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A Method for the Stress Analysis of Helicopter Blades

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ABSTRACT

The method of analysis presented herein is applicable to both fixed and hinged rotors. Although the equations developed apply directly to rectangular blades, the method may be extended to cover blades having other shapes.[4]

Up to this time, no "exact" general solution has been found for even the simplest form of the differential equation governing rotor bending moments, although solutions can be determined for certain combinations of constants (these solutions are not general). Because of the lack of a general exact solution, various approximate methods such as series, perturbation methods, etc., were tried, but convergence was poor in most cases, necessitating the use of an unwieldy number of terms.

The solution presented in this paper is based on the method of finite differences and depends on the evaluation of a number of simultaneous linear equations. The method leads to a rapid determination of the bending moment at any point on the blade.

SYMBOLS

E = Young's modulus (lbs. per sq.in.)
I = moment of Inertia (in.^{4})
A = cross-section Area of Blade (sq.in.)
w = unit weight (lbs. per in.^{3})
g = 386 in. per sec.^{2}
ω = rotational velocity (rad. per sec.)
C.F. = centrifugal force
dm = differential mass
θ = cone angle
Z = axis of rotation
r = axis normal to Z
ξ = undeflected blade axis
ζ = axis normal to ξ
S = length of blade
P = axial load
S_s = shear
M = moment
x, v = running coordinates
B = $(Aw/g)\omega^2 \cos\theta$, (lbs. per sq.in.)
C = $\dfrac{1}{EI}\left(\dfrac{w_t}{6S} - \dfrac{B\sin\theta}{6.}\right)$, (1/in.^{4})
M_d = hinge damping moment
D = $B\cos\theta/EI$, (1/in.^{4})
w_t = tip loading (lbs. per in.)
x = (ξ/S)
φ = $d\zeta/d\xi$
B.C. = boundary conditions
F = $w_t/12EI\,S^6$, (1/in.^{6})
G = $B\sin\theta/6EI$, (1/in.^{4})
α = $3CS^4$
β = $DS^4/2$
γ = $4FS^6$
λ = $3GS^4$
$f(x)$ = value of function at x
f_x' = first derivative of f relative to x
η = $S^4\,nAw/EI$
n = load factor

INTRODUCTION

THE EQUATIONS DEVELOPED BELOW hold for blades of rectangular plan form; the development for other plan forms follow the same general outline.[4] At this point it is noted that the assumption normally made in beam theory that $EId^2\zeta/d\xi^2 = M$, is also made here. This assumption implies small deflections and negligible lengthening of the blade under load. Because of the small deflection notion, the angle for freely hinged blades must be chosen carefully (see Eq. 24).

In applying this method, certain preliminary work is required, namely:

(1) Determine the blade principal axes.

(2) Resolve the air load relative to these axes (the total thrust may be assumed to cause bending about the principal axis of minimum inertia).

In line with the assumption generally made in propeller analysis, only the air load normal to the principal axis of minimum inertia need be assumed as acting.[1]

Derivation of the Differential Equation

Referring to Fig. 1 and terminology:

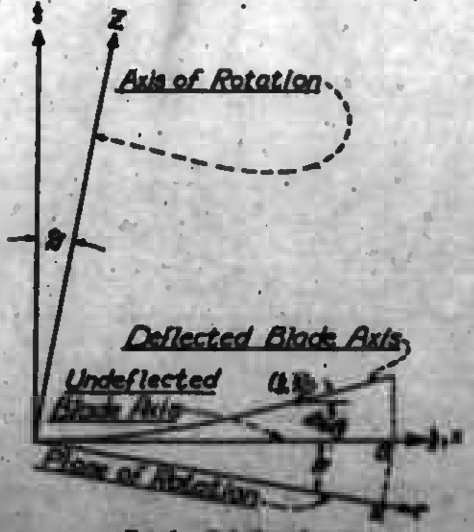

FIG. 1. Relation of axes.

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$$dm = (Aw/g)d\xi$$
$$dC.F. = r\omega^2 dm$$
$$dC.F. = (Aw/g)r\omega^2 d\xi$$

Let $B = (Aw/g)\omega^2 \cos\vartheta$, then, since $r = \xi \cos\vartheta$, we have:

$$dC.F. = B\xi d\xi$$

At any point ξ, the total C.F. is

$$C.F._\xi = B\int_S^\xi \xi d\xi = B/2(\xi^2 - S^2) \qquad (1)$$

where the negative value of C.F. is due to the point S being the origin of integration.

MOMENT DUE TO C.F. ABOUT ANY POINT

To determine the moment about the point (ξ, ζ) due to C.F., the C.F. may be broken into components normal to and parallel to the ξ axis. (See Fig. 2.)

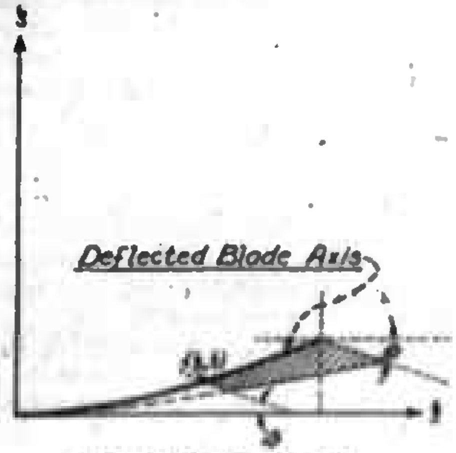

FIG. 2. Centrifugal loading.

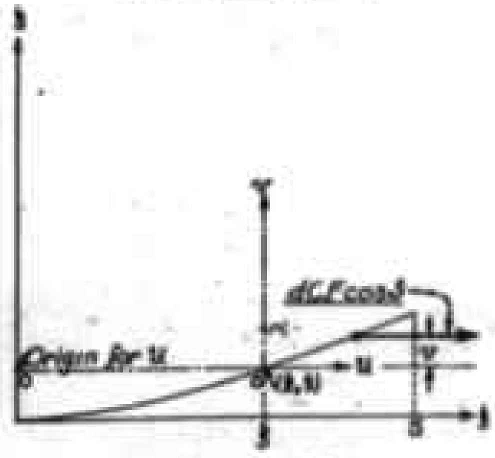

FIG. 3. u and v coordinates.

