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A HEIGHT DATA SMOOTHING MECHANISM

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A HEIGHT DATA SMOOTHING MECHANISM

The schematic diagram of a new type of height data smoothing mechanism is shown in Figure 1. The discontinuous height data e(t) is fed into the input shaft at intervals. This drives a differential, connected also to the ball carriage and roller of an integrator whose disk is turned by a constant speed motor. A correcting handwheel and the integrator roller feed another differential whose output is the output of the device. The output and input of the machine are compared through a differential feeding dial. The operator is supposed to turn the handwheel in such a way that the positive and negative oscillations of the dial about zero are equal.

The actual height of the target h(t) is a continuous function of time and we may assume that just after each reading e(t) is an approximation to this. Thus h(t) and e(t) might be as shown in Figure 2.

The shaft y(t) clearly satisfies the equation

$$y + \frac{1}{m} y' = e(t) \quad . \tag{1}$$

The x shaft satisfies

$$x(t) = y(t) + c(t) \tag{2}$$

and the dial reads

(3) $$D(t) = e(t) - x(t) \quad .$$

During the period between height readings the position of the e(t) shaft is constant, say $e(t_n)$, the reading taken at t_n,

$$y + \frac{1}{m} y' = e(t_n)$$

$$y = e(t_n) + A_n e^{-m(t - t_n)} \qquad t_n \leq t < t_{n+1} \quad .$$

Since y is obviously continuous, it will follow a curve consisting of a series of connected exponentials, each with the same time constant, $\frac{1}{m}$. The continuity of the curve implies

$$e(t_n) + A_n e^{-m(t_{n+1} - t_n)} = e(t_{n+1}) + A_{n+1} e^{-m.0}$$

$$A_{n+1} = A_n e^{-m(t_{n+1} - t_n)} - e(t_{n+1}) + e(t_n) \quad .$$

Assuming the intervals between readings the same, say a seconds, the response y for two different time constants $m_1 a = \ln 2$ and $m_2 a = \ln 10$ are shown in Figure 3.

The larger the time constant, the more the lag in response of y(t), but the smoother the curve. This may be seen another way: the e to y system is equivalent to an R, L circuit with position of shafts analogous to voltage as shown in Figure 4. With $\frac{L}{R}$ small y follows e closely including the irregularities. With $\frac{L}{R}$ large y(t) is smooth compared to e but lags considerably.

Movement of the handwheel does not affect y(t) but shifts x(t) up or down with respect to y. If the operator turns the wheel to give equal positive and negative movements of the dial, it may be seen that in the "steady state" (say with f(t) = at) there is a constant lag even when the damping is low and the interpolation nearly linear. In this case the system bridges linearly between the mid-ordinates of the steps, while actually it should bridge between the points $(t_n + 0)$. With higher damping the shape becomes worse but the interpolated exponentials are nearer to the true curve most of the time. We shall find a formula for the best time constant of the system under the following assumptions

1. That the "best" time constant is the one making the actual error least in the mean square sense.
2. That we may take as the true curve, so far as our knowledge goes, the linear interpolation between the points $t_n + 0$. This may be justified by the fact that the device cannot in any way perform higher order interpolation - the curve y(t) is convex upward whenever e(t) increased in its last step over the final value of y from the preceding step, and this is quite independent of the <u>curvature</u> of e(t).

3. That the system is in a "steady state", that is, that in the step under consideration y(t) ends at the same distance below e(t) as it was just before the step.

4. That the steps come at approximately equal intervals of <u>a</u> seconds.

An interval under these conditions is shown in Figure 5. Here we assumed that the handwheel was turned to give a ratio of $\frac{c}{b-c}$ as deflection of the dial just after to just before a step.

We have

$$y = A\, e^{-mt}$$

with

$$y(o) - b = y(a)$$

$$A - b = A\, e^{-am}$$

$$A = \frac{b}{1 - e^{-am}}$$

Hence

$$y = \frac{b\, e^{-mt}}{1 - e^{-am}}$$

also

$$x = y - y(o) + c$$

$$= - b\, \frac{1 - e^{-mt}}{1 - e^{-am}} + c$$

The integral of the squared error per second is then

$$I^2 = \frac{1}{a}\int_o^a \left[- b\,\frac{1 - e^{-mt}}{1 - e^{-am}} + c + \frac{b}{a}\,t\right]^2 dt$$

$$\frac{I^2}{b^2} = \frac{1}{a}\int_o^a \left[k + \frac{t}{a} - \frac{1 - e^{-mt}}{1 - e^{-am}}\right]^2 dt$$

where $k = \frac{c}{b}$

letting $D = ma, \qquad u = \frac{t}{a}$

$$\left(\frac{I}{b}\right)^2 = \frac{1}{a}\int_o^1 \left[k + u - \frac{1 - e^{-Du}}{1 - e^{-D}}\right]^2 du$$

$$= \frac{1}{a}\int_o^1 \left[k^2 + u^2 + \left(\frac{1 - e^{-Du}}{1 - e^{-D}}\right)^2 + 2\,ku\right.$$

$$\left.- 2\,k\,\frac{1 - e^{-Du}}{1 - e^{-D}} - 2\,u\,\frac{1 - e^{-Du}}{1 - e^{-D}}\right] du$$

$$= \frac{1}{a}\left[k^2u + \frac{u^3}{3} + \frac{1}{(1-e^{-D})^2}\left\{u + \frac{2}{D}\,e^{-Du} - \frac{e^{-2\,Du}}{2\,D}\right\}\right.$$

$$+ k u^2 - \frac{2k}{1 - e^{-D}} \left\{ u + \frac{1}{D} e^{-Du} \right\}$$

$$\left. - \frac{2}{1 - e^{-D}} \left[\frac{u^2}{2} + e^{-Du} \left(\frac{u}{D} + \frac{1}{D^2} \right) \right] \right]_0^1$$

$$= \frac