense tanks were simulated, the higher-numbered tanks shown in Figure 2-1 were dropped in descending order.

A. EFFECTIVENESS MEASURES

The generalized methodology used to make quantity-versus-quality trade-offs was to plot a family of iso-effectiveness curves on a grid of quantity versus the quality of forces committed on one side of the battle, for a given quantity and quality of forces committed by the opposing side. Therefore, the first problem was to define an appropriate effectiveness

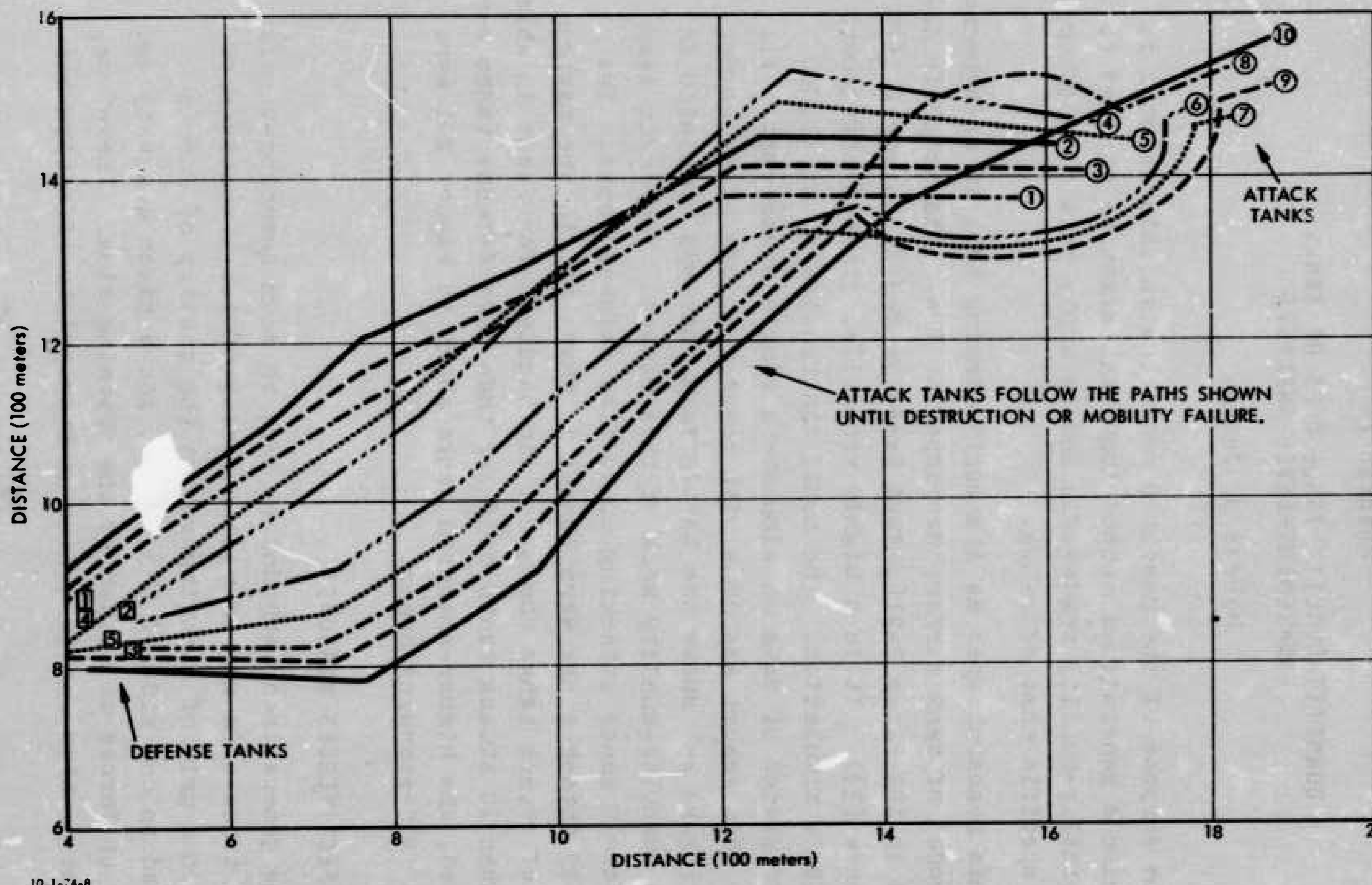

Figure 2-1. TANK BATTLEFIELD (SIMULATED)

measure that may be used to make quantity-quality trade-offs. Since the numerical value of the effectiveness measure would be empirically derived from the data generated by the tank-exchange model, certain effectiveness measures that might be ideal from a decision-maker's point of view may have undesirable statistical properties when derived. A total of five different effectiveness measures were examined in detail and are discussed below.

(1) $E(\widehat{\dot{A}/\dot{D}})$

(2) $E(\hat{\dot{A}})/E(\hat{\dot{D}})$

(3) $E(\hat{\dot{A}})/A \;/\; E(\hat{\dot{D}})/D$

(4) $E(\widehat{\dot{A}-\dot{D}})$

(5) $E(\hat{\dot{A}})/A - E(\hat{\dot{D}})/D$,

where

$E(\cdot)$ = expectation of the parenthesized event;

$\dot{A}$ = number of attack tanks destroyed in a given engagement;

$\dot{D}$ = number of defense tanks destroyed in a given engagement;

A = number of attack tanks committed to a given engagement;

D = number of defense tanks committed to a given engagement; and

^ over a parameter means that the parameter is being estimated.

Only two of the five measures (3 and 5) were found appropriate for making quantity-quality trade-offs. Note that for measures 3 and 5 there are companion measures $E(\hat{\dot{A}}/A)/E(\hat{\dot{D}}/D)$ and

$E(\dot{A}/A\ \hat{-}\ \dot{D}/D)$, respectively. These measures were not considered because of statistical problems that are discussed in Section B (below).

1. $E(\hat{\dot{A}}/\dot{D})$

The first measure (the average of the exchange ratios of the individual battles) was dismissed immediately, since--as long as there was any possibility that the defense would suffer no casualties in any single engagement (a definite occurrence in our data)--the expectation of the exchange ratio would always be infinity.

2. $E(\hat{\dot{A}})/E(\hat{\dot{D}})$

The second measure (the ratio of the expected losses) is an exchange ratio of sorts, but it does not go to infinity if occasionally the defense suffer zero losses in an engagement. Even if the defense suffer no losses in 99 percent of the battles, $E(\hat{\dot{D}}) \neq 0$. However, for making quantity-quality trade-offs in which the intent is to improve the quality of the weapon system so that the total quantity of the force required to win a war may be reduced, the ratio of the expected losses on each side is an inappropriate and misleading measure. The reason that this measure is inappropriate is that, as one side improves its quality so that it can commit fewer tanks to an engagement while holding the effectiveness constant (ratio of the expected losses) on each side, it also reduces the number of casualties that it inflicts on the other side. This phenomenon is best illustrated by an example: Assume that Red has 2,000 tanks and will attack Blue in limited engagements with groups of eight attack tanks. Blue planners desire to make trade-offs between the quantity and quality of tanks required to defeat Red's 2,000 tanks. Assume further that Blue would like to have a tank that would maintain an effectiveness ratio (of the expected loss of attack tanks to the expected loss of defense tanks) of 2. Then Blue would need at least 1,001 tanks to destroy all of Red's 2,000 tanks. The only trade-off between quantity and quality that can be made directly using the ratio of the expected losses as the effectiveness measure is how many tanks Blue must commit to each engagement, since maintaining a constant effectiveness measure (ratio of the expected losses) of 2 will always require Blue to have at least 1,001 tanks to defeat Red.

Figure 2-2 illustrates the limited quantity-quality trade-offs that can be made by using the ratio of the expected losses as the effectiveness measure. Figure 2-2 is a plot of a family of iso-effectiveness lines on a grid of number of defensive tanks committed to an engagement versus defense lethality measure, for eight tanks on the attack. (For the moment, let us ignore how the curves have been derived and the exact definition of lethality measure; these matters are described in subsequent sections of this report.) Figure 2-2 shows how the number of defensive tanks committed to an engagement can be decreased if the quality of the tank is improved for a constant effectiveness. Figure 2-2 shows that, if Blue wants to defend against Red's eight tank assaults with five defensive tanks while maintaining an expected loss ratio of 2, Blue's lethality measure would have to be 0.95. If, on the other hand, it were only desired to commit four defense tanks while still maintaining an exchange ratio of 2, the defense lethality measure would have to be increased from 0.95 to 1.08.

The misleading aspect of the exchange ratio as an effectiveness measure is that although we have increased the quality of the defensive tank so that we can maintain a constant exchange ratio when the defense is outnumbered 8 to 4 instead of 8 to 5, we have not decreased the total number of defensive tanks that are required to win the war.

The second problem with using exchange ratios as an effectiveness measure is a statistical problem. Because $\hat{E}(D)$ is an

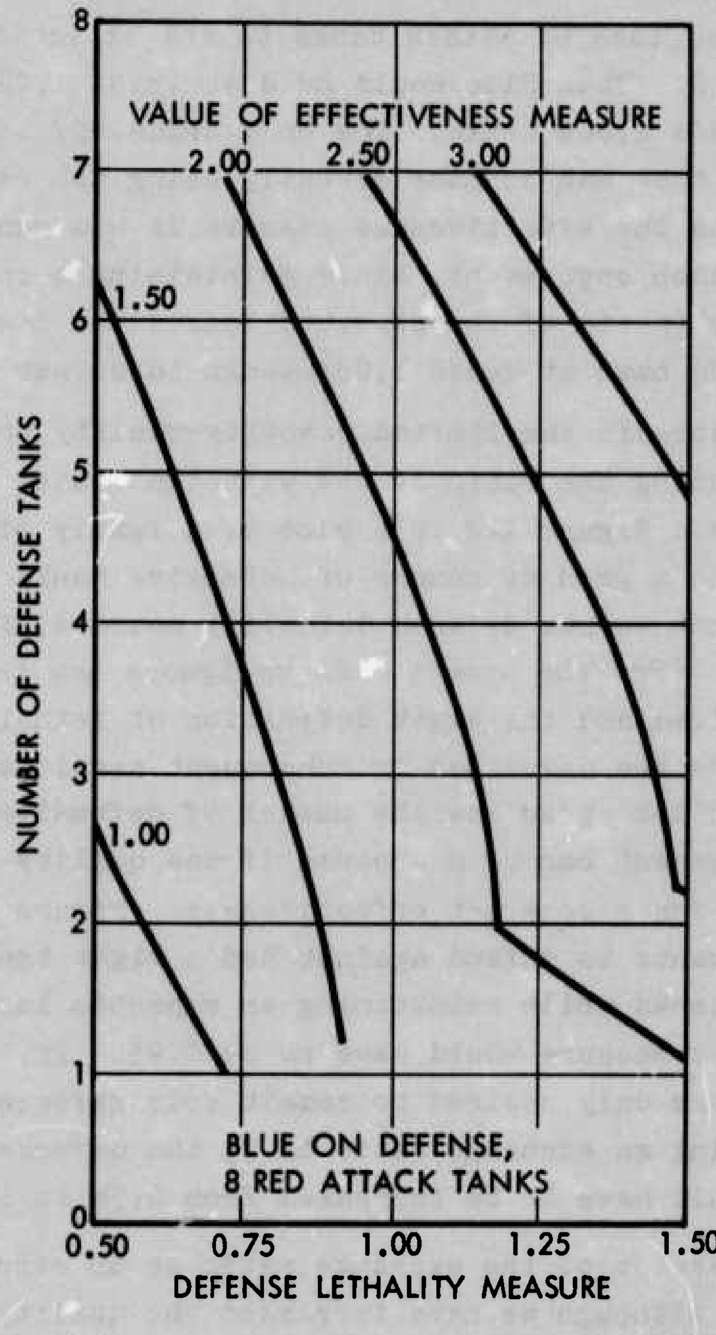

Figure 2-2. EFFECTIVENESS MEASURE (EXCHANGE RATIO) AS A FUNCTION OF NUMBER OF DEFENSE TANKS AND DEFENSE LETHALITY MEASURE

empirically derived estimate, there is some uncertainty about the true value of $E(\dot{D})$. That is, we are sure with some level of confidence only that

$$E(\dot{D}) - U \leq E(\hat{\dot{D}}) \leq E(\dot{D}) + U ,$$

where $E(\dot{D})$ is the true mean of the number of defensive tanks destroyed and $E(\hat{\dot{D}})$ is an estimator of $E(\dot{D})$. The exact confidence-band equations that are complex are presented in Appendix 2-B and need not be explored here. The point to be concerned here is that, unless $E(\hat{\dot{D}}) >> U$, the uncertainty band about the estimated exchange ratio becomes very large and approaches infinity as $E(\hat{\dot{D}})$ approaches U.

3. $E(\hat{\dot{A}})/A \,/\, E(\hat{\dot{D}})/D$

This effectiveness measure is the ratio of the loss rates. Because this effectiveness measure contains the number of forces committed on each side of the engagement, it is well suited for making quantity-quality trade-offs directly. Again, assume that Red has 2,000 tanks and that the war will be made up of a series of limited engagements in which a group of eight Red tanks attack the defending Blue tanks. Figure 2-3 (which is identical in form to Figure 2-2 except that the effectiveness measure is the ratio of the expected loss rates rather than the ratio of the expected losses), which may be used in making quantity-quality trade-offs, is presented to show how the ratio of the expected loss rates can be used to make quantity-quality trade-offs directly for the entire force--assuming that Blue desires to maintain an effectiveness measure of 1. Then, if Blue wishes to defend against Red with five tanks in each engagement, it can be seen from Figure 2-3 that Blue will have to design a tank with a lethality measure of 0.7. If, on the other hand, Blue wishes to commit only four

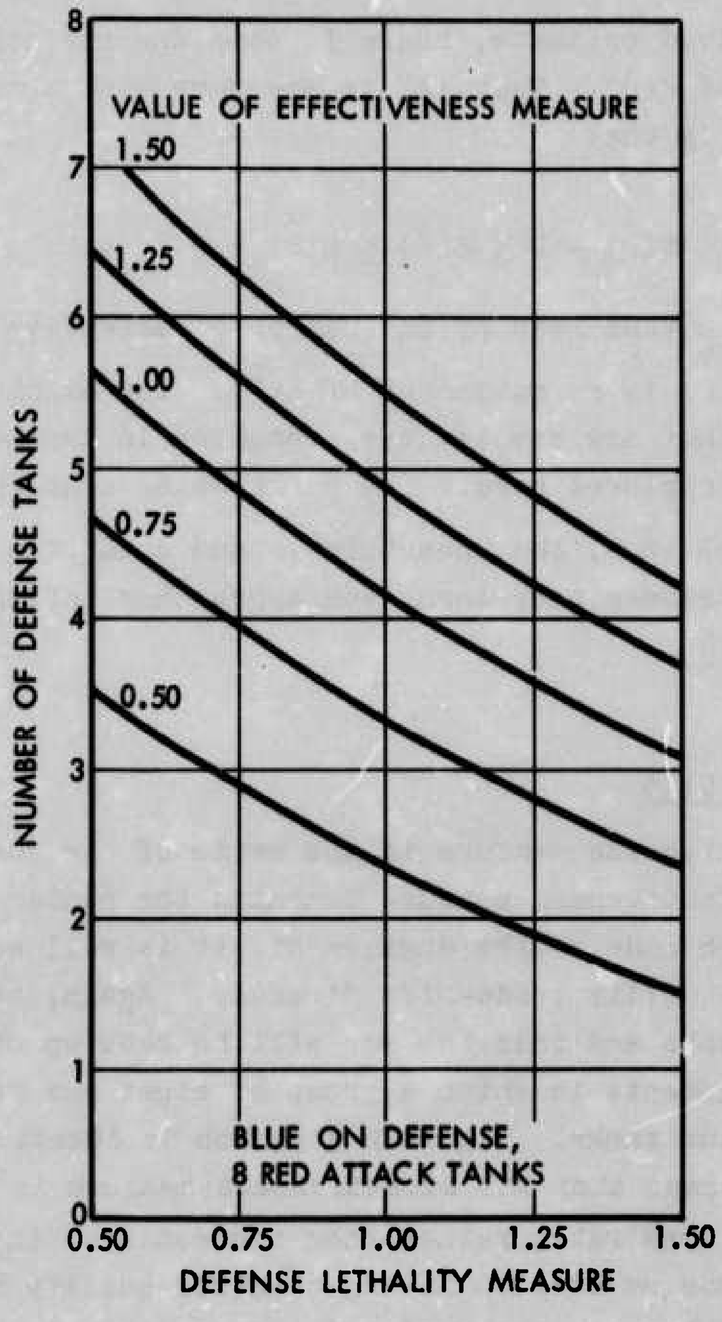

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Figure 2-3. EFFECTIVENESS MEASURE (RATIO OF LOSS RATES) AS A FUNCTION OF NUMBER OF DEFENSE TANKS AND DEFENSE LETHALITY MEASURE tanks to each engagement, Blue will have to design a tank with a lethality measure of 1.08. However, the important thing to note with this effectiveness measure is that quantity-quality trade-offs can be made directly. Thus, when Blue commits five tanks each with a lethality measure of 0.7 to defend against eight Red tanks, Blue will require at least 1,250 tanks to destroy all of Red's 2,000 tanks; but, if Blue designed a tank that had a lethality measure of 1.08, Blue would require only a minimum of 1,001 tanks to defeat all Red's 2,000 tanks. Similarly, it can be seen (by extrapolating Figure 2-3) that if Blue designed a tank with a lethality measure of 1.55 and committed only three to each engagement, Blue would need only 750 tanks to destroy all of Red's 2,000 tanks. Further, this effectiveness measure has the desirable property of indicating which side on average is expected to win each engagement. If the measure is greater than 1, the assault wins; if less than 1, the defense wins.

While the expected loss-rate ratio has some desirable properties, it suffers from two serious shortcomings. First, just like the exchange ratio, it has $E(\hat{\dot{D}})$ in the demoninator. Thus, the previous discussion about confidence limits on the empirically derived ratio apply equally to the expected loss-rate ratio. Second, the measure is an asymmetric measure. When the defense forces are totally destroyed, the measure is bounded by 0 and 1. However, when the assault forces are totally destroyed, the measure is bounded by 0 and infinity.

4. $\underline{E(\widehat{\dot{A} - \dot{D}})}$

This measure is the absolute difference in the losses. Its one major advantage is that, unlike the other effectiveness measures, it can be estimated directly, while the other effectiveness measures require algebraic manipulation of estimators. However, from a practical standpoint, this is not a good measure on which to make trade-offs between quantity and quality.

Since it, like effectiveness measure 2 (above), does not include the forces committed on each side, it is difficult to interpret in a quantity-quality context. As the quality of one side is improved and few tanks are committed, the number of casualties inflicted on the other side also changes. Therefore, while curves were derived and are presented for this measure in Appendix 2-A, it is felt that it is a poor effectiveness measure for quantity-quality trade-offs. For comparison purposes, Figure 2-4 is presented.

5. $E(\hat{A})/A - E(\hat{D})/D$

This effectiveness measure has all desirable properties of the exchange-rate ratio without certain of its shortcomings. Unlike the exchange-rate ratio, this effectiveness measure (being bounded between 1 and -1) is symmetric. When the defense are totally destroyed, the measure will be between -1 and 0; when the assault are totally destroyed, the measure will be between 1 and 0. When the loss rates of the forces are exactly equal, the measure will be zero. Because the measure is a difference rather than a ratio, it does not have the statistical problem of the confidence bands about the measure becoming infinite (or even particularly large) as $E(\hat{D})$ approaches zero.

Figure 2-5 presents iso-effectiveness curves using the difference in expected loss rates as the effectiveness measure. Note that the line where the difference in expected loss rates is equal to zero in Figure 2-4 is identical to the line in Figure 2-3, where the expected loss-rate ratio is equal to 1. This is the case where each side is destroying at the same rate and $E(A_k)/A = E(D_k)/D$. Thus, $E(\dot{A})/A - E(\dot{D})/D = 0$, and $E(\dot{A})/A/E(\dot{D})/D = 1$. Note, too, that the family of both curves in Figures 2-3 and 2-5 are essentially parallel lines. However, the curves in Figure 2-5 (because of the symmetric property of its

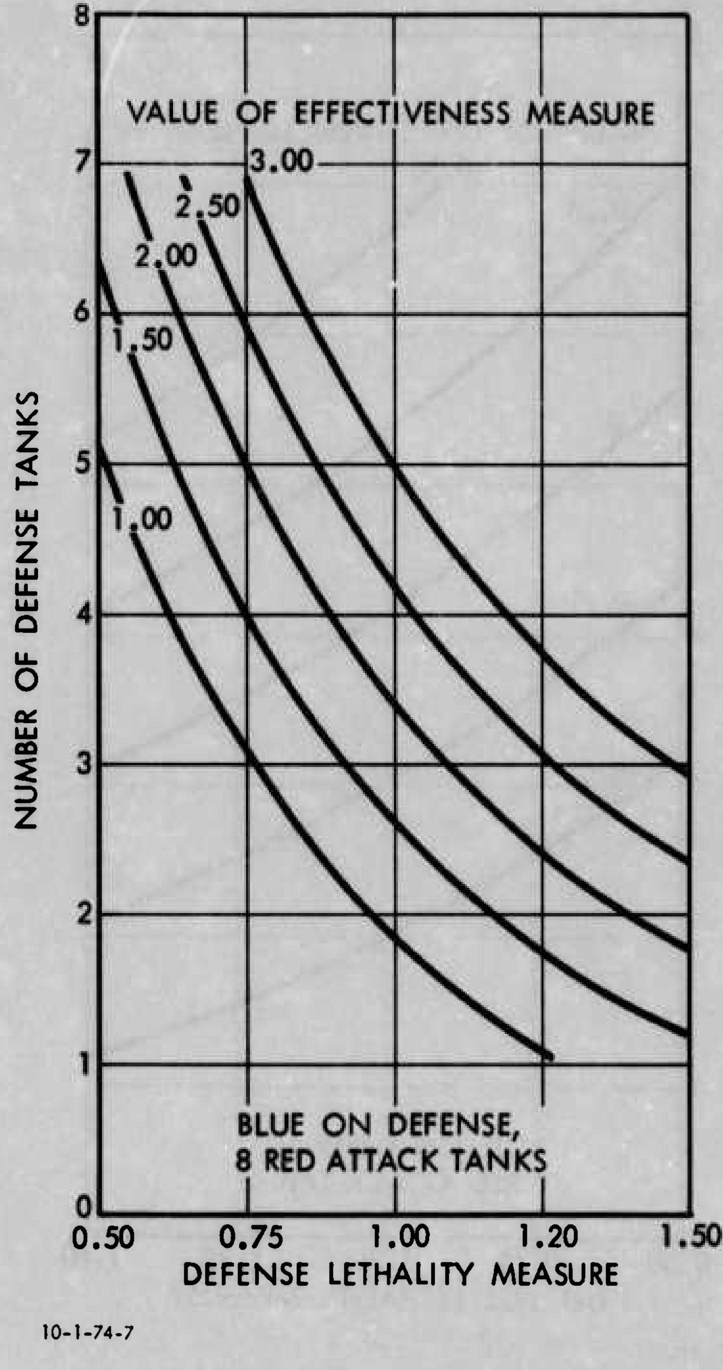

Figure 2-4. EFFECTIVENESS MEASURE (LOSS DIFFERENCE) AS A FUNCTION OF NUMBER OF DEFENSE TANKS AND DEFENSE LETHALITY MEASURE

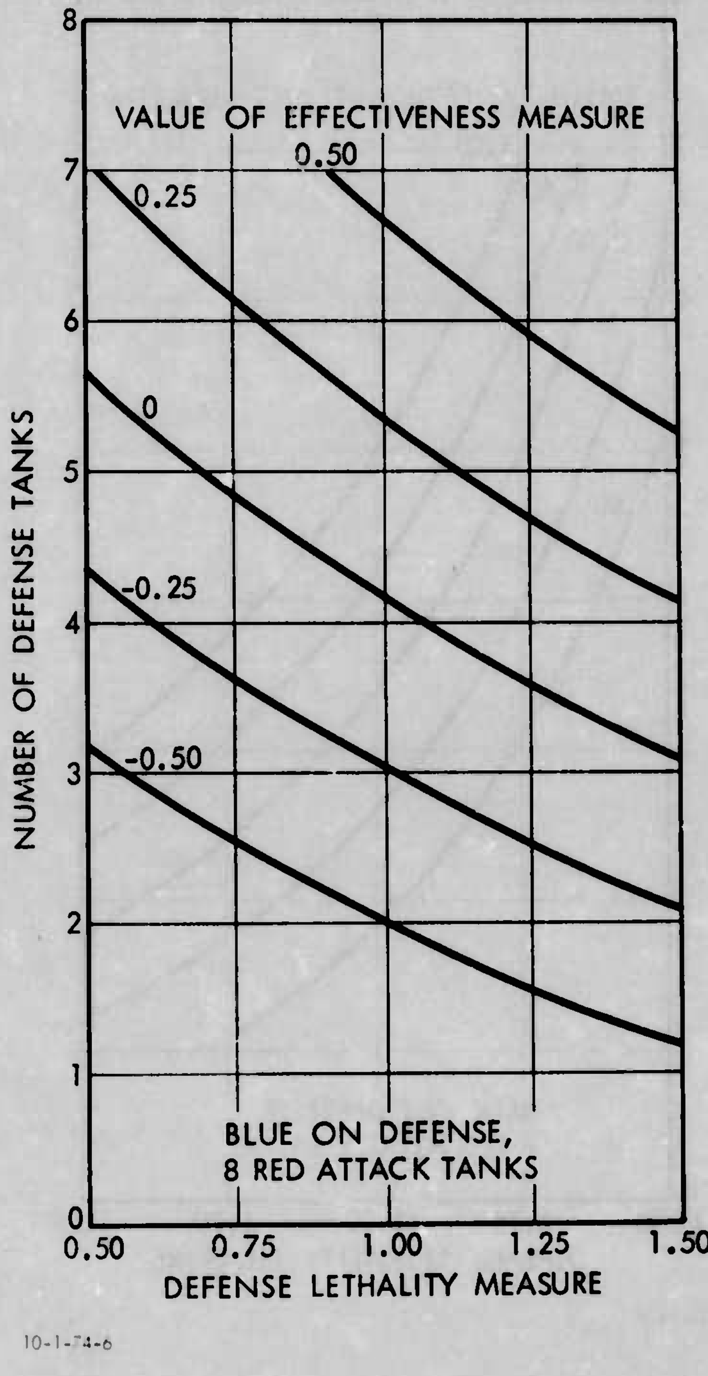

Figure 2-5. EFFECTIVENESS MEASURE (LOSS-RATE DIFFERENCE) AS A FUNCTION OF NUMBER OF DEFENSE TANKS AND DEFENSE LETHALITY MEASURE

effective measure) are equally spaced for a constant increment in the effectiveness measure, while the curves in Figure 2-3 are more tightly spaced for a constant increment in the effectiveness measure as the effectiveness measure increases.

Because the difference in the expected loss rates is a symmetric bounded measure (with a stable confidence bandwidth, even when the expected loss rate of the defense approaches zero) that lends itself directly to making quantity-quality trade-offs, it is felt that it is the preferred measure to use in making quantity-quality effectiveness trade-offs.

B. METHODOLOGY

Having defined the effectiveness measures for which quantity-quality trade-offs could be made, our next problem was to develop a methodology by which the appropriate variables could be estimated.

Ideally, we would like to derive a set of equations such as the following:

$$\dot{A} = f(A, D, Q_{1A}, \ldots, Q_{KA}, Q_{1D}, \ldots, Q_{JD}) \quad (1a)$$

$$\dot{D} = g(A, D, Q_{1A}, \ldots, Q_{KA}, Q_{1D}, \ldots, Q_{JD}) \quad (1b)$$

$$C_D = H(D, Q_{1D}, \ldots, Q_{JD}) \quad (1c)$$

(for the case in which Blue is on defense and similar set where on attack), where

$\dot{A}$ = the number of attack tanks destroyed in a given engagement;

$\dot{D}$ = the number of defense tanks destroyed in a given engagement;

C_D = the total life-cycle cost in dollars of the defense force;

A = the number of assault tanks committed to an engagement;

D = the number of defense tanks committed to an engagement;

Q_{iA} = the i^{th} quality parameter of assault tanks; and

Q_{jD} = the j^{th} quality parameter of defense tanks.

As the first step in developing a quantity-versus-quality methodology, almost all the efforts involved in this paper have been expended in defining Equations (1a) and (1b).

One question that may occur to a reader is that, since our preferred effectiveness measure is of the form $E(\hat{A}/A) - E(\hat{D}/D)$, why don't we try to develop one equation for the effectiveness measure directly instead of developing one for $\dot{A}$ and $\dot{D}$ separately and then algebraically manipulating the two equations to derive the effectiveness measure? The reason is that the equations are to be derived using ordinary least-squares regression techniques. In order to utilize some of the statistical tests we will be using, certain assumptions with respect to probabilistic distributions of the dependent variable must be met. One is that the error term in the regression is independent of all the independent variables in the equation. If we were to regress the effectiveness measure directly, the quantities A and D would appear as both parts of the dependent variable and as independent variables--violating the assumption that the error term was independent of all the independent variables.

Examining Equation (1a), we see that what we are going to attempt to do is to fit a surface in 2+K+J dimensional space. The methods used to explore this surface are referred to in the statistical literature as response surface methodology. Since the mathematical form of f in Equation (1a) is not known, Myers [5] suggests that a good approximation of f may be a first- (or higher-) order polynomial in the variables included in Equation (1a).

The inputs to the tank model were then examined to determine what quality variables might be varied. Table 2-1 lists 30 variables that could be varied as input to the tank exchange model and that were felt to fall into the category of quantity variables or quality variables. The first step was to examine these 30 variables in a first-order polynomial--i.e.,

$$\dot{A} = B_0 + B_1X_1 + B_2X_2 + \ldots, B_{30}X_{30} , \qquad (2)$$

where

X_i = the i^{th} quality or quantity variable; and

B_i = the coefficient associated with the i^{th} variable.

Equation (2) was fit by least-squares regression. At what value to set each of the 30 X_i for each observation was nontrivial. Just selecting two values for each one of the 30 variables and then simulating the output for all combinations of the 30 variables would require 2^{30} or 1×10^9 different battles to be simulated. A single simulation of one battle runs approximately 4.2 seconds of computer time. Thus, to simulate a complete 2^K factorial for 30 variables would take over 12×10^6 computer hours.[1] While there are sampling techniques to develop fractional factorials that reduce the number of observations down to manageable proportions, they are extremely complicated to set up for a problem involving as many as 30 variables. Both Cochran and Cox [2] and Hufschmidt [4] suggest that an alternative method to reduce the number of sample points required (while not as efficient from an estimating point of view as some systemized design of experiments sampling

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[1]The computer used for the simulation was a CD 6400.

Table 2-1. EFFECTIVENESS PARAMETERS INITIALLY INVESTIGATED (BLUE ON DEFENSE)

Variable Investigated	Coefficients Found Statistically Significant	
	Attack Losses	Defense Losses
Number of Attack Tanks	0.26	0.28
Number of Defense Tanks	0.73	0.31
Attack Vulnerability Measure	1.50	-0.84
Defense Vulnerability Measure	-0.38	0.28
Attack Error Measure	0.88	-0.44
Defense Error Measure	-1.26	0.61
Attack Tank Height	1.25	-- [a]
Defense Tank Height	--	--
Attack Load-Time Measure	0.38	--
Defense Load-Time Measure	-0.41	--
Attack Aim-Time Measure	--	--
Defense Aim-Time Measure	--	0.27
Probability Attack Classified K|H[b]	0.92	-0.53
Probability Defense Classified K|H[b]	--	--
Attack Probability of Misfire	--	--
Defense Probability of Misfire	--	--
Attack Time to Clear Misfire	--	--
Defense Time to Clear Misfire	--	--
Acceleration	--	--
Deceleration	-0.39	--
Attack Maximum Detection Range	-0.47	--
Defense Maximum Detection Range	--	--
Attack Detection-Range Multiplier	--	--
Defense Detection-Range Multiplier	0.35	-0.24
Attack Maximum Detection-Range Weapon Sign	--	0.22
Defense Maximum Detection-Range Weapon Sign	--	--
Attack Probability Defense-Weapon Signature Detected	-1.29	0.68
Defense Probability Defense-Weapon Signature Detected	--	--
Attack Probability Target Detected, given Weapon Signature Detected	-1.16	0.83
Defense Probability Target Detected, given Weapon Signature Detected	--	-0.23
Constant Found in Regression Equation	0.31	1.81
Coefficient of Determination (R^2)	0.67	0.79
Residual MSS (Mean Sum-of-Squares Error)/ Pure Error SS (Sum of Squares)	4.48[c]	3.70[c]

a. Everywhere it is not shown, the calculated t statistic < 2.
b. Killed, given a hit.
c. Has an F distribution.

plans) would be to select the numerical values for all the independent variables randomly for each observation. This is what we did. For each observation, the value for each independent variable was selected randomly from a uniform distribution that (with the exception of the quantity variables) had a range of 0.5-1.5 times each variable's nominal value.[1] The range of the quantity variable was 1-5 for the defense and 1-10 for the assault. A total of 396 different values were selected for each variable. Ten battles were fought for each set of input, so that a total of 3,960 battles were simulated. The average number of assault tanks destroyed for each set of inputs (10 battles) was then used as the dependent variable in a stepwise-regression routine. The same was done for the number of defensive tanks destroyed. Table 2-1 (above) presents the results of the regressions. All the numerical values shown in this table are the regression coefficients that were found to be statistically significant (i.e., calculated t values exceed 2). The fact that a variable's coefficient had a t value less than 2 does not necessarily mean that the variable might not be significant. The variable could be significant as a cross-product term between two or more variables.

Because we simulated each battle 10 times while holding all of the inputs to the model constant, we are able to estimate the pure-error mean square. This is the variation that is inherent in the process and that would be observed even if the equation were a perfect fit. This pure-error mean square can be compared with the residual mean square of the regression equation to test the equation for lack of fit. The ratio of the pure-error mean square and residual mean square of the regression has an F distribution, and the hypothesis that there is a significant lack of fit can be tested. Using an F table, we found that the critical F at the 0.05 level is approximately

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[1]The nominal value was taken as the existing tank's current value.

1.2. This compares with the calculated F of 4.48 and 3.70 for the $\hat{\dot{A}}$ and $\hat{\dot{D}}$ regression equations, respectively. Therefore, we reject the hypothesis and conclude that there is a significant lack of fit when using a first-order polynomial. The formula for the number of coefficients that must be considered for a higher-degree polynomial is $\binom{K+d}{d}$, where K is the number of independent variables and d is the degree of the polynomial. To go to a second-degree polynomial with 30 variables means that we would have to consider 466 cross-product terms. This number of cross-product terms was considered to be unmanageable.

The next step was to investigate a much smaller number of variables in detail. A total of six variables were chosen for detailed investigation. These six variables were chosen on the basis that they had significant and relatively large coefficients in Table 2-1 and that they were variables that might be possible to relate to specific design specifications.

Because the equations that are derived differ depending on which side is on assault, it is now convenient to switch symbol notation from assault and defense to Red and Blue. The six variables investigated in detail were--

B = number of Blue tanks committed to an engagement;

R = number of Red tanks committed to an engagement;

V_B = Blue vulnerability measure;

L_B = Blue lethality measure;

E_B = Blue error measure; and

E_R = Red error measure.