Considering the normal (relative to ξ) component of C.F.,

$$(S_{\xi,v})_v = B\sin\vartheta \int_S^\xi \xi d\xi$$
$$(M_{\xi,v})_v = B\sin\vartheta \int_S^\xi \int_S^\xi \xi d\xi d\xi$$

$$(S_{\xi,v})_v = (B\sin\vartheta/2)(\xi^2 - S^2) \qquad (2)$$
$$(M_{\xi,v})_v = (B\sin\vartheta/6)(2S^3 + \xi^3 - 3S^2\xi) \qquad (3)$$

Considering now the parallel component of C.F.

$$P = \text{axial load} = C.F. \cos\vartheta = (B\cos\vartheta/2)(\xi^2 - S^2) \qquad (4)$$

Moment about (ξ, ζ) of $dC.F. \cos\vartheta$ is (see Fig. 3):

$$dM_h = dC.F. \cos\vartheta(v)$$

but in terms of u and v, $dC.F. = Budu$, hence:

$$dM_h = B\cos\vartheta\, vudu$$
$$M_h = B\cos\vartheta \int_S^\xi vudu \qquad (5)$$

MOMENT DUE TO AIR LOAD

The moment due to a known air-load distribution may readily be found. Depending on the down-wash distribution, blade twist, etc., the air loading may vary from an approximately triangular distribution to some nth degree parabolic distribution or may be defined by a trigonometric or power series. Two simple cases are considered here: (1) triangular and (2) second degree parabolic.

(1) *Triangular Distribution (See Fig. 4)*

If tip load $= w_0$, then $w_\xi = (w_0/S)\xi$, and

$$S_A = \frac{w_0}{S}\int_S \xi d\xi = \frac{w_0}{2S}(\xi^2 - S^2) \qquad (6)$$

$$M_A = \frac{w_0}{S}\int_S\int_S \xi d\xi d\xi = \frac{w_0}{6S}(2S^3 + \xi^3 - 3S^2\xi) \qquad (7)$$

(2) *Parabolic Distribution*

Again let w_0 = tip load, then $w_\xi = (w_0/S^2)\xi^2$, and

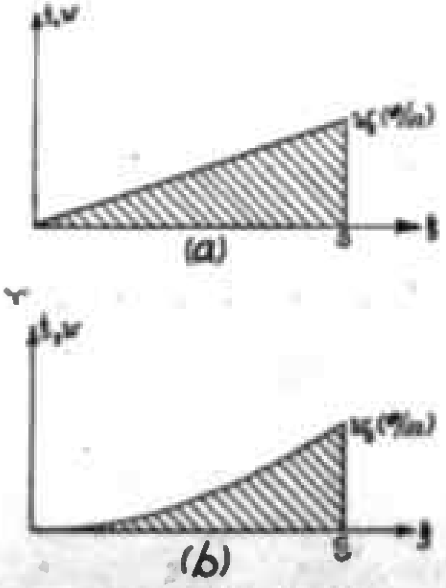

FIG. 4. Air loadings considered.

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$$S_\Delta = \frac{w_t}{S^2}\int_S^\xi \xi^2 d\xi = \frac{w_t}{3S^2}(\xi^3 - S^3) \qquad (6a)$$

$$M_\Delta = \frac{w_t}{S^2}\int_S^\xi\int_S^\xi \xi^2 d\xi d\xi = \frac{w_t}{12S^2}(3S^4 + \xi^4 - 4S^3\xi) \qquad (7a)$$

TOTAL MOMENT

(1) Triangular Air-Load Distribution

$$M_{\xi,\zeta} = -(3) + (5) + (7)$$

$$M = \left(\frac{w_t}{6S} - \frac{B\sin\vartheta}{6}\right)(2S^3 + \xi^3 - 3S^2\xi) + B\cos\vartheta\int_S^\xi vudu \qquad (8)$$

Since $M/EI = (d^2\zeta/d\xi^2)$, we have

$$\frac{d^2\zeta}{d\xi^2} = \frac{1}{EI}\left(\frac{w_t}{6S} - \frac{B\sin\vartheta}{6}\right)(2S^3 + \xi^3 - 3S^2\xi) + \frac{B\cos\vartheta}{EI}\int_S^\xi vudu \qquad (9)$$

Let

$$C = \frac{1}{EI}\left(\frac{w_t}{6S} - \frac{B\sin\vartheta}{6}\right) \qquad (10)$$

$$D = \frac{B\cos\vartheta}{EI} \qquad (11)$$

Then:

$$\frac{d^2\zeta}{d\xi^2} = C(2S^3 + \xi^3 - 3S^2\xi) + D\int_S^\xi vudu \qquad (12)$$

(2) Parabolic Air-Load Distribution

$$M = -(3) + (5) + (7a)$$

$$M = \frac{w_t}{12S^2}(3S^4 + \xi^4 - 4S^3\xi) - \frac{B\sin\vartheta}{6}\times(2S^3 + \xi^3 - 3S^2\xi) + B\cos\vartheta\int_S^\xi vudu \qquad (8a)$$

$$\frac{d^2\zeta}{d\xi^2} = \frac{w_t}{12EIS^2}(3S^4 + \xi^4 - 4S^3\xi) - \frac{B\sin\vartheta}{6EI}(2S^3 + \xi^3 - 3S^2\xi) + \frac{B\cos\vartheta}{EI}\int_S^\xi vudu \qquad (9a)$$

Let

$$F = w_t/(12EIS^2);\ G = \frac{B\sin\vartheta}{6EI} \qquad (10a)$$

Then:

$$\frac{d^2\zeta}{d\xi^2} = F(3S^4 + \xi^4 - 4S^3\xi) - G(2S^3 + \xi^3 - 3S^2\xi) + D\int_S^\xi vudu \qquad (12a)$$

TRANSFORMATION

Prior to reduction of Eqs. (12) and (12a), note that the B.C. are as follows:

(A) Fixed Blades

$$\zeta(0) = \zeta_\xi'(0) = \zeta_\xi''(0) = 0$$

(B) Hinged Blades

$$\zeta(0) = \zeta_\xi''(0) = \zeta_\xi''(S) = 0$$

no hinge damping.

$$\zeta(0) = \zeta_\xi''(S) = 0;\ \zeta_\xi''(0) = M_d/EI$$

with hinge damping.

Eqs. (12) and (12a) may be simplified by considering the following:[3]

$$\frac{d}{d\xi}\int_S^\xi vudu = \int_S^\xi \frac{d}{d\xi}(vu)du + [v(\xi)\xi]\frac{d\xi}{d\xi} - [v(S)\xi]\frac{dS}{d\xi}$$

Now: $v = \zeta(u) - \zeta(\xi)$. Hence, $v(\xi) = \zeta(\xi) - \zeta(\xi) = 0$; also, $dS/d\xi = 0$, since S is constant.

Since u does not depend on ξ,

$$(d/d\xi)(vu)du = udu(dv/d\xi)$$

$$dv/d\xi = d[\zeta(u) - \zeta(\xi)]/d\xi = -d\zeta/d\xi$$

Hence,

$$\frac{d}{d\xi}\int_S^\xi vudu = -\frac{d\zeta}{d\xi}\int_S^\xi udu = -\left(\frac{d\zeta}{d\xi}\right)\tfrac{1}{2}(\xi^2 - S^2) \qquad (13)[*]$$

Making use of Eq. (13) and differentiating Eqs. (12) and (12a), one obtains:

$$\frac{d^3\zeta}{d\xi^3} = 3C(\xi^2 - S^2) - D/2(\xi^2 - S^2)\frac{d\zeta}{d\xi} \qquad (14)$$

$$\frac{d^3\zeta}{d\xi^3} = 4F(\xi^3 - S^3) - 3G(\xi^2 - S^2) - D/2(\xi^2 - S^2)\frac{d\zeta}{d\xi} \qquad (14a)$$

Let

$$\varphi = d\zeta/d\xi \qquad (15)$$

Then:

$$\frac{d^2\varphi}{d\xi^2} + D/2(\xi^2 - S^2)\varphi = 3C(\xi^2 - S^2) \qquad (16)$$

$$\frac{d^2\varphi}{d\xi^2} + D/2(\xi^2 - S^2)\varphi = 4F(\xi^3 - S^3) - 3G(\xi^2 - S^2) \qquad (16a)$$

Let

$$\xi/S = x \qquad (17)$$

Note:

$$(d\varphi/d\xi)(d\xi/dx) = d\varphi/dx = S(d\varphi/d\xi)$$

$$(d/dx)S(d\varphi/d\xi) = (d/dx)(d\varphi/dx) = d^2\varphi/dx^2$$

But

$$S(d/dx)(d\varphi/d\xi) = S^2(d^2\varphi/d\xi^2)$$

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[*] This equation, when multiplied by $B\cos\vartheta$, gives the equivalent shear due to the centrifugal load component parallel to the ξ axis.

Hence,

$$(d^2\varphi/dx^2)(1/S^2) = d^2\varphi/d\xi^2 \qquad (18)$$

Let

$$\varphi'' = d^2\varphi/dx^2 \qquad (19)$$

Then Eqs. (16) and (16a) become

$$\varphi'' + (DS^4/2)(x^2 - 1)\varphi = 3CS^4(x^2 - 1) \qquad (20)$$

$$\varphi'' + (DS^4/2)(x^2 - 1)\varphi = 4FS^3(x^3 - 1) - 3GS^4(x^2 - 1) \qquad (20a)$$

Finally, let

$$\left.\begin{matrix} \beta = DS^4/2, & \gamma = 4FS^3 \\ \alpha = 3CS^4, & \lambda = 3GS^4 \end{matrix}\right\} \qquad (21)$$

And Eqs. (20) and (20a) become

$$\varphi'' + \beta(x^2 - 1)\varphi = \alpha(x^2 - 1) \qquad (22)$$

$$\varphi'' + \beta(x^2 - 1)\varphi = \gamma(x^3 - 1) - \lambda(x^2 - 1) \qquad (22a)$$

The B.C. on $\varphi(x)$ are:

(*A*) ***Fixed Blades***

$$\varphi(0) = \varphi'(1) = 0$$

(*B*) ***Hinged Blades***

$$\varphi'(0) = \varphi'(1) = 0$$

no hinge moment.

$$\varphi'(0) = SM_d/EI; \; \varphi'(1) = 0$$

with hinge damping.

Note that

$$M = EI\frac{d^2\zeta}{d\xi^2} = EI\frac{d\varphi}{d\xi} = \frac{EI}{S}\frac{d\varphi}{dx} \qquad (23)$$

Also

$$\varphi = d\zeta/d\xi; \; \zeta = \int_0^\xi \varphi\, d\xi = S\int_0^x \varphi\, dx$$

At this point it is noted that the angle ϑ is given for fixed blades as the initial dihedral angle (if any); for freely hinged blades, ϑ is determined by the condition that the moment at the root must be zero. The angle, ϑ, which satisfies this condition is given by Eqs. (24) and (24a).

(*1*) ***Triangular Loading***

$$\sin 2\vartheta = 2gw_s/SwA\omega^2 \qquad (24)$$

(*2*) ***Parabolic Loading***

$$\sin 2\vartheta = {}^{9}/_{8}(w_s g/SwA\omega^2) \qquad (24a)$$

It is interesting to note that Eq. (24) automatically gives zero bending over the entire blade. This follows from the fact that the assumed air-load distribution in this case is geometrically similar to the distribution of centrifugal load and it is possible for the blade to align itself with the total load resultant without flexure.

DEVELOPMENT OF FINITE DIFFERENCE EQUATIONS[9]

A number of methods were tried in an effort to determine an "exact" solution to Eqs. (22) and (22a). However, except for the fictitious case of $\beta = -1$ (implies negative dihedral, very stiff rotor), none was found. Series solutions were found but the convergence was generally considered too poor in the working range of β to make the series solution of practical interest.

As a consequence, the method of finite differences was tried, and it was found that this method gave accurate results with minimum labor.

As an aid to those not acquainted with difference solutions of differential equations, the development of the difference equations is outlined below. (See Fig. 5.)

The slope at the point φ_n for any continuous function $\varphi(x)$ may be presented by

$$\varphi_n' = (\Delta\varphi/\Delta x)_n \cong (\varphi_{n+1} - \varphi_{n-1})/2\Delta x \qquad (25)$$

The rate of change of slope, $\Delta^2\varphi/\Delta x^2$, is then expressed:

TABLE 1

1 Point	2 x	3 x^2	4 $x^2 - 1$	5 ④ × Δx^2	6 $-\alpha$ × ⑤ × 10^3	7 $-\beta$ × ⑤	8 ⑦ − 2
0	0	0	−1.0000	−0.00250000	5.00000	0.1750000	−2.1750000
1	0.05	0.0025	−0.9975	−0.00249375	4.98750	0.1745625	−2.1745625
2	0.10	0.0100	−0.9900	−0.00247500	4.95000	0.1732500	−2.1732500
3	0.15	0.0225	−0.9775	−0.00244375	4.88750	0.1710625	−2.1710625
4	0.20	0.0400	−0.9600	−0.00240000	4.80000	0.1680000	−2.1680000
5	0.25	0.0625	−0.9375	−0.00234375	4.68750	0.1640625	−2.1640625
6	0.30	0.0900	−0.9100	−0.00227500	4.55000	0.1592500	−2.1592500
7	0.35	0.1225	−0.8775	−0.00219375	4.38750	0.1535625	−2.1535625
8	0.40	0.1600	−0.8400	−0.00210000	4.20000	0.1470000	−2.1470000
9	0.45	0.2025	−0.7975	−0.00199375	3.98750	0.1395625	−2.1395625
10	0.50	0.2500	−0.7500	−0.00187500	3.75000	0.1312500	−2.1312500
11	0.55	0.3025	−0.6975	−0.00174375	3.48750	0.1220625	−2.1220625
12	0.60	0.3600	−0.6400	−0.00160000	3.20000	0.1120000	−2.1120000
13	0.65	0.4225	−0.5775	−0.00144375	2.88750	0.1010625	−2.1010625
14	0.70	0.4900	−0.5100	−0.00127500	2.55000	0.0892500	−2.0892500
15	0.75	0.5625	−0.4375	−0.00109375	2.18750	0.0765625	−2.0765625
16	0.80	0.6400	−0.3600	−0.00090000	1.80000	0.0630000	−2.0630000
17	0.85	0.7225	−0.2775	−0.00069375	1.38750	0.0485625	−2.0485625
18	0.90	0.8100	−0.1900	−0.00047500	0.95000	0.0332500	−2.0332500
19	0.95	0.9025	−0.0975	−0.00024375	0.4875	0.0170625	−2.0170625
20	1.00	1.0000	0.0000	0.00000000	0	0	0