V_B Blue vulnerability measure is defined in the following way:

$$P_{\dot{B}_1} = \begin{cases} P_{\dot{B}_{1_o}} V_B, & \text{for } 0 < V_B < 1 \text{ ;} \\ V_B - 1 + 2P_{\dot{B}_{1_o}} - V_B P_{\dot{B}_{1_o}} \text{ ,} & \text{for } 1 \leq V_B \leq 2 \text{ ,} \end{cases}$$

subject to the constraint

$$P_{\dot{B}_i} + P_{B\dot{M}_i} + P_{B\dot{F}_i} \leq 1$$

where

$P_{\dot{B}_{i_o}}$ = the probability that the existing Blue tank will be destroyed, given a hit in the i^{th} target area by the existing Red tank weapon. Vulnerability tapes of actual tanks are used. Each tank is divided into one-foot squares. Each one-foot square has a probability that the tank will be destroyed, given that it is hit in this square.

$P_{\dot{B}_i}$ = the probability that the modified Blue tank is destroyed, given that Blue is hit in the i^{th} target square.

$P_{B\dot{M}_i}$ = the probability of a Blue mobility failure, given a hit in the i^{th} target square. Note that (because of the way the model is formulated) total destruction, mobility failure, and firepower failure are mutually exclusive events.

$P_{B\dot{F}_i}$ = the probability of a Blue firepower failure, given a hit on the i^{th} target square.

Note that when $V_B = 0$, $P_{\dot{B}_i} = 0$ and the tank is completely invulnerable; when $V_B = 1$, $P_{\dot{B}_i} = P_{\dot{B}_{i_o}}$ and the modified tank is the existing tank; when $V_B = 2$, $P_{\dot{B}_i} = 1$ and the tank is completely vulnerable. Similarly, L_B Blue lethality measure is defined as

$$P_{\dot{R}_i} = \begin{cases} P_{\dot{R}_{i_o}} L_B, & \text{for } 0 < L_B < 1 \,; \\ L_B - 1 + 2P_{\dot{R}_{i_o}} - L_B P_{\dot{R}_{i_o}} \,, & \text{for } 1 \leq L_B \leq 2 \,, \end{cases}$$

subject to the constraint

$$P_{\dot{R}_i} + P_{R\dot{M}_i} + P_{R\dot{F}_i} \leq 1$$

where

$P_{\dot{R}_i}$ = the modified probability that the Red tank is destroyed, given a hit on the i^{th} target square.

$P_{\dot{R}_{i_o}}$ = the probability that the Red tank is destroyed, given a hit on the i^{th} target square by the Blue tank weapon.

$P_{R\dot{M}_i}$ = the probability of a Red mobility failure, given a hit on the i^{th} target square.

$P_{R\dot{F}_i}$ = the probability of a Red fire-power failure given a hit on the i^{th} target square.

E_B Blue error measure is defined as

$$TIE_B = TIE_{B_o} E_B ,$$

TIE_B = the modified total impact error of Blue firing on Red. This includes all aiming, gun-peculiar, and shell-peculiar errors.

TIE_{B_o} = the existing Blue tanks total impact error.

Similarly, for $TIE_R + TIE_{R_o} E_R$,

TIE_R = the modified total impact error of Red firing on Blue. This includes all aiming, gun-peculiar, and shell-peculiar errors.

TIE_{R_o} = the existing Red tanks total impact error.

The problems of deriving a response surface with six variables are several orders of magnitude simpler than deriving a response surface with 30 variables. With six variables, only 27 coefficients must be considered in a second-order polynomial. In order to derive a second-order polynomial, it is necessary to sample each variable at at-least three levels. A complete 3^k factorial for six variables requires only 729 different observations. In order to preclude having to extrapolate the derived equations, two of the three levels selected for each of the variables were their extreme points. The third point was the midpoint of.their range. Further to supplement the full factorial, especially in the center region, an additional 500 observations were generated by random selection of the input variables. For each set of inputs, each engagement was simulated only once--for a total of 1,229 individual tank engagements. A stepwise regression routine was used to fit the data from the 1,229 battles. Equations (2a), (2b), and (2c), below, are the results of the derived regression equation when Red is on attack. The parenthesized numbers are t statistics for each coefficient.

$$\dot{R} = \underset{(10.2)}{1.49} - \underset{(4.3)}{0.43}V_B E_R - \underset{(4.4)}{0.60}L_B^2 + \underset{(2.5)}{0.16}L_B B + \underset{(11.6)}{0.37}L_B R$$
$$- \underset{(4.1)}{0.06}B^2 + \underset{(19.0)}{0.15}RB + \underset{(1.8)}{0.09}BE_R - \underset{(12.7)}{0.05}R^2 + \underset{(5.5)}{0.15}RE_R$$
$$- \underset{(10.6)}{0.18}RE_B \;. \qquad \text{(2a)}$$

$$\dot{B} = \underset{(0.42)}{0.048} + \underset{(3.7)}{0.36}B + \underset{(6.2)}{0.13}V_B B - \underset{(11.4)}{0.24}L_B B - \underset{(6.2)}{0.09}B^2$$
$$+ \underset{(36.3)}{0.09}BR - \underset{(7.0)}{0.15}BE_R + \underset{(8.8)}{0.18}BE_B \;. \qquad \text{(2b)}$$

$$(\dot{R} - \dot{B}) = \underset{(0.75)}{0.25} + \underset{(2.4)}{1.05}E_B + \underset{(2.4)}{0.60}V_B^2 - \underset{(2.7)}{0.24}V_B B - \underset{(2.0)}{0.09}V_B R$$
$$- \underset{(2.4)}{0.70}V_B E_R - \underset{(3.5)}{0.70}L_B^2 + \underset{(5.1)}{0.45}L_B B + \underset{(8.0)}{0.37}L_B R$$
$$+ \underset{(5.5)}{0.06}BR + \underset{(3.6)}{0.29}BE_R - \underset{(4.0)}{0.37}BE_B - \underset{(5.6)}{0.03}R^2$$
$$+ \underset{(3.8)}{0.16}RE_R - \underset{(5.2)}{0.25}RE_B \;. \qquad \text{(2c)}$$

As can be seen from Equation (2), one of the problems with polynomials is that they are complex and difficult to interpret intuitively, particularly since a number of the terms enter into the equations with both positive and negative signs as different cross-product terms. The interpretation of the equations will be left to the next section (since the purpose of this section is to explain the statistical methodology).

The coefficients of determination, which indicate what fraction of the total variance is explained by the regression equations for Equations (2a-c), are 0.62, 0.61, and 0.36, respectively. The reasons that the coefficients of determination are so low are that each data point represented only one battle and that the natural variation inherent in the data is high. In order to estimate the pure-error term that could be used to test the hypothesis that Equations (2a-c) adequately fit the data--and used to generate an independent set of data on which to test Equations (2a-c)--an additional 200 sets of input data were generated by randomly selecting values of each of the six variables. These 200 input sets were then used as input to the tank-exchange model. Each set of inputs was used to simulate 10 tank battles. Thus, a total of 2,000 additional battles were simulated. Since for each fixed set of inputs 10 battles were simulated, the pure-error term can be estimated. The ratios of the residual mean-square error to the pure mean-square error for Equations (2a-c) are 1.03, 0.999, and 0.9677, respectively. This ratio has an F distribution with 1,200; 1,800 degrees of freedom as entering arguments to the table. While tables with this number of degrees of freedom are not readily available, the critical F at the 0.05 level for ∞;1,000 degrees of freedom is 1.11. Thus, we would reject the hypothesis that there was a significant lack of fit. Because of the large number of degrees of freedom, this is a fairly precise test. Further, when Equations (2a-c) were used to predict the outcome of the second set of data, the 200-battle case, the equations explained 88 percent, 87 percent, and 69 percent of the total output variance, respectively. The reason that Equations (2a-c) explain a greater percent of the variation when applied to the second set of data (the 200-battle case) than they do when applied to the first set of data (from which the equations were derived) is that the second set of data represents the average of the results of 10 battles (where the inputs were held constant), whereas the data in the first data set were the results of individual battles. Theoretically, the inherent variance in the second set of data should be $1/\sqrt{10}$ times the inherent variance in the first set of data; and, thus, it should be expected that Equations (2a-c) should produce higher coefficients of determination when applied to the first set of data than when applied to the second set of data.

Once Equations (2a-b) have been derived, they are easily combined algebraically and manipulated into Equation (3), which has the preferred effectiveness measure as the dependent variable. Note that Equations (2a-b) can easily be combined and manipulated so that any other of the effectiveness measures discussed is the dependent variable.

$$\begin{aligned}\frac{E(\hat{\dot{R}})}{R} - \frac{E(\hat{\dot{B}})}{B} = &- \frac{0.06B^2}{R} + 0.24B + (0.09E_R + 0.16L_B)\frac{B}{R} \\ &- \frac{0.048}{B} - 0.14R - (0.60L_B^2 + 0.43V_BE_R - 1.49)\frac{1}{R} \\ &+ 0.61L_B - 0.13V_B - 0.36E_B + 0.30E_R - 0.36 \; . \qquad (3)\end{aligned}$$

Equation (3) can be algebraically manipulated into Equation (4), a cubic in B, which for a given level of effectiveness can be used to make quantity-quality trade-offs.

$$\begin{aligned}&\left[-\frac{0.06}{R}\right]B^3 + \left[\left(\frac{0.09E_R + 0.16L_B}{R}\right) + 0.24\right]B^2 \\ &\quad - \left[0.14R + \left(\frac{0.60L_B^2 + 0.43V_BE_R - 1.49}{R}\right)\right. \\ &\quad \left. - 0.61L_B + 0.13V_B + 0.36E_B - 0.30E_R + 0.36\right]B \\ &\quad - \left[0.048\right] = 0 \; . \qquad (4)\end{aligned}$$

A similar procedure of generating 1,229 data points was used to derive a set of equations when Blue was on attack. This resulted in Equations (5a-c), displayed below. These equations were then algebraically combined and manipulated so that quantity-quality trade-offs could be made.

$$\begin{aligned}\dot{B} = {} & \underset{(1.6)}{-0.77} + \underset{(2.1)}{1.68}L_B + \underset{(5.3)}{0.84}R + \underset{(4.5)}{0.35}B + \underset{(1.7)}{0.08}L_B R \\ & - \underset{(6.0)}{0.14}L_B B - \underset{(2.7)}{0.94}V_B^2 + \underset{(8.1)}{0.24}V_B B - \underset{(3.5)}{0.73}V_B E_B \\ & + \underset{(2.3)}{0.59}V_B E_R - \underset{(10.6)}{0.22}R^2 + \underset{(26.1)}{0.19}RB + \underset{(2.0)}{0.13}RE_R \\ & - \underset{(10.2)}{0.04}B^2 + \underset{(5.8)}{0.16}BE_B - \underset{(5.1)}{0.15}BE_R + \underset{(1.8)}{0.36}E_B E_R \\ & - \underset{(3.2)}{0.59}E_R^2 \, . \end{aligned} \tag{5a}$$

$$\begin{aligned}\dot{R} = {} & \underset{(2.8)}{-0.57} + \underset{(3.7)}{0.41}R + \underset{(5.0)}{0.19}B - \underset{(2.6)}{0.20}L_B^2 + \underset{(3.5)}{0.13}L_B R \\ & + \underset{(4.1)}{0.07}L_B B + \underset{(2.5)}{0.26}B^2 - \underset{(6.0)}{0.23}V_B R - \underset{(6.6)}{0.12}V_B B \\ & + \underset{(4.1)}{0.46}V_B E_B - \underset{(2.4)}{0.28}V_B E_R - \underset{(4.7)}{0.06}R^2 + \underset{(6.2)}{0.03}RB \\ & - \underset{(4.7)}{0.16}RE_B + \underset{(3.2)}{0.11}RE_R - \underset{(5.0)}{0.08}BE_B + \underset{(2.9)}{0.05}BE_R \, . \end{aligned} \tag{5b}$$

$$\begin{aligned}(\dot{B} - \dot{R}) = {} & \underset{(2.5)}{0.85} + \underset{(2.1)}{0.47}R - \underset{(8.2)}{0.17}L_B B + \underset{(2.3)}{0.18}V_B R + \underset{(11.8)}{0.39}V_B B \\ & - \underset{(5.0)}{1.07}V_B E_B - \underset{(6.0)}{0.17}R^2 + \underset{(17.3)}{0.17}BR + \underset{(2.3)}{0.17}RE_B \\ & - \underset{(9.5)}{0.04}B^2 + \underset{(8.3)}{0.27}BE_B - \underset{(8.4)}{0.18}BE_R \, . \end{aligned} \tag{5c}$$

C. INTERPRETATION OF RESULTS

Having derived predictive equations for $\dot{B},\dot{R}$, we are in a position to manipulate them analytically so that quantity-versus-quality trade-offs can be made. In this section, quantity-quality trade-offs using the difference in loss rates $(E(\hat{A})/A - E(\hat{D})/D)$ as the effectiveness measure will be made. Similar curves that will be presented here are presented in Appendix 2-A for other effectiveness measures. Before exploring quantity-quality trade-offs directly, let us examine the partial derivatives of the effectiveness measure with respect to each of the six independent variables, by varying each one individually and seeing whether the results correspond with intuition. Remember that the higher the effectiveness measure is, the better it is for the defense (e.g., when the effectiveness measure is 1, the assault is completely destroyed and the defense suffers no casualties; when it is -1, the defense is totally destroyed and the assault suffers no casualties).

Figure 2-6 presents the effectiveness measure $(E(\hat{A})/A - E(\hat{D})/D)$ as a function of the number of Red attack tanks for a family of curves that represent the number of Blue tanks on defense.

Figure 2-6 shows that just by massing more tanks, without making any quality changes, the effectiveness of one side improves dramatically. For example, as Blue adds defensive tanks against an assault of five Red tanks, the loss-rate difference changes from -0.32 to 0.62 for one and five defensive tanks, respectively. Similarly, as Red attacks with

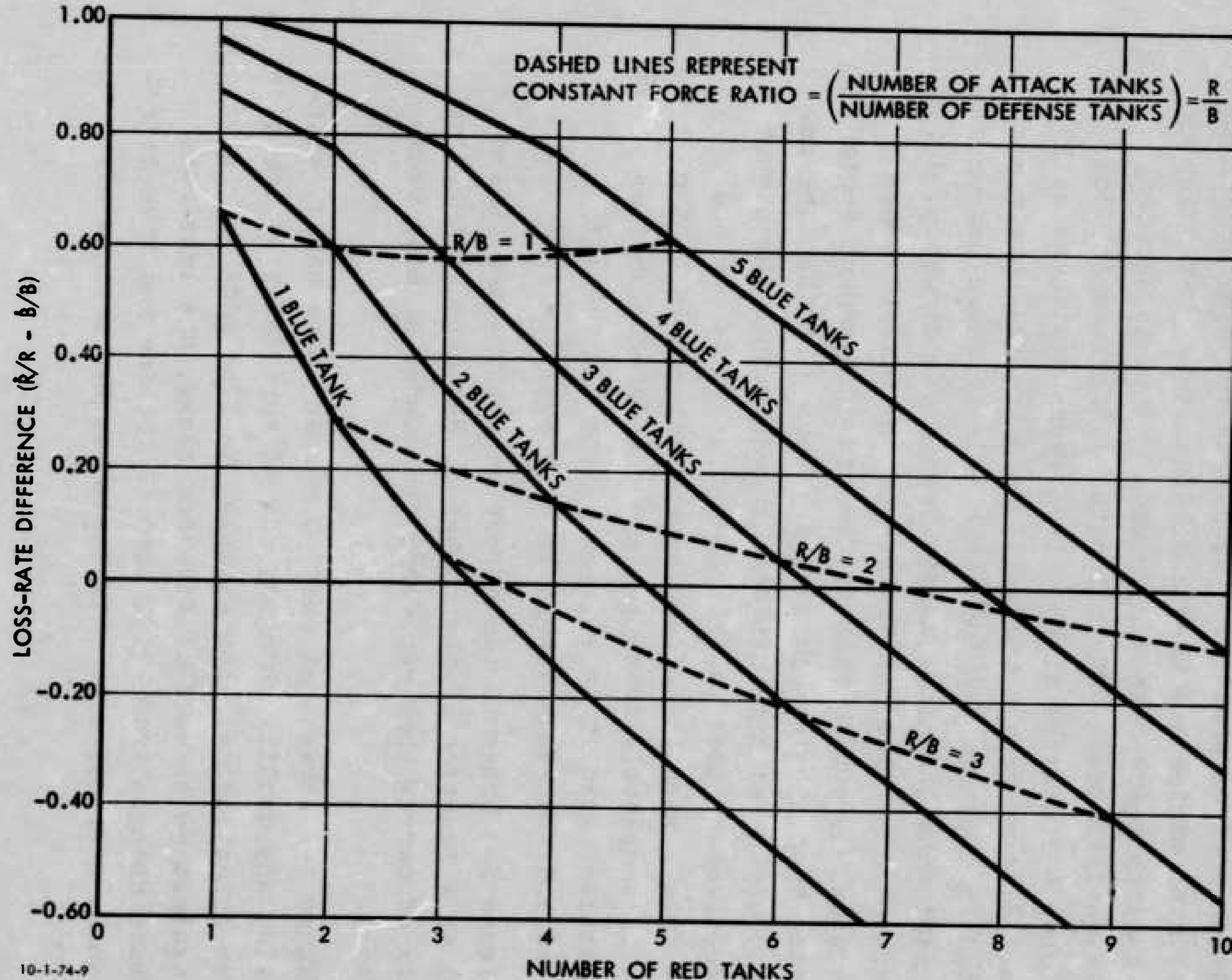

Figure 2-6. LOSS-RATE DIFFERENCE AS A FUNCTION OF NUMBER OF RED ATTACK TANKS

more tanks against a fixed number of defense tanks, the loss-rate difference shifts in Red's favor--going from 1 to -0.1 as Red assaults five defense tanks with 1 and 10 tanks, respectively. The dotted lines in Figure 2-6 are lines of constant force ratio. As can be seen by the dashed lines, the difference in expected loss rates decreases for a constant force ratio. That is, as the absolute size of the battle gets bigger (even though the force ratio on the two sides remains the same), the outcome of the battle shifts in favor of the attack. The line where the force ratio is 1 suggests that there may even be an optimum-size battle for the attack, given a force ratio, since the line for force ratio equal to 1 is U-shaped. The other lines may be U-shaped too, but the minimum was not reached for the maximum-size battles depicted in Figure 2-6. Figure 2-7 is identical to Figure 2-6, except that Blue is on attack and Red is on defense. Although the constant-force lines are flatter in Figure 2-7 than in Figure 2-6, they too show the same tendency: the effectiveness measure shifts in favor of the assault as the absolute size of the battle increases, even though the ratio of the forces on each side is constant. The force-ratio line of 1 in Figure 2-7 also is U-shaped. Since the effectiveness shifts in favor of the defense as the absolute size of the battle increases when either Red or Blue is on defense, the phenomenon does not appear to be tank-related. We do not at this time have any explanation for this phenomenon. It certainly warrants further examination. Not too surprising, as will be seen in Appendix 2-A, this same phenomenon occurs for the other effectiveness measures.

Returning to comparing Figures 2-6 and 2-7, it can be seen that in both cases when the force ratios are equal the effectiveness measure is highly positive (in the defense's favor). This conforms to intuition, which says that the defense, being in defilade, should have a large

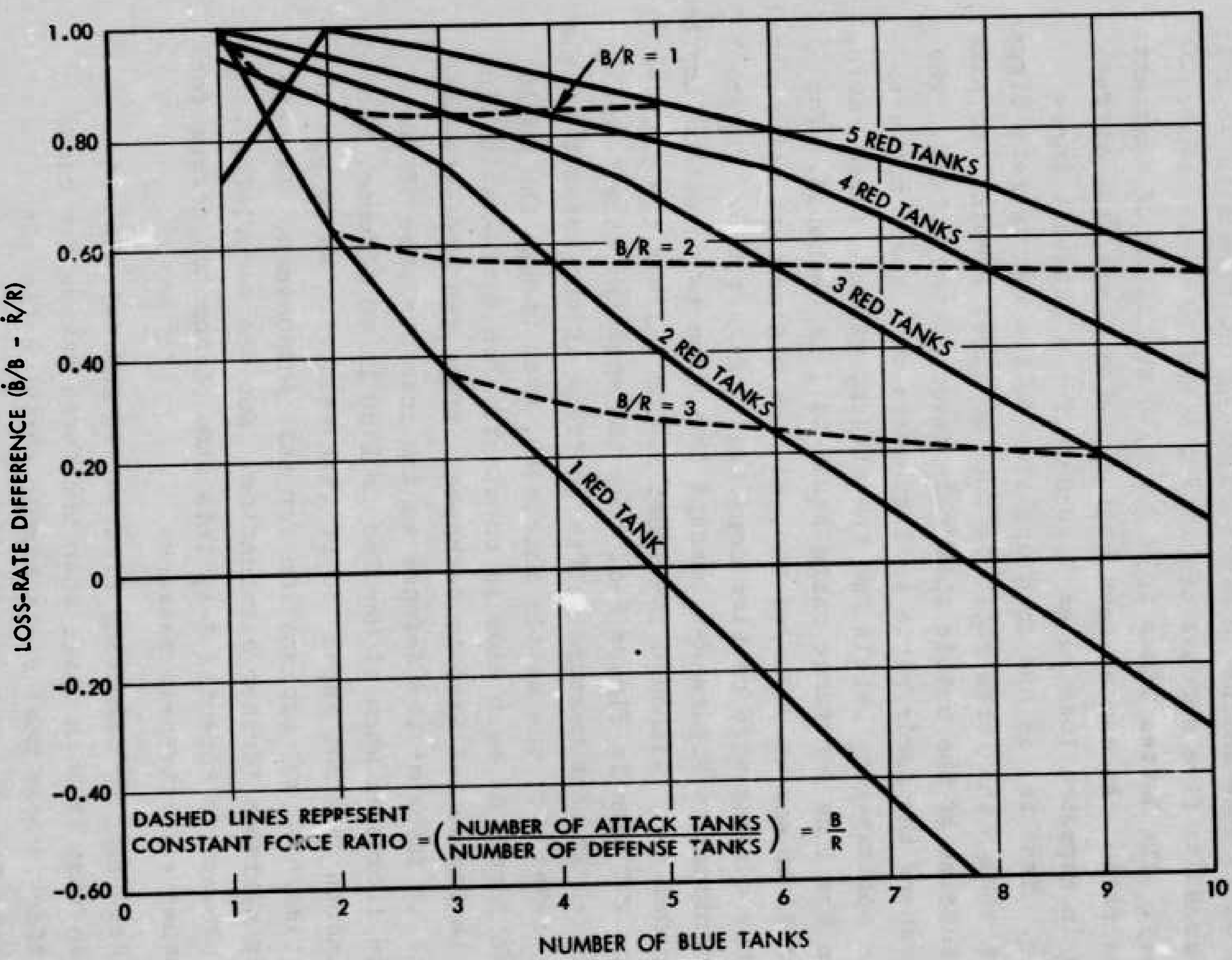

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Figure 2-7. LOSS-RATE DIFFERENCE AS A FUNCTION OF NUMBER OF BLUE ATTACK TANKS advantage over the attack. Note, too, that all of the lines in Figure 2-7 are above the corresponding line in Figure 2-6--suggesting that, on the average, the existing Red tank is better than the existing Blue tank. This conforms to conventional wisdom. In Figure 2-7, there appears to be an aberration for the effectiveness when one tank assaults five defense tanks. This is at the edge of the data used to derive the response surface and is probably due to some lack of fit in the equations. Since the one attack tank versus five defense tanks is not a situation likely to be of interest, it does not appear to be too serious an error.

The next series of curves shows effectiveness as a function of one of the quality parameters for a family of curves that represent the number of defensive tanks. In all cases, the number of attack tanks equals twice the number of defense tanks; and all other quality parameters except the one explicitly shown are set at their existing values.

Figure 2-8 is the effectiveness measure as a function of the defense vulnerability measure for Blue on defense. The curve matches intuition and shows that the effectiveness measure moves in favor of the assault as the defense becomes more vulnerable.

Figure 2-9 is the effectiveness measure as a function of the defense lethality measure for Blue on defense. The curves for the most part match intuition and show that the effectiveness measure shifts in the defense's favor as the defense lethality is increased. The one exception is the curve corresponding to the situation in which two attack tanks attack one defense tank. While this curve behaves as would be expected (with the effectiveness measure increasing as the defense lethality is increased up to 1.25), it then degenerates--turning down as the defense lethality increases beyond 1.25 and crosses the other curves. The case of one defense tank is at the edge of the data used to derive the response surface; thus, the

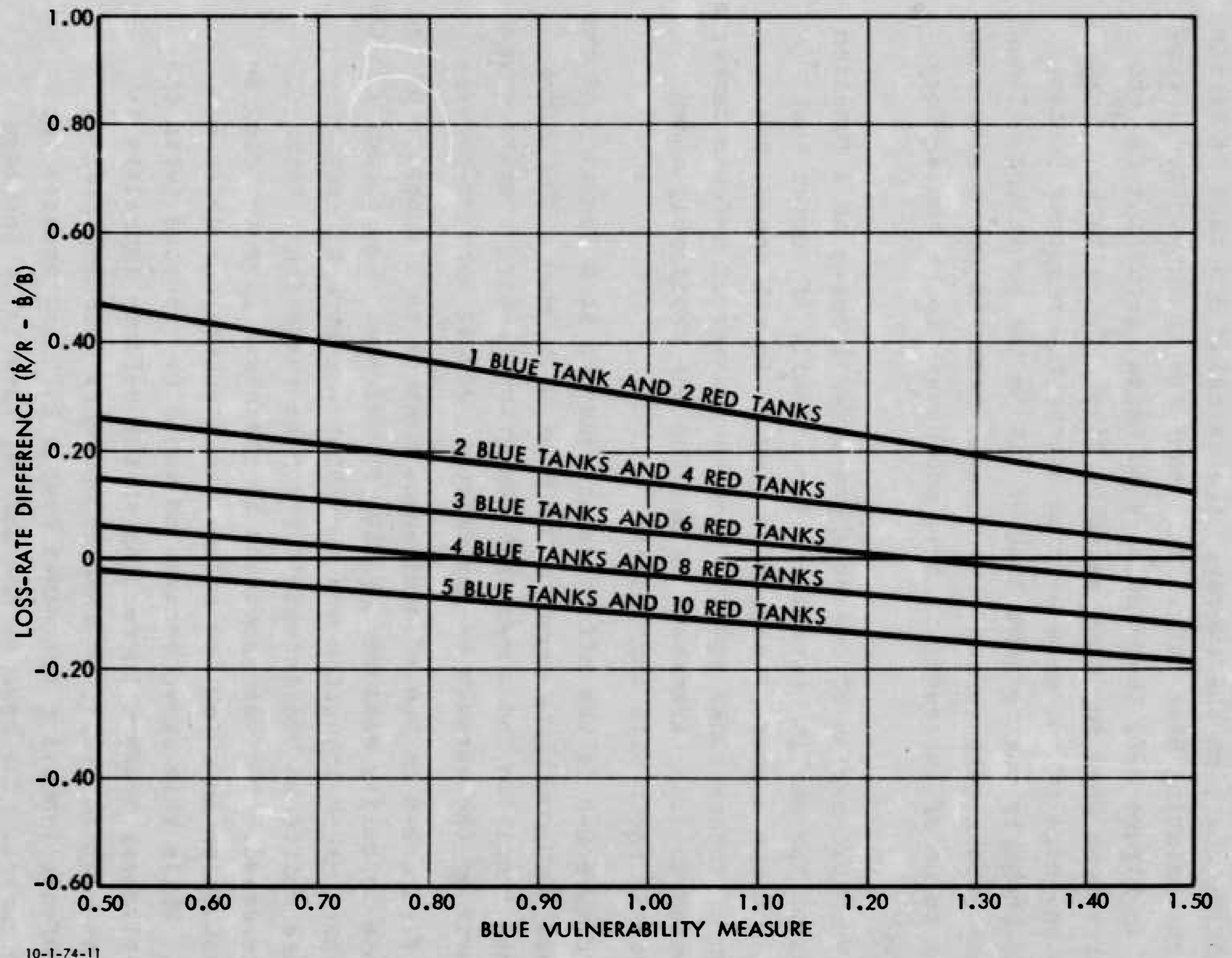

Figure 2-8. LOSS-RATE DIFFERENCE AS A FUNCTION OF BLUE VULNERABILITY MEASURE (BLUE ON DEFENSE)

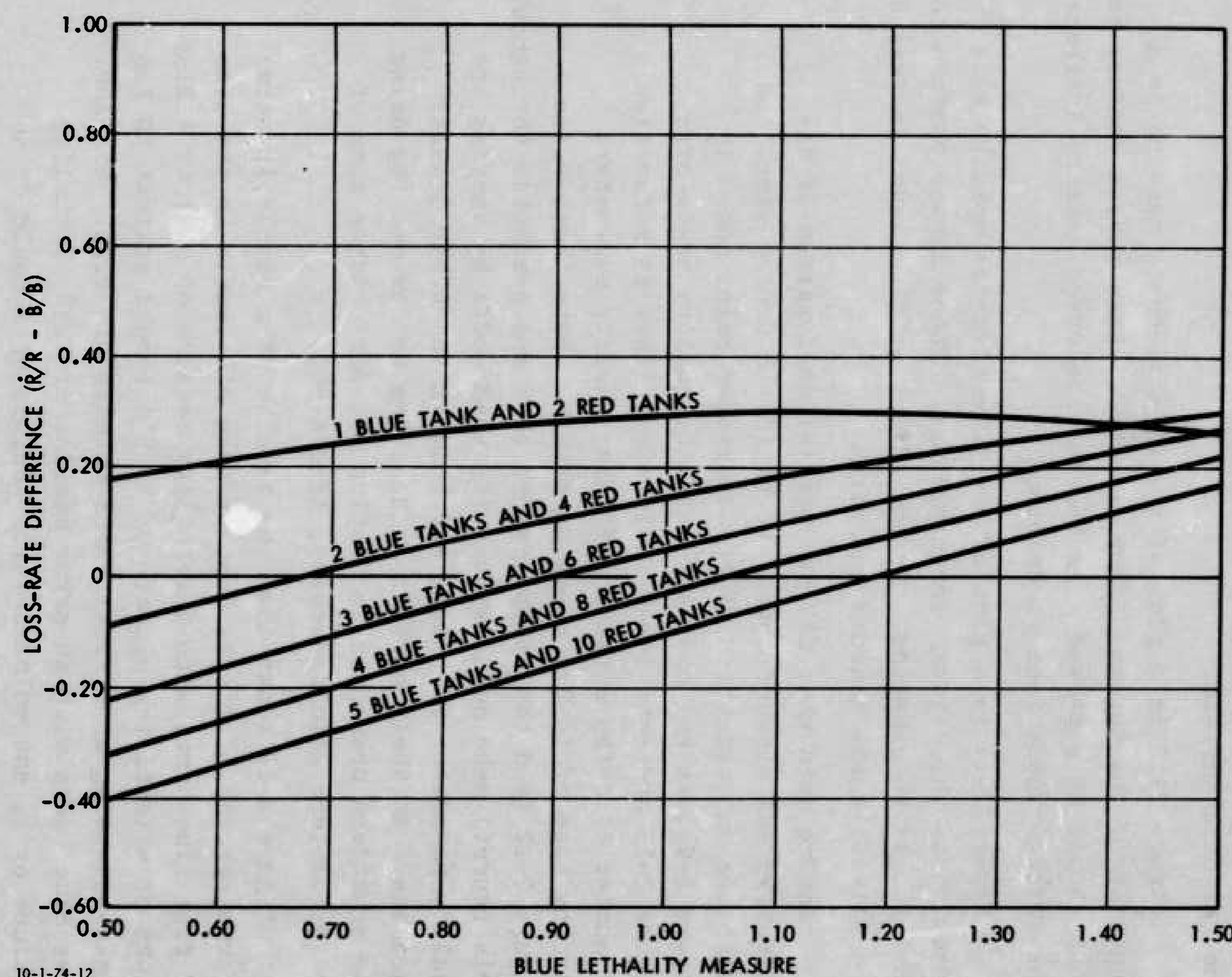

Figure 2-9. LOSS-RATE DIFFERENCE AS A FUNCTION OF BLUE LETHALITY MEASURE (BLUE ON DEFENSE)

seeming discrepancy from what would normally be expected could be attributed to lack of fit of the equations at the edge of the response surface.

Figure 2-10 is a plot of the effectiveness measure as a function of the defense error measure. These curves conform to what should be expected. As the defense error measure increases, the effectiveness measure decreases.

Figure 2-11 is a plot of the effectiveness measure as a function of the attack error measure. These curves conform to what should be expected. As the attack error measure increases, the effectiveness measure increases.

Having seen that all the partial derivatives of the effectiveness measure appear to be in the proper direction (with the one exception of the crossover point noted in Figure 2-9), we can now make quantity-quality trade-offs. Figure 2-12 presents iso-effectiveness lines as a function of number of defense tanks and three quality parameters for the case of 10 Red tanks on attack. Note that, while Figure 2-12 (and the other figures that are presented throughout this report) make quantity-quality trade-offs by varying one quality parameter at a time and holding the other quality parameters at their nominal values, the reader can (by using the equations presented in Section B), vary two or more of the parameters simultaneously, if desired.