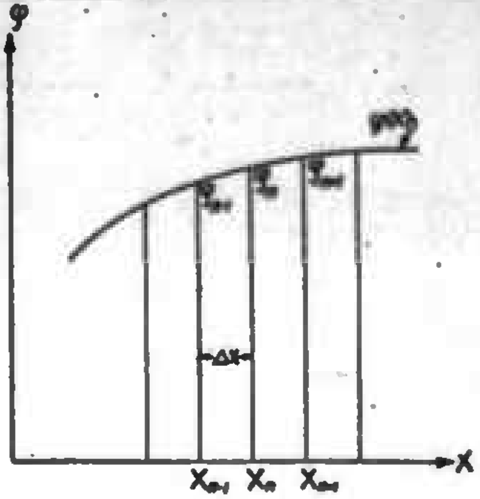

FIG. 5.

$$\varphi_n'' = \left(\frac{\Delta^2\varphi}{\Delta x^2}\right)_n \cong \left(\frac{\varphi_{n+1} - \varphi_n}{\Delta x} - \frac{\varphi_n - \varphi_{n-1}}{\Delta x}\right)/x\Delta$$

$$(\Delta^2\varphi/\Delta x^2)_n = (\varphi_{n+1} - 2\varphi_n + \varphi_{n-1})/\Delta x^2 \qquad (26)$$

As Δx → 0, Eqs. (25) and (26) approach the exact derivatives; however, for a continuous function without singularity, even large values of Δx give excellent approximations to the exact solution.

Noting that $\varphi'' \cong \Delta^2\varphi/\Delta x^2$, Eqs. (22) and (22a) may be written:

$$\varphi_{n+1} + \varphi_{n-1} + \varphi_n[\beta\Delta x^2(x^2 - 1) - 2] = \alpha\Delta x^2(x^2 - 1) \qquad (27)$$

$$\varphi_{n+1} + \varphi_{n-1} + \varphi_n[\beta\Delta x^2(x^2 - 1) - 2] = \gamma\Delta x^2(x^3 - 1) - \lambda\Delta x^2(x^2 - 1) \qquad (27a)$$

The method of solving these equations is illustrated below by means of three related problems.

ILLUSTRATIVE EXAMPLES

Consider the following rotor blade.

S = 20 ft. = 240 in.
chord = 2 ft. = 24 in.
thickness (maximum) = 15 per cent C = 3.6 in. = b
$A^1 = 0.74\,bh = 0.74(3.6)(24) = 63.94$ sq.in.
$I^1 = 0.0472\,bh^3 = 0.0472(24)(3.6)^3 = 52.85$ in.^{4}
Solid wood construction, E = 1,500,000 lbs. per sq.in.
w_s = 700 lbs. per ft. = 58.2 lbs. per in. (assumed)
$w = 0.60(62.4) = 37.44$ lbs. per ft.^{3} = 0.0216 lbs. per in.^{3}
$g = 32.2(12) = 386$ in. per sec.^{2}
$w_s/S = 58.2/240 = 0.242$ lbs. per sq.in. (may vary from 0 to 0.25 approximately)
ω = 300 r.p.m./60 $(2\pi) \cong 30$ rad. per sec.
$\omega^2 = 900$
sin 2ϑ(0 to 0.5); ϑ, 0° to 15°

$$B = \frac{Aw}{g}\omega^2\cos\vartheta \cong \frac{(63.94)(0.0216)}{386.0}(900)(1) = 3.22$$
lbs. per sq.in.

$$C = (1/EI)[(w_s/6S) - (B\sin\vartheta/6)]$$

$$\frac{1}{EI} = \frac{1}{(1.5)(5.285)} \times 10^{-6} = 0.1261 \times 10^{-6}$$

Since w_s/S may vary from 0 to 0.25 and $B\sin\vartheta$ from 0 to 0.483 (approximately 0.5),

C ranges from $+0.53 \times 10^{-8}$ to -1.1×10^{-8}, 1/in.^{4}

$$D = \frac{B\cos\vartheta}{EI} = 3.22(1)(0.1261)(10^{-6}) = 40.6 \times 10^{-8}$$
$$\alpha = 3CS^4;\ S^4 = (2.40)^4 \times 10^8 = 33.20 \times 10^8$$
$$\alpha = +1.76 \text{ to } -3.65$$
$$\beta = \frac{DS^4}{2} = \frac{40.6}{2} \times 10^{-8} \times 3.32 \times 10^9 = 67.4$$

For a hinged blade:

$$\vartheta = \text{arc sin } (0.242/3.22) = \text{arc sin } 0.0752 = 4°\,19'$$

Example 1

It is desired to determine the bending moment in a fixed blade having $\beta = 70$, $\alpha = 2$, and loaded by a triangular air-load distribution.

The quantities $[\alpha(\Delta x^2)(x^2 - 1)]$ and $[\beta(\Delta x^2) \times (x^2 - 1) - 2]$ are calculated below for $\Delta x = 0.05$.

From Table 1 and Eq. (27), the following set of linear equations is obtained.

(1) $\varphi_0 + \varphi_2 - 2.174563\varphi_1 = -0.00498750$
(2) $\varphi_1 + \varphi_3 - 2.173250\varphi_2 = -0.00495000$
(3) $\varphi_2 + \varphi_4 - 2.171063\varphi_3 = -0.00488750$

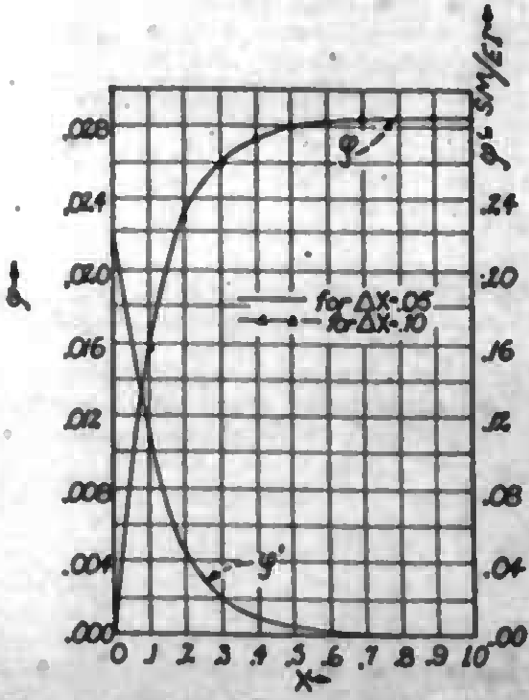

FIG. 6. Curves showing relation between φ and x and φ' and x.