Figure 2-12 shows that the results of a battle in terms of the percent of forces lost on each side would be the same if five Blue tanks with a lethality measure of 0.77 or 4 Blue tanks of a lethality measure of 1.15 defended against 10 Red tanks--assuming an effectiveness measure of -0.25. Similarly, five Blue tanks with an error measure of 1.38, lethality measure of 1, and vulnerability measure of 1 could do as effective a job as four Blue tanks with an error measure of 0.76. As can be seen from Figure 2-12, the effectiveness

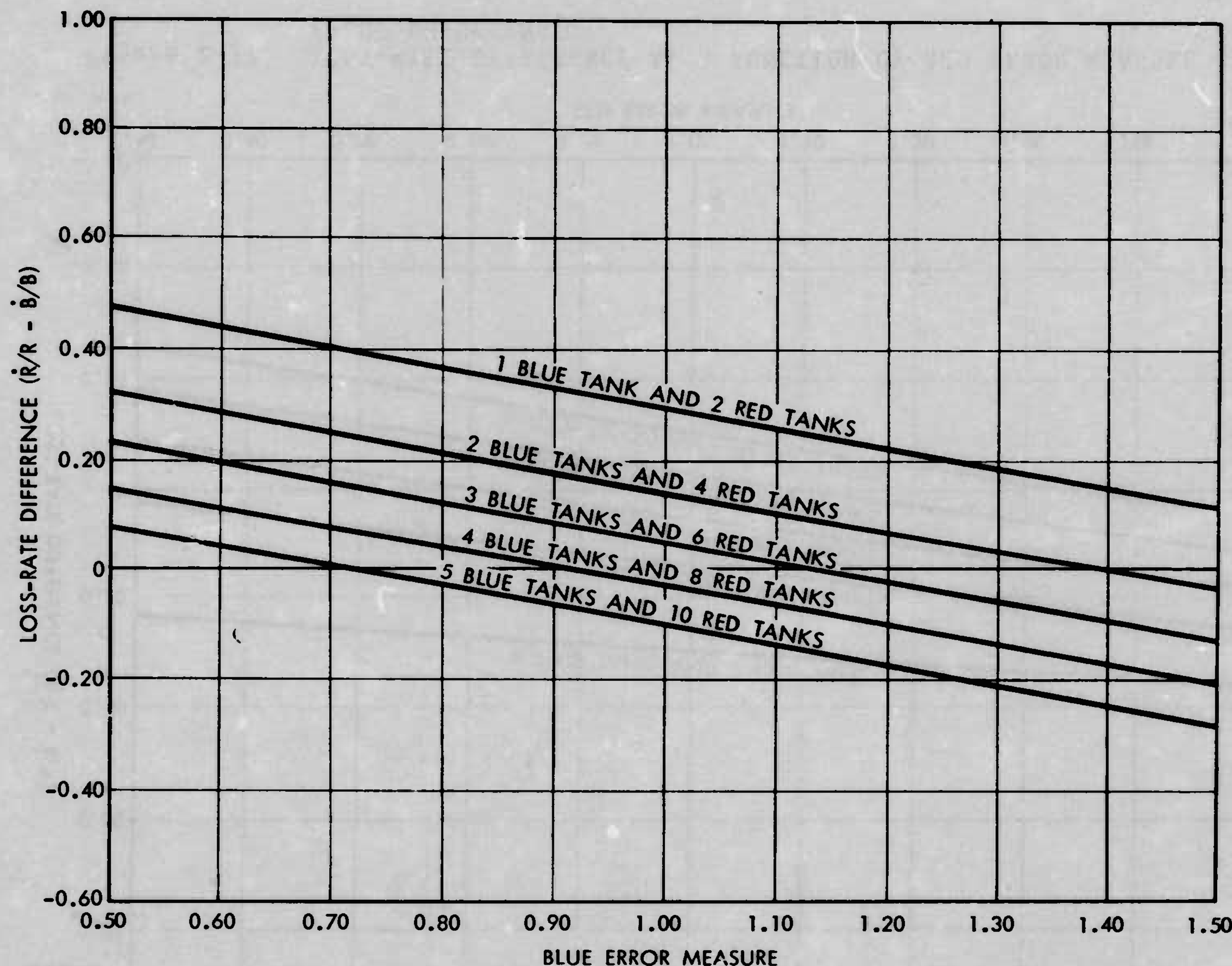

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Figure 2-10. LOSS-RATE DIFFERENCE AS A FUNCTION OF BLUE ERROR MEASURE (BLUE ON DEFENSE)

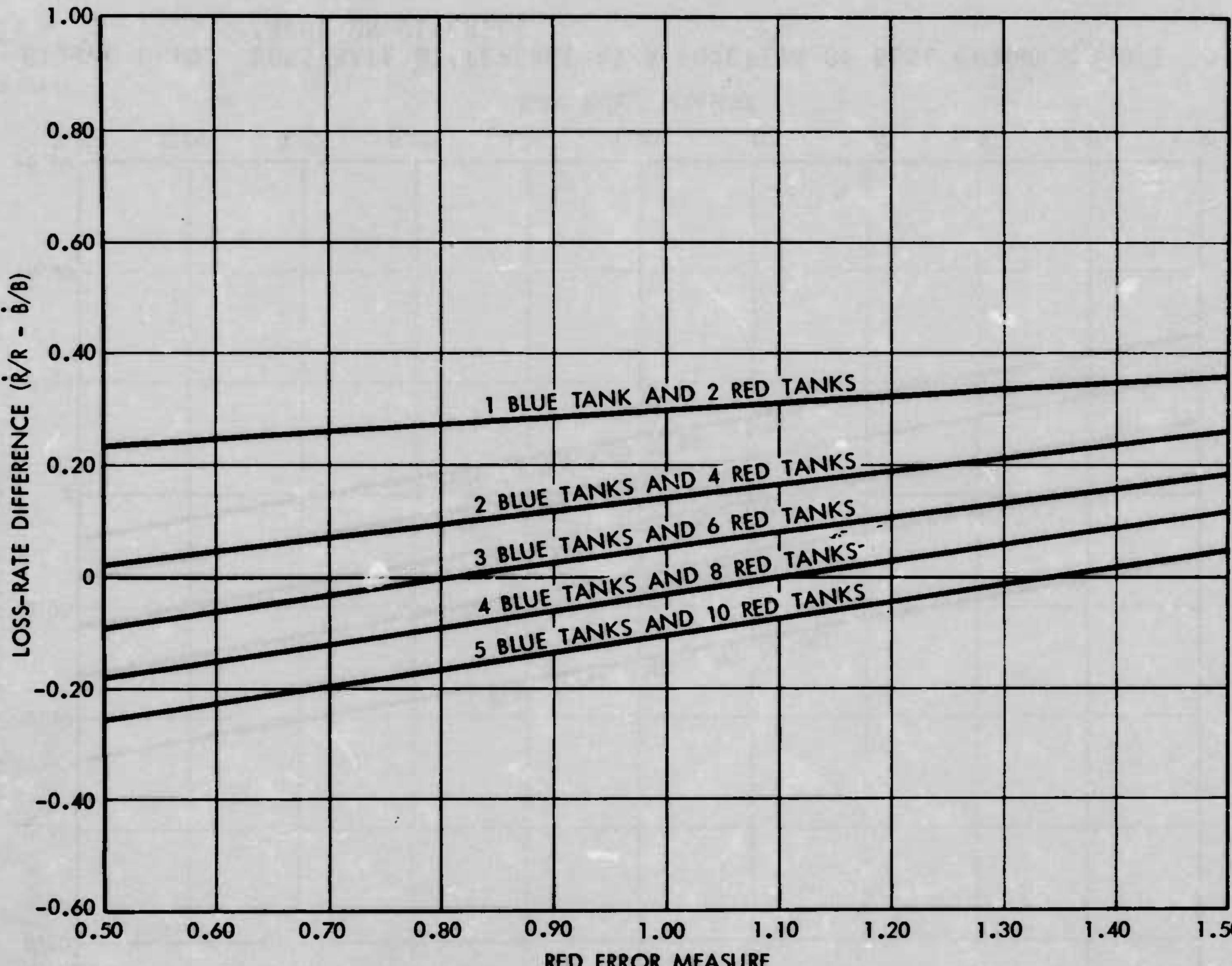

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Figure 2-11. LOSS-RATE DIFFERENCE AS A FUNCTION OF RED ERROR MEASURE (BLUE ON DEFENSE)

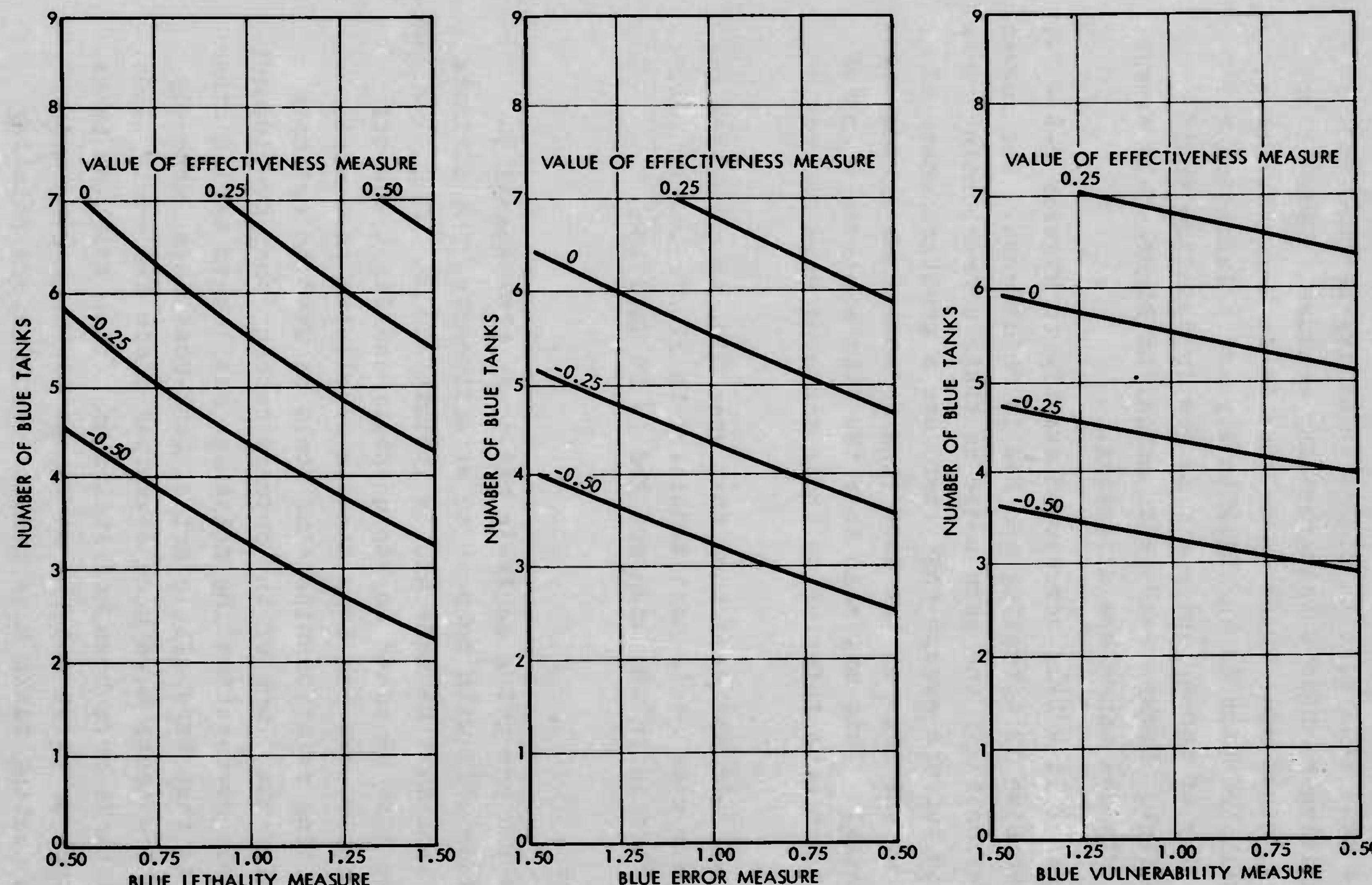

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Figure 2-12. EFFECTIVENESS MEASURE (LOSS-RATE DIFFERENCE) AS A FUNCTION OF NUMBER OF BLUE TANKS AND QUALITY PARAMETERS - 10 RED ATTACK TANKS measure is most sensitive to Blue's lethality measure and least sensitive to Blue's vulnerability measure. This is not surprising, since Blue being on defense is in defilade; thus, only a small fraction of the tank is exposed. Improving the vulnerability of those portions that are in defilade would have no effect. Figure 2-13 is identical to Figure 2-14 except that only six Red tanks are attacking.

Figures 2-14 - 2-19 correspond exactly to Figures 2-8 - 2-13 except that Blue is attacking and Red is on defense. The curves all have essentially the same shape as their corresponding curve for Blue on defense except that there are a greater number of anomalies at the edge of the data (one defense tank against two assault tanks). This may mean that the fits achieved when Red was on defense were inferior to those achieved when Red was on attack.

Figures 2-18 and 2-19 show that when Blue is on attack the effectiveness measure is most sensitive to Blue's vulnerability, since now Blue is in the open and Red is in defilade.

D. COSTS

While the preceding analysis has dealt with quantity-quality trade-offs with respect to effectiveness, the ultimate trade-off that must be made has to include costs. Thus, constant-cost curves must be added to the quantity-quality trade-off curves. Little work has been done in developing parametric cost-estimating relationships that could be used to estimate the cost of tanks. One of the primary reasons that development of parametric cost-estimating relations has lagged behind other equipment is that very little actual tank-cost data currently exist. No new tanks have been developed since the early 1960s. All recent tanks have been modifications of the existing tanks. The data that do exist and the work that has been done with respect to costing tanks have derived cost factors relating

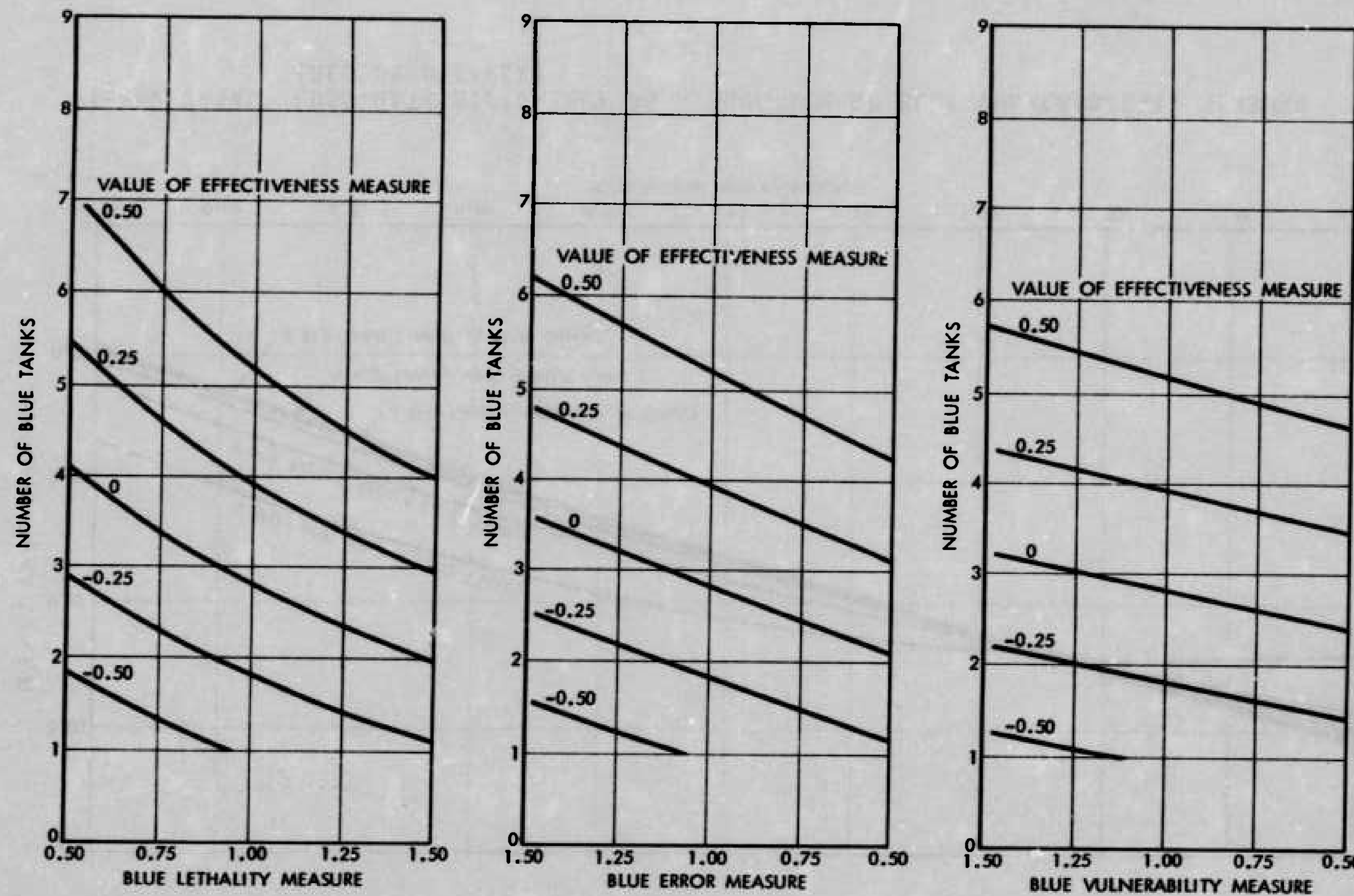

Figure 2-13. EFFECTIVENESS MEASURE (LOSS-RATE DIFFERENCE) AS A FUNCTION OF NUMBER OF BLUE TANKS AND QUALITY PARAMETERS - 6 RED ATTACK TANKS

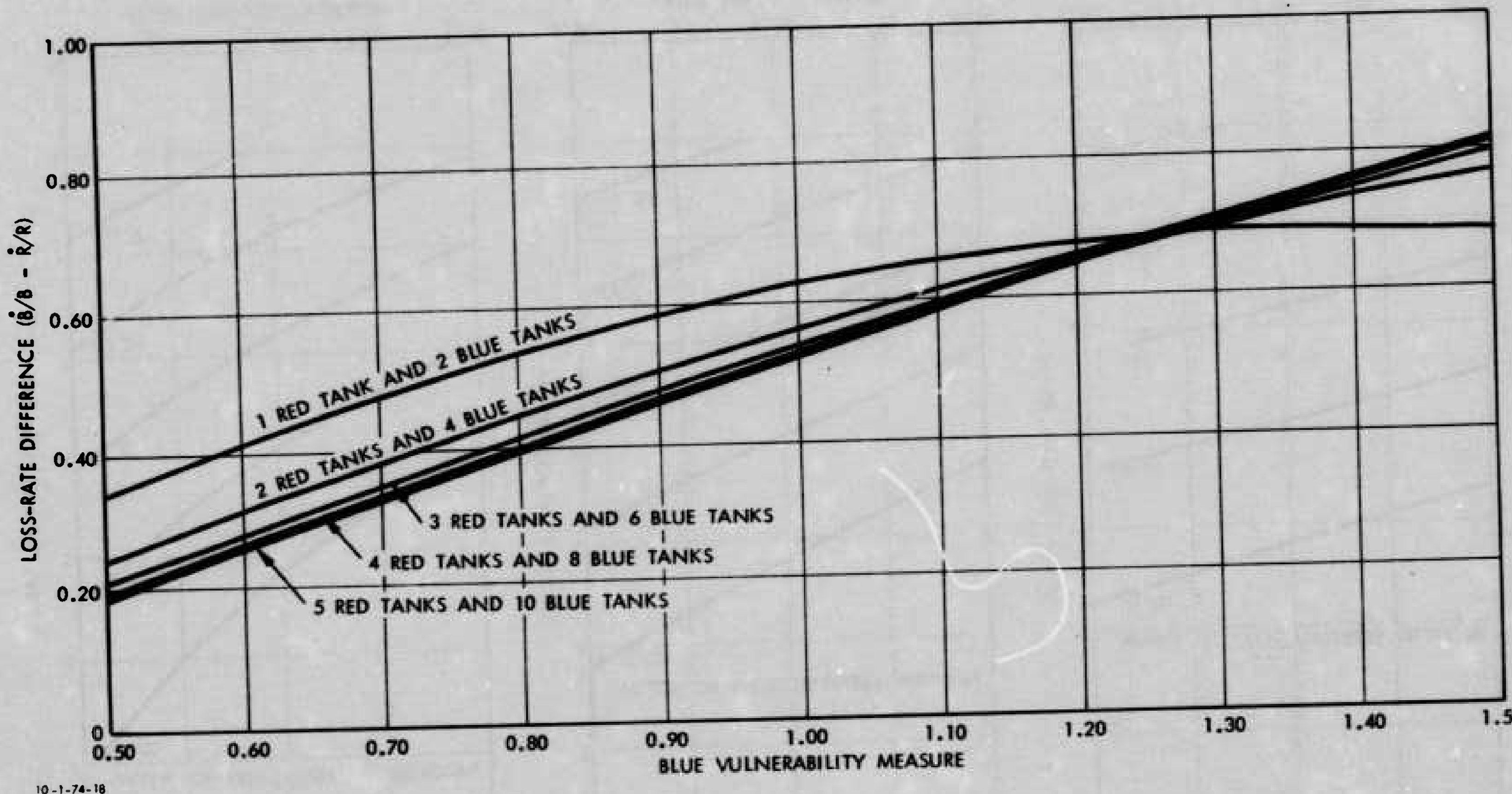

Figure 2-14. LOSS-RATE DIFFERENCE AS A FUNCTION OF BLUE VULNERABILITY MEASURE (RED ON DEFENSE)

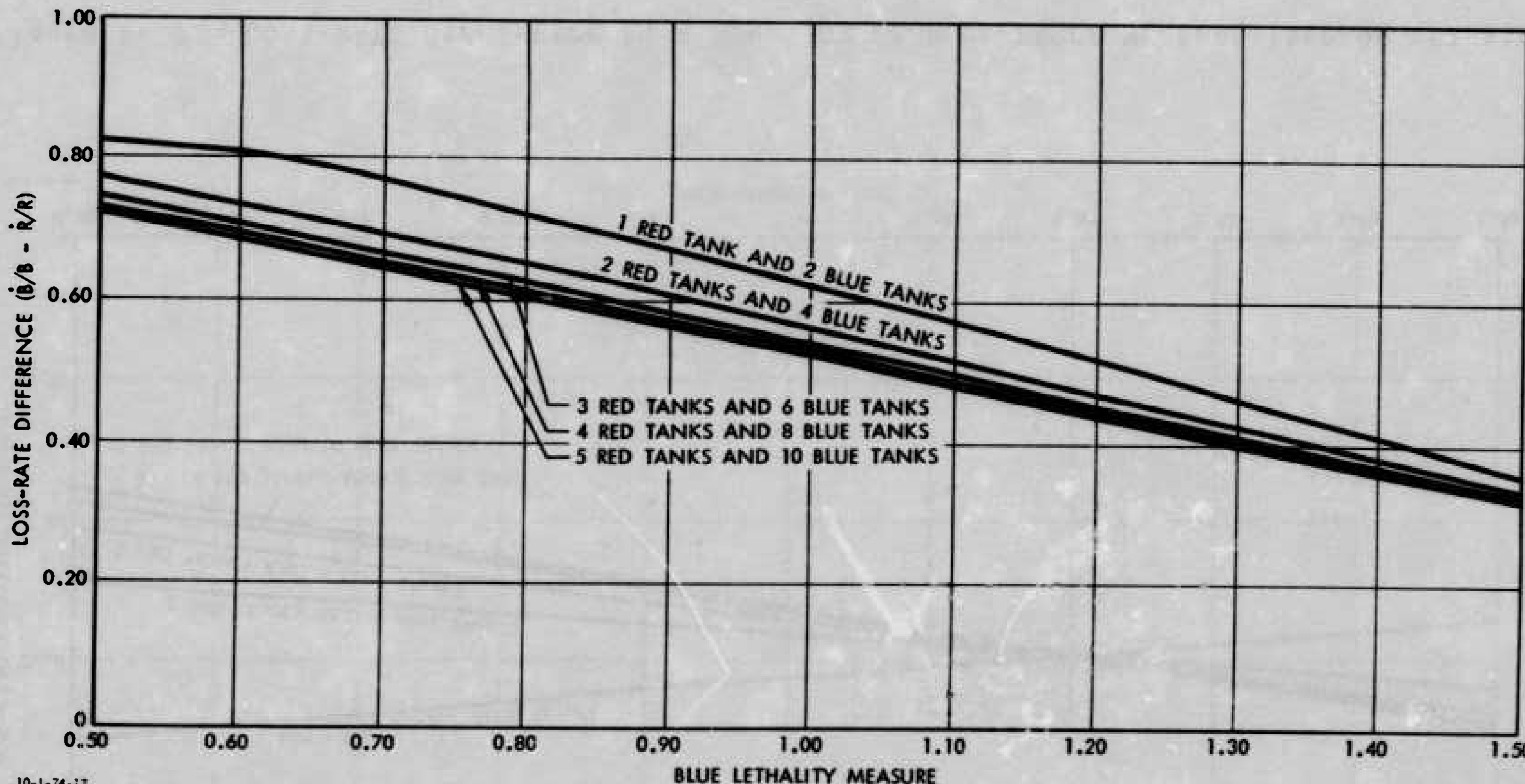

Figure 2-15. LOSS-RATE DIFFERENCE AS A FUNCTION OF BLUE LETHALITY MEASURE (RED ON DEFENSE)

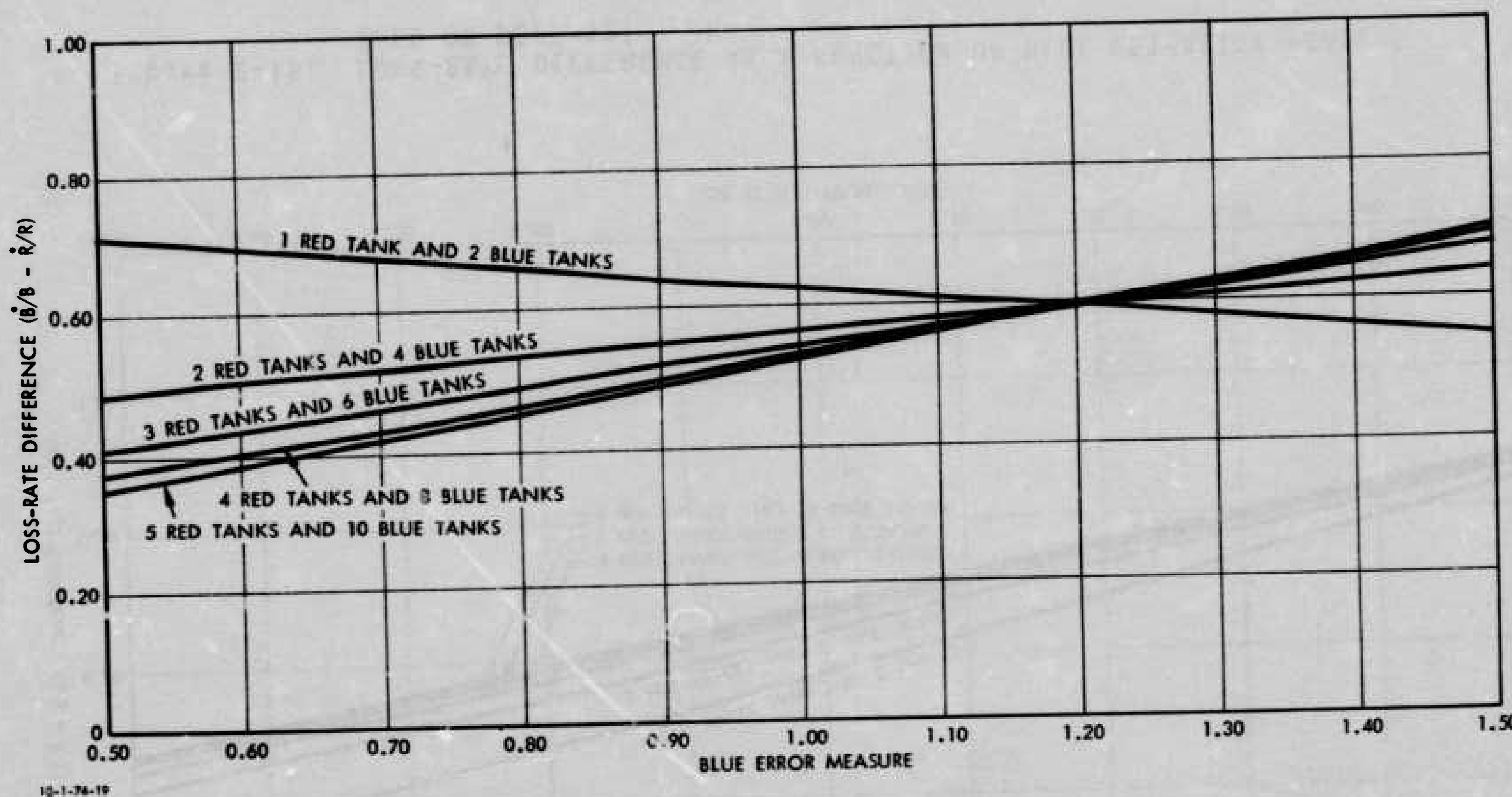

Figure 2-16. LOSS-RATE DIFFERENCE AS A FUNCTION OF BLUE ERROR MEASURE (RED ON DEFENSE)

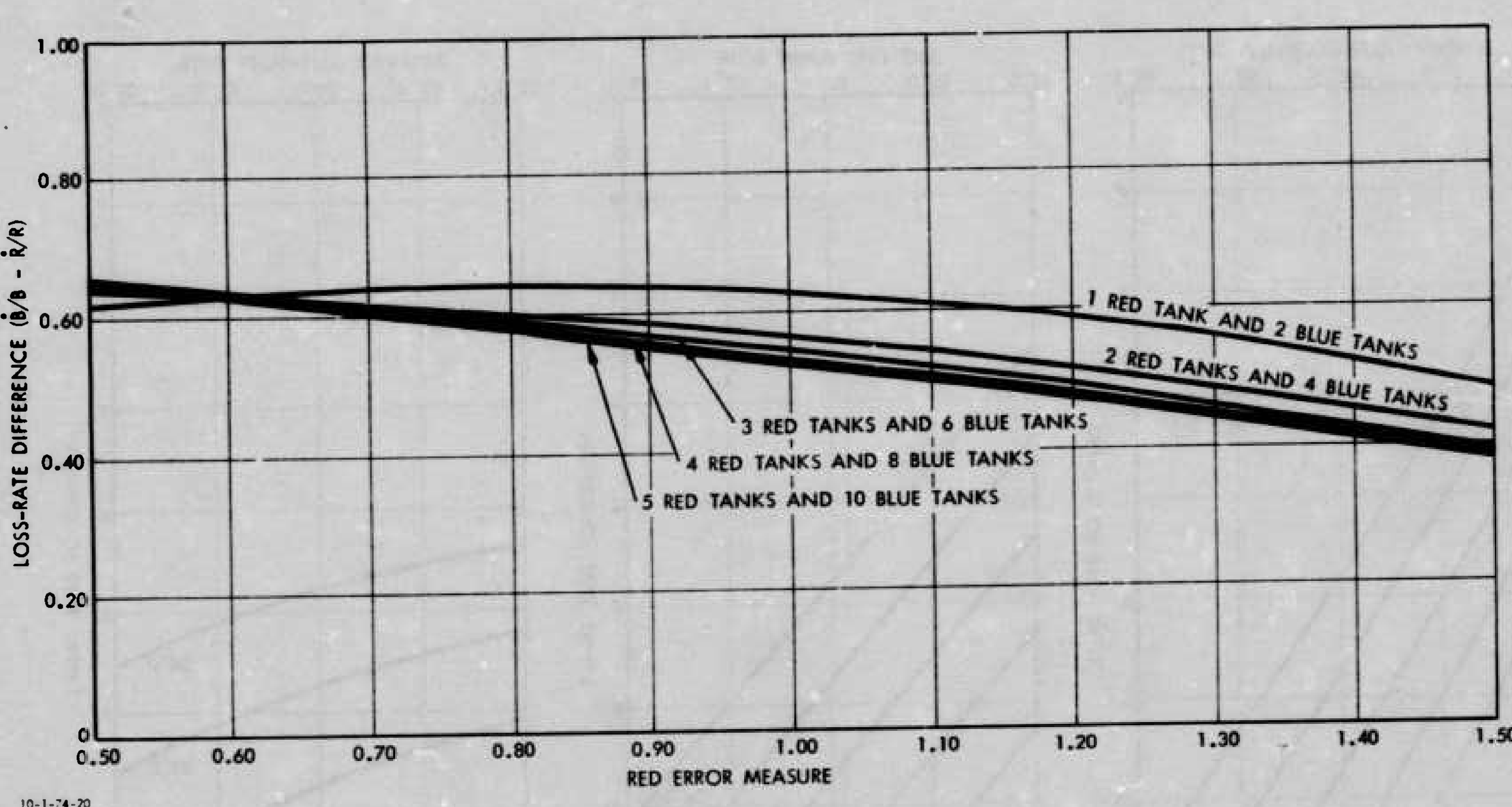

Figure 2-17. LOSS-RATE DIFFERENCE AS A FUNCTION OF RED ERROR MEASURE (RED ON DEFENSE)

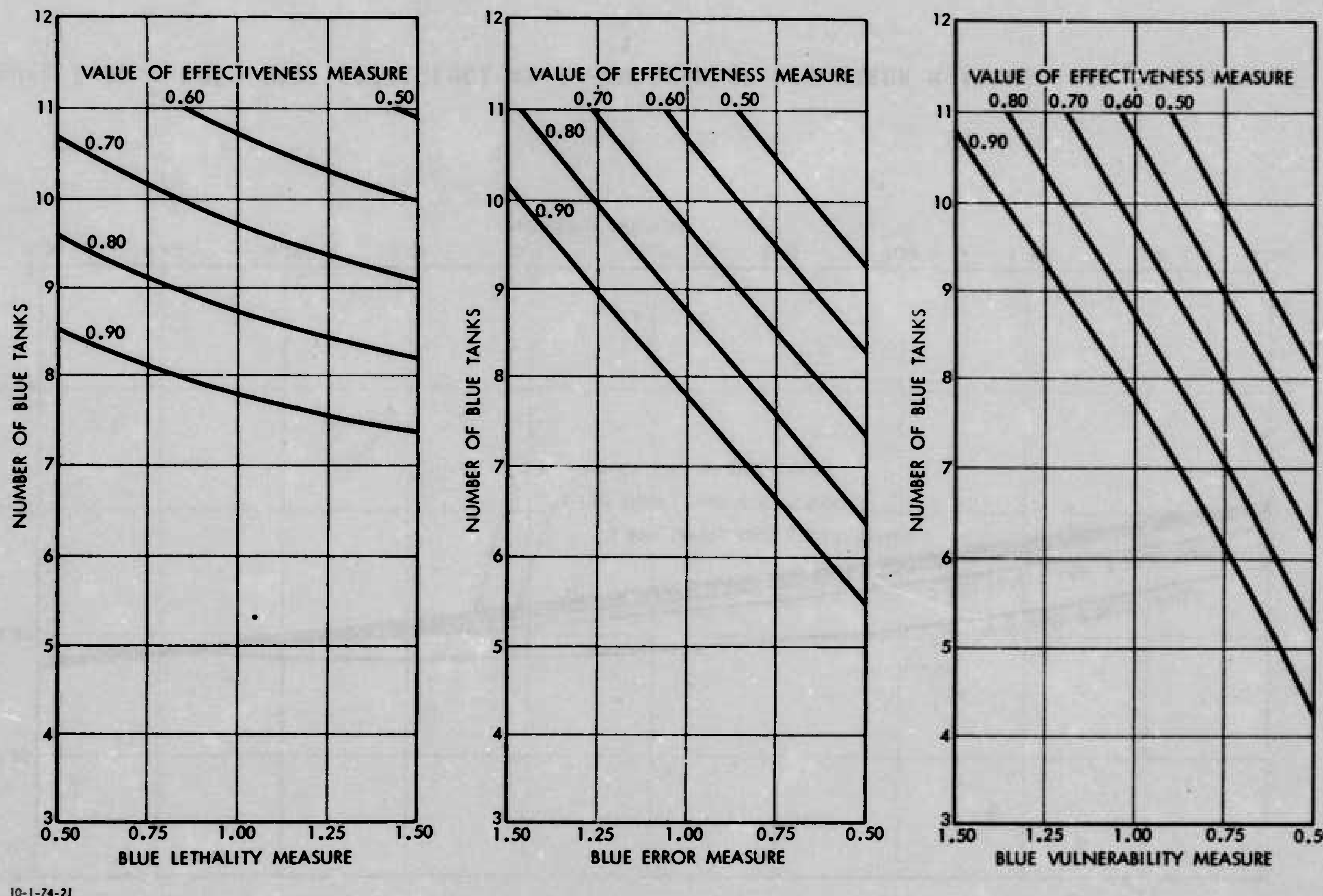

Figure 2-18. EFFECTIVENESS MEASURE (LOSS-RATE DIFFERENCE) AS A FUNCTION OF NUMBER OF BLUE TANKS AND QUALITY PARAMETERS - 5 RED DEFENSE TANKS

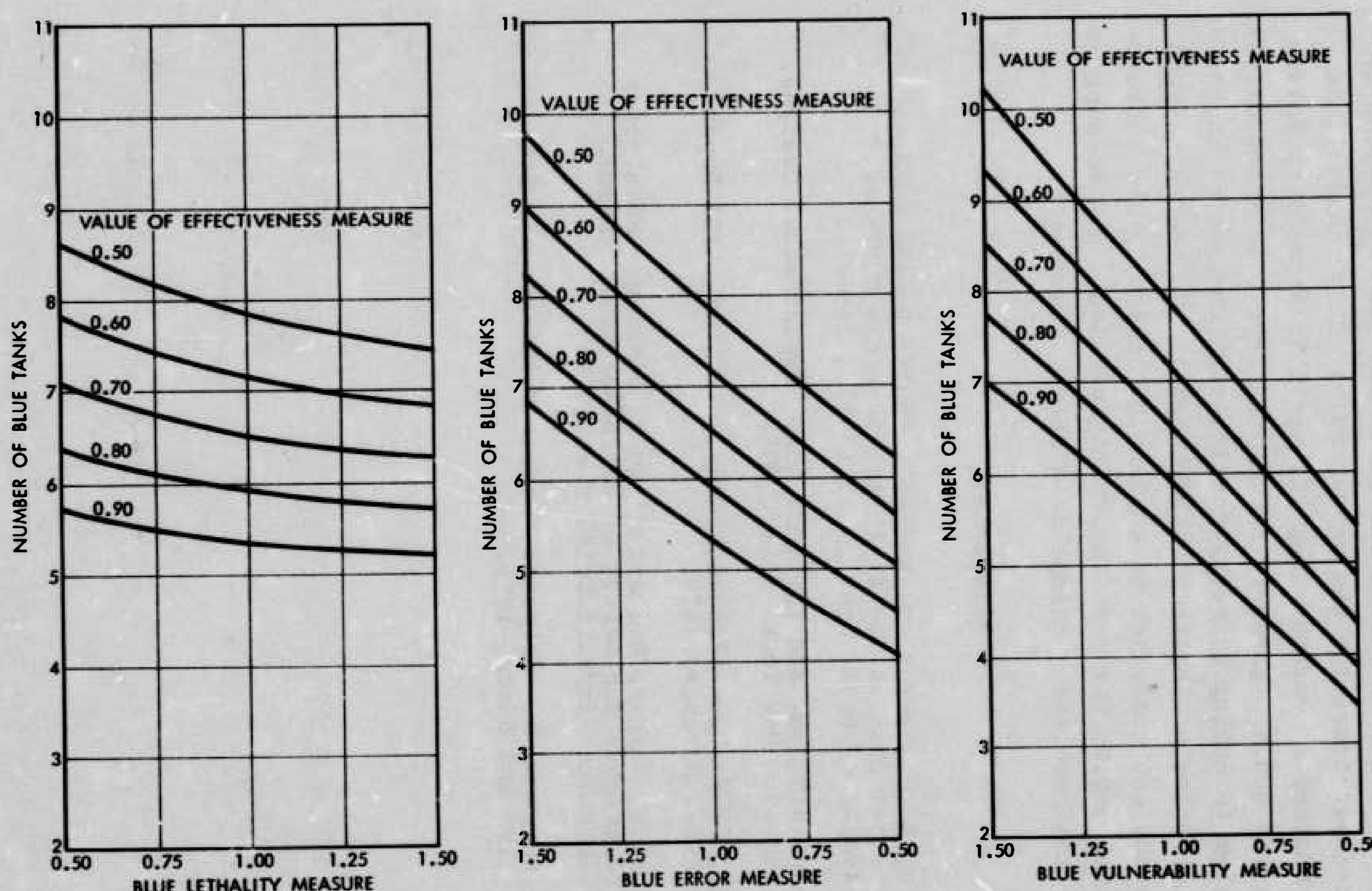

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Figure 2-19. EFFECTIVENESS MEASURE (LOSS-RATE DIFFERENCE) AS A FUNCTION OF NUMBER OF BLUE TANKS AND QUALITY PARAMETERS - 3 RED DEFENSE TANKS the cost on a per-pound basis for different tank subsystems or on a horsepower basis for the drive train. Unfortunately, it would be another monumental task (if possible at all) to relate to weight the quality parameters examined here. A second difficult problem in making quantity-quality cost trade-offs is the perennial problem of indirect support costs. No one has a good handle on support cost, yet in trading off quantity the support-cost changes may be the most important question. How do support and other indirect costs change as quantity changes?