(4) $\varphi_5 + \varphi_3 - 2.168000\varphi_4 = -0.00480000$
(5) $\varphi_6 + \varphi_4 - 2.164063\varphi_5 = -0.00468750$
(6) $\varphi_7 + \varphi_5 - 2.159250\varphi_6 = -0.00455000$
(7) $\varphi_8 + \varphi_6 - 2.153563\varphi_7 = -0.00438750$
(8) $\varphi_9 + \varphi_7 - 2.147000\varphi_8 = -0.00442000$
(9) $\varphi_{10} + \varphi_8 - 2.139563\varphi_9 = -0.00398750$
(10) $\varphi_{11} + \varphi_9 - 2.131250\varphi_{10} = -0.00375000$
(11) $\varphi_{12} + \varphi_{10} - 2.122063\varphi_{11} = -0.00348750$
(12) $\varphi_{13} + \varphi_{11} - 2.112000\varphi_{12} = -0.00320000$
(13) $\varphi_{14} + \varphi_{12} - 2.101063\varphi_{13} = -0.00288750$
(14) $\varphi_{15} + \varphi_{13} - 2.089250\varphi_{14} = -0.00255000$
(15) $\varphi_{16} + \varphi_{14} - 2.076563\varphi_{15} = -0.00218750$
(16) $\varphi_{17} + \varphi_{15} - 2.063000\varphi_{16} = -0.00180000$
(17) $\varphi_{18} + \varphi_{16} - 2.048563\varphi_{17} = -0.00138750$
(18) $\varphi_{19} + \varphi_{17} - 2.033250\varphi_{18} = -0.00095000$
(19) $\varphi_{20} + \varphi_{18} - 2.017063\varphi_{19} = -0.00048750$
(20) $\varphi_{21} + \varphi_{19} - 2\varphi_{20} = 0$

The B.C. are $\varphi(0) = 0$; $\varphi' = 0$

$$\varphi(0) = 0 \rightarrow \varphi_0 = 0$$
$$\varphi'(1) = 0 \rightarrow (\varphi_{21} - \varphi_{19})/2\Delta x = 0$$
$$\varphi_{21} = \varphi_{19}$$

Eq. (20) gives $2\varphi_{19} - 2\varphi_{20} = 0$; or $\varphi_{19} = \varphi_{20}$.

The above system of equations is easily solved as follows:

(1) $\varphi_2 = 2.174563\varphi_1 - 0.0049875$
(2) $\varphi_3 = -\varphi_1 + 2.173250(2.174563\varphi_1 - 0.0049875) - 0.0049500$
$\varphi_3 = 3.725869\varphi_1 - 0.0157891$
(3) $\varphi_4 = -2.174563\varphi_1 + 0.0049875 + 2.171063(3.725869\varphi_1 - 0.0157891) - 0.00488750$
$\varphi_4 = 5.914514\varphi_1 - 0.0341791$
(4) $\varphi_5 = -3.725869\varphi_1 + 0.0157891 + 2.168000(5.914514\varphi_1 - 0.0341791) - 0.0048000$
$\varphi_5 = 9.096797\varphi_1 - 0.0631112$
(5) $\varphi_6 = -5.914514\varphi_1 + 0.0341791 + 2.164063(9.096797\varphi_1 - 0.0631112) - 0.0046875$
$\varphi_6 = 13.771528\varphi_1 - 0.107085$
(6) $\varphi_7 = -9.096797\varphi_1 + 0.0631112 + 2.159250(13.771528\varphi_1 - 0.107085) - 0.0045500$
$\varphi_7 = 20.639375\varphi_1 - 0.1726621$
(7) $\varphi_8 = -13.771528\varphi_1 + 0.107085 + 2.153563(20.639375\varphi_1 - 0.1726621) - 0.0043875$
$\varphi_8 = 30.676666\varphi_1 - 0.2691412$
(8) $\varphi_9 = -20.639375\varphi_1 + 0.1726621 + 2.147000(30.676666\varphi_1 - 0.2691412) - 0.0042000$
$\varphi_9 = 45.223427\varphi_1 - 0.4093841$
(9) $\varphi_{10} = -30.676666\varphi_1 + 0.2691412 + 2.139563(45.223427\varphi_1 - 0.4093841) - 0.0039875$
$\varphi_{10} = 66.081705\varphi_1 - 0.6107494$
(10) $\varphi_{11} = -45.223427\varphi_1 + 0.4093841 + 2.131250(66.081705\varphi_1 - 0.6107494) - 0.0037500$
$\varphi_{11} = 95.613207\varphi_1 - 0.8960256$
(11) $\varphi_{12} = -66.081705\varphi_1 + 0.6107494 + 2.122063(95.613207\varphi_1 - 0.8960256) - 0.00348750$
$\varphi_{12} = 136.815544\varphi_1 - 1.2941609$
(12) $\varphi_{13} = -95.613207\varphi_1 + 0.8960256 + 2.112000(136.815544\varphi_1 - 1.2941609) - 0.0032000$
$\varphi_{13} = 193.341222\varphi_1 - 1.8404422$
(13) $\varphi_{14} = -136.815544\varphi_1 + 1.2941609 + 2.101063(193.341222\varphi_1 - 1.8404422) - 0.0028875$
$\varphi_{14} = 269.406554\varphi_1 - 2.5750116$
(14) $\varphi_{15} = -193.341222\varphi_1 + 1.8404422 + 2.089250(269.406554\varphi_1 - 2.5756116) - 0.0025500$
$\varphi_{15} = 369.516421\varphi_1 - 3.5432043$
(15) $\varphi_{16} = -269.406554\varphi_1 + 2.5750116 + 2.076563(369.516421\varphi_1 - 3.5432043) - 0.00218750$
$\varphi_{16} = 497.917574\varphi_1 - 4.7842629$
(16) $\varphi_{17} = -369.516421\varphi_1 + 3.5432043 + 2.063000(497.917574\varphi_1 - 4.7842629) - 0.0018000$
$\varphi_{17} = 657.687534\varphi_1 - 6.3285301$
(17) $\varphi_{18} = -497.917574\varphi_1 + 4.7842629 + 2.048563(657.687534\varphi_1 - 6.3285301) - 0.0013875$
$\varphi_{18} = 849.396774\varphi_1 - 8.1815172$
(18) $\varphi_{19} = -657.687534\varphi_1 + 6.3285301 + 2.033250(849.396774\varphi_1 - 8.1815172) - 0.00095000$
$\varphi_{19} = 1069.348457\varphi_1 - 10.3074897$
(19) $849.396774\varphi_1 - 8.1815172 - 1.017063(1069.348457\varphi_1 - 10.3074897) + 0.0004875 = 0$

$$\varphi_1 = \frac{2.3023367}{238.197976} = 0.0096656$$

Note: It can be shown that for this type of loading φ_1 is a direct function of α, and the following can be written.