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Appendix 2-A

ALTERNATIVE EFFECTIVENESS MEASURES IN QUANTITY-QUALITY TRADE-OFF

This appendix presents variations of the quantity-versus-quality curves that were presented in the main section of the chapter, where the curves used the difference of the expected loss rates as the effectiveness measure. Three sets of curves are presented in this appendix; each set uses a different effectiveness measure: The first set (Figures 2-A-1 - 2-A-7) uses the ratio of the expected loss rates; the second set (Figures 2-A-8 and 2-A-9), the ratio of expected losses; and the third set (Figures 2-A-10 and 2-A-11), the difference in losses.

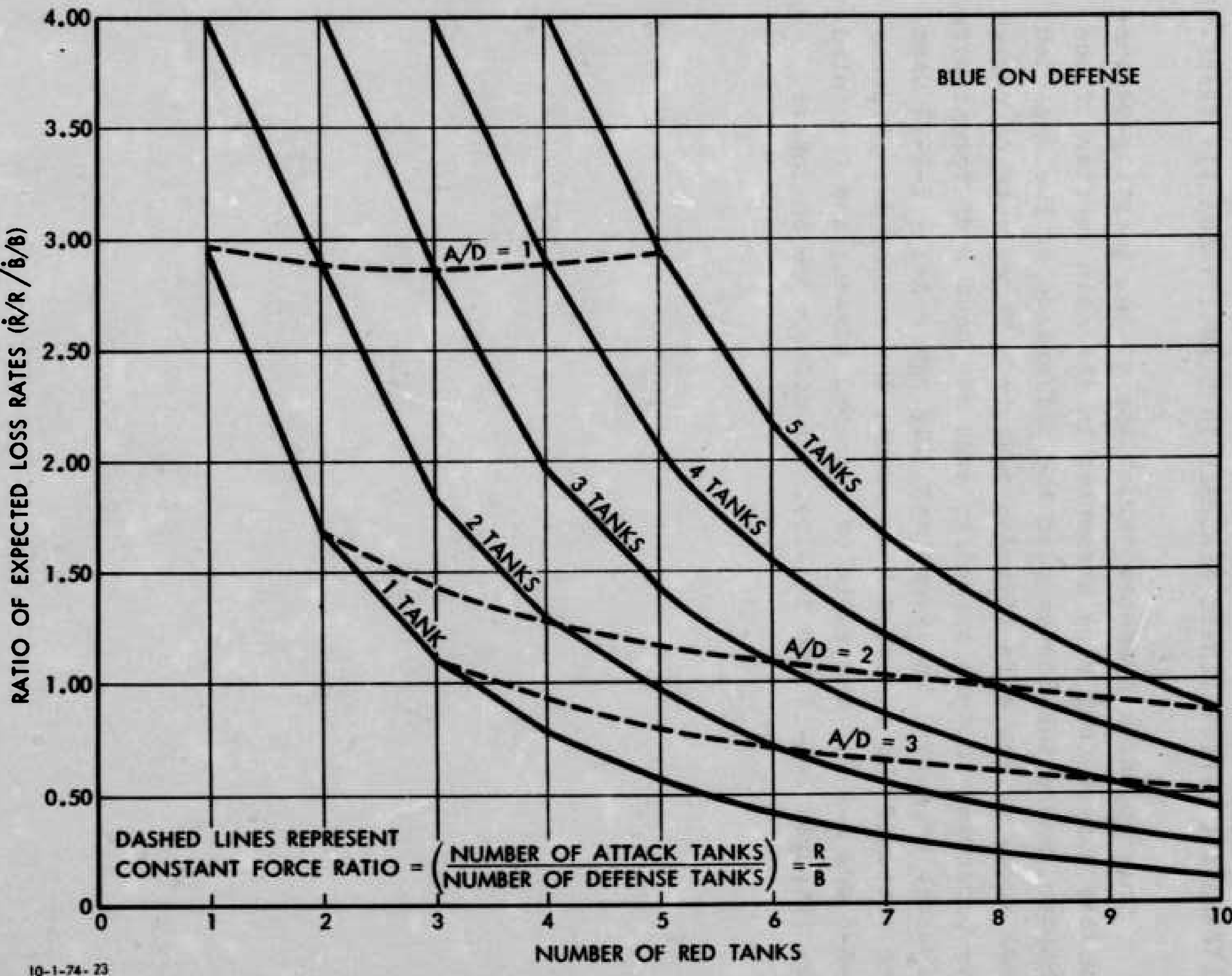


Figure 2-A-1. RATIO OF EXPECTED LOSS RATES VERSUS NUMBER OF BLUE TANKS AND RED TANKS

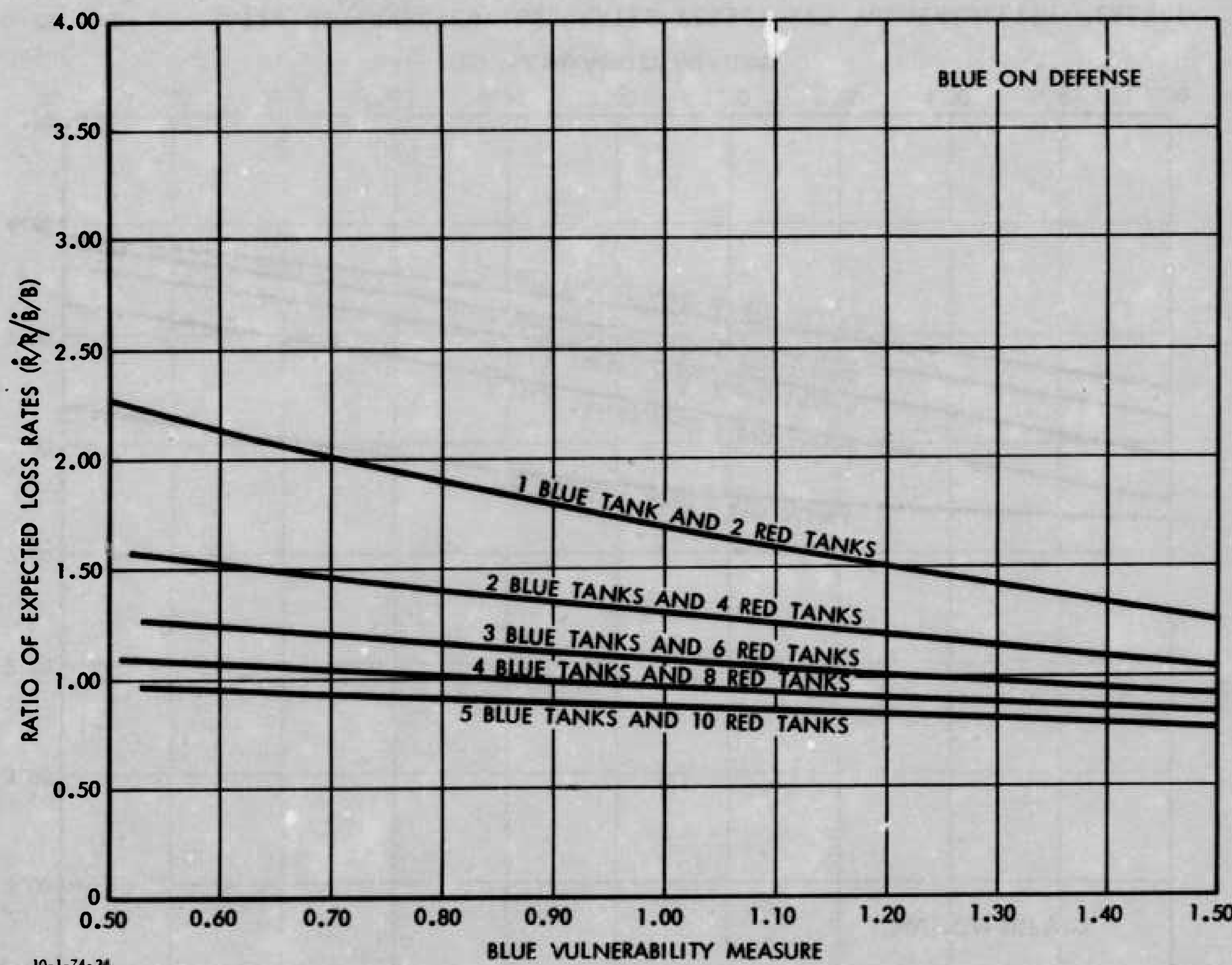


Figure 2-A-2. RATIO OF EXPECTED LOSS RATES VERSUS BLUE VULNERABILITY MEASURE

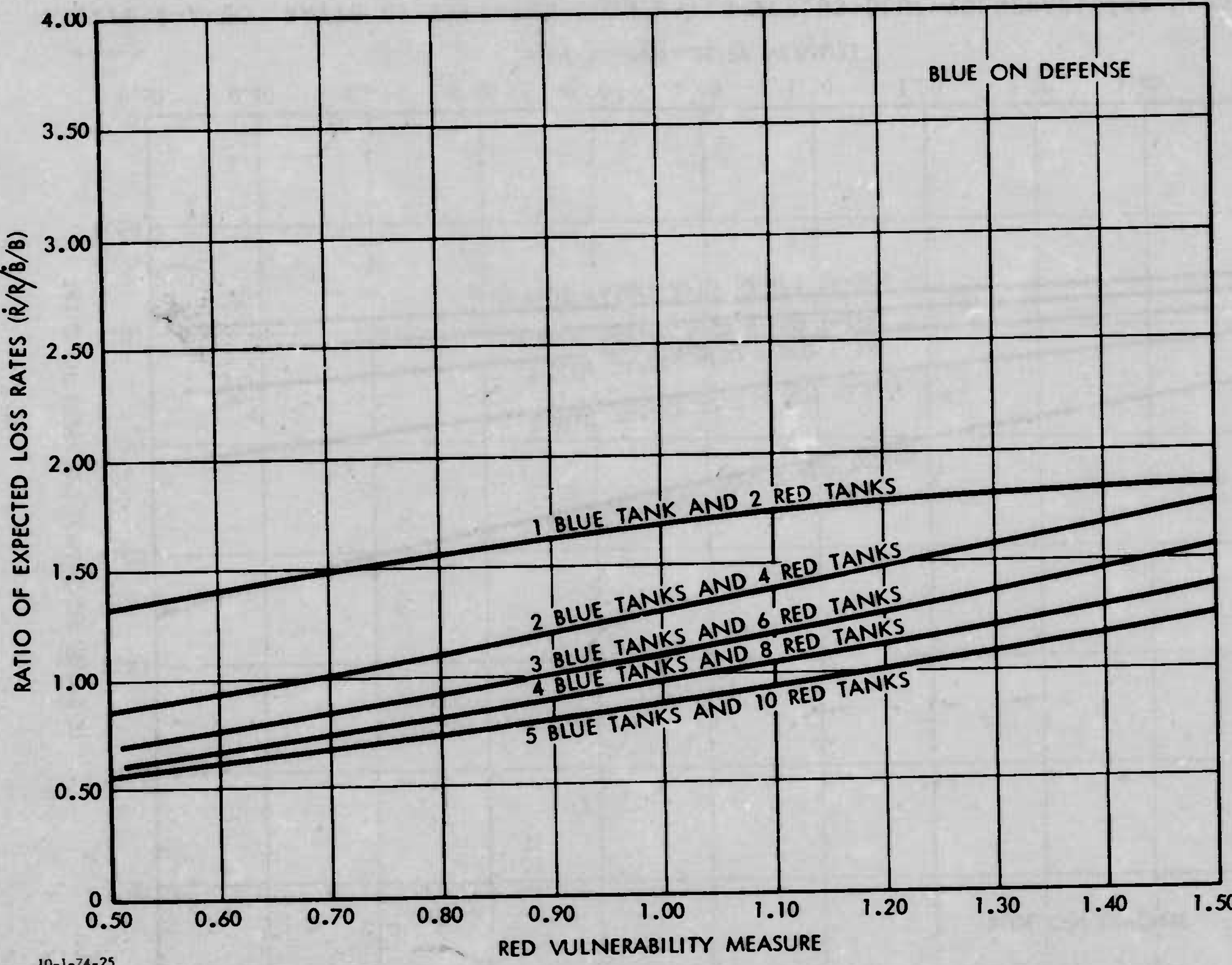


Figure 2-A-3. RATIO OF EXPECTED LOSS RATES VERSUS RED VULNERABILITY MEASURE

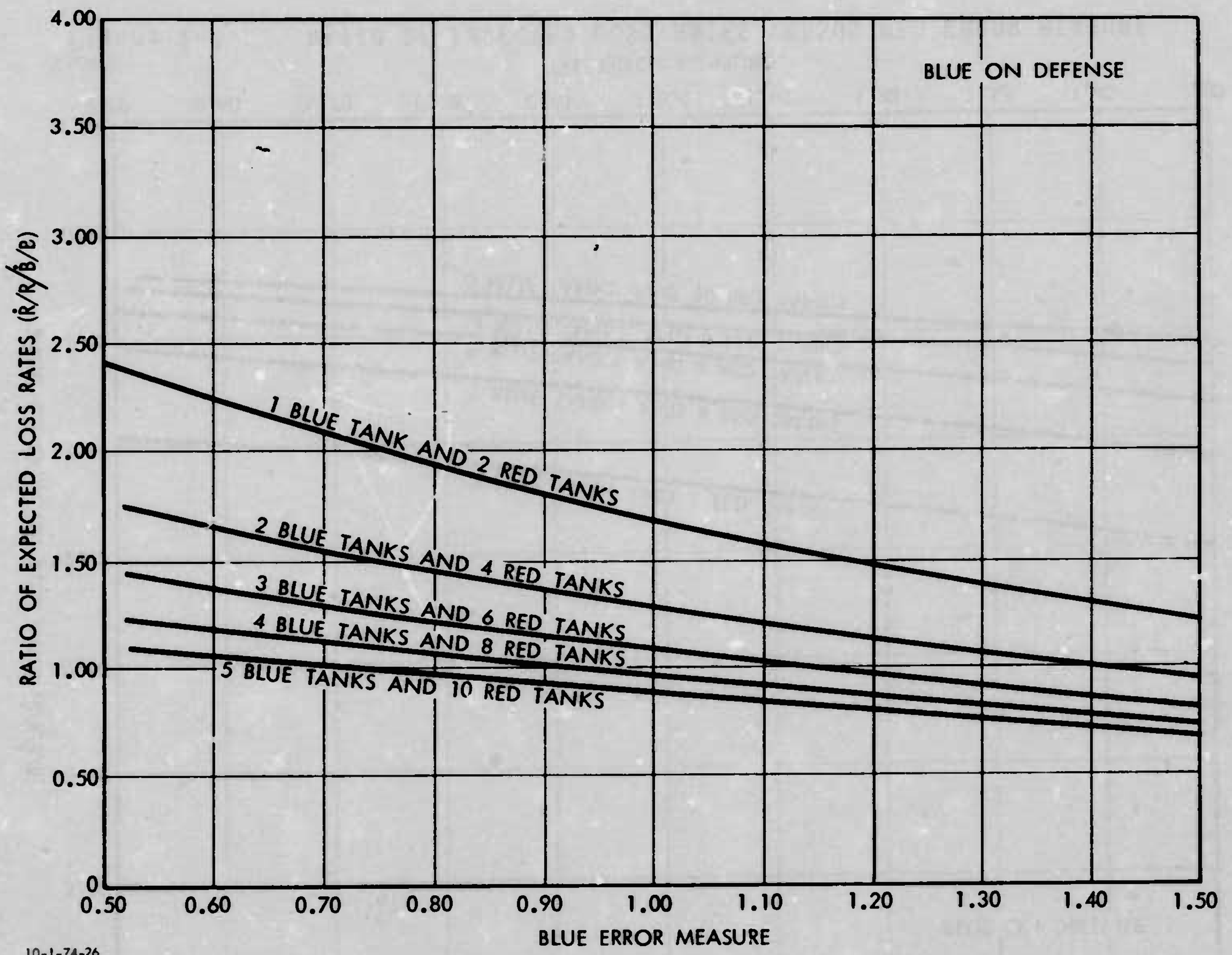


Figure 2-A-4. RATIO OF EXPECTED LOSS RATES VERSUS BLUE ERROR MEASURE

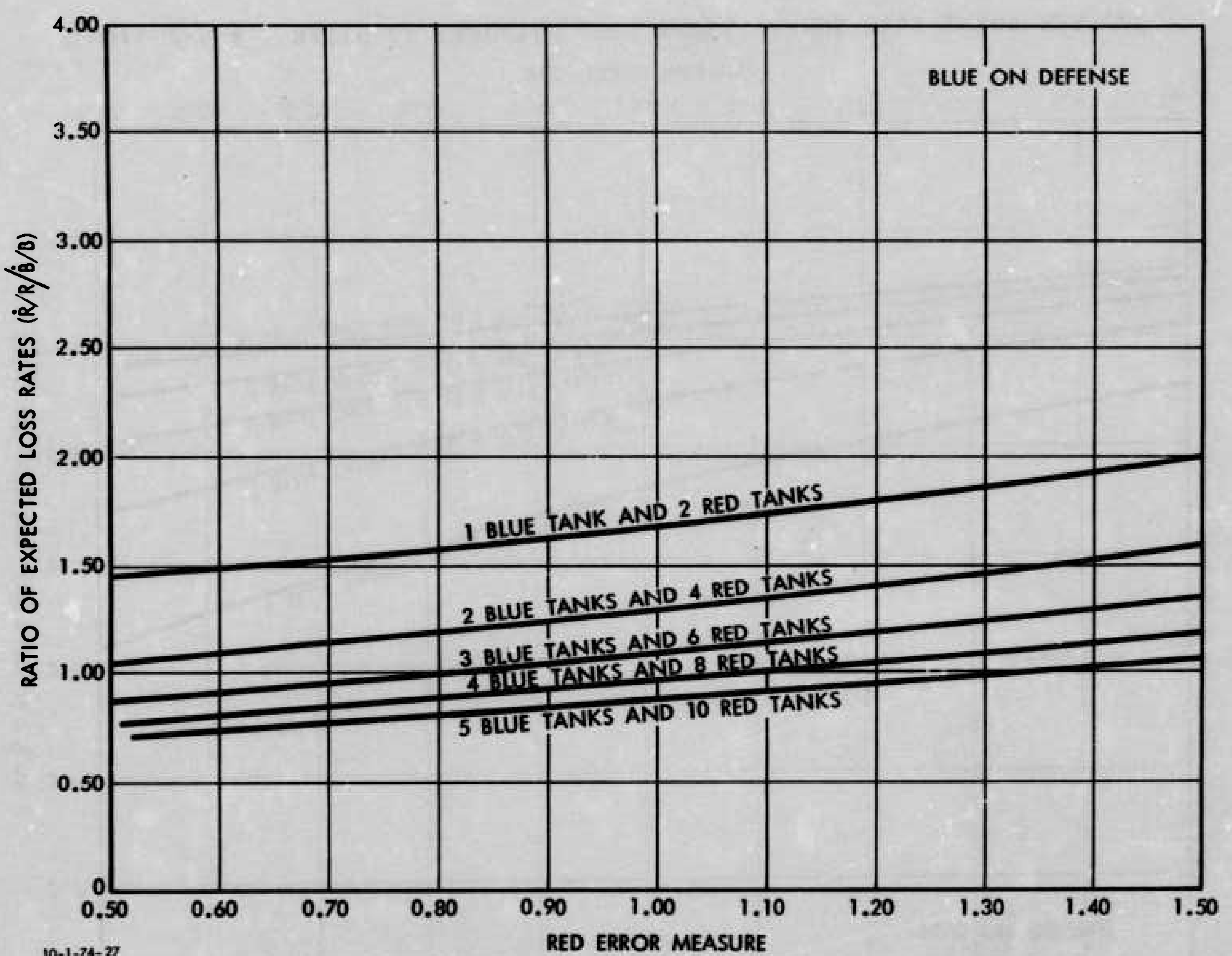


Figure 2-A-5. RATIO OF EXPECTED LOSS RATES VERSUS RED ERROR MEASURE

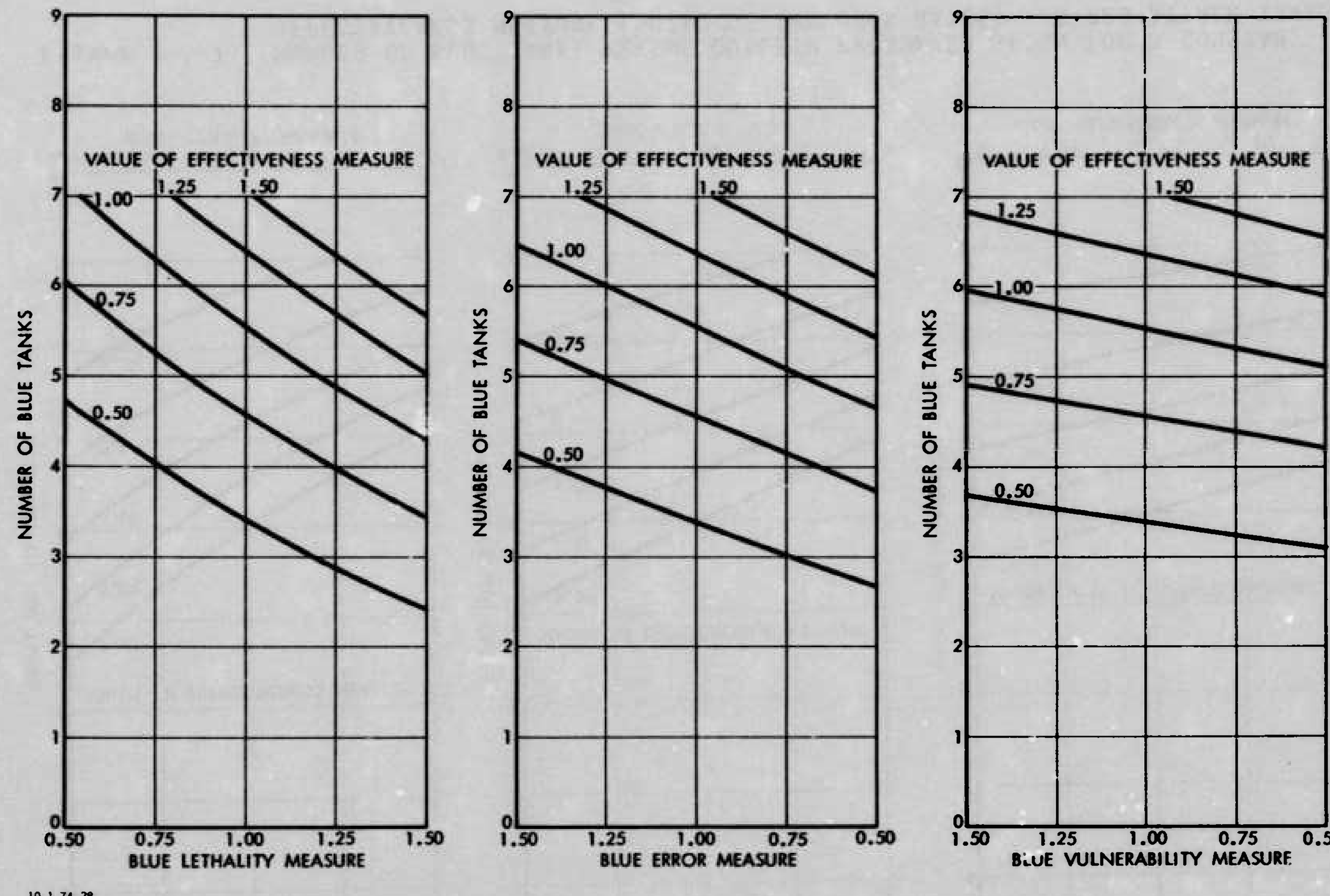


Figure 2-A-6. NUMBER OF BLUE TANKS VERSUS QUALITY PARAMETER SHOWN FOR A CONSTANT EFFECTIVENESS MEASURE (RATIO OF THE LOSS RATES) - 10 RED ATTACK TANKS

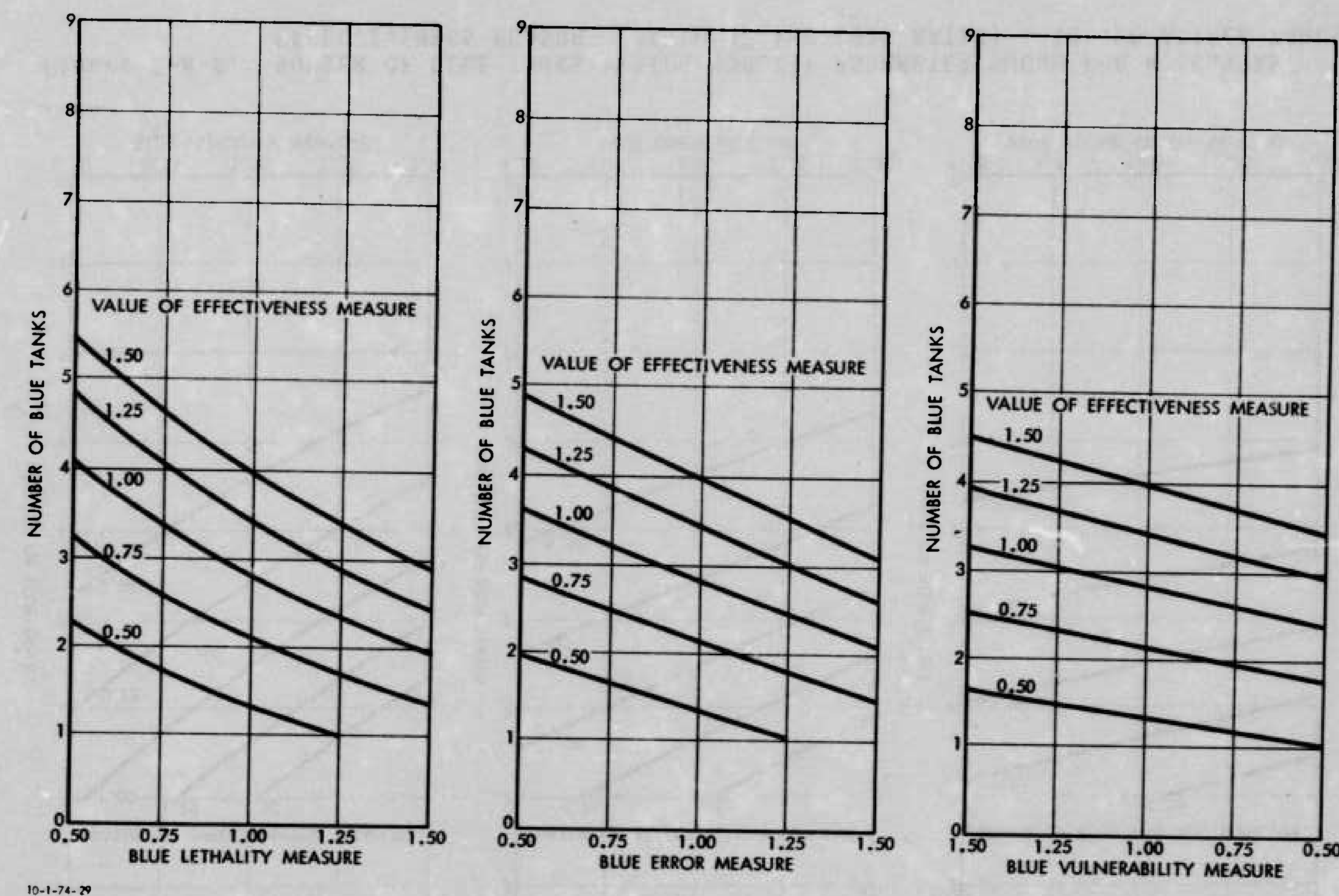


Figure 2-A-7. NUMBER OF BLUE TANKS VERSUS QUALITY PARAMETER SHOWN FOR A CONSTANT EFFECTIVENESS MEASURE (RATIO OF THE LOSS RATES) - 6 RED ATTACK TANKS

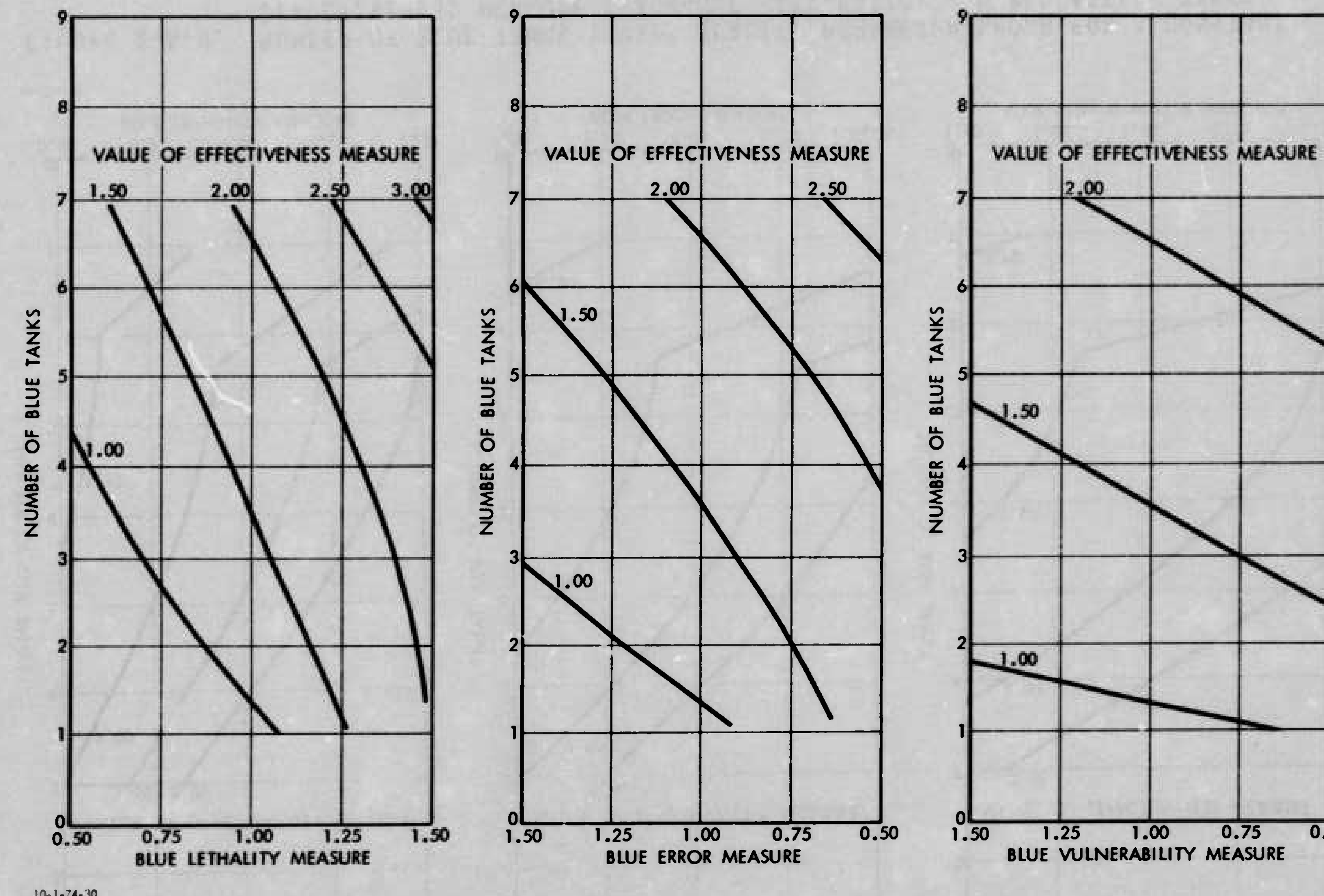
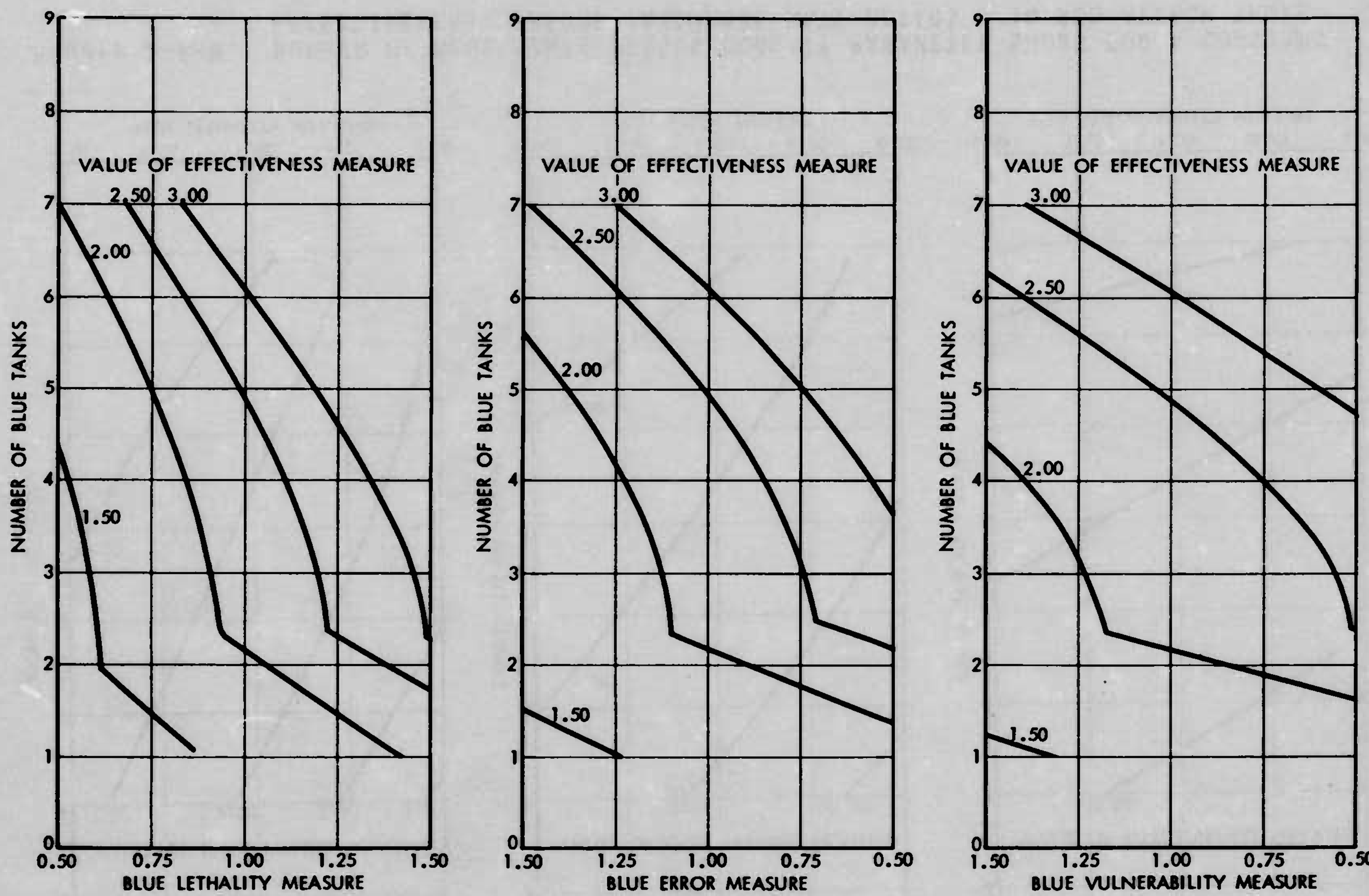