$$\varphi_1/\alpha = 0.0048328 \quad (\beta = 70)$$

Further, φ at any point is linearly dependent on α, with the consequence that φ' and M are also linearly dependent on α. It follows that, knowing M for one α, M can be determined directly for any other α.

Recapitulating:

$\varphi_1 = 0.0096656$
$\varphi_2 = 2.174563(0.0096656) - 0.0049875 = 0.016031$
$\varphi_3 = 3.725869(0.0096656) - 0.0157891 = 0.020224$
$\varphi_4 = 5.914514(0.0096656) - 0.0341791 = 0.022988$
$\varphi_5 = 9.096797(0.0096656) - 0.0631112 = 0.024815$
$\varphi_6 = 13.771528(0.0096656) - 0.107085 = 0.026025$
$\varphi_7 = 20.639375(0.0096656) - 0.1726621 = 0.026830$
$\varphi_8 = 30.676666(0.0096656) - 0.2691412 = 0.027367$
$\varphi_9 = 45.223427(0.0096656) - 0.4093841 = 0.027728$
$\varphi_{10} = 66.081705(0.0096656) - 0.6107494 = 0.027970$
$\varphi_{11} = 95.613207(0.0096656) - 0.8960256 = 0.028133$
$\varphi_{12} = 136.815544(0.0096656) - 1.2941609 = 0.028243$
$\varphi_{13} = 193.341222(0.0096656) - 1.8404422 = 0.028317$
$\varphi_{14} = 269.406554(0.0096656) - 2.5750116 = 0.028364$
$\varphi_{15} = 369.516421(0.0096656) - 3.5432043 = 0.028394$
$\varphi_{16} = 497.917574(0.0096656) - 4.7842629 = 0.028409$

TABLE 2

1 Point	2 x	3 x^2	4 $x^2 - 1$	5 ④ × Δx^2	6 α × ⑤	7 β × ④	8 ⑦ − 2
0	0	0	−1	−0.100	−0.0200	−0.7000	−2.7000
1	0.1	0.01	−0.99	−0.0099	−0.0198	−0.6930	−2.6930
2	0.2	0.04	−0.96	−0.0096	−0.0192	−0.6720	−2.6720
3	0.3	0.09	−0.91	−0.0091	−0.0182	−0.6370	−2.6370
4	0.4	0.16	−0.84	−0.0084	−0.0168	−0.5880	−2.5880
5	0.5	0.25	−0.75	−0.0075	−0.0150	−0.5250	−2.5250
6	0.6	0.36	−0.64	−0.0064	−0.0128	−0.4480	−2.4480
7	0.7	0.49	−0.51	−0.0051	−0.0102	−0.3570	−2.3570
8	0.8	0.64	−0.36	−0.0036	−0.0072	−0.2520	−2.2520
9	0.9	0.81	−0.19	−0.0019	−0.0038	−0.1330	−2.1330
10	1.0	1.00	0	0.0000	0.0000	+0.0000	−2.0000

$$\varphi_{17} = 657.687534(0.0096656) - 6.3285301 = 0.028415$$

$$\varphi_{18} = 849.396774(0.0096656) - 8.1815172 = 0.028412$$

$$\varphi_{19} = \varphi_{20} = 1069.348457(0.0096656) - 10.3074897 = 0.028405$$

The moments at the stations considered will be determined later; however, in order to determine the effect of larger Δx, φ will be determined first for $\Delta x = 0.1$.

Example 2

Same as Example 1 except for $\Delta x = 0.1$.
From Table 2, the following system of equations results:

(1) $\varphi_2 + \varphi_0 - 2.693\varphi_1 = -0.0198$
(2) $\varphi_3 + \varphi_1 - 2.672\varphi_2 = -0.0192$
(3) $\varphi_4 + \varphi_2 - 2.637\varphi_3 = -0.0182$
(4) $\varphi_5 + \varphi_3 - 2.588\varphi_4 = -0.0168$
(5) $\varphi_6 + \varphi_4 - 2.525\varphi_5 = -0.0150$
(6) $\varphi_7 + \varphi_5 - 2.448\varphi_6 = -0.0128$
(7) $\varphi_8 + \varphi_6 - 2.357\varphi_7 = -0.0102$
(8) $\varphi_9 + \varphi_7 - 2.252\varphi_8 = -0.0072$
(9) $\varphi_{10} + \varphi_8 - 2.133\varphi_9 = -0.0038$
(10) $\varphi_{11} + \varphi_9 - 2.000\varphi_{10} = -0.0000$

The B.C. give $\varphi_0 = 0$ and $\varphi_{11} = \varphi_9 = \varphi_{10} = 0$.
Solving:

(1) $\varphi_2 = 2.693\varphi_1 - 0.0198$
(2) $\varphi_3 = -\varphi_1 + 2.672(2.693\varphi_1 - 0.0198) - 0.0192$
$\varphi_3 = 6.195696\varphi_1 - 0.0721056$
(3) $\varphi_4 = -2.693\varphi_1 + 0.0198 + 2.637(6.195696\varphi_1 - 0.0721056) - 0.0182$
$\varphi_4 = 13.645050\varphi_1 - 0.1885425$
(4) $\varphi_5 = -6.195696\varphi_1 + 0.0721056 + 2.588(13.645050\varphi_1 - 0.1885425) - 0.0168$
$\varphi_5 = 29.117693\varphi_1 - 0.4326424$
(5) $\varphi_6 = 13.64505\varphi_1 + 0.1885425 + 2.525(29.117693\varphi_1 - 0.4326424) - 0.0150$
$\varphi_6 = 59.877125\varphi_1 - 0.9188796$
(6) $\varphi_7 = -29.117693\varphi_1 + 0.4326424 + 2.448(59.877125\varphi_1 - 0.9188796) - 0.0128$
$\varphi_7 = 117.461509\varphi_1 - 1.8295749$
(7) $\varphi_8 = -59.877125\varphi_1 + 0.9188796 + 2.357(117.461509\varphi_1 - 1.8295749) - 0.0102$
$\varphi_8 = 216.979652\varphi_1 - 3.4036284$
(8) $\varphi_9 = -117.461509\varphi_1 + 1.8295749 + 2.252(216.979652\varphi_1 - 3.4036284) - 0.0072$
$\varphi_9 = 371.176667\varphi_1 - 5.8425963$
(9) $216.979652\varphi_1 - 3.4036284 - 1.133(371.176667\varphi_1 - 5.8425963) + 0.0038 = 0$

$$\varphi_1 = \frac{3.2198332}{203.563512} = 0.0158173;$$

$$\frac{\varphi_1}{\alpha} = 0.00790865$$

Recapitulating:

$\varphi_1 = 0.0158173$
$\varphi_2 = 2.693(0.0158173) - 0.0198 = 0.02280$
$\varphi_3 = 6.195696(0.0158173) - 0.0721056 = 0.02589$
$\varphi_4 = 13.645050(0.0158173) - 0.1885425 = 0.02729$
$\varphi_5 = 29.117693(0.0158173) - 0.4326424 = 0.02792$
$\varphi_6 = 59.877125(0.0158178) - 0.9188796 = 0.02821$
$\varphi_7 = 117.461509(0.0158173) - 1.8295749 = 0.02835$
$\varphi_8 = 216.979652(0.0158173) - 3.4036284 = 0.02840$
$\varphi_9 = 371.176667(0.0158173) - 5.8425963 = 0.02842$
$\varphi_{10} = \varphi_9 = 0.02842$

Comparing the above with the results obtained for $\Delta x = 0.05$, one notes that there is a maximum error of $1^1/_8$ per cent in φ caused by an increase in Δx from 0.05 to 0.1 (considerably less work is involved in using $\Delta x = 0.1$).

Note: φ_1 for $\Delta x = 0.1 = \varphi_2$ for $\Delta x = 0.05$, etc.

A plot of φ versus x for $\Delta x = 0.05$ and $\Delta x = 0.1$ is given below.

Since the two curves for φ fall almost on top of one another, $\varphi' = SM/EI$ is determined for $\Delta x = 0.05$ only and plotted in Fig. 6.

Eq. 25 gives

$$\varphi_n' = (\varphi_{n+1} - \varphi_{n-1})/2\,\Delta x$$

φ_0' is taken directly from the plot of φ.

$$\varphi_0' = \frac{0.004}{0.018} = 0.222 \left(\text{check } \varphi_0' \cong \frac{\varphi_1}{\Delta x} = 0.0096656 \div 0.05 = 0.193312\right)$$

$$\varphi_1' = \frac{\varphi_2 - \varphi_0}{2\Delta x} = \frac{0.016031}{0.1} = 0.16031$$

$$\varphi_2' = \frac{\varphi_3 - \varphi_1}{2\,\Delta x} = \frac{0.020224 - 0.0096656}{0.1} = 0.10558$$

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$$\varphi_3' = \frac{\varphi_4 - \varphi_2}{2\,\Delta x} = \frac{0.022988 - 0.016031}{0.1} = 0.06957$$

$$\varphi_4' = \frac{\varphi_5 - \varphi_3}{2\,\Delta x} = \frac{0.024815 - 0.020224}{0.1} = 0.04591$$

$$\varphi_5' = \frac{\varphi_6 - \varphi_4}{2\,\Delta x} = \frac{0.026025 - 0.022988}{0.1} = 0.03037$$

$$\varphi_6' = \frac{\varphi_7 - \varphi_5}{2\,\Delta x} = \frac{0.026830 - 0.024815}{0.1} = 0.02015$$

$$\varphi_7' = \frac{\varphi_8 - \varphi_6}{2\,\Delta x} = \frac{0.027367 - 0.026025}{0.1} = 0.01342$$

$$\varphi_8' = \frac{\varphi_9 - \varphi_7}{2\,\Delta x} = \frac{0.027728 - 0.026830}{0.1} = 0.00898$$

$$\varphi_9' = \frac{\varphi_{10} - \varphi_8}{2\,\Delta x} = \frac{0.027970 - 0.027367}{0.1} = 0.00603$$

$$\varphi_{10}' = \frac{\varphi_{11} - \varphi_9}{2\,\Delta x} = \frac{0.028133 - 0.027728}{0.1} = 0.00405$$

$$\varphi_{11}' = \frac{\varphi_{12} - \varphi_{10}}{2\,\Delta x} = \frac{0.028243 - 0.027970}{0.1} = 0.00273$$

$$\varphi_{12}' = \frac{\varphi_{13} - \varphi_{11}}{2\,\Delta x} = \frac{0.028317 - 0.028133}{0.1} = 0.00184$$

$$\varphi_{13}' = \frac{\varphi_{14} - \varphi_{12}}{2\,\Delta x} = \frac{0.028364 - 0.028243}{0.1} = 0.0012$$

$$\varphi_{14}' = \frac{\varphi_{15} - \varphi_{13}}{2\,\Delta x} = \frac{0.028394 - 0.028317}{0.1} = 0.00077$$

$$\varphi_{15}' = \frac{\varphi_{16} - \varphi_{14}}{2\,\Delta x} = \frac{0.028409 - 0.028364}{0.1} = 0.00045$$

$$\varphi_{16}' = \frac{\varphi_{17} - \varphi_{15}}{2\,\Delta x} = \frac{0.028415 - 0.028394}{0.1} = 0.00021$$

$$\varphi_{17}' = \frac{\varphi_{18} - \varphi_{16}}{2\,\Delta x} = \frac{0.028412 - 0.028409}{0.1} = 0$$

$$\varphi_{18}' = \frac{\varphi_{19} - \varphi_{17}}{2\,\Delta x} = \frac{0.028405 - 0.028415}{0.1} = 0$$

$$\varphi_{19}' = \frac{\varphi_{20} - \varphi_{18}}{2\,\Delta x} = \frac{0.028405 - 0.028412}{0.1} = 0$$

$$\varphi_{20}' = 0$$

*Example 3**

Same as Example 2 except for freely hinged rotor. The system of equations is the same as that given in Example 2; however, the B.C. are: $\varphi'(0) = \varphi'(1) = 0$; these give $\varphi_0 = \varphi_1$; $\varphi_9 = \varphi_{10}$.

Solving:

(1) $\varphi_2 = +1.693\varphi_1 - 0.0198$

(2) $\varphi_3 = -\varphi_1 + 2.672(1.693\varphi_1 - 0.0198) - 0.0192$

$\varphi_3 = 3.523696\varphi_1 - 0.0721056$

(3) $\varphi_4 = -1.692\varphi_1 + 0.0198 + 0.0198 +$
$2.637(\,3.523096\varphi_1 - 0.0721056) - 0.0182$

$\varphi_4 = 7.5989864\varphi_1 - 0.1885425$

(4) $\varphi_5 = -3.523696\varphi_1 + 0.0721056 +$
$2.588(7.598986\varphi_1 - 0.1885425) - 0.0168$

$\varphi_5 = 16.142480\varphi_1 - 0.4326424$

(5) $\varphi_6 = -7.598986\varphi_1 + 0.1885425 +$
$2.525(16.142480\varphi_1 - 0.4326424) - 0.0150$

$\varphi_6 = 33.160776\varphi_1 - 0.9188796$

(6) $\varphi_7 = -16.142480\varphi + 0.4326424 +$
$2.448(33.160776\varphi_1 - 0.9188796) - 0.0128$

$\varphi_7 = 65.035010\varphi_1 - 1.8295749$

(7) $\varphi_8 = -33.160776\varphi_1 + 0.9188796 +$
$2.357(65.035010\varphi_1 - 1.8295749) - 0.0102$

$\varphi_8 = 120.126743\varphi_1 - 3.4036284$

(8) $\varphi_9 = -65.035010\varphi_1 + 1.8295749 +$
$2.252(120.126743\varphi_1 - 3.4036284) - 0.0072$

$\varphi_9 = 205.490415\varphi_1 - 5.8425963$

(9) $120.126743\varphi_1 - 3.4036284 -$
$1.133(205.490415\varphi_1 - 5.8425963) - 0.0038$

$\varphi_1 = 3.2198332/112.693897 = 0.0285715$

$\varphi_{1/2} = 0.0142857$

Recapitulating:

$\varphi_0 = \varphi_1 = 0.0285715$

$\varphi_2 = 1.693000(0.0285715) - 0.0198 = 0.0285715$

$\varphi_3 = 3.523696(0.0285715) - 0.0721056 = 0.0285717$

$\varphi_4 = 7.598986(0.0285715) - 0.1885425 = 0.0285719$

$\varphi_5 = 16.142480(0.0285715) - 0.4326424 = 0.028725$

$\varphi_6 = 33.160776(0.0285715) - 0.9188796 = 0.0285735$

$\varphi_7 = 65.035010(0.0285715) - 1.8295749 = 0.0285729$

$\varphi_8 = 120.126743(0.0285715) - 3.4036234 = 0.0285728$

$\varphi_{10} = \varphi_9 = 205.490415(0.0285715) - 5.8425963 = 0.0285731$

The above result is interesting, since it indicates that an incorrect choice of ϑ did not give an incorrect answer. In the above problem flapping inertia, coriolis and gyroscopic loads were not considered. If the net distribution of these loads combined with air load were geometrically similar to the centrifugal load distribution, the condition of zero moment all along the blade would still exist even if these loads were considered. If the load distributions were not similar, small bending moments would exist, since, in this case, the blade would have to flex in order to satisfy the boundary conditions. The effect of a hinge damping moment is considered in the Appendix.

CONCLUSIONS

(1) The method of finite differences leads to a rapid determination of rotor bending moments.

(2) For constant chord (rectangular plan form) blades of uniform section and weight, loaded with a

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* Eq. (24) for this case gives $\alpha = 0$; however, to determine the effect of an incorrect choice of ϑ, α is taken as 2. Note that $\alpha = 0$, with tha B.C. $\varphi'(0) = \varphi'(1)$ automatically gives $\varphi'' = 0$ at all points.

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triangular load, the bending moments are directly proportional to α and, hence, to the tip loading.

Appendix

I. *Summary of Formulas for Untapered Blades*

Axial Load:

$$P_\xi = \frac{B\cos\vartheta}{2}(\xi^2 - S^2) - \frac{B\sin\vartheta}{2}(\xi^2 - S^2)\frac{d\zeta}{d\xi}$$

Shear with triangular air load:

$$S_\xi = \left(\frac{w_t}{2S} - \frac{B\sin\vartheta}{2}\right)(\xi^2 - S^2) - \frac{B\cos\vartheta}{2}(\xi^2 - S^2)\frac{d\zeta}{d\xi}$$

Shear with parabolic air load:

$$S_\xi = \frac{w_t}{3S^2}(\xi^3 - S^3) - \frac{B\sin\vartheta}{2}(\xi^2 - S^2) - \frac{B\cos\vartheta}{2}(\xi^2 - S^2)\frac{d\zeta}{d\xi}$$

Bending Moment:

$$M = (EI/S)(d\varphi/dx);\ \varphi' = d\varphi/dx$$

Deflection:

$$\zeta = S\int_0^x \varphi\, dx$$

II. *Effect of Blade Dead Weight*

For a rectangular blade of uniform section and weight, the dead-weight inertia loading is uniform and has a running value of: ndW, where n = load factor and dW = differential blade weight; but, $dW = Aw d\xi$; hence, $ndW = nAw d\xi$.

$$S_n = \text{inertia shear} = nAw\int_S^\xi d\xi = nAw(\xi - S)$$

$$M_\xi = nAw\int^\xi\int_S^\xi d\xi d\xi = \frac{nAw}{2}(\xi^2 + S^2 - 2S\xi)$$

The inertia moment acts in opposition to the air-load moment; hence Eqs. (22) and (22a) become:

$$\varphi'' + \beta(x^2 - 1)\varphi = \alpha(x^2 - 1) - \eta(x - 1) \qquad \text{(II—22)}$$

$$\varphi'' + \beta(x^2 - 1)\varphi = \gamma(x^3 - 1) - \lambda(x^2 - 1) - \eta(x - 1) \qquad \text{(II—22a)}$$

where $\eta = S^3nAw/EI$.

III. *Effects of Hinge Damping*

Since the magnitude of a hinge damping moment is dependent on the angular flapping velocity and not alone on the blade position, the angle, ϑ (and, hence, α), is indeterminate when hinge moments due to damping are present. The moment distribution can be determined, however, if an assumed value of ϑ is used.

The method of determining the bending moment at any point with hinge damping follows exactly the procedure of the illustrative examples with the exception of consideration of the B.C.—in this case,

$$\frac{EI}{S}\varphi_0' = M_d;\quad \varphi_0' = SM_d/EI$$

Assuming a straight line between φ_1 and φ_0:

$$(\varphi_1 - \varphi_0)/\Delta x = SM_d/EI$$
$$\varphi_1 = SM_d\Delta x/EI + \varphi_0$$

Knowing the relation between φ_0 and φ_1, the remaining φ_n are readily determined.

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ABSTRACT:

This method is applicable to both fixed and hinged rotors. It may be extended to cover all blade shapes, although developed equations apply only to rectangular blades. Solution is based on method of finite differences and depends on evaluation of a number of simultaneous linear equations. Bending moment at any point on the blade can be determined rapidly by use of this method, but certain careful preliminary work must be done.

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