Figure 2-A-8. NUMBER OF BLUE TANKS VERSUS QUALITY PARAMETER SHOWN FOR A CONSTANT EFFECTIVENESS MEASURE (EXCHANGE-RATE RATIO) - 10 RED ATTACK TANKS



10-1-74-31

Figure 2-A-9. NUMBER OF BLUE TANKS VERSUS QUALITY PARAMETER SHOWN FOR A CONSTANT EFFECTIVENESS MEASURE (EXCHANGE-RATE RATIO) - 6 RED ATTACK TANKS

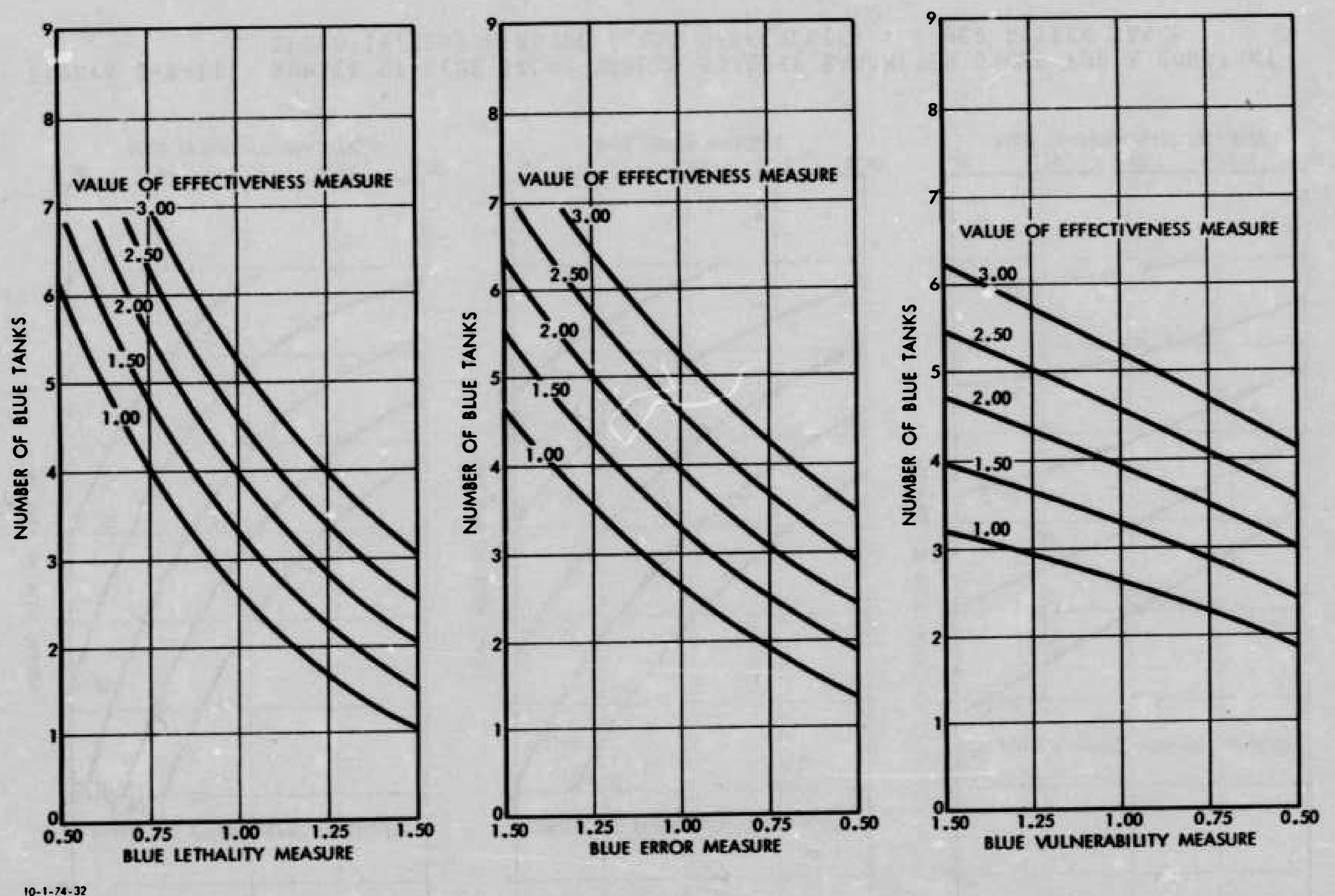
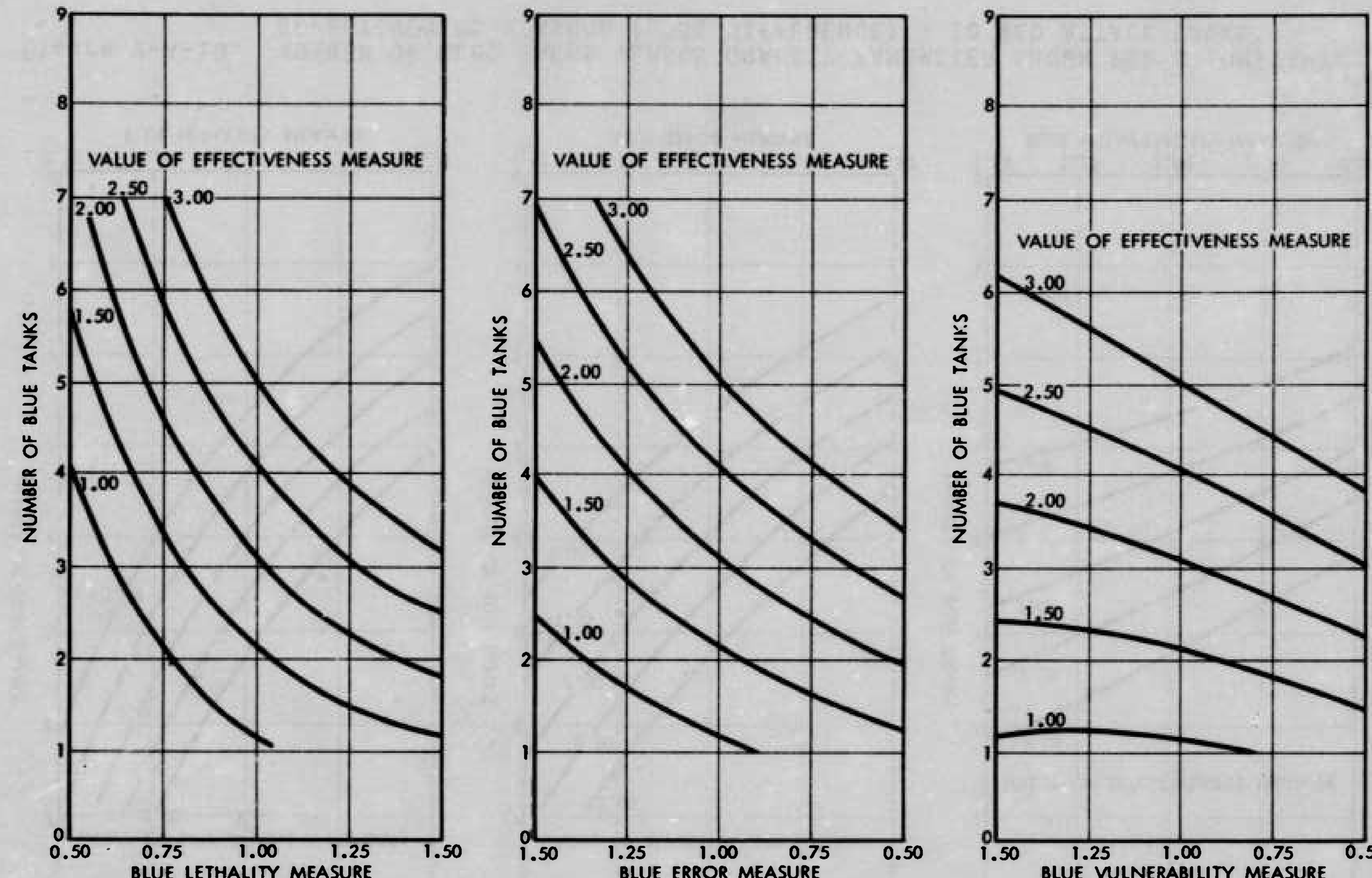


Figure 2-A-10. NUMBER OF BLUE TANKS VERSUS QUALITY PARAMETER SHOWN FOR A CONSTANT EFFECTIVENESS MEASURE (LOSS DIFFERENCE) - 10 RED ATTACK TANKS



10-1-74-33

Figure 2-A-11. NUMBER OF BLUE TANKS VERSUS QUALITY PARAMETER SHOWN FOR A CONSTANT EFFECTIVENESS MEASURE (LOSS DIFFERENCE) - 6 RED ATTACK TANKS

Appendix 2-B

STATISTICAL METHODOLOGY[1]

This appendix is in two parts: The first part derives the equations necessary to calculate a confidence band about an estimate that is the ratio of two empirically derived expectations. The second part deals with different sampling plans that can be used in deriving a response surface.

1. Confidence-Band Calculation

To obtain a confidence interval for $E(Y_1)/E(Y_2)$ for a particular set of values for the independent variables, let

$$R = \frac{E(Y_1)}{E(Y_2)} \ .$$

Then $\hat{Y}_1 - R\hat{Y}_2$ is normally distributed with mean zero and

$$\begin{aligned} \text{Var}\ (\hat{Y}_1 - R\hat{Y}_2) &= \text{Var}\ \hat{Y}_1 + R^2\ \text{Var}\ (\hat{Y}_2) \\ &\quad - 2R\ \text{Cov}\ (\hat{Y}_1, \hat{Y}_2) \\ &= \sigma_1^2 + R^2\sigma_2^2 - 2R\sigma_{12} \ . \end{aligned}$$

This simply relabels the variances and the covariance for notational convenience.

The quantity

$$\frac{\hat{Y}_1 - R\hat{Y}_2}{\sqrt{\sigma_1^2 + R^2\sigma_2^2 - 2R\sigma_{12}}}$$

has a normal distribution with mean zero and variance 1.

---

[1]This appendix was written by Hubert Lilliefors.

In order to find a $1-\alpha$ level confidence interval for R, we find $Z_{\alpha/2}$ from a table of the normal distribution (e.g., 1.96 for $\alpha = 0.05$); and the confidence interval will consist of all values for R for which the following inequality is satisfied:

$$\frac{(\hat{Y}_1 - R\hat{Y}_2)^2}{\sigma_1^2 + R^2\sigma_2^2 - 2R\sigma_{12}} \leq (Z_{\alpha/2})^2 .$$

Equivalently, this gives

$$R^2[\hat{Y}_2^2 - Z_{\alpha/2}^2\sigma_2^2] + R[-2\hat{Y}_1\hat{Y}_2 + 2\sigma_{12}Z_{\alpha/2}^2] + \hat{Y}_1^2 - \sigma_1^2 Z_{\alpha/2}^2 \leq 0 .$$

The end points of the interval would be found by setting the left-hand side equal to zero and solving the resultant quadratic.

Note that in some cases a confidence interval of this form may not exist.

For Y_1, we have a set of observations:

$$Y_{1i} = \alpha_0 + \alpha_1 X_{1i} + \alpha_2 X_{2i} + \ldots + \alpha_p X_{pi} + \varepsilon_{1i} , \quad i = 1, \ldots, n .$$

In matrix notation, this set of equations is written

$$\underline{Y}_1 = X_1\underline{\alpha} + \underline{\varepsilon}_1 ,$$

where

$$\underline{Y}_1 = \begin{bmatrix} Y_{11} \\ Y_{12} \\ \vdots \\ Y_{1n} \end{bmatrix} ;$$

$$\underline{\alpha} = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_p \end{bmatrix} ;$$

$$\underline{X}_1 = \begin{bmatrix} 1 & X_{11} & X_{21} & \cdots & X_{p1} \\ 1 & X_{12} & X_{22} & \cdots & X_{p2} \\ & \vdots & & & \\ 1 & X_{1n} & X_{2n} & \cdots & X_{pn} \end{bmatrix} ; \text{ and}$$

$$\underline{\varepsilon}_1 = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{12} \\ \vdots \\ \varepsilon_{1n} \end{bmatrix} .$$

The least-squares estimates for the parameters $\alpha_0, \ldots, \alpha_p$ are found from the equation

$$\underline{a} = (\underline{X}'\underline{X})^{-1}\underline{X}'\underline{Y}_1 ,$$

where

$$\underline{a} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_p \end{bmatrix} .$$

The predictive equation for a particular set of values for the independent variables is given by

$$\hat{Y}_1 = a_0 + a_1X_1 + \ldots + a_pX_p = \underline{a}'\underline{X}_1^*$$

where

$$\underline{X}_1^* = \begin{bmatrix} 1 \\ X_1 \\ \vdots \\ X_p \end{bmatrix} .$$

Similarly for Y_2, we have a set of observations:

$$Y_{2i} = \beta_0 + \beta_1 Z_{1i} + \beta_2 Z_{2i} + \ldots + \beta_q Z_{qi} + \varepsilon_{2i} ,$$
$$i = 1, \ldots, n .$$

In matrix notation, these equations are

$$\underline{Y}_2 = \underline{Z\beta} + \underline{\varepsilon}_2 ,$$

where

$$\underline{Y}_2 = \begin{bmatrix} Y_{21} \\ Y_{22} \\ \vdots \\ Y_{2n} \end{bmatrix} ;$$

$$\underline{Z} = \begin{bmatrix} 1 & Z_{11} & Z_{21} & \ldots & Z_{q1} \\ 1 & Z_{12} & Z_{22} & \ldots & Z_{q2} \\ & \vdots & & & \\ 1 & Z_{1n} & Z_{2n} & \ldots & Z_{qn} \end{bmatrix} ;$$

$$\underline{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_q \end{bmatrix}; \text{ and}$$

$$\underline{\varepsilon}_2 = \begin{bmatrix} \varepsilon_{21} \\ \varepsilon_{22} \\ \vdots \\ \varepsilon_{2n} \end{bmatrix}.$$

The least-squares estimate for the parameters β_0, β_1, ..., β_q is given by

$$\underline{b} = (\underline{Z}'\underline{Z})^{-1}\, \underline{Z}'\underline{Y} ,$$

where

$$\underline{b} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_q \end{bmatrix}.$$

The predictive equation for a particular set of values for the independent variables is given by

$$\hat{Y}_2 = b_0 + b_1 Z_1 + \ldots + b_q Z_q = \underline{b}'\underline{Z}^* ,$$

where

$$\underline{Z}^* = \begin{bmatrix} 1 \\ Z_1 \\ \vdots \\ Z_q \end{bmatrix}.$$

Let

σ_1^2 = variance of Y_{1i} [= variance of ε_{1i}];

σ_2^2 = variance of Y_{2i} [= variance of ε_{2i}]; and

σ_{12} = covariance of Y_{1i}, Y_{2i} [= cov $(\varepsilon_{1i}, \varepsilon_{2i})$].

Then

$$\text{Var}\ (\hat{Y}_1) = \sigma_1^2\ \underline{X}^{*\prime} (\underline{X}'X)^{-1}\ \underline{X}^*$$

$$\text{Var}\ (\hat{Y}_2) = \sigma_2^2\ \underline{Z}^{*\prime} (\underline{Z}'Z)^{-1}\ \underline{Z}^*$$

$$\text{Cov}\ (\hat{Y}_1, \hat{Y}_2) = \sigma_{12}\ \underline{X}^{*\prime} (\underline{X}'X)^{-1}\ \underline{X}'\underline{Z}(\underline{Z}'Z)^{-1}\ \underline{Z}^*\ .$$

(The expressions for Var $(\hat{Y}_1)$ and Var $(\hat{Y}_2)$ are found in Draper and Smith;[1] the expression for Cov $(\hat{Y}_1, \hat{Y}_2)$ can be derived in a similar fashion.)

Since σ_1^2, σ_2^2, and σ_{12} are not known, we need to use estimates for them.

σ_1^2 can be estimated by the pure-error mean square or by the residual mean square. Also,

$$\sigma_1^2\ (\underline{X}'X)^{-1} = \begin{bmatrix} \text{Var}\ (b_0) & \text{Cov}\ (b_0,b_1) & \ldots & \text{Cov}\ (b_0,b_q) \\ \text{Cov}\ (b_1,b_0) & \text{Var}\ (b_1) & \ldots & \text{Cov}\ (b_1,b_q) \\ & \vdots & & \\ \text{Cov}\ (b_q,b_0) & \text{Cov}\ (b_q,b_1) & \ldots & \text{Var}\ (b_q) \end{bmatrix}$$

[1]N. R. Draper and H. Smith, *Applied Regression Analysis* (New York: Wiley, 1966).

(the variance-covariance matrix for the coefficients b_0, b_1, ..., b_q).

Thus, if all the variances and covariances are given in the computer output, we could find the $(X'X)^{-1}$ matrix.

A number of regression routines will give the $(X'X)^{-1}$ matrix as part of the output.

Since the data are on cards, the $(X'X)^{-1}$ matrix could be calculated directly.

σ_2^2 and $(Z'Z)^{-1}$ would also be obtained as above.

The estimate for σ_{12}, the covariance, might be obtained from the sets of repeated observations.

Suppose that for the i^{th} data set there are J repetitions; then, for Y_{1i}, we have the J data points (for the case J = 10)

$$Y_{1i1}, \ldots, Y_{1iJ} ;$$

and, similarly for Y_{2i}, we have

$$Y_{2i1}, \ldots, Y_{2iJ} .$$

Then

$$E\left[\sum_{j=1}^{J} (Y_{1ij} - \overline{Y}_{1i\cdot})(Y_{2ij} - \overline{Y}_{2i\cdot})\right] = (J - 1)\sigma_{12} .$$

To obtain an estimate of σ_{12}, we calculate

$$\hat{\sigma}_{12} = \frac{\sum_{i=1}^{n} \sum_{j=1}^{J} (Y_{1ij} - \overline{Y}_{1j\cdot})(Y_{2ij} - \overline{Y}_{2i\cdot})}{n(J - 1)} ;$$

or, in a more readily calculable form,

$$\hat{\sigma}_{12} = \frac{\sum_{i=1}^{n} \sum_{j=1}^{J} (Y_{1ij}Y_{2ij} - J \sum_{i=1}^{n} \overline{Y}_{1i\cdot}\overline{Y}_{2i\cdot})}{n(J - 1)} .$$

where

$$\bar{Y}_{1i\cdot} = \sum_{j=1}^{J} Y_{1ij}/J$$

$$\bar{Y}_{2i\cdot} = \sum_{j=1}^{J} Y_{2ij}/J \ .$$

Another estimate for $\hat{\sigma}_{12}$ might be obtained from the set of n observations as follows:

$$\hat{\sigma}_{12} = \sum (Y_{1i} - \hat{Y}_{1i})(Y_{2i} - \hat{Y}_{2i})/(n - p) \ .$$

Note that if the same set of independent variables had been used in each equation then all of the quantities needed could have been obtained from the Biomed program for the Multivariate General Linear Hypothesis.

2. Sampling Plan Methods

Relative merits of--

(a) Full Factorial Design

(b) Central Composite Design

(c) Random Selection of Independent Variables.

The comparison will be between (a) and (b) followed by a brief discussion of (c).

In what follows, most of the information is taken from three references:

(1) William Cochran and Gertrude Cox, *Experimental Designs*, 2nd ed. (New York: Wiley, 1957);

(2) G. E. P. Box and J. S. Hunter, "Multi-Factor Experimental Designs for Exploring Response Surfaces" (*Annals of Math. Stat.*, 28 [1957], 195-241); and

(3) Raymond H. Myers, *Response Surface Methodology* (Boston: Allyn and Bacon, 1971).

In any discussion of factorial designs, we assume that the X variables have been coded (transformed) so that the levels are 1,-1 for a two-level factorial and 1,0,-1 for a three-level factorial. An experimental design for fiting a second-order response surface must involve at least three levels of each variable in order to estimate the coefficients. Thus, in using a factorial design, we need a 3^K design if there are K variables. If there are K variables, then there will be $\binom{K+2}{K} = \frac{(K+2)(K+1)}{2}$ coefficients.

This immediately presents one difficulty with the full factorial design. If there are a large number of variables, then the number of observations becomes quite large. In this case we had six variables, which for a 3^6 factorial required 729 observations to estimate the $\frac{8x7}{2} = 28$ coefficients.

If we had wanted to use eight variables, then we would have needed a total of 6,561 observations.

Thus, the full factorial would be most appropriate with only a few variables.

As pointed out in Reference (2), the variances of the estimates of the quadratic coefficients are twice as large as those for the interaction coefficients.

The use of a fractional factorial design could reduce considerably the number of observations required, but "the device of fractional replication is not very effective in generating from the higher level factorials satisfactory designs of order greater than one" (2).

The Central Composite Design (CCD) makes use of a 2^K factorial or fractional factorial augmented by additional points that allow estimation of the coefficients of a second-order surface.

$$
\text{Additional points}\left\{\begin{array}{ccccc}
X_1 & X_2 & X_3 & \cdots & X_K \\
0 & 0 & 0 & \cdots & 0 \\
-\alpha & 0 & 0 & \cdots & 0 \\
\alpha & 0 & 0 & \cdots & 0 \\
0 & -\alpha & 0 & \cdots & 0 \\
0 & \alpha & 0 & \cdots & 0 \\
\vdots & & & & \\
0 & 0 & 0 & \cdots & -\alpha \\
0 & 0 & 0 & \cdots & \alpha
\end{array}\right.
$$

Thus, we need $2^K + 2K + 1$ observations, which for six variables requires $(64 + 12 + 1) = 77$ points. The number could be reduced still further by using a fractional replication of the 2^K factorial.

Thus, we have reduced considerably the number of points required.

The choice of α is up to the experimenter, and different choices can give different properties to the design.

We could, for instance, obtain an orthogonal CCD that would yield uncorrelated estimates of the coefficients.

In Reference (3), tables are given that compare the efficiciences of orthogonal CCDs and other CCDs with the full factorial design on a per-observation basis (see Tables X-Z, below).

The efficiencies are ratios of the variances of the estimated coefficients normalized to a per-observation basis.

An efficiency greater than 1 would imply a greater precision for the Central Composite Design.

From the first table, it appears that the orthogonal CCD is at least as good as the 3^K design for $K \leq 6$.

In the case of $K = 6$, the superiority is not too marked. The choice of α can be made so as to have a rotatable design.

Table X. EFFICIENCIES OF ORTHOGONAL CCD wrt 3^K FACTORIAL

	K = 2	K = 3	K = 4	K = 5[1]	K = 6[1]
b_{st}[2]	1.00	1.00	1.00	1.00	1.000
b_{ss}[3]	1.00	1.07	1.00	1.43	1.106

Table Y. EFFICIENCIES OF CCD FOR α = 1.5 wrt 3^K FACTORIAL

	K = 2	K = 3	K = 4	K = 5[1]
b_{st}[1]	0.4983	0.7680	0.9518	1.0280
b_{ss}[3]	0.8561	1.0561	1.0770	1.3483

Table Z. EFFICIENCIES OF CCD FOR α = 2.0 wrt 3^K FACTORIAL

	K = 2	K = 3	K = 4	K = 5[1]
b_{st}[1]	0.25	0.4688	0.6944	0.75
b_{ss}[3]	1.60	1.2980	0.9800	1.50

---

[1] 1/2 fraction of 2^K was used.

[2] b_{st} is the coefficient of $X_s X_t$.

[3] b_{ss} is the coefficient of X_s^2.

A design is rotatable if Var $(\hat{Y})$ is a function only of the distance from the center of the design and not on the direction.

In general, the precision will be greatest at the center of the design and will decrease as the distance from the center increases.

No numerical comparison of the efficiency for a rotatable CCD was available.

The value of α can also be chosen so as to reduce the bias if the surface is not a quadratic--see Reference (3).

__Random Selection of Independent Variables__. We could find nothing on this subject. Clearly, it has the one advantage that we could, if desired, go to a higher-order response surface without taking additional data (as was necessary with the other designs).

We presume that it is less efficient than the other designs (but by how much?). With more time, we might have investigated this question.

One might compare the variances for the coefficients from runs with the full factorial and the 500 random points--e.g.,

$$\text{Eff (F to R)} = \frac{\text{Var}_R(b_1)}{\text{Var}_F(b_1)} \times \frac{500}{729} \text{ --}$$

and see whether this is reasonably stable over similar coefficients.

Chapter 3

ON MATHEMATICAL MODELING OF COMBAT ATTRITION PROCESSES

Alan F. Karr

In this chapter we describe the results of three investigations taking place in the course of the project. The first concerns the development of a scheme for classifying physical processes of combat in terms of characteristics applicable to the choice of a mathematical model of attrition. This classification applies only in the context of theater-level models such as IDAGAM I, and it is presented in Section A (below). The second and third efforts have led to two new and rather promising attrition models--one a generalized stochastic Lanchester process (in which the square-law/linear-law distinction is considerably clarified) and the other a Markov model of barrier-penetration interactions. Section B contains detailed descriptions and interpretations of these new processes. We also present the attrition models recommended for two other classes of combat processes, which are not newly derived, however.

A. CLASSIFICATION OF COMBAT PROCESSES

In this section we present a rather general and heuristic scheme for classifying (from the standpoint of attrition modeling) physical combat processes. The general classification scheme clarifies, we hope, the distinctions that can be made among different combat situations, as well as suggesting for inclusion in attrition models factors not currently treated. In effect, each of the large family of combat situations delineated by our classification scheme should be described by its own attrition model (otherwise, there would be no particular reason to distinguish such situations). Unfortunately, our knowledge of theater-level attrition modeling is currently such that only one of the several classifications presented is actually operative.

The context of this discussion is that of computerized, theater-level combat models such as IDAGAM I [3], in which the primary outputs of interest are casualties (to personnel and to equipment) and FEBA position. In particular, this context influences the level of resolution of the distinctions made in the characterization scheme. Further, only nonnuclear combat is considered. We have sought, however, to make the characterization structure sufficiently general to include ground, air, and naval combat.

In this scheme for the classification and characterization of combat processes, the categories are largely heuristic and connote mostly an attempt to clarify the ways in which one can think about combat processes. Some secondary purposes are to elucidate the assumptions made in choosing a mathematical model of combat attrition, to suggest areas for further research, and to stimulate development of more precise classifications. Indeed, a good classification of combat processes is the key step in choosing attrition models on the basis of their underlying assumptions.

The main purpose of mathematical models of attrition is to describe (as accurately as possible, subject to the constraints of the context within which we are working) the time history of the position, size, and structure of the two opposing forces. Seen in this light, actual losses of personnel, equipment, and supplies are almost a secondary output of the model. In the context of IDAGAM I, position is represented only by a FEBA separating the two sides and the division of the territory held by each side into smaller regions and sectors.

Following tradition, we refer to the two opposing sides as Red and Blue, respectively. The FEBA represents the line separating the opposing forces, and the calculation of its position (and the variation over time thereof) is a principal goal of many combat models. (One can also be interested in the modeling of combat interactions--guerrilla warfare and air-to-air combat being examples--in which there exists no FEBA at all; the present comments, but not the whole paper, are not relevant to this case). Orthogonal to the FEBA, there exists for each side a partition of its area into some sectors (not necessarily the same number on both sides), and the level of resolution of the model is to maintain time-evolving totals of the number of each type of resource in each sector. The existence of rear regions of some sort is also possible--but of less interest and importance in terms of computation of the immediate effects of combat interactions.

The role of sectors is nontrivial: they are, as surrogates for more detailed positional information, determinants of *interaction eligibility*. That is, the sectors provide as inputs to mathematical models the numbers of resources participating in interactions that are not, in general, further divisible, except possibly for purposes of modeling and computation. The interactions determined by these sectors proceed independently of one another; and their effects are aggregated linearly (except for that on movement of the FEBA, which is computed separately in each sector, independently of the others).

For the purposes of characterizing combat situations, we limit attention to indecomposable interactions of the type discussed in the preceding paragraph. Hence, there is no explicit representation at all of the positions of the resources involved in the interaction; and all resources present are assumed to be vulnerable (to some extent) to one another.

In Figure 3-1, we present a general set of descriptors of combat processes. A discussion follows the figure. The choice of these descriptive characteristics is largely on the basis of

I. Qualitative Characteristics

A. Relative Positions of the Two Opposing Forces

1. On Opposite Sides of FEBA
2. Essentially Completely Intermingled

B. Nature of the Interaction

1. Each Side Searching to Find the Other
2. One Side Maintaining a Barrier Through Which the Other Attempts to Penetrate
3. One Side Attempting to Destroy Passive Targets on the Other

C. Objectives of the Two Sides

1. Infliction of Casualties
2. FEBA Movement
3. Limitation of Losses and Loss Rates of Own Resources

II. Principal Quantifiable Characteristics

A. Scale

1. Numbers of Objects of Each Type Participating in the Interaction and Numbers of Newly Arriving Objects as Functions of Time
2. Geographic Dimensions of the Set Over Which the Interaction Takes Place
3. Length of Time

B. Interaction-Independent Characteristics of Weapons

1. Range
2. Mobility
3. Effect of Shots: single or multiple kill (and, in the latter case, how many); also various suppressive effects
4. Personnel Requirements
5. Supply Requirements

(continued on next page)

Figure 3-1. CHARACTERIZATION OF COMBAT PROCESSES

III. Important Factors That Are Possibly Neither Quantifiable Nor Easy to Deal With Qualitatively

A. Organizational Characteristics

1. Command and Control
2. Communications
3. Morale

B. Environmental Factors

1. Weather
2. Terrain

C. Synergistic Effects

D. Exogenous Events

Figure 3-1 (concluded)

their use in determining the possible applicability of certain mathematical models of attrition to the physical processes being studied and, in particular, that of the four models discussed in the next section.

We remark that the classification scheme is also intended to be valid for processes (e.g., naval combat) not treated explicitly in IDAGAM I, provided that the levels of resolution and aggregation be the same.

Some remarks on the classification scheme are in order. It seems rather clear that combat situations in which the two sides are separated by a clearly defined FEBA are qualitatively different from those (e.g., air-to-air combat or guerrilla warfare) in which they are quite intermingled. Few models yet developed, however, appear to do a good job of making this distinction; thus, further research may be merited.

The category I.B (Nature of the Interaction) is crucial. In fact, it forms the basis for deciding which of the four models discussed in the next section is applicable to a given combat process. In other words, the state of mathematical models of attrition is such that only this part of the classification is operative.

The situation (I.B.1) is characterized by symmetry: elements of each side are seeking to engage those of the other. It is also an ongoing process, as opposed to that of barrier penetration (I.B.2), which occurs over a relatively short period of time and wherein one side seeks to evade the other, which is seeking to detect and engage it. The attack of passive targets (I.B.3) is self-explanatory.

Objectives of the two sides are important, but we seem not to know how to incorporate them into models. Even the notion of breakpoints is still rather primitive, in terms of potential incorporation into combat simulations.

The quantifiable characteristics listed under II.A (Scale) are self-explanatory. Geographical scale is relevant even in the absence of explicit information concerning positions (indeed, more so in this case); it clearly affects the values of certain input parameters.

"Interaction-Independent Characteristics of Weapons" is intended to connote those properties that are the same for all interactions in which a given (type of) weapon is involved and can be measured in a noncombat situation. In some sense, a weapon possesses no truly interaction-independent characteristics except physical dimensions, yet it seems reasonable that the characteristics listed are also roughly interaction-independent. It turns out that the present state of attrition modeling also requires that such properties as method and rate of engagement initiation also be taken to be interaction independent (see the next section for details).

The factors in Category III are important, but no model yet devised comes near to handling any of them adequately. IDAGAM I makes at least an attempt to include synergistic effects.

B. SOME RECOMMENDED ATTRITION MODELS

We begin by classifying a number of combat situations into the three categories:

(I.B.1) Mutual Search and Engagement,
(I.B.2) Barrier Penetration, and
(I.B.3) Attack of Passive Targets;

and we present a recommended form of attrition model for each of these three classes of situations.

The combat situations are classified and presented in Figure 3-2, which is not complete and to which we hope others will add. We shall next discuss, in some detail, the

(I.B.1)	Processes of Mutual Search and Engagement
(a)	Ground Combat With Objective of Either FEBA Movement or Casualties
(b)	Ground-Air Combat
(c)	Aircraft Combat for Maintenance of Air Superiority
(d)	Combat Among Submarines
(e)	Combat Among Naval Surface Forces
(I.B.2)	Processes of Barrier Penetration
(a)	Bomber Aircraft Through Interceptors
(b)	Aircraft Through SAMs and AAA
(c)	Submarines Through Minefields
(d)	Submarines Through Submarine Barriers
(e)	Submarines Through Search Aircraft
(f)	Submarines Through Convoy Escort Ships
(I.B.3)	Processes of Attack of Passive Targets
(a)	Attacks of Various Targets by Aircraft
(b)	Attacks of Merchant Ships by Submarines

Figure 3-2. A CLASSIFICATION OF CERTAIN COMBAT SITUATIONS

attrition models we recommend for use in each of these categories.

1. Processes With Both Sides Searching

For processes of the class (I.B.1), we recommend a new, mixed-mode, stochastic Lanchester-type attrition process.

The analysis leading to the mixed-mode (or, as we have called it, generalized) Lanchester attrition process is relevant to the previously rather obscure distinction between Lanchester square combat and Lanchester linear combat, to which we now address ourselves. Lanchester himself [6] sought to distinguish the two according as the numerically superior side is able to make full use of its superiority (square law) or not (linear law). This is one interpretation of the distinction as we have come to understand it, at least in the case of homogeneous forces. For heterogeneous forces, however, the problem is considerably more subtle; and we feel that the explanation to be presented below is the most cogent yet constructed.

In this process, weapons are grouped into four classes on the basis of two qualitative distinctions. One of these is a single-kill or multiple-kill (per shot) distinction, which we shall consider below. The second distinction is based on the qualitative nature of the rates of engagement initiation (i.e., rates at which shots are fired). A particular type of weapon is said to have *square law engagement initiation* if the total mean rate at which it fires shots at opposition weapons is independent of the numerical size of the opposition force (as well as of its precise structure). In other words, the mean firing rate depends only on the type of weapon. This may be interpreted to the effect that the weapon simply fires at its own mean rate, independent of the behavior of the opposition (e.g., an artillery piece) or that it can move and that it advances in such a manner as to maintain a constant rate of contact with the opposition. The possibility that the rates at which shots directed at particular types of opposition weapons are fired may depend on the numbers of various kinds of opposition weapons present is not inconsistent with our first statement, which concerned only the total mean rate of fire.

On the other hand, a type of weapon is said to possess *linear law engagement initiation* if the mean rate at which it fires shots at a target force $y = (y_1, \ldots, y_n)$ (assuming that the opposition has n different types of weapons) is of the form $\sum_{i=1}^{n} \alpha_i y_i$, where the α_i are nonnegative constants. A word of warning is in order here concerning our usage of the term "mean rate." What we intend is that it be interpreted in the sense of the infinitesimal generator of a continuous-time Markov process: that is, the instantaneous rate at which the event in question tends to occur, given the current state of the process. It seems to us that the most appealing interpretation of linear-law engagement initiation is that each given opposition weapon requires an exponentially distributed random time to detect; different opposition weapons are detected independently; and an engagement occurs if and only if an opposition weapon is detected. We refer the reader to Reference [4] for further details; an understanding of the mathematics there is not essential to the present development. In particular, the rate at which shots are fired at opposition weapons of any given type is directly proportional to the number of weapons of that type currently surviving.

From the above analysis, possibly the most important conclusion to be drawn is that the square-law/linear-law distinction *applies not to the combat as a whole but to individual types of weapons*. In other words, while a particular type of weapon may be said to have square-law or linear-law engagement initiation, the combat itself cannot be said to possess either property. Even if all weapons present belonged to one of the two classes (in this case one would have essentially Process S3 or Process L3 of [4]), it would still be a misnomer to say that the combat itself is of that type, because the distinction simply is not of that nature. The square-law/linear-law distinction applies only to individual types of weapons.

Now what determines whether a particular type of weapon possesses square-law or linear-law engagement initiation? It seems to us that Lanchester's original differentiation is fairly close to the truth. A weapon type has square-law engagement initiation if all weapons of that type are able simultaneously to bring their fire to bear on the opposition. Two ways in which this can be envisioned are that shots are simply fired (at a rate determined only by the shooting weapon) at an area in which the opposition is known (or thought) to be located, or that the weapons are mobile and push forward in such a way as to maintain a rate of contacts with enemy weapons that is independent of the number of enemy weapons present. Linear-law engagement initiation, on the other hand, requires a prior detection and proceeds essentially in a one-on-one fashion.

The problem of deciding whether a particular type of weapon has square-law or linear-law engagement initiation seems to us crucial and difficult in terms of attrition modeling. The implications in terms of computed levels of attrition and FEBA movement are likely to be substantial; experiments with simplified homogeneous models (essentially discretized versions of Lanchester's original differential equations) have shown this to be so. Hence, this classification should not be undertaken lightly or in ignorance, as it may have overwhelming effects on the outputs of combat-simulation models. For certain types of weapons, such as artillery (square law) or small arms (linear law), the choice seems fairly clear.

But, for some other types (e.g., tanks), the choice is not at all clear; and it appears that to which category a tank belongs may be the result of tactical decisions by the two sides, may change during the course of a battle, and is more properly an output of the attrition model rather than a prescribed input. We cannot refute all these criticisms. We note, however, that no attrition model yet devised correctly addresses any of the difficulties raised. The possibility that engagement initiation is in its qualitative nature the result of tactical decisions is particularly interesting, however; the weaker side would seek to create linear-law engagement initiation and the stronger side would desire square-law initiation.

The foregoing remarks, we hope, have clarified the square-law/linear-law distinction and will serve as a stimulus to further research in this important area.

Our second classification characterizes weapons as *single-kill* or *multiple-kill* according as one shot can kill at most one (or, possibly, more than one) opposition weapon. An artillery piece is an example of a multiple-kill weapon, while an antitank weapon exemplifies the notion of a single-kill weapon. It seems that whether a weapon be single-kill or multiple-kill should depend on the type of target (an artillery piece fired at infantrymen is clearly multiple-kill, but is single-kill when fired at tanks); the mathematical structure we describe here is in fact sufficiently general to represent this phenomenon--but only with somewhat of a loss of clarity.

According to our scheme of classification, there are thus four qualitative classes of weapons:

SS - Weapons with square-law engagement initiation and single kill per shot;

LS - Weapons with linear-law engagement limitation and single kill per shot;

SM - Weapons with square-law engagement initiation and multiple kills per shot; and

LM - Weapons with linear-law engagement initiation and multiple kills per shot.

The classical term "area fire" seems best represented by SM weapons; small-arms point fire involves weapons of class LS. The interested reader can devise his own interpretations for all four categories.

For each class of weapon, certain input parameters will be required for the attrition process to be discussed below; our new process generalizes the processes presented in Reference [4], from which the qualitative nature of the input parameters is determined. In particular, SS weapons behave as described in the Processes S3 and S3a of Reference [4], LS weapons as in Process L3, SM weapons as in Process A1 (suitably generalized), and LM weapons as in the linear-law analogue of Process A1, which does not appear in Reference [4]. The reader is referred to Reference [4] for the statement and interpretations of these specialized attrition processes.

Here are the assumptions and notations for the new generalized Lanchester attrition process. The combat is bilateral, involving two heterogeneous forces (Red and Blue); weapons are not distinguished except by the qualitative classes given above and, within each class, only by numerical values of parameters.

a. **Assumptions**

(1) The Blue force consists of M_1 types of weapons of the class SS, M_2 weapon types of class LS, M_3 weapon types of the class SM, and M_4 weapon types of the class LM. Hence, Blue has altogether $M = M_1 + M_2 + M_3 + M_4$ *types* of weapons. The analogous numbers for the Red side are N_1, N_2, N_3, N_4; and $N = N_1 + N_2 + N_3 + N_4$.

The notation used below is of the following form: The letter i will be used as a generic index for Blue weapon types. Types 1, ..., M_1 are the SS weapons; $M_1 + 1, \ldots, M_1 + M_2$ are LS weapons; weapon types

$M_1 + M_2 + 1, \ldots, M_1 + M_2 + M_3$ are of class SM; and the remaining types belong to the class LM. Certain parameters below are defined only for Blue weapons belonging to only one class; if, for example, that class is SM, the relevant index i ranges over 1, ..., M_3; when all Blue weapon types are considered, the index i ranges over 1, ..., M. For Red, a similar situation holds for the index j.

Blue forces are denoted by vectors $x \in \underset{\sim}{N}^M$, where $\underset{\sim}{N} = \{0, 1, 2, \ldots\}$; x_i is the number of type-i weapons (i = 1, ..., M) currently present. Red forces are, analogously, denoted by vectors $y \in \underset{\sim}{N}^N$.

(2) Times between shots fired by a surviving Blue type-i SS weapon and independent and identically exponentially distributed with mean $1/r_B(i)$; i = 1, ..., M_1.

(3) When a Blue type-i SS weapon fires a shot, it is directed at exactly one Red weapon, chosen from among *all* Red weapons currently surviving, according to a uniform distribution, independently of the past history of the process.

(4) The conditional probability that a Blue type-i SS weapon kills a Red type-j weapon, given a shot fired at that weapon, is $p_B(i,j)$ for i = 1, ..., M_1 and j = 1, ..., N.

(5) Assumptions (3), (4), and (5) hold for Red SS weapons with parameters $r_B(j)$, j = 1, ..., N_1 and $p_R(j,i)$, j = 1, ..., N_1; i = 1, ..., M.

(6) The time required for a particular, surviving Blue type-i LS weapon to detect a particular, surviving Red type-j weapon is exponentially distributed with mean $1/d_B(i,j)$ for i = 1, ..., M_2 and j = 1, ..., N. Each Blue LS weapon detects different Red weapons independently of one another.

(7) A Blue LS weapon attacks every Red weapon it detects. The conditional probability that a Blue type-i LS weapon kills a Red type-j weapon, given detection and attack, is $k_B(i,j)$ for i = 1, ..., M_2; j = 1, ..., N.

(8) Red LS weapons satisfy Assumptions (6) and (7) with mean detection times $1/d_R(j,i)$ and kill probabilities $k_R(j,i)$, defined for j = 1, ..., N_2 and i = 1, ..., M.

(9) Times between shots fired by a surviving Blue type-i SM weapon ($i = 1, \ldots, M_3$) are independent and identically exponentially distributed with mean $1/r_B^*(i)$.

(10) Given that a Blue type-i SM weapon fires a shot at a currently surviving Red force y (recall the notation introduced above), the probability that the surviving target force has the composition z is $\varphi_B(i,y;z)$. Symbolically,

$$\varphi_B(i,y;z) = P\{\text{Surviving Red force is } z \mid \text{shot is fired at Blue type-i SM weapon at Red force with composition } y\}.$$

Here $y,z \in \underset{\sim}{N}^N$.

(11) Red SM weapons obey Assumptions (9) and (10), with mean firing rates $r_R^*(j)$, $j = 1, \ldots, N_3$ and kill distributions $\varphi_R(j,x;w)$ defined for $j = 1, \ldots, N_3$; $x,w \in \underset{\sim}{N}^M$.

(12) The time required for a particular surviving Blue type-i LM weapon ($i = 1, \ldots, M_4$) to detect a particular, surviving Red type-j weapon ($j = 1, \ldots, N$) is exponentially distributed with mean $1/d_B^*(i,j)$.

Each Blue LM weapon detects different Red weapons independently of one another.

(13) At the instant of each detection, a Blue LM weapon fires a shot.

(14) Each shot is independent of the previous history of the attrition process. If a Blue type-i LM weapon fires at a Red force of composition y after having detected a Red type-j weapon ($j = 1, \ldots, N$), the probability that the Red force survives the shot is z, is denoted by $\mu_B(i,j,y;z)$.

(15) Red weapons of the class (not type!) LM satisfy Assumptions (12), (13), and (14), with parameters $d_R^*(j,i)$, $\mu_R(j,i,x;w)$, defined for $j = 1, \ldots, N_4$, $i = 1, \ldots, M$ and $x,w \in \underset{\sim}{N}^M$.

(16) The detection, attack, and kill processes of all weapons are mutually independent.

b. Remarks on the Assumptions

The spirit and meaning of the assumptions is that of Reference [4]. In particular, the terminology used, although suggestive and frequently (we feel) the most plausible interpretation, need not be adhered to exactly. Especially, the apparent dichotomy between square- and linear-law engagement initiations (on the basis of no detections or individual detection) can be interpreted differently; further comments can be found in Reference [4, p. 39] and in the beginning of the subsection (B.1).

The derivation of the kill distributions φ_B, φ_R, μ_B, and μ_R is a problem that we have not yet treated in any depth. In any application of our model to computerized simulations, this would be the problem most in need of attention. Here we have indicated abstractly what functional dependencies seem plausible and, hence, those that we feel can safely be ignored. For example, the kill distributions do not depend on the structure of the force to which a shooting weapon belongs, although in principle they could. Indeed, a reasonable way of describing such dependence would allow representation of the synergistic effects of weapons on the same side. Similarly, the kill distributions μ_B and μ_R for LM weapons indicate that the distribution oi weapons killed can depend on the particular (type of) weapon detected; differing ammunition or tactics used against different detected weapons can thus be modeled. Further comments appear in Subsection c (below), where we also give some examples.

For certain parameters, to avoid unnecessary proliferation of notation, asterisks denote multiple-kill weapons and no asterisk denotes single-kill weapons.

That all "engagements" occur instantaneously, with ensuing total loss of contact, is admittedly an unrealistic aspect of the model (although no other theater-level models seem to have successfully addressed the difficulty either). One way of including binary engagements with exponentially distributed lengths is discussed in the Process L2 of Reference [4].

c. Results

We can now describe and characterize the stochastic attrition process engendered by Assumptions (1-16), above. Let $E = \underset{\sim}{N}^{M+N}$ consist of states denoted by $\alpha = (x,y)$; α is to be thought of as a possible pair of surviving forces at some instant, with x corresponding to the Blue force and y the Red. As a measurable space, E is assumed to be endowed with the σ-algebra of all its subsets. The sample space for our attrition process is the family Ω of functions from $\underset{\sim}{R}+$ to E that are right-continuous and have limits from the left with respect to the discrete topology on E. The coordinate (vector-valued) stochastic process $\big((B_t,R_t)\big)_{t\geq 0}$ (here $B_t:\Omega \to \underset{\sim}{N}^M$ and $R_t:\Omega \to \underset{\sim}{N}^N$ for each t) has the usual interpretation: B_t is the surviving Blue force at time t, and R_t is the Red force at time t. We further define

$$\underline{\underline{F}}_t = \sigma\big((B_u,R_u);\ 0 \leq u \leq t\big) ,$$

which is the history of the attrition process until time t, and

$$\underline{\underline{F}} = \sigma\big((B_u,R_u);\ u \geq 0\big) ,$$

which is the entire history of the process. For each $\alpha \in E$, we denote by P^{α} the probability law on $(\Omega,\underline{\underline{F}})$ of the attrition process governed by Assumptions (1-16), subject to the initial condition

$$P^{\alpha}\{P_0,R_0) = \alpha\} = 1 .$$

One then has the following characterization of the stochastic attrition process defined by Assumptions (1-16); the reader is referred to Reference [4] for details concerning the role of continuous-time Markov processes in attrition modeling and interpretation of the quantities involved.

THEOREM. Subject to Assumptions (1-16) listed above, the stochastic process

$$(\underset{\sim}{B},\underset{\sim}{R}) = \left(\Omega,\underline{\underline{F}}_t,\underline{\underline{F}}_t,(B_t,R_t),P^{\alpha}\right)$$

is a regular Markov process with state space E. The infinitesimal generator A of the process is of the following form: consider two states $\alpha = (x,y)$ and $\alpha' = (x',y')$ such that $y' = y$ and $x_k' = x_k$ for all k except $k = i$, where i is a fixed index, and $x_i' = x_i - 1$ (the new state α' is reached from α by the destruction of *exactly one* Blue type-i weapon). Then

$$\begin{aligned} A(\alpha,\alpha') = {} & \frac{x_i}{\sum_{k=1}^{M} x_k} \sum_{j=1}^{N_1} r_R(j)p_R(j,i)y_j \\ & + x_i \sum_{j=1}^{N_2} d_R(j,i)k_R(j,i)y_{N_1+j} \\ & + \sum_{j=1}^{N_3} r_R^{*}(j)\varphi_R(j,x;x')y_{N_1+N_2+j} \\ & + \sum_{k=1}^{M} x_k \sum_{j=1}^{N_4} d_R^{*}(j,k)\mu_R(j,k,x;x')y_{N_1+N_2+N_3+j} \; . \end{aligned} \tag{1}$$

For any state $\alpha' = (x',y')$ with $y' = y$, $x_i' \leq x_i$ for all i, and $\sum_i (x_i - x_i') \geq 2$ (this corresponds to the simultaneous destruction of more than one Blue weapon and can be effected only by Red weapons of classes SM and LM), we have

$$\begin{aligned} A(\alpha,\alpha') = {} & \sum_{j=1}^{N_3} r_R^{*}(j)\varphi_R(j,x;x')y_{N_1+N_2+j} \\ & + \sum_{k=1}^{M} x_k \sum_{j=1}^{N_4} d^{*}_R(j,k)\mu_R(j,k,x;x')y_{N_1+N_2+N_3+j} \; . \end{aligned} \tag{2}$$

Similarly, for $\alpha' = (x', y')$ such that $x' = x$, $y'_\ell = y_\ell$ for all $\ell \neq j$ and $y'_j = y_j - 1$,

$$A(\alpha,\alpha') = \frac{y_j}{\sum_{\ell=1}^{N} y_\ell} \sum_{i=1}^{M_1} r_B(i) p_B(i,j) x_i$$

$$+ y_j \sum_{i=1}^{M_2} d_B(i,j) k_B(i,j) x_{M_1+i}$$

$$+ \sum_{i=1}^{M_3} r_B^*(i) \varphi_B(i,y;y') x_{M_1+M_2+i}$$

$$+ \sum_{\ell=1}^{N} y_\ell \sum_{i=1}^{M_4} d_B^*(i,\ell) \mu_B(i,\ell,y;y') x_{M_1+M_2+M_3+i} ,$$

while for any other state $\alpha' = (x', y')$ such that $x' = x$, $y'_j \leq y_j$ for all j and $\sum_j (y_j - y'_j) \geq 2$, we have

$$A(\alpha,\alpha') = \sum_{i=1}^{M_3} r_B^*(i) \varphi_B(i,y;y') x_{M_1+M_2+i}$$

$$+ \sum_{\ell=1}^{N} y_\ell \sum_{i=1}^{M_4} d_B^*(i,\ell) \mu_B(i,\ell,y;y') x_{M_1+M_2+M_3+i} .$$

Moreover, for all $\alpha' \neq \alpha$ and not of the forms above, we have

$$A(\alpha,\alpha') = 0 ;$$

and, finally,

$$A(\alpha,\alpha) = - \sum_{\alpha' \neq \alpha} A(\alpha,\alpha') .$$

The expressions for the generator A have probabilistic interpretations and are written in what we feel is the most revealing form. Consider for example, the term $A(\alpha,\alpha')$ of Equation (1). Here the first summand on the right (of the four) is the (instantaneous) rate at which Blue type-i weapons are being killed by Red SS weapons when the two forces have compositions x and y, respectively, and the second term is a similar rate arising from Red LS weapons. For kills caused by single-shot weapons, "rate of kill" and "rate of kill one at a time" are the same notion, but not for multiple-shot weapons. Hence, the third summand on the right-hand side of Equation (1) must be interpreted as the instantaneous rate at which Blue type-i weapons are being killed *exactly one at a time* by Red SM weapons. The interpretation of the fourth summand is then analogous; note that the kill of a type-i weapon can in principle arise from the detection of any type of weapon. Hence, $A(\alpha,\alpha')$ is, when the structure of the two forces is $\alpha = (x,y)$, the instantaneous rate at which Blue type-i weapons are being destroyed precisely one at a time. Note that single kills can, in general, arise from multiple-shot weapons, as indicated by the presence of the third and fourth summands in Equation (1).

For the term of A given by Equation (2), only multiple-kill weapons need be considered; and interpretations are similar to those given above.

We next discuss in this section the potential application of the attrition model derived above as the assessment mechanism in a theater-level deterministic combat simulation (such as IDAGAM I [3]). There are three principal problems to be considered: implementation of the attrition model on a computer, derivation of the qualitative form of the kill distributions φ_B, φ_R, μ_B, μ_R for multiple-shot weapons, and selection of the exact values of input parameters. We shall discuss mostly the first two problems, with brief consideration of the third and of other, minor problems.

Within an iterative deterministic simulation like IDAGAM I, the most difficult problem is the proper handling of expected values of random variables. At present there is no justified method of doing so in this context. Consider the attrition model derived above. When the initial forces are (the deterministic point) (x,y), what is the expected attrition to a given type of weapon in a unit interval of time? Denote by $(P_t)_{t \geq 0}$ the transition function of the attrition process $(\underset{\sim}{B},\underset{\sim}{R})$; that is,

$$P_t(\alpha,\alpha') = P^{\alpha}\{(B_t,R_t) = \alpha'\}$$

for all α, $\alpha' \in E$. The expected number of Blue type-i weapons surviving at the end of one time period is then

$$E^{(x,y)}[B_1(i)] = \sum_{k=1}^{x_i} k\, P^{(x,y)}\{B_1(i) = k\}$$

$$\sum_{k=1}^{x_i} k \sum_{\alpha'=(x',y'):x_i'=k} P_1((x,y),\alpha') \ ,$$

when initial conditions are (x,y). From this the expected attrition is easily computed. Hence proper expected attritions for deterministic initial conditions can be computed once the transition function is known and, in fact, once the Markov matrix P_1 is known.

Computation of the transition function from the generator A, however, is not a simple matter in practice. The relevant expression is

$$P_t = e^{tA} = \sum_{n=0}^{\infty} \frac{t^n A^n}{n!} \ . \qquad (3)$$

Uniformly good finite approximations may be possible, but they would entail enormous requirements for storage and computation time for forces of theater- (or even sector-) level magnitude. The only approximation involving no such difficulties is the first-order approximation given by

$$E^{(x,y)}[B_1(i)] \sim \sum_{k=1}^{x_i} k \sum_{(x',y'):x_i'=k} A\big((x,y);(x',y')\big) , \quad (4)$$

which is quite feasible for implementation. Note that in addition we have

$$P^{(x,y)}\{B_1(i) = k\} \sim \sum_{(x',y'):x_i=k} A\big((x,y);(x',y')\big) , \quad (5)$$

which gives the approximate probability distribution of the number of surviving Blue type-i weapons. Joint distributions of number of surviving weapons may be similarly approximated.

IDAGAM I and similar models employ an iterative method of calculation for representing the evolution of time. In such a scheme, the outputs of the attrition calculation for one time-period constitute a portion of the inputs for the computation corresponding to the next time-period. In particular, if a deterministic equation of the form given in Equation (4)--or even a correct evaluation of expected numbers of survivors--is used to compute numbers of survivors, then the inputs for the second period's calculations are the expected number of survivors from the first; and therefore the result of that calculation will not, in general, be the expected number of survivors at the end of the second period. What has happened, of course, is that the random number of survivors at the end of the first period was replaced by its expected value for use as an initial condition for the second period. Within the current structure of IDAGAM, there is no way to circumvent this difficulty.

Rather drastic alterations to IDAGAM might allow the program to carry from one period to the next the joint distribution of the numbers of surviving weapons, in which case Equation (5) is the relevant approximation and no further approximations or simplifications are required.

In a somewhat different setting, the generator A contains sufficient data to perform Monte Carlo simulations designed to improve the approximations in Equations (4) and (5) without performing the computations required to obtain Equation (3) or one of the approximating partial sums. This possibility appears worthy of further investigation.

The storage and bookkeeping problems associated with implementation of Equation (4) as an attrition equation would be significant, but probably not overwhelming. Most of the difficulties would arise from the large size of the generator matrix A and the large number of kill distributions required to be stored for possible use in computations.

We discuss next the problem of derivation of the form of the kill distributions for multiple-shot weapons. First of all, each type of weapon must be placed in one of the four classes of weapons defined earlier. Then for each multiple-kill weapon the form of the kill distributions must be determined. This is evidently an arduous and lengthy task, and there is no clear conception as to how one should even begin. We offer here only some plausible examples and interpretive comments.

We first show how single-shot weapons can be looked upon as special cases of multiple-shot weapons. Consider a Blue type-i SM weapon and the associated kill distribution $\varphi_B(i, \cdot;\cdot)$. If for each y the probability measure $\varphi_B(i,y;\cdot)$ is of the form

$$\varphi_B(i,y;y^{(j)}) = \frac{y_j}{\sum_{\ell=1}^{N} y_\ell} p_B'(i,j) \ ,$$

where

$$y_k^{(j)} = \begin{cases} y_k, & \text{if } k \neq j \\ y_j - 1, & \text{if } k = j \end{cases}$$

and $0 \leq p_B'(k,j) \leq 1$, then the weapon becomes an SS weapon satisfying Assumptions (2-4), provided $\varphi_B(i,y;y') = 0$ for y' not of the form above nor equal to y.

More generally, suppose that whenever y' is such that $y_\ell' = y_\ell$ for all $\ell \neq j$ one has

$$\varphi_B(i,y;y') = \frac{y_j}{\sum_{\ell=1}^{N} y_\ell} \varphi_B'(i,j,y_j;y') \ ,$$

where $\varphi_B'(i,j,y_j;y')$ is the probability that if the weapon in question fires at a type-j target when there are y_j such targets present, y_j' of them survive. In this case, the target type is chosen by the uniform fire allocation of Assumption (3), but possibly (although not necessarily) more than one of the class of target weapons can be destroyed. This structure admits, therefore, the situation in which a weapon is single-kill against some kinds of targets but multiple-kill against others, answering the potential criticism raised previously.

The binomial distributions of the Process A1 of Reference [4] could be used for the kill distributions φ_B' (above).

For weapons of class LM, it might be reasonable to allow only weapons of the same type as that detected to be killed.

Further work on deriving reasonably simple kill distributions based on rigorous but plausible hypotheses is the aspect of this model in greatest need of further research.

Data-gathering and interpretation problems could also prove troublesome. Firing rates, for example, must be averaged to account for periods of time in which no interaction occurs, which may possibly preclude use of existing data. This problem, however, is irrelevant to the internal consistency and plausibility of the attrition model.

2. Barrier Penetration Processes

We next describe a new attrition model, which we propose for the description of processes of barrier penetration (Class I.B.2). While related to the binomial attrition model discussed below, this new model is different in several important respects, the most crucial of which is that the families of underlying assumptions are different (and such differences, as we have previously noted, are the most sensible basis for comparing and distinguishing different models). Secondly, the new model separates explicitly the detection and attack phases of combat interaction, which other models of comparable simplicity do not. Moreover, the model permits simple computations of the probability distributions (rather than just expectations) of relevant random variables. This property is especially important in view of the desirability of developing combat simulations in which random variables are not incorrectly and unjustifiably replaced by their expectations.

Our model describes a bilateral combat attrition process involving a set of defenders and a set of penetrators (or attackers). The assumptions we give here attempt to be as free of restrictive physical interpretation as possible, but obviously cannot be entirely so; roughly speaking, one should envision the defenders as protecting some target that the penetrators wish to attack.

a. **Assumptions**

(1) Penetrators attempt to penetrate the defenses and reach the target successively, one after another.

(2) An attacker attempting penetration of the defenses is detected by each defender present with probability d, independent of detections by any other defenders.

(3) If the penetrator is detected by one or more defenders, then exactly one defender is assigned to engage the penetrator in a one-on-one duel.

(4) An engagement between a penetrator and a defender ends in one of the following outcomes (with the respective probabilities shown), independent of the past history of the process.

Outcome of Engagement	Probability
Destruction of both	p_1
Destruction of penetrator only	p_2
Destruction of defender only	p_3
Destruction of neither	p_4

Clearly, $0 \leq p_i \leq 1$ ($i = 1, 2, 3, 4$) and $p_1 + p_2 + p_3 + p_4 = 1$.

(5) A defender that survives a duel is unable to return to the set of active defenders; a penetrator that survives a duel must turn back without attempting to attack the target.

Assumptions (1), (3), and (5) are those that are distinctive to this model. All action is supposed to occur within some fixed period of time.

We now define some quantities of interest in this attrition model, whose probability distributions and expectations we shall then proceed to compute. Let

D_k = number of active defenders remaining after k penetrators have attempted penetrations;

B_k = number of engagements involving one of the first k penetrators;

X_k = number of defenders destroyed by the first k penetrators;

Y_k = number of the first k penetrators destroyed;

U_k = number of defenders engaged, but not destroyed, by the first k penetrators; and

V_k = number of the first k penetrators that are engaged by defenders, but not destroyed.

Obviously,

$$
\begin{aligned}
B_k + D_k &= D_0 \\
X_k + U_k &= B_k \\
Y_k + V_k &= B_k
\end{aligned}
\tag{6}
$$

for each k.

b. Results

The stochastic process $D = (D_k)_{k \geq 0}$ describing the evolution of the set of active defenders is the key to the analysis of this attrition model and may be characterized in the following manner.

THEOREM. The stochastic process D is a Markov process with state space $\underset{\sim}{N} = \{0,1,2,\cdots\}$ transition matrix P given by

$$P(0,0) = 1 ,$$

while for $i \geq 1$,

$$P(i,j) = \begin{cases} 1 - (1 - d)^i, & \text{if } j = i - 1; \\ (1 - d)^i, & \text{if } j = i . \end{cases}$$

The transition matrix P has the following simple form:

$$
P = \begin{array}{c|cccccccc}
 & 0 & 1 & 2 & \cdots & i-1 & i & \cdots \\
\hline
0 & 1 & & & & & & \\
1 & d & (1-d) & & & & & \\
2 & 0 & 1-(1-d)^2 & (1-d)^2 & & & & \\
\vdots & & & \ddots & \ddots & & & \\
i & & & & & 1-(1-d)^i & (1-d)^i & \\
\vdots & & & & & & \ddots & \ddots
\end{array} ,
$$

which should allow efficient calculation on a computer of the powers of P used in the following further computations; further details appear below.

The engagement process B is somewhat harder to describe. Because of the presence of the random variable D_0 in Equation (6), this process is not a Markov process. But we can state two descriptions; the first is interesting in an abstract sense, and the second is of practical computational value.

THEOREM. Conditioned on the random variable D_0, B is a Markov process with transition matrix $\hat{P}$ given by

$$\hat{P}(i,j) = P(D_0 - i, D_0 - j) .$$

THEOREM. For each k and for $j \leq k$ we have

$$P\{B_k = j\} = \sum_{\ell} P\{D_0 = \ell\} P^k(\ell, \ell - j) .$$

We next compute the probability distributions for the attrition processes $X = (X_k)_{k \geq 0}$ and $Y = (Y_k)_{k \geq 0}$.

THEOREM. For $\ell \leq k$, we have

$$P\{X_k = \ell\} = \sum_{m=\ell}^{k} \binom{m}{\ell} (p_1 + p_3)^{\ell} (p_2 + p_4)^{m-\ell} P\{B_k = m\}$$

and

$$P\{Y_k = \ell\} = \sum_{n=\ell}^{k} \binom{n}{\ell} (p_1 + p_2)^{\ell} (p_3 + p_4)^{n-\ell} P\{B_k = n\} .$$

By exactly the same methods, we obtain probability distributions for the processes U and V, included here for the sake of completeness.

THEOREM. Provided that $\ell \leq k$,

$$P\{U_k = \ell\} = \sum_{m=\ell}^{k} \binom{m}{\ell} (p_2 + p_4)^{\ell} (p_1 + p_3)^{m-\ell} P\{B_k = m\}$$

$$P\{V_k = \ell\} = \sum_{n=\ell}^{k} \binom{n}{\ell} (p_3 + p_4)^{\ell} (p_1 + p_2)^{n-\ell} P\{B_k = n\} .$$

There are other quantities of interest, whose probability distributions one would like to obtain. Let

R_k = number of defenders surviving (but not necessarily active) after k attempted penetrations;

S_k = number of the first k penetrators that are *not* detected and engaged by defenders; and

T_k = number of the first k penetrators surviving interactions (if any) with defenders.

Clearly,

$$R_k = D_0 - X_k;$$

$$S_k = k - B_k; \text{ and}$$

$$T_k = k - Y_k .$$

Thus, we have the following result:

THEOREM. For appropriate values of ℓ, one has the following expressions:

$$P\{R_k = \ell\} = \sum_{i,j} \binom{j}{\ell-i+j} (p_2 + p_4)^{\ell-i+j} (p_1 + p_3)^{i-\ell} P^k(i,i-j) P\{D_0=i\};$$

$$P\{S_k = \ell\} = P\{B_k = k - \ell\}; \text{ and}$$

$$P\{T_k = \ell\} = P\{Y_k = k - \ell\}.$$

So far we have considered fixed numbers of penetrators. What happens to all this if the number of penetrators attempting to penetrate the defense within the time period under consideration is a random variable A? Essentially, all the necessary mathematics is done. For example, suppose we wish to compute the probability distribution of the number of defenders killed, which in this case is the random variable X_A. Then we have (assuming that A is independent of D_0 and the attrition process)

$$\begin{aligned} P\{X_A = q\} &= \sum_k P\{X_A = q, \ A = k\} \\ &= \sum_k P\{X_k = q, \ A = k\} \\ &= \sum_k P\{X_k = q\} P\{A = k\} , \end{aligned}$$

which is immediately calculable in terms of the probability distribution of A and quantities given above. Similar comments apply to the other stochastic processes we have defined.

In Figure 3-3, we summarize how all relevant probability distributions can be computed in terms of the probability distributions of the numbers of defenders and penetrators, the detection probability d and the kill probabilities p_1, p_2, p_3, and p_4. We also include the expectation of each random variable.

In some cases, identities given below allow simplification of the expressions for expectations. Indeed, all expectations depend only on $E[D_0]$, $E[A]$, and $E[D_A]$.

Given: $\mu(i) = P\{D_0 = i\}$ (initial number of defenders)

$\lambda(j) = P\{A = j\}$ (number of penetrators)

P = transition matrix of (D_k)

p_1, p_2, p_3, p_4 = engagement-outcome probabilities

Computations:

(1) D_A = number of remaining active defenders after all attempted penetrations.

$$P\{D_A = k\} = \sum_{i,j} \mu(i)\lambda(j)P^j(i,k) .$$

$$E[D_A] = \sum_{i,j,k} \mu(i)\lambda(j)kP^j(i,k) .$$

(2) B_A = number of one-on-one engagements occurring.

$$P\{B_A = k\} = \sum_{i,j} \mu(i)\lambda(j)P^j(i,i-k) .$$

$$E[B_A] = E[D_0] - E[D_A]$$

$$= \sum_i i\mu(i) - \sum_{i,j,k} \mu(i)\lambda(j)kP^j(i,k).$$

(3) X_A = number of defenders destroyed.

$$P\{X_A = k\} = \sum_{i,j,\ell} \mu(i)\lambda(j)P^j(i,i-\ell)\binom{\ell}{k}(p_1+p_3)^k(p_2+p_4)^{\ell-k} .$$

$$E[X_A] = (p_1+p_3)E[B_A]$$

$$= (p_1+p_3)\left(\sum_i i\mu(i) - \sum_{i,j,k} \mu(i)\lambda(j)kP^j(i,k)\right) .$$

(concluded on next page)

Figure 3-3. COMPUTATION OF PROBABILITY DISTRIBUTIONS

(4) Y_A = number of penetrators destroyed.

$$P\{Y_A = k\} = \sum_{i,j,\ell} \mu(i)\lambda(j)P^j(i,i-\ell)\binom{\ell}{k}(p_1+p_2)^k(p_3+p_4)^{\ell-k} .$$

$$E[Y_A] = (p_1+p_2)E[B_A]$$

$$= (p_1+p_2)\left(\sum_i i\mu(i) - \sum_{i,j,k} \mu(i)\lambda(j)kP^j(i,k)\right) .$$

(5) R_A = number of defenders surviving at the end of the time period under consideration.

$$P\{R_A = k\} = \sum_{i,j,m} \mu(i)\lambda(j)\binom{m}{k-i+m}(p_2+p_4)^{k-i+m}(p_1+p_2)^{i-m} \times P^j(i,i-m).$$

$$E[R_A] = E[D_0] - E[X_A]$$

$$= (p_2+p_4)\sum_i i\mu(i) + (p_1+p_3)\sum_{i,j,k} \mu(i)\lambda(j)kP^j(i,k).$$

(6) S_A = number of penetrators able to attack the target.

$$P\{S_A = k\} = \sum_{i,j} \mu(i)\lambda(j)P^j(i,i-j+k) .$$

$$E[S_A] = E[A] - E[B_A]$$

$$= \sum_j j\lambda(j) - \sum_i i\mu(i) + \sum_{i,j,k} \mu(i)\lambda(j)kP^j(i,k) .$$

(7) T_A = number of penetrators surviving interactions (if any) with defenders.

$$P\{T_A = k\} = \sum_{i,j,m} \mu(i)\lambda(j)\binom{m}{j-k}(p_1+p_3)^{j-k}(p_2+p_4)^{m-j+k}P^j(i,1-m) .$$

$$E[T_A] = E[A] - E[Y_A]$$

$$= \sum_j j\lambda(j) - (p_1+p_2)\sum_i i\mu(i) + (p_1+p_2)\sum_{i,j,k} \mu(i)\lambda(j)kP^j(i,k).$$

Details concerning the assumptions of this model, their interpretations, and their validity can be found in Reference [2].

Implementation of this model in the context of a computerized combat simulation requires an efficient way to compute powers of the transition matrix P, a problem to which we next briefly turn. Such computations are facilitated by the special form of P.

P is a lower triangular matrix of the following form (when restricted to a finite subset of $\{0, 1, \ldots, n\}$ of the full state space $\underset{\sim}{N} = \{0, 1, 2, \ldots\}$):

$$P = \begin{bmatrix} P(0,0) & & & & & \\ P(1,0) & P(1,1) & & & & \\ & P(2,1) & P(2,2) & & & \\ & & \ddots & \ddots & & \\ & & & \ddots & \ddots & \\ & & & & P(n,n-1) & P(n,n) \end{bmatrix}$$

While one can make use of this structure in order to diagonalize the matrix P and thus obtain its powers, those powers may be found explicitly, by using a transform agrument.

THEOREM. For each n, i, and j, we have

$$P^n(i,j) = \begin{cases} \prod_{\ell=j}^{i-1} \left(1-(1-d)^{\ell}\right) \sum_{k=1}^{i-1} \sum_{m=1}^{n} \dfrac{(1-d)^{km-k+ik-im}}{\prod_{\substack{q=j \\ q \neq k}}^{i-1} \left((1-d)^k - (1-d)^q\right)}, & \text{if } j < i,\ i - j \leq n; \\ (1-d)^{in}, & \text{if } j = i; \\ 0, & \text{otherwise.} \end{cases}$$

When the number of attempted penetrations is small, recursive calculations of the quantities of interest may be carried out in the following manner. Recall that B_ℓ is the number of the first ℓ penetrators that are detected. Then it can be shown that

$$P\{B_\ell = k|D_0 = M\} = P\{B_{\ell-1} = k - 1|D_0 = M\}[1 - (1-d)^{M-k+1}] + P\{B_{\ell-1} = k|D_0 = M\}(1-d)^{M-k} .$$

It is also true that

$$P\{B_\ell = 0|D_0 = M\} = (1-d)^{\ell M} .$$

Consequently, given

$$P\{B_\ell = i|D_0 = M\} \quad \text{for} \quad \ell = i, \ldots, M ,$$

one can easily compute

$$P\{B_\ell = i + 1|D_0 = M\} \quad \text{for} \quad \ell = i + 1, \ldots, M .$$

From this, $E[B_\ell|D_0 = M]$ can be computed directly.

Little computer storage is required to implement these calculations. For notation purposes, let

$$f(k, \ell, M) = P\{B_\ell = k|D_0 = M\} .$$

Supressing the index M, we see that only values of $f(i, \ell-1, \cdot)$ for $\ell = i, \ldots, M$ are needed to calculate $f(i+1, \ell, \cdot)$ for $\ell = i + 1, \ldots, M$. Thus, only a single vector need be maintained to calculate entries of the array f. Maintaining, additionally, a vector $E[B_\ell]$ enables the computation of an expected number of detections, given ℓ attempted penetrations.

Some preliminary computational testing of this model has shown it to be rather efficient; it has as yet received no implementation or testing in the context of computerized combat simulation models.

3. Processes of Attack of Passive Targets

Finally, for processes of attack of passive targets, we recommend binomial attrition equations of either single-shot or multiple-shot form, as appropriate, which we now proceed to describe. The reader can find full details in Reference [5]. We begin with the homogeneous, single-shot case.

Consider a one-sided combat between two homogeneous forces, a force of R indistinguishable "targets" and a force of B indistinguishable "searchers." We make the following assumptions concerning this combat.

a. Assumptions

(1) At a fixed time, all R targets become vulnerable to detection and attack by the B searchers.

(2) The probability that the i^{th} searcher detects the j^{th} target is d for all $i = 1, \ldots, B$ and $j = 1, \ldots, R$. Each particular searcher detects different targets independently of one another.

(3) A searcher that makes no detections makes no attack. A searcher that makes one or more detections chooses one target to attack according to a uniform distribution over the set of targets detected, independent of the detection process.

(4) The conditional probability that a searcher kills a target (given detection and attack) is k, for all searchers and targets.

(5) No searcher may attack more than one target.

(6) Detection and attack processes of different searchers are mutually independent.

b. Results

We begin by computing the expected number of targets killed.

PROPOSITION. Assume that Assumptions (1-6) above are satisfied. Then, if K denotes the number of targets killed, we have

$$E[K] = R\{1 - (1 - \frac{k}{R}[1 - (1 - d)^R])^B\} .$$

To compute the probability distribution of the number of targets killed, which is needed to extend this static model to a discrete-time dynamic model, we take an alternative approach. It is not true, as one might conjecture, that the number of targets killed is binomially distributed with parameters R and q, where

$$q = 1 - (1 - \frac{k}{R}[1 - (1 - d)^R])^B .$$

The reason for this is that even though searchers operate independently of one another, different targets do not die independently of one another. Since each searcher can attack at most one target, knowing that a particular target was killed means that some other target was not attacked by the searcher that killed the former target and is, hence, less likely to have been killed.

PROPOSITION. For m such that $m \le R$ and $m \le B$, we have

$$P\{K = m\} = \binom{R}{m} \sum_{r=0}^{m} (-1)^{m-r}\binom{m}{r}\left[(1-q_R) + \frac{q_R r}{R}\right]^B ,$$

where

$$q_R = k[1 - (1 - d)^R] .$$

Use of the approximation

$$E[K] \sim R\left[1 - \exp\left(- \frac{Bk}{R}[1 - (1 - d)^R]\right)\right]$$

or the further approximation

$$E[K] \sim R\left[1 - \exp\left(- \frac{Bk}{R}[1 - e^{-dR}]\right)\right]$$

is valid in the sense that

$$\lim_{n\to\infty, a\downarrow 0} |e^{-an} - (1 - a)^n| = 0 ;$$

but such use is largely unnecessary, especially within the context of computerized combat simulations. The correct expression given above can be computed as quickly (or perhaps even more quickly, depending on how a specific computer performs exponentiations) as either approximation. Moreover, the approximations may be rather poor for small values of B or R or for moderately large values of k and d.

We can extend the model above to the case in which there are several types of targets and searchers, with detection and kill probabilities dependent on the type of target and type of searcher. The physics of the process, however, remains unchanged.

Let us assume that there are M types of searchers, B_i type-i searchers (i = 1, ..., M), N types of targets, and R_j type-j targets (j = 1, ..., N). We will impose the following hypotheses:

(1) At a fixed time all targets become vulnerable to detection and attack.

(2) The probability that a given, fixed type-i searcher detects a given, fixed type-j target is d_{ij}.

(3) Of the targets (of all kinds) detected by a given searcher, he chooses one to fire upon, according to a uniform distribution.

(4) Given that he detects and fires upon a type-j target, a given type-i searcher destroys that target with probability k_{ij}.

(5) A given searcher detects different targets independently of one another.

(6) No searcher may fire more than once.

(7) The detection and firing processes of all the searchers are mutually independent.

We wish to compute, under these assumptions, the expected number of targets of each type destroyed. First, we give an analytical solution. Let

$$B = B_1 + \dots + B_M$$

denote the total number of searchers; and let

$$R = R_1 + \dots + R_N$$

be the total number of targets. Let K_j be the number of type-j targets destroyed.

THEOREM. Under the Assumptions (1-7) above, the expected number of type-j targets destroyed is given by

$$E[K_j] = R_j\left(1 - \prod_{i=1}^{M} [1 - d_{ij}k_{ij}P(i,j)]^{B_i}\right), \tag{7}$$

where

$$P(i,j) = \sum_{r_1=0}^{R_1} \cdots \sum_{r_{j-1}=0}^{R_{j-1}} \sum_{r_j=0}^{R_j-1} \sum_{r_{j+1}=0}^{R_{j+1}} \cdots \sum_{r_N=0}^{R_N} \left\{ \frac{1}{1 + \sum_{p=1}^{N} r_p} \binom{R_j - 1}{r_j} d_{ij}^{r_j} (1 - d_{ij})^{R_j - 1 - r_j} \times \prod_{\substack{1 \le q \le N \\ q \ne j}} \binom{R_q}{r_q} d_{iq}^{r_q} (1 - d_{iq})^{R_q - r_q} \right\}. \tag{8}$$

Instead of the Assumptions (1) and (3) concerning the physics of the process of detection and attack, we could assume that targets become vulnerable to a given searcher sequentially in a randomly chosen order, with all R! orders equally probable, that each target is either detected or not, and that the first detected target is fired upon. This process occurs (once for each searcher) independently of the processes corresponding to other searchers, but no targets are destroyed until the end of the entire process. This interpretation is useful in considering allocation-of-fire problems.

It should be noted that this model has no provision for searcher determination of whether a given detected target is to be attacked or not. In reality, a searcher might be tempted to pass over low-value targets in the hope of detecting a high-value target. Or, if all targets are vulnerable simultaneously, a given searcher would not choose among them uniformly, but would instead choose the target whose destruction entails the largest reward to him.

While the model makes no explicit provision for such a choice mechanism, it could be incorporated in the "simultaneously vulnerable" interpretation. It develops in the derivation of Equation (7) that the uniform target choice is manifested only in the conditional probabilities

$$P\{A_{ij}(k,\ell) \mid G_{ij}(r_1, \ldots, r_N; k,\ell)\} = \frac{1}{1 + \sum_{p=1}^{N} r_p},$$

where $A_{ij}(k,\ell)$ is the event that the ℓ^{th} type-j target is attacked by k^{th} type-i searcher and $G_{ij}(r_1, \ldots, r_N; k,\ell)$ is the event that the k^{th} type-i searcher detects, in addition to the ℓ^{th} type-j target, r_j other type-j targets and r_p targets of type $p \neq j$.

Hence, a different rule for target choice can be incorporated simply by changing the form of this conditional probability, subject to some obvious regularity conditions.

EXAMPLE. Suppose that the value to a type-i searcher of destroying a type-j target is u_{ij}. Then,

$$v_{ij} = k_{ij} u_{ij}$$

is the expected return from an attack upon a type-j target.

The rule "Of the targets detected, fire at one whose destruction entails maximal expected return" leads to

$$P\{A_{ij}(k,\ell)\,|\,G_{ij}(r_1, \ldots, r_N;k,\ell)\}$$

$$= \begin{cases} \dfrac{1}{1 + r_j}\,, & \text{if } v_{ij} \geq v_{im} \text{ for all } m \neq j \text{ such that } r_m > 0\,; \\ 0\,, & \text{otherwise.} \end{cases}$$

Here we have assumed for simplicity that $v_{ij} \neq v_{im}$ whenever $j \neq m$. Other examples can be described similarly.

To implement Equation (7) as the attrition equation in a computerized combat model would be laborious and would lead to a cumbersome result. Let us consider the difficulties more carefully, beginning with some simple cases. First suppose that

$$N = 3$$

and

$$R_1 = R_2 = R_3 = 1\,.$$

In this case, Equation (8) yields

$$p(i,1) = \sum_{\ell=0}^{1} \sum_{m=0}^{1} \frac{1}{1 + \ell + m} \left[\binom{1}{\ell} d_{12}^{\ell}(1 - d_{12})^{1-\ell} \binom{1}{m} d_{13}^{m}(1 - d_{13})^{1-m}\right]$$

$$= 1[(1 - d_{12})(1 - d_{13})]$$

$$+ \tfrac{1}{2}[(1 - d_{12})d_{13} + d_{12}(1 - d_{13})]$$

$$+ \tfrac{1}{3}[d_{12}d_{13}]$$

$$= 1 - \tfrac{1}{2}(d_{12} + d_{13}) + \tfrac{1}{3}(d_{12}d_{13})\,.$$

Analogous expressions for p(i,2) and p(i,3) clearly exist. If

$$N = 4$$

and

$$R_1 = R_2 = R_3 = R_4 = 1 ,$$

we have

$$p(i,1) = 1 - \frac{1}{2}(d_{i2} + d_{i3} + d_{i4}) + \frac{1}{3}(d_{i2}d_{i3} + d_{i2}d_{i4} + d_{i3}d_{i4}) - \frac{1}{4} d_{i2}d_{i3}d_{i4} .$$

The general pattern is by now evident; we summarize it in the following result.

PROPOSITION. For any N, if

$$R_1 = R_2 = R_3 = \cdots = R_N = 1 ,$$

then for $1 \leq i \leq M$ and $1 \leq j \leq N$,

$$p(i,j) = 1 + \sum_{m=2}^{N} \frac{(-1)^m}{m} \sum_{\substack{j_1 \\ j_1 \neq j}} \sum_{\substack{j_2 > j_1 \\ j_2 \neq j}} \cdots \sum_{\substack{j_m > j_{m-1} \\ j_m \neq j}} d_{i,j_1} d_{i,j_2} \cdots d_{i,j_m} .$$

The Proposition is not so useless as it first appears: in a computer implementation, one could assume that there is only one target of each type--even though some targets have the same probabilities of detection and kill given detection--then carry out the attrition calculation using Equation (7) and the Proposition. If targets were grouped into classes of indistinguishable (except by artificial notation) objects, attrition ΔR_k to objects in the k^{th} class would then be given by

$$\Delta R_k = \sum_{j \varepsilon C_k} \left(1 - \prod_{i=1}^{M} [1 - d_{ij}k_{ij}p(i,j)]^{B_i}\right) , \qquad (9)$$

where C_k is the set of indices j such that the j^{th} object belongs to the k^{th} class. For fixed i and k, d_{ij} and k_{ij} are independent of $j \in C_k$.

While Equation (9) could be implemented on a computer, its use might involve long and time-consuming computations, especially if R were of the magnitude of the number of soldiers in a moderate-sized land battle. Hence it is worthwhile to seek simplified versions of Equation (7), several of which we now proceed to consider.

The following result, which aids in developing simplified versions of Equation (7), shows that the complicated form of Equation (7) arises from having more than one type of target, rather than from having more than one type of searcher.

PROPOSITION. If $N = 1$, then

$$E[K] = R\left(1 - \prod_{i=1}^{M} \left(1 - \frac{k_i}{R}[1 - (1 - d_i)^R]\right)^{B_i}\right),$$

where subscripts i denote the type of searcher.

We next discuss three simplified versions of Equation (7) for the case $N > 1$. The notation $\Delta R_j = E[K_j]$ is used hereafter.

Successive Application of All Searchers to Each Type of Target. If there were only type-j targets, then the expected number of such targets destroyed would be

$$\Delta R_j = R_j\left[1 - \prod_{i=1}^{M} \left(1 - \frac{k_{ij}}{R_j}[1 - (1 - d_{ij})^{R_j}]\right)^{B_i}\right]. \quad (10)$$

One means of attrition assessment would calculate each ΔR_j by Equation (10), but this method clearly gives an advantage to searchers which is unwarranted in terms of Assumption (1). In effect, this method allows each searcher one shot at each type of target, rather than one shot altogether.

Calculation With Weighted Probabilities. For each i, the quantity

$$\bar{d}_i = \frac{1}{R} \sum_{j=1}^{N} d_{ij} R_j$$

represents a detection probability for type-i searchers averaged with respect to the numbers of targets present, while

$$\bar{k}_i = \frac{1}{R} \sum_{j=1}^{N} k_{ij} R_j$$

is a similarly averaged conditional probability of kill, given detection.

We could then approximate the original attrition process by one with R targets of a single type and B_i type-i searchers with detection and kill probabilities $\bar{d}_i$ and $\bar{k}_i$, respectively. Attrition to these R targets would be

$$\Delta R = R\left(1 - \prod_{i=1}^{M} \left(1 - \frac{\bar{k}_i}{R}\left[1 - (1 - \bar{d}_i)^R\right]\right)^{B_i}\right) .$$

Against type-j targets would then be assessed the fraction R_j/R of this attrition ΔR, so that

$$\Delta R_j = R_j\left(1 - \prod_{i=1}^{M} \left(1 - \frac{\bar{k}_i}{R}\left[1 - (1 - \bar{d}_i)^R\right]\right)^{B_i}\right) . \quad (11)$$

In Equation (11) it was unnecessary to use the averaged conditional probabilities of kill given detection and attack, and we could more accurately have written

$$\Delta R_j = R_j\left(1 - \prod_{i=1}^{M} \left(1 - \frac{k_{ij}}{R}\left[1 - (1 - d_i)^R\right]\right)^{B_i}\right) . . \quad (11')$$

It is possible, using one further averaging step, to use the homogeneous equation directly. The numbers

$$d = \frac{1}{B} \sum_{i=1}^{M} B_i \bar{d}_i$$

and

$$k = \frac{1}{B} \sum_{i=1}^{M} B_i \bar{k}_i$$

are, respectively, probabilities of detection and kill, given detection, averaged with respect to both targets and searchers.

We may approximate the original attrition process by an attrition process with R targets of a single type, B searchers of a single type, and parameters d and k as computed just above. The expected attrition in such a process is given by

$$\Delta R = R\left(1 - \left(1 - \frac{k}{R}[1 - (1 - d)^R]\right)^B\right) .$$

Using the same attrition apportionment as in Equation (11), these three equations yield the following values for the expected attrition to type-j targets:

$$\Delta R_j = R_j\left(1 - \left(1 - \frac{k}{R}[1 - (1 - d)^R]\right)^B\right) . \qquad (12)$$

Prior Apportionment of Searchers. We might also model the attrition process as a number of smaller engagements by (prior to attrition assessment) dividing the searchers among different types of targets on the basis of relative numbers of targets present and vulnerable. That is, type-j targets would be vulnerable only to

$$B(i,j) = B_i \cdot \frac{R_j}{R}$$

type-i searchers, rather than to all B_i type-i searchers. One can interpret this method as assigning to each searcher one and only one type of target.

Then attrition to type-j targets is given in this case by

$$\Delta R_j = R_j\left(1 - \prod_{i=1}^{M}\left(1 - \frac{k_{ij}}{R_j}\left[1 - (1 - d_{ij})^{R_j}\right]\right)^{B(i,j)}\right). \quad (13)$$

But there are other methods of prior searcher allocation that are more closely related to our model and to methods developed by L. B. Anderson [1]. Here one computes explicitly the searcher allocation for a "typical" target set (we leave the term purposely vague) and uses this allocation to allocate searchers in each attrition computation, as described below.

Suppose that t_j is the proportion of a "typical" target set that are type-j targets and that a_{ij}^t is the fraction of the shots fired by a type-i searcher that are directed at type-j targets when the target set is "typical." If we put

$$\alpha_{ij} = \frac{a_{ij}^t}{t_j}, \quad (14)$$

then the fraction of shots fired (by a type-i searcher) that is aimed at type-j targets, when the target set consists of R_j type-j targets ($j = 1, \ldots, N$), is given by

$$\frac{\alpha_{ij}R_j}{\sum_k \alpha_{ik}R_k};$$

and we could then define

$$B(i,j) = B_i \frac{\alpha_{ij}R_j}{\sum_k \alpha_{ik}R_k},$$

to be used in Equation (13).

An interesting question at this point is, Can the a_{ij}^t be derived from other given quantities using the assumptions set forth above? To begin, we note that the object is to derive a distribution of the fire of each searcher. For the model described by Assumptions (1-7), we have the following result:

PROPOSITION. For each i and j, the conditional probability that a fixed type-i searcher attacks some type-j target, given that the searcher makes an attack, is

$$\frac{R_j d_{ij} p(i,j)}{1 - \prod_{k=1}^{N} (1 - d_{ik})^{R_k}} ,$$

with p(i,j) defined by Equation (8).

Suppose now that a "typical" target set consists of R_j^t type-j targets (j = 1, ..., N). We can then define

$$t_j = \frac{R_j^t}{\sum_{k=1}^{N} R_k^t} ,$$

and define

$$a_{ij}^t = \frac{R_j^t d_{ij} p^t(i,j)}{1 - \prod_{k=1}^{N} (1 - d_{ik})^{R_k^t}} ,$$

with $p^t(i,j)$ computed by Equation (8), with the R_k there replaced by R_k^t. Then, to compute attrition in an engagement with arbitrary numbers R_1, ..., R_N of targets, we would compute the α_{ik} by Equation (14) and apply Equation (13). In use, this procedure would require only one computation of the p(i,j)--that for the "typical" target set.

The rationale behind this procedure is that, in a statistical sense of the long run, the probability that a given type of target is fired upon, given that there is a shot fired, is the same as the fraction of shots fired at type-j targets if the attrition process were carried out experimentally a large number of times.

While we feel that for most situations of attack against passive targets the one-shot-per-searcher hypothesis is plausible, there are situations in which it is not. Hence, we have also developed an attrition model in which each searcher has essentially unlimited firing capability and attacks each target it detects. If we replace Assumption (3) by (3′)

> Each searcher has sufficient firing capability to fire once and only once at each target he detects,

then we have the following results:

PROPOSITION. Under Assumptions (1), (2), (3′) and (4-7),

$$E[K_j] = R_j\left(1 - \prod_{i=1}^{M} (1 - d_{ij}k_{ij})^{B_i}\right) \qquad (15)$$

for each j.

Equation (15) is a reasonable and easily computable equation for use in modeling situations in which the one-shot hypothesis (Assumption (3)) seems unrealistic.

As in the other cases, exponential approximations are, in the context of computerized simulations, superfluous.

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Chapter 4

METHODOLOGICAL DEVELOPMENT OF PROCEDURES TO RELATE WEAPON-SYSTEM EFFECTIVENESS TO PHYSICAL CHARACTERISTICS

Roy J. Shanker

This chapter discusses the development of two methodological approaches to the problem of quantifying the trade-off between the quality of a specific weapon system and the quantity of that weapon system available--this, of course, being under the constraint of a fixed budget. A third area relevant to the general problem of data development is discussed in the final section of the report.

Conceptually, the solution to the problem of quality-versus-quantity (Q-Q) is rather easy. Let us assume that the global goal of this endeavor is force planning. Let us also assume that we have some black box (attrition relationships, war-game simulator, etc.) that can assess the relative worth of different forces. In turn, the forces are usually described by their number (quantity) and some aggregate measures of effectiveness (quality--e.g., for an aircraft this may be a set of parameters describing performance such as the aircraft's probability of kill). Finall., let us assume that the cost of the weapon system can be specified as a function of the measure of quality[1] and that the aggregate budget for the total force is specified. Then the solution to the Q-Q problem may be derived from the straightforward use of techniques of mathematical programming.

There is, however, an impasse in the above methodology. Measures of effectiveness are usually stated in abstractions that are compatible with the black boxes used for force

[1] An example of this costing is presented in RAND Report R-761-PR, *Cost Estimating Relationships for Aircraft Airframes*.

assessment, while cost relationships are usually stated in terms of overt physical characteristics. Thus, if the two-parameter single-engagement binomial attrition equation--see Equation (3), below--was used for aircraft-force planning, we would describe these aircraft in terms of their probability of detection and probability of kill. However, in actuality, an aircraft is most likely to be described in terms of X dollars for a better rocket or radar system. The necessary link, which supplies the conversion of a change in probability of kill (a force-planning parameter) to physical characteristics, is missing.

The following two methodologies developed for this project are directed towards the establishment of a quantitative specification of that link--i.e., the statement of "force-planning parameters" as a function of cost or some direct surrogate that can be costed (such as physical characteristics).

It should be made clear that the intent of this chapter is methodological development, not the derivation of empirical results. The author recognizes the many limitations in the derived data base. Also, several technical issues of aerodynamics are approached without extensive attention to the physical meaning of aeronautical engineering parameters. However, the conceptual structure offers a potentially desirable approach to a difficult problem and may warrant a careful empirical implementation on contemporary weapon systems. The detailed analysis necessary for such empirical work is beyond the scope and resources of this study.

A. PROCESSING OF EMPIRICAL OBSERVATIONS

The first methodology addresses the use of empirical information. This procedure attempts to use this information to estimate the parameters for some force-planning model and, in turn, to find a functional relationship between these parameters and the physical attributes of the weapon system. To match the empirical example, the methodology is presented in the context of the fighter aircraft.

Let B_i ($i = 1, \ldots, I$) denote the I different types of Blue aircraft; and $\dot{B}_i$, the Blue aircraft losses. For each B_i, there will be a physical-quality-attribute set $\{X^b_{ik}\}$ ($k = 1, \ldots, K^b_i$) describing the K^b_i attributes of the i^{th} aircraft. Similarly, define R_j ($j = 1, \ldots, J$), $\dot{R}_j$ and $\{X^r_{jk}\}$ ($k = 1, \ldots, K^r_j$) for the various Red aircraft types. For each of the at-most 2(IxJ) possible pairings of aircraft (B_i versus R_j), there will be S_{ij} observations of combat encounters.[1]

We then select N attrition models Q_n ($n = 1, \ldots, N$). For each of the IxJ conflict sets B_i versus R_j, a parameter set $\{P^n_{ij}\}$ ($n = 1, \ldots, N$) composed of elements $h = 1, \ldots, H^n$ will be estimated. The problem is then to find a functional form, $f^n_{ij}(X^b_{i1}, \ldots, X^b_{iK^b_i}, X^r_{j1}, \ldots, X^r_{jK^r_j}; h)$ that "best" describes each

$$P^n_{ij}(h) = f^n_{ij}(X^b_{i1}, \ldots, X^b_{iK^b_i}, X^r_{j1}, \ldots, X^r_{jK^r_j}; h),$$

$$h = 1, \ldots, H^n . \qquad (1)$$

This functional relationship thus specifies the force-planning parameters as a function of aircraft and aircraft-opponent attributes. Allowing that we can differentially cost aircraft of varying attribute sets, the above supplies the missing segment for the solution of the Q-Q problem. Of course, each solution will be dependent on the attrition equation selected. The procedure does not indicate the "best" attrition relationship.

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[1]Due to the symmetry of most attrition situations, there are 2(IxJ) pairings: i attacking j and j attacking i. While stated for a single i x j pair, all comments are consistent with all pairs.

It is important to notice that in this methodology the "quantity" side of the Q-Q problem is submerged in the set of force-planning parameters $\{P^n_{ij}\}$. When we use an attrition or simulation model, it has been assumed that the set $\{P^n_{ij}\}$ fully determines this model. Thus, if we are able to cost the weapon system and if we have a budget constraint, we may determine the quantity available. Given f^n_{ij}, we can also assess the quality of the system as stated by P^n_{ij}. Finally, with the force-planning model Q_n, we can measure the effectiveness of the force. The Q-Q trade-off is visible as we change either the quality $\{P^n_{ij}\}$ or quantity (as reflected by the budget constraint and the cost of the weapon system associated with P^n_{ij}). The quantification of the effect of these trade-offs is manifest in the changes in the output of the planning model Q_n.

Given the above, three modes of comparison are of interest. First, the comparison of the difference in the forms of f^n_{ij} as n varies allows a differentiation between the various attrition (force-planning) models that may be of interest. The second type of comparison is between various classifications of R_i,B_j pairs (i.e., the results that would occur from considering only such pairs as Red fighters and Blue bombers). This comparison allows for the estimation of Q-Q in specific combat roles.

The third mode of comparison is the most subtle and is a method of investigating hypothesized effects of tactics in the above model.

Assume that as an alternative to Equation (1) we had chosen the two-equation system

$$P^n_{ij}(h) = g^n_{ij}(\text{TACTICS}, X^b_{i1}, \ldots, X^b_{iK^b_i}, X^r_{j1}, \ldots, X^r_{jK^r_j}; h),$$

$$h = 1, \ldots, H^n \quad \text{(2a)}$$

$$TACTICS(h) = g^{*n}_{ij}(X^b_{i1}, \ldots, X^b_{iK^b_i}, X^r_{j1}, \ldots, X^r_{jK^r_j}; h),$$

$$h = 1, \ldots, H^n . \quad (2b)$$

That is, tactics and physical attributes determine the parameter set P, and tactics are determined by relations among the physical attributes of the two weapons systems involved. It is clear that a relationship like Equations (2a-b) is very difficult (if not impossible) to specify. However, if Equations (2a-b) are the "true" model, the use of a "compressed" model such as Equation (1) leads to a concatenation of the information held in the parameters of the estimated function in Equation (1).

Recognition of the above possibility allows for the investigation of hypotheses concerning the contribution of tactics. Consider a simple form for f,

$$P^n_{ij} = \alpha + \beta(s_i - s_j) .$$

That is, the parameter of force planning is a linear function of the difference in speed of the Red and Blue aircraft. If this is a proper expression for P^n_{ij}, we might consider that (1) when $s_i - s_j$ is positive, the tactics used by the B_i aircraft would exploit speed; and (2 when $s_i - s_j$ is negative, alternative tactics would be used to de-emphasize speed. We could test this inferential measure of tactics by segmenting our data into two groups ($\Delta s > 0$, and $\Delta s \leq 0$) and then estimating β for each group. If our hypothesis about tactics is correct, we would expect $\beta(\Delta s > 0) > \beta(\Delta s \leq 0)$.

1. U.S. Versus German Aircraft in World War II

A very simple example of this methodology was worked out, using historical data. These data were collected from group records of World War II. The summary of engagements and appropriate comments are shown in Appendix 4-A (below). A subset of these encounters was selected, consisting only of fighters flying bomber-escort missions. This selection was made so as to simplify estimation of a P_{ij}^n.

Again, it must be remembered that this application constitutes an exercise on a data base of convenience (although even so limited an investigation was quite time-consuming). Several discrepancies exist in the records available. It was impossible to identify the specific model types of the aircraft involved, and no attempt was made to integrate the operations into the full scope of the war (decreasing German logistic supplies may have limited intensity of encounters). The objective was conceptual development, not the exactness of the empirical data base.

The exponential form of the single-engagement binomial attrition equation is as follows:

$$\dot{T} = T\left[1 - e^{-\frac{kS}{T}(1 - e^{-dT})}\right], \qquad (3)$$

where

- T = number of targets;
- S = number of shooters;
- d = probability that a particular shooter detects a particular target;
- k = probability that a shooter kills a target, given detection and attack; and
- $\dot{T}$ = number of targets killed.

In any conflict, both parties assume the roles of "shooter" and "target," and thus two relationships are reported for each pairing. As the missions were bomber escorts and the enemy sought the engagement, d was set equal to 1. The results of estimating k from Equation (3) and the encounters displayed in Appendix 4-A are shown in Table 4-1.[1]

Table 4-1. DERIVED PROBABILITIES OF KILL

Paired Encounter (Shooter vs Target)	Number of Observations	Probability of Kill
P-51 versus FW190	7	0.250
P-51 versus Me109	25	0.220
P-47 versus Me109	8	0.070
P-47 versus FW190	18	0.080
FW190 versus P-51	7	0.060
Me109 versus P-51	25	0.040
Me109 versus P-47	8	0.058
FW190 versus P-47	18	0.058

A summary of the physical attributes of the aircraft was made (see Appendix 4-B); and maximum speed, service to ceiling, wing loading, power loading, and armament were selected to be used as X^b_{ik} and X^r_{jk}. These values are denoted in Table 4-2. A compilation of differences and ratios of the five characteristics between combatant pairs is shown in Table 4-3. Finally, various function forms, f^n_{ij}, easily estimable by linear techniques, were tried--with quite satisfactory results. (See Table 4-4).

As can be seen in Table 4-4, several of the relations have a very high estimating ability (R^2 adjusted; see starred Equations 7, 19, 36, 41, 45, 51) and may be considered as a statistically significant measure of the correlation between probability of kill and the physical attributes of the two planes involved. Some caution is needed in interpreting the estimated relationships. Due to the high degree of multi-

[1]A grid search routine programmed by J. Bracken was used.

Table 4-2. BASIC DATA

Aircraft	Wing Loading (lbs/sq ft)	Power Loading (lbs/hp)	Maximum Speed (mph)	Maximum Ceiling (ft)	Number of Machine Guns	Number of Cannon
P-51	42.9	6.37	445	42,000	6	0
P-47	41.6	7.69	430	40,000	8	0
Me109	34.8	4.62	390	37,000	2	1
FW190	42.3	5.25	408	37,000	2	2

Table 4-3. VARIABLES BY PAIRED CONFLICT

Pair	Number of Observations	ΔWL	$\frac{WL_1}{WL_2}$	ΔPL	$\frac{PL_1}{PL_2}$	Δs	$\frac{s_1}{s_2}$	Δc	$\frac{c_1}{c_2}$	Armament Ratio $\frac{MG_1 + \alpha c_1}{MG_2 + \alpha c_2}$		
										AA1 $\alpha = 1$	AA05 $\alpha = 0.5$	AA2 $\alpha = 2$
P-51 versus 190	7	0.6	1.01	1.12	1.21	37	1.09	5,000	1.14	1.500	2.000	1.00
P-51 versus 109	25	8.6	1.23	1.75	1.38	55	1.14	5,000	1.14	2.000	2.400	1.50
P-47 versus 109	8	6.8	1.19	3.07	1.66	40	1.10	3,000	1.08	2.660	3.200	2.00
P-47 versus 190	18	-0.7	0.98	2.44	1.46	22	1.05	3,000	1.09	2.000	2.660	1.33
FW190 versus 51	7	-0.6	0.99	-1.12	0.83	-37	0.92	-5,000	0.88	0.660	0.500	1.00
Me109 versus 51	25	-8.1	0.81	-1.75	0.72	-55	0.88	-5,000	0.88	0.500	0.417	0.66
Me109 versus 47	8	-6.8	0.84	-3.07	0.60	-40	0.91	-3,000	0.93	0.375	0.300	0.50
FW190 versus 47	18	0.7	1.02	-2.44	0.68	-22	0.95	-3,000	0.93	0.500	0.375	0.75

Table 4-4. REGRESSION RESULTS - PROBABILITY OF KILL AS A FUNCTION OF AIRCRAFT ATTRIBUTES

ADDITIVE FORMS							
No.	Model PK =	R^2	R^2(adjusted)	$K(t_k)$	$\beta_1(t_1)$	$\beta_2(t_2)$	ρ_{12}*
1.	$K+\beta_1\Delta s$	.53	.45	.105(4.88)	.0014(2.59)	--	--
2.	$K+\beta_1\Delta c$	.59	.53	.105(5.29)	.00001(3.0)	--	--
3.	$K+\beta_1\Delta s+\beta_2 AA1$	.78	.69	.209(4.47)	.0026(3.98)	-.087(-2.38)	.80
4.	$K+\beta_1\Delta c+\beta_2 AA1$	.71	.59	.16(3.54)	.00002(3.27)	-.046(-1.34)	.67
5.	$K+\beta_1\Delta WL+\beta_2 AA1$	.38	.14	.20(2.05)	.017(1.58)	-.08(-1.02)	.88
6.	$K+\beta_1\Delta PL+\beta_2 AA1$	.26	-	.16(1.16)	.028(1.09)	-.04(-.57)	.86
★ 7.	$K+\beta_1\Delta s+\beta_2 AA05$	.84	.78	.22(5.51)	.003(4.69)	-.087(-3.1)	.88
8.	$K+\beta_1\Delta c+\beta_2 AA05$	.78	.69	.18(4.49)	.00002(3.82)	-.05(-2.03)	.81
9.	$K+\beta_1\Delta WL+\beta_2 AA05$	.26	-	.12(1.56)	.0087(8.89)	-.008(-.176)	.82
10.	$K+\beta_1\Delta PL+\beta_2 AA05$	.23	-	.15(1.25)	.03(.78)	-.04(-.42)	.94
11.	$K+\beta_1\Delta s+\beta_2 AA2$	.65	.51	.16(3.27)	.0016(3.02)	-.05(-1.3)	.34
12.	$K+\beta_1\Delta c+\beta_2 AA2$	.63	.48	.13(2.69)	.00001(2.89)	-.02(-.638)	.12
13.	$K+\beta_1\Delta WL+\beta_2 AA2$	.73	.63	.28(4.62)	.019(3.70)	-.15(-3.0)	.75
14.	$K+\beta_1\Delta PL+\beta_2 AA2$	.29	.01	.18(2.14)	.02(1.42)	-.04(-.78)	.41
15.	$K+\beta_1 \frac{s_1}{s_2}+\beta_2 AA1$	.82	.75	-.97(-4.2)	1.18(4.54)	-.094(-2.8)	.81
16.	$K+\beta_1 \frac{c_1}{c_2}+\beta_2 AA1$	.73	.63	-.62(-3.12)	.78(3.52)	-.046(-1.42)	.66
17.	$K+\beta_1 \frac{w_1}{w_2}+\beta_2 AA1$	.37	.11	-.47(-1.3)	.67(1.5)	-.08(-1.0)	.89

(continued on next page)

*ρ_{12} is the correlation between the independent variables.

Table 4-4 (continued)

No.	Model PK =	R^2	R^2(adjusted)	$K(t_k)$	$\beta_1(t_1)$	$\beta_2(t_2)$	ρ_{12}
18.	$K+\beta_1 \frac{p_1}{p_2}+\beta_2 AA1$	.23	--	-.02(-.18)	.18(1.02)	-.05(-.59)	.89
★19.	$K+\beta_1 \frac{s_1}{s_2}+\beta_2 AA05$	.88	.83	-1.26(-5.25)	1.49(5.54)	-.09(-3.77)	.88
20.	$K+\beta_1 \frac{c_1}{c_2}+\beta_2 AA05$	.81	.73	-.81(-3.8)	.98(4.17)	-.05(-2.18)	.80
21.	$K+\beta_1 \frac{w_1}{w_2}+\beta_2 AA05$	.24	--	-.20(-.61)	.31(.82)	-.006(-.137)	.82
22.	$K+\beta_1 \frac{p_1}{p_2}+\beta_2 AA05$	.21	--	-.06(-.3)	.22(.66)	-.05(-.42)	.97
23.	$K+\beta_1 \frac{s_1}{s_2}+\beta_2 AA2$	.67	.53	-.54(-2.5)	.70(3.17)	-.06(-1.4)	.35
24.	$K+\beta_1 \frac{c_1}{c_2}+\beta_2 AA2$	.66	.52	-.46(-2.3)	.58(3.07)	-.02(.65)	.11
25.	$K+\beta_1 \frac{w_1}{w_2}+\beta_2 AA2$	.73	.62	-.46(-2.8)	.73(3.67)	-.16(-3.03)	.76
26.	$K+\beta_1 \frac{p_1}{p_2}+\beta_2 AA2$	.27	--	.037(.40)	.11(1.3)	-.04(-.79)	.44
27.	$K+\beta_1 \frac{c_1}{c_2}+\beta_2 \frac{s_1}{s_2}$	.64	.50	-.41(-1.8)	.94(1.2)	-.43(-.50)	.97

(continued on next page)

Table 4-4 (continued)

No.	Model PK =	R^2	R^2(adjusted)	$K(t_k)$	$\beta_1(t_1)$	$\beta_2(t_2)$	ρ_{12}
28.	$K+\beta_1\Delta c+\beta_2\Delta s$	.61	.46	.10(4.91)	.00002(1.04)	-.0009(-.41)	.97
29.	$K+\beta_1\Delta PL+\beta_2\Delta WL$	.27	--	.10(3.59)	.007(.36)	-.005(.662)	.72
30.	$K+\beta_1\Delta c+\beta_2\Delta PL$	.69	.56	.10(5.30)	.00002(2.6)	-.02(-1.18)	.85
31.	$K+\beta_1\Delta s+\beta_2\Delta WL$	.57	.40	.10(4.68)	.002(1.93)	-.005(-.72)	.85
	LOGARITHMIC FORMS						
No.	log PK =			log K +	β_1 log X_1 +	β_2 log X_2	
32.	$K\left(\frac{c_1}{c_2}\right)^{\beta_1}$	.71	.66	-2.49(-18.2)	4.96(3.84)	--	--
33.	$K\left(\frac{s_1}{s_2}\right)^{\beta_1}$	.65	.59	-2.48(-16.5)	5.32(3.34)	--	--
34.	$K\left(\frac{WL_1}{WL_2}\right)^{\beta_1}$	.34	.23	-2.48(-12)	2.66(1.76)	--	--
35.	$K\left(\frac{PL_1}{PL_2}\right)^{\beta_1}$	.32	.21	-2.47(-11.9)	.96(1.7)	--	--
★ 36.	$K\left(\frac{PL_1}{PL_2}\right)^{\beta_1}\left(\frac{c_1}{c_2}\right)^{\beta_2}$	.79	.71	-2.48(-19.5)	-1.52(-1.03)	-1.87(3.36)	.86
37.	$K\left(\frac{WL_1}{WL_2}\right)^{\beta_1}\left(\frac{s_1}{s_2}\right)^{\beta_2}$	.66	.53	-2.48(-15.3)	6.64(2.14)	-1.28(-.58)	.84
38.	$K\left(\frac{s_1}{s_2}\right)^{\beta_1}\left(\frac{PL_1}{PL_2}\right)^{\beta_2}$	.77	.68	-2.48(-18.6)	9.92(3.12)	-1.4(-1.68)	.9

(continued on next page)

Table 4-4 (continued)

No.	Model PK =	R^2	R^2(adjusted)	$K(t_k)$	$\beta_1(t_1)$	$\beta_2(t_2)$	ρ_{12}
39.	$K\left(\frac{C_1}{C_2}\right)^{\beta_1}\left(\frac{WL_1}{WL_2}\right)^{\beta_2}$	.69	.57	-2.48(-16.1)	4.70(2.38)	.12(.076)	.69
40.	$K\left(\frac{C_1}{C_2}\right)^{\beta_1}\left(\frac{S_1}{S_2}\right)^{\beta_2}$	.70	.58	-2.48(-16.2)	5.82(.98)	-1.16(.18)	.97
★41.	$K\left(\frac{S_1}{S_2}\right)^{\beta_1} AA1^{\beta_2}$	.80	.72	-2.48(-20.12)	9.46(3.83)	-.74(-1.97)	.85
42.	$K\left(\frac{C_1}{C_2}\right)^{\beta_1} AA1^{\beta_2}$	.74	.63	-2.49(-17.5)	5.95(3.13)	-.24(-.74)	.71
43.	$K\left(\frac{WL_1}{WL_2}\right)^{\beta_1} AA1^{\beta_2}$	.36	.11	-2.47(-11.18)	4.34(1.05)	-.40(-.42)	.92
44.	$K\left(\frac{PL_1}{PL_2}\right)^{\beta_1} AA1^{\beta_2}$	.33	.07	-2.47(-10.9)	1.37(.90)	-.27(-.30)	.91
★45.	$K\left(\frac{S_1}{S_2}\right)^{\beta_1} AA05^{\beta_2}$	.82	.75	-2.48(-21.1)	12.3(3.6)	-.84(-2.2)	.93
46.	$K\left(\frac{C_1}{C_2}\right)^{\beta_1} AA05^{\beta_2}$	.77	.68	-2.49(-18.7)	7.40(3.0)	-.36(-1.2)	.86
47.	$K\left(\frac{WL_1}{WL_2}\right)^{\beta_1} AA05^{\beta_2}$	.38	.14	-2.48(-11.3)	1.27(.45)	.27(.59)	.83
48.	$K\left(\frac{PL_1}{PL_2}\right)^{\beta_1} AA05^{\beta_2}$	.39	.15	-2.48(-11.5)	-2.31(.53)	1.4(.76)	.99
49.	$K\left(\frac{S_1}{S_2}\right)^{\beta_1} AA2^{\beta_2}$	.73	.63	-2.48(-17.3)	6.03(3.7)	-.38(-1.2)	.35

(concluded on next page)

Table 4-4 (concluded)

No.	Model PK =	R^2	R^2(adjusted)	$K(t_k)$	$\beta_1(t_1)$	$\beta_2(t_2)$	ρ_{12}
50.	$K\left(\frac{C_1}{C_2}\right)^{\beta_1} AA2^{\beta_2}$	.72	.60	-2.49(-16.8)	5.01(3.6)	-.10(-.33)	.11
★51.	$K\left(\frac{WL_1}{WL_2}\right)^{\beta_1} AA2^{\beta_2}$	.81	.74	-2.47(-20.6)	6.44(4.66)	-1.37(-3.55)	.77
52.	$K\left(\frac{PL_1}{PL_2}\right)^{\beta_1} AA2^{\beta_2}$	.39	.14	-2.48(-11.4)	1.15(1.78)	-.35(-.72)	.42

colinearity, the coefficients should not be interpreted separately as marginal contributions to P_k. (See Table 4-5 for correlation table.) Rather, the independent variables in aggregate should be considered as a system that makes changes in P_k--which is not only a proper statistical but also a proper physical interpretation, as many of the aircraft's attributes are interrelated (due to physical laws manifest in the plane's design).

The intent here is not to claim that these specific relationships are a reasonable portrayal of the causality of P_k. What is intended, rather, is to show the full conceptual process. In no way would the author offer these relationships as accurate depictions of the structure of P_k. To hope to make such claims validly, a more formal aeronautical analysis is needed as the preliminary stages of data collection, variable definition, and model structure.

2. Application to Design of Current Aircraft

In fact, the interrrelation of the physical attributes suggests a method for the search and selection of new aircraft designs, given that an appropriate f^n_{ij} has been developed for contemporary aircraft.

Consider a multidimensional space R^{k_i}, where each dimension represents a physical attribute of the Blue aircraft. There exists a region $W^{k_i} \subset R^{k_i}$, which describes the combinations of attributes $X_i, \ldots, X_{ik_i}$ for a physically feasible aircraft. For a specified opponent, we then seek the vector $X^* \varepsilon W^{k_i}$ which optimizes[1] f^n_{ij}:

---

[1]The term "optimize" is used rather than "maximize" or "minimize," because the nature of optimization is dependent on the force planning parameter being estimated and, in turn, is dependent on n, the index of the planning model used.

Table 4-5. CORRELATION OF AIRCRAFT ATTRIBUTES

Attribute	PKILL	AA1	AA05	AA2	ΔWL	$\frac{WL_1}{WL_2}$	ΔPL	$\frac{PL_1}{PL_2}$	Δs	$\frac{s_2}{s_2}$	Δc
AA1	.28										
AA05	.37	.96									
AA2	-.08	.76	.56								
ΔWL	.50	.88	.82	.75							
$\frac{WL_1}{WL_2}$	.49	.89	.82	.76	1.00						
ΔPL	.46	.86	.94	.41	.72	.71					
$\frac{PL_1}{PL_2}$	.43	.89	.97	.44	.75	.75	.99				
Δs	.73	.80	.88	.34	.85	.84	.88	.89			
$\frac{s_1}{s_2}$	.73	.81	.88	.35	.86	.84	.88	.89	1.00		
Δc	.77	.67	.81	.12	.71	.69	.84	.85	.97	.97	
$\frac{c_1}{c_2}$	.79	.66	.80	.11	.70	.69	.83	.84	.97	.97	1.00

$$\operatorname{opt}_{\substack{k_i \\ X^* \epsilon W}} f^n_{ij}(X^* | X_{j1}, \ldots, X_{jk_j}) . \qquad (4)$$

These results, based on obsolete World War II aircraft, are not relevant to present planning functions, except as a potentially attractive conceptual design to solve a very difficult problem.

It would be desirable to develop a data base relevant to modern aircraft (such as the current SED/ASSIF project) and then to implement this methodology. It would also be beneficial to have an aeronautical engineer involved in the development of the relevant models concerning the physical attributes of the aircraft.

The application of this methodology for current aircraft might occur in the following manner. Initially, test flights or simulations of aerial combat would be run over the spectrum of USAF-Soviet aircraft pairs. The results of these conflicts would be used to estimate a set of attrition parameters for each pair of aircraft. In turn, functional relationships would be sought that specify these attrition or force-planning parameters as a function of the physical attributes of the two aircraft. The final step is then to use this functional relationship as an objective function in the design of possible aircraft as denoted above in Equation (4).

B. PROCESSING OF SUBJECTIVE DATA

A second procedure was also considered for providing a link between physical attributes and quality. This procedure is a modification of the psychometric technique of conjoint measurement.

Conjoint measurement takes as an input a preference order on a set of objects that represents the concatenation of several independent attributes. For example, in Figure 4-1 the nine cells represent the objects; the order is represented by the notations A to I (low to high); and there are two attributes (X,Y), each having three levels.

	Y_3	D	F	I
Y	Y_2	B	E	H
	Y_1	A	C	G
		X_1	X_2	X_3
			X	

Figure 4-1. TWO-ATTRIBUTE CONJOINT MEASUREMENT MATRIX

The problem, which can be generalized for multiple attributes, is to find a set of functions $f^1(x_i), f^2(y_j)$ such that their sum (product) is the "best" fit to the original order, A to I. The desirable property of this procedure is that it transforms ordinal preferences on a concatenation into interval measures for each level of the separate attributes. These interval values of each attribute level are unique within a linear transformation and may be interpreted as utilities.

One of the major drawbacks of this procedure is the difficulty in collecting data when more than two attributes are involved. People seem to find a two-dimensional preference ordering trivial, while anything greater than two dimensions becomes difficult if not impossible. Thus, the major effort for this report was to develop a technique for the simultaneous estimation of multiple-attribute utilities while using only pairwise preference orderings as input. This has been accomplished and is now described.

Consider the general case where we have n attributes X_i, $i = 1, \ldots, n$. Each attribute in turn has m_i levels (X_{i1},

... X_{ip}, ... X_{im_i}). Then the total number of possible cells to be ordered is $M = \prod_{i=1} m_i$. Clearly, M becomes quite large even when the number of attributes is relatively small. As well, as mentioned earlier, it is quite difficult for people to make subjective orderings when $n > 2$. Thus, an approach based on the $n(n - 1)/2$ possible attribute pairs and their associated orderings was developed. Consider for a moment a single attribute pair (X_i, X_j) of m_i and m_j levels, respectively.

The subjective data gives an order X:

$$X = \{(p,q) 1 \leq p \leq m_i \quad 1 \leq q \leq m_j\}, \text{ for a given } i \text{ and } j.$$

If we assume an additive relationship between the attributes $\Phi_{pq} = f(X_{ip}) + g(X_{jq})$ and give arbitrary starting values to the discrete levels of $f(X_{ip})$ and $g(X_{jq})$, we can solve for the "best" values of f and g using a gradient technique. From the initial order X, we can derive a series of values of Φ_{pq} using the initialized values of f and g. As we wish the order to be increasing in value, we then perform a monotonic transformation on the value Φ_{pq} of the order. That is

$$\min \sum_{pq \varepsilon X} (\Phi_{pq} - \hat{\Phi}_{pq})^2 = S_{ij} , \qquad (5)$$

subject to

$$\hat{\Phi}_{pq} \leq \hat{\Phi}_{kl} \qquad pq \leq^{*} kl ,$$

where the notation $<^{*}$ implies the ordering based on X.

Similarly, we can derive an S for all i,j pairs. Let $\mathcal{S} = \sum S_{ij}$. Then we wish to take the gradient of $\mathcal{S}$, $\overline{\nabla}\mathcal{S}$, with respect to all the levels of all the attributes.[1] In other

[1]The usual procedure is to normalize S with respect to the total deviation. This gives Equation (5) a standardized range from 0 to 1 and will allow comparisons with other applications. However, for our purposes, the normalization only complicates the mathematics--at no gain to understanding--and is thus left out.

words, there will be $\sum_i m_i$ elements in the gradient.

The form of each element in the gradient is

$$\frac{\partial \mathcal{J}}{\partial f(X_{ipo})} = \sum_X 2[(f(X_{ipo}) + g(X_{jq}) - \hat{\Phi}p_o q)] \ .$$

(The form is slightly more complex but simplifies to the above.)

Finally, we adjust each function value $f(X_{ip})$, $g(X_{jq})$ in the direction of $-\nabla \mathcal{J}$ and iterate the procedure until $\nabla \mathcal{J} \approx 0$. The final values for each $f(X_{ip})$ and $g(X_{jq})$ can then be interpreted as interval measures of utility for each attribute level.

It should be noted that one need not have all n-1 pairs for each attribute. Thus, if cost limits the number of paired orders to be obtained, then the pairs may be allocated on some *a priori* measure of significance. Also, individual elements of $\mathcal{J}$ may be inspected separately to see if any particular pair is causing an unusually large contribution to $\mathcal{J}$. These elements may have several useful interpretations, dependent on the problem context.

The question now arises as to the worth of this procedure in solving the Q-Q problem. If we consider the attributes to be physical characteristics of a weapon system and the cells to be the specified system, then the subjective rank orderings of competent judges can be used to derive as utilities of the various attributes. Given interval utilities and cost, we now have the information necessary for the solution of the Q-Q problem.

Presumably, there is a set of people whose function is to establish the design and performance specifications for new weapons systems. Ideally, these are the individuals whose judgment should be used to establish the preference ordering. The benefits of this approach are two-fold. First, it accomplishes the solution of the Q-Q problem. Second, it makes manifest to these planners the inherent utility that they have placed on certain attributes, which may be of considerable importance in their future work.

It should also be noted that there is no constraint to make the preference order subjective. If adequate data is available, the ordering of cells may be based on actual performance such as combat performance, when the ordering is based on the dominance of one weapon system over another.

C. CALIBRATION AND ESTIMATION OF HISTORICAL DATA

A problem that arose in the search for appropriate data to use with the first methodology seems worth discussing at this time, due to its relevance to all facets of research that used historical combat data. A desirable trait for such data is the existence of corroborative evidence on both sides of the conflict. In fact, so little credibility is given to uncorroborated data that in many instances only the corroborated data are deemed admissible--an unfortunate situation, since the corroborated data exist only for a very small (and possibly biased) portion of historic engagements.

The following suggests a method of using this pool of corroborated data as a calibration for other data, which would allow a much broader spectrum of engagements to be used. Assume that for each engagement i (i = 1, ..., n) we have two sets of observations:

$$\{O_{B_i}\},\ \{O_{R_i}\},$$

the former are the observations by Blue of engagement i (O_{B_i}); the latter, by Red, O_{R_i}. For simplicity, let us say these sets contain only four elements each:

$$\left\{O_{B_i}\right\} = \left\{\dot{B}_{B_i}, B_{B_i}, \dot{R}_{B_i}, R_{B_i}\right\}$$

$$\left\{O_{R_i}\right\} = \left\{\dot{B}_{R_i}, B_{R_i}, \dot{R}_{R_i}, R_{R_i}\right\} ,$$

where $\dot{B}$, B are Blue losses and initial forces and $\dot{R}$, R are Red losses and initial forces. The subscript then denotes which side estimated B, $\dot{B}$, R, $\dot{R}$. If we believe that R_{R_i}, B_{B_i}, $\dot{B}_{B_i}$ are the accurate descriptors of the engagement then our goal is to find some way of estimating $\dot{R}_{R_i}$, R_{R_i} (or $\dot{B}_{B_i}$, B_{B_i}) when we are given only $\dot{R}_{B_i}$, R_{B_i} (or $\dot{B}_{R_i}$, B_{r_i}).

In general, we seek some function f such that

$$\left\{R_{R_i}, \dot{R}_{R_i}\right\} = f(O_{B_i}, X) ,$$

where X is a vector of attributes that details the environment of the engagement. Typical elements of X may be weather, observer's rank, etc. Thus, whenever we are presented with an uncorroborated data base, we could then use the function f to aid us in estimating enemy forces and casualties.

References

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Appendix 4-A

AIR-COMBAT DATA

These data are based on group combat records received through the office of the United States Air Force Historian (acting director, Mr. M. Rosenberg) from the archives of Maxwell Air Force Base, Montgomery, Alabama.

Information was received for the following groups: 4th (England WWII), 354th (England WWII), 33rd (North Africa WWII), and 23rd (Pacific WWII).

This initial selection is somewhat biased, since it represents very famous and successful U.S. fighter groups. While difficult to quantify, such biases may be eliminated by simply researching more representative group records.

The information contained in these records varied widely. Within a group (and even within a squadron), record-keeping format and procedure was constantly changing. In general, even simple observations such as "5 P-51's engaged 6 Me-109's with 2 kills each" were not available and, thus, had to be deduced from the records.

Aside from lack of consistency in reporting, two major problems prevent complete usage of the records. The first was simply a lack of definitive reporting and incomplete description of an encounter. The second was the recurring problems of squadrons or sections reporting only their own actions without acknowledging the presence of other allied aircraft--which made it difficult to discern the full extent of an engagement, disallowing a considerable amount of observations.

Within these limitations, three general types of data observations were made to which a "reasonable" degree of confidence (of course, all kills are estimates) is given.

(1) *Total group or squadron encounters.* Here only a summary of the total encounter was available, and the group or squadron fought independently.

(2) *Section encounters.* This is a relatively better source of data. When flying bomber escort, U.S. aircraft were assigned to incoming enemy aircraft a section (4 planes) at a time. Then separable encounters offer a good data base.

(3) *Individual or pair encounters.* From pilot summaries, it was possible to deduce the outcome of 1-on-1, 2-on-1, 2-on-2 (etc.) encounters. When these seemed separable from a larger engagement, they were recorded.

The presence of very thorough statistical reviews by the group statistical-control section for the group and wing were alluded to in the records. However, in conversations with the archivist at Maxwell AFB, it was determined that, aside from an innumerative search through all group and wing records, there would be no way of finding such reviews. The table lists the "most reasonable" summary of aircraft in an encounter, with associated kills listed below.

8th AIR FORCE (8th FIGHTER COMMAND, 65th WING, 4th GROUP, 334th, 335th, and 336th SQUADRONS) - Europe

Date	Encounter			Mission
31 Jan 44	16 P-47s 0	Versus	20 Me-109s 0	Bomber Escort*
6 Feb 44	4 P-47s 0	Versus	5 FW-190s 1/2**	
	3 P-47s 0	Versus	5 FW-190s 1 down	

*All unmarked missions are Bomber-escorted.
**1/2 denotes aircraft damaged. Multiple damaged aircraft denoted by n x 1/2

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Date	Encounter			Mission
9 Feb 44	8 P-47s 0	Versus	5 Me-109s 1 loss	
	5 P-47s 1/2	Versus	3 FW-190s 1-1/2	
20 Feb 44	4 P-47s 0	Versus	2 FW-190s 1	
	4 P-47s 0	Versus	2 FW-190s 1	
21 Feb 44	15 P-47s 0	Versus	15 FW-190s 3-1/2	
25 Feb 44	4 P-47s 0	Versus	2 FW-190s 1	
2 Nov 43	48 P-47s 2-1/2	Versus	24 Me-109s 2	
4 Nov 43	16 P-47s 0	Versus	12 FW-190s 1/2	
	16 P-47s 0	Versus	8 FW-190s 1	
8 Nov 43	1 P-47 0	Versus	2 FW-190s 1/2	
26 Nov 43	1 P-47 0	Versus	1 Me-109 1	
28 Nov 43	1 P-47 0	Versus	1 Me-109 1/2	
12 Aug 43	4 P-47s 0	Versus	4 FW-190s 1	
	4 P-47s 1/2	Versus	8 Me-109s 2-1/2	

(continued on next page)

Date	Encounter	Mission
28 Aug 43	16 P-47s Versus 8-10 FW-190s 0 2	
26 Jul 43	4 P-47s Versus 4 FW-190s 0 ≃ 1	Strafing
28 Jul 43	2 P-47s Versus 6 FW-190s 0 3-1/2	
26 Jun 43	8 P-47s Versus 6 Me-109s 0 2	
16 May 43	36 P-47s Versus 15 FW-190s 2 x 2-1/2 3 x 1/2	
	24 P-47s Versus 20 Me-109s 1 2	
29 May 43	38 P-47s Versus 6 FW 190s 1/2 0	
10 Apr 43	16 P-47s Versus 5 FW 190s 0 3-1/2	Cover for Strafing
8 Mar 43	25 SF (Last SF)Versus 7 FW-190s 0 1	Escort
11 Mar 43	4 P-47s Versus 2 FW-190s 1 1	
1 Apr 44	16 P-51s Versus 4 Me-109s 0 3	
8-30 Apr 44	30 Aircraft down, no good record	
11 Apr 44	16 P-51s Versus 4 Me-109s 0 4	
13 Apr 44	28 P-51s Versus 20+ FW-190s 1 2	
22 Apr 44	24 P-51s Versus 20 Me-109s 0 2	
30 Apr 44	3 P-51s Versus 1 Me-109 0 1	
8 Apr 44	2 P-51s Versus 1 Me-109 0 1	

(continued on next page)

Date	Encounter			Mission
24 Apr 44	2 P-51s 1	Versus	1 Me-109 1	
	1 P-51 1	Versus	1 Me-109	
19 Apr 44	2 P-51s 0	Versus	1 Me-109 1	
8 Apr 44	3 P-51s 0	Versus	1 Me-109 1	
	3 P-51s 0	Versus	3 FW-190s 2	
	3 P-51s 1	Versus	6 Me-109s	
1 Apr 44	16 P-51s 0	Versus	4 Me-109s 1	
12 Apr 44	12 P-51s 1/2	Versus	4 Me-109s 4	
8 May 44	4 P-51s 0	Versus	4 Me-109s 2 x 1/2	
19 May 44	16 P-51s 0	Versus	3 Me-109s 1	
	16 P-51s 0	Versus	6 Me-109s 5	
18 Aug 44	8 P-51s 1	Versus	6 Me-109s 0	Bombing/Strafing
	4 P-51s 2	Versus	15 Me-109s 2	
5 Aug 44	2 P-51s 0	Versus	1 Me-109 1	Strafing
16 Aug 44	1 P-51 0	Versus	1 Me-109 1	
25 Aug 44	14 P-51s 0	Versus	11 Me-109s 3 + 2 x 1/2	

(concluded on next page)

Date	Encounter			Mission
12 Sep 44	19 P-51s 1	Versus	14 FW-190s 2	
17 Sep 44	16 P-51s 2	Versus	15 FW-190s 6	Patrol
5 Mar 44	7 P-51s 0	Versus	1 Me-109 1	
6 Oct 44	12 P-51s 0	Versus	8 Me-109s 0	
14 Oct 44	1 P-51 1	Versus	10 Me-109s 1	
6 Oct 44	2 P-51s 0	Versus	1 Me-410 1	
26 Oct 44	1 P-51 0	Versus	6 FW-190s 2	
21 Nov 44	4 P-51s 1/2	Versus	24 Me-109s 5	
20 Feb 45	2 P-51s 0	Versus	2 FW-190s 2	
19 Mar 45	12 P-51s 0	Versus	1 Me-109 1	Strafing
22 Mar 45	48 P-51s 0	Versus	15 FW-190s 11-1/2	
4 Apr 45	12 P-51s 0	Versus	8 Me-262s 3-1/2	
25 Apr 45	4 P-51s 0	Versus	1 Me-262 1	

33rd GROUP - North Africa

Date	Encounter	Mission
12 Jan 43	2 P-40s Versus 4 Me-109s 0 1	Defensive Patrol
14 Jan 43	4 P-40s Versus 4 Me-109s 0 0	Defensive Patrol
2 Feb 43	4 P-40s Versus 4 Me-109s 1-1/2 1	Patrol
30 Mar 43	48 P-40s Versus 12 Me-109s 1 6	Escort
3 Feb 43	1 P-40 Versus 12 Me-109s 1 0	Base Defense

23rd FIGHTER GROUP - Asia

30 Jul 43	1 P-40 Versus 1 Zero 1	Escort
31 Jul 43	7 P-40s Versus 23 Zeros 0 2	Base Defense
	1 P-40 Versus 2 Zeros 1	Base Defense
5 Aug 43	8 P-40s Versus 16 Zeros 3 2	Base Defense
8 Aug 43	7 P-40s Versus 6 Zeros 2	Escort
2 Nov 43	2 P-40s Versus 12 Zeros 1 0	Base Defense
	3 P-40s Versus 1 Zero 1	Base Defense
	1 P-40 Versus 6 Zeros 0 0	Base Defense
9 Nov 43	2 P-40s Versus 1 Zero 1	Base Defense

(continued on next page)

23rd FIGHTER GROUP - Asia (continued)

Date	Encounter	Mission
12 Nov 43	2 P-40s Versus 1 Zero 0 1	Base Defense
	1 P-40 Versus 1 Zero 1	Base Defense
23 Nov 43	2 P-40s Versus 1 I-97 0 0	Bomber
1 Nov 43	4 P-40s Versus 1 Zero 1	Escort
	1 P-40 Versus 1 Zero 1	Escort
	1 P-40 Versus 1 I-97 1	Escort
23 Dec 43	5 P-40s Versus 2 I-97s 0 0	Bomber Escort
26 Dec 43	1 P-40 Versus 1 Zero 1	Base Defense
	1 P-40 Versus 1 I-45 0 1	Escort
	1 P-40 Versus 1 Zero 1 0	

Appendix 4-B

A PHYSICAL REVIEW OF WORLD WAR II FIGHTERS

The following tables represent summaries of the physical attributes of fighter aircraft active in World War II. This information was gathered from several sources, and in some instances the data shown represent a composite of several production models of the aircraft. Sources used were the following:

(1) *Jane's All the World's Aircraft*, 1940-45.

(2) William Green, *Famous Fighters of the Second World War*, 2 vols. (New York: Doubleday, 1962).

(3) *Luftwaffe, Avalon-Hill Bookcase Game* (Baltimore: Avalon-Hill, 1971).

(4) *Spitfire, an Historical Simulations Game* (New York: Simulations Publications, 1973).

Plane	Country	HP	Armament	Ceiling	Weight		Climb		Maximum Speed		Wing Loading		Power Loading (lbs/HP)	Range (miles)	
					Gross	Empty	Ft/Min	Alt.	MPH	Alt.	Gross Wt (lb/ft²)	Area (ft)		w/tanks	w/o tanks
Boulton Paul Defiant (2-seat)	Great Britain		4 0.303 MGs (turret)		8,350	6,078			313	19,000					
Hawker Typhoon	Great Britain		12 0.303 MGs (Browning)												
Hawker Hurricane II (1942)	Great Britain	1,185	4 20mm cannon 2 40mm cannon + 2 0.303 MGs		7,400	5,600	3,000	40,000	340	21,000	26.5		6.25		
Hurricane I (1940)	Great Britain	1,030	8 0.303 MGs	36,000	6,665	5,265	2,520		330	17,500	25.9		6.47		
Super Marine Spitfire I	Great Britain								342						
Mark V & IV Spitfire (composite)	Great Britain	1,730	4 0.303 MGs 2 or 4 20mm cannon	36,000	6,750	5,065	3,000		369	19,500			3.90		500
Foche-Wulf FW-190	Germany	1,600	2 7.92mm MGs 4 20mm cannon	37,000	8,580	6,270	3,050 3,280	Low 17,000	376 408	18,000 21,000	42.3		5.25	807	520
ME-109F (1943 and after)	Germany	1,475	1 20mm cannon or 15mm high-speed 2 7.9mm MGs	37,000	6,054		3,320 2,370	5,000 20,000	390	21,000	34.8		4.62	615	348
ME-262	Germany		4 30mm cannon	39,360	15,500				540						652
Macchi C-202	Italy	1,200	2 12.7mm MGs 2 7.7mm MGs		6,300				330	34,500	35	180	5.25		
Reggiane RE-2001	Italy	1,150	2 12.7mm MGs 2 7.7mm MGs		7,000		348	22,000			31.82	220	6.09		
LA-5	Russia	1,600	2 20mm cannon						370			188			400
LAGG-3	Russia	1,100	20mm cannon + 2 12.7mm MGs	29,520	7,040	5,764	348	16,000			37.2	188	6.37		400
MIG-3	Russia	1,200	1 12.7mm MGs 2 7.7mm MGs						360						500
YAH-1	Russia	1,100	20mm cannon						335						
Kawanishi NIK1 (Rex) (Fighter seaplane)	Japan	1,550													
Kawanishi NIK2-J (George)	Japan	2,000	4 20mm cannon (fighter) 2 7.7mm MGs						400						
Kawasaki KI61 (Tony) (Fighter)	Japan	1,200	2 20mm cannon + 2 12.7mm cannon or 7.7mm MGs or 2 12.7 MGs + 2 12.7mm cannon or 7.7mm MGs						356						

Plane	Country	HP	Armament	Ceiling	Weight		Climb		Maximum Speed		Wing Loading		Power Loading (lbs/HP)	Range (miles)	
					Gross	Empty	Ft/Min.	Alt.	MPH	Alt.	Gross Wt (lb/ft²)	Area (ft)		w/tanks	w/o tanks
Mitsubishi J2M2 (Jack)	Japan	1,850	2 20mm cannon + 2 7.7mm MGs						400						
Mitsubishi A6M5 (Zeke-Zero) (Carrier fighter)	Japan	1,020	2 20mm cannon + 2 7.7mm MGs		5,750	3,920	3,020	8,000	340	19,700	24.2	238	5.63		
Nakajima Ki-84 (Frank)	Japan	1,900	2 12.7mm MGs 2 20mm cannon		7,940				426	23,000	35.13	226	4.18		
Nakajima Ki-44 (Tojo)	Japan	1,500	2 7.7mm MGs + 2 20mm cannon + 1 40mm cannon		6,100	4,300	3,940	36,000	383	17,000	39.1	169	4.06		
Nakajima Ki 43 (Oscar)	Japan	1,150	2 12.7mm MGs		5,500				333	19,680			4.78	1,000	
North American P-51 Mustang III and IV	United States	1,570	4 0.30-cal MGs + 4 0.50-cal MGs or 4 0.2mm cannon or 6.5-cal MGs	30,000-42,000	10,000		3,200		370-445 (after 1943)		42.9	233	6.37	2,300	950
Republic P-470 (Thunderbolt)	United States	1,625	8 0.5mm MGs	40,000	12,500		3,000		430	29,000	41.6	300	7.69	1,800	600
Vought F4U-1,4 Corsair (Navy)	United States	2,000/2,250	6 0.50-cal MGs or 4 20mm cannon	37,900/41,000	12,700	8,700	2,900	3,800/3,120	425	20,000				386	500
Bell P-39 (Airocobra)	United States	1,200	20mm cannon + 2 0.30-cal MGs (RAF) (later 37mm cannon + 2 0.50-cal MGs)	35,000	8,052	5,968	3,333		385	11,000	37.76	213	6.71	1,050	525
Curtis P-40 (Warhawk)	United States	1,240	2 0.50-cal MGs + 2 0.3-cal MGs or 4 0.5-cal MGs or 6 0.5-cal MGs	31,000	8,720	6,550			364		36.95	236	7.03	1,200	610
Grumman F4 (Wildcat)	United States	1,200	6 0.5-cal MGs	34,800	7,412				318		28.5	260	6.18		925
Lockheed P38-F-5 (Lightning)	United States	2 1400 hp engines	1 23mm cannon 4 0.50-cal MGs	35,000	15,500	12,700			414	25,000	47.4	327	5.53		460
Grumman F6F-3 (Hellcat)	United States	2,000	6 0.50-cal MGs		12,100	9,238	3,2[illegible]0		376	21,000		334	6.05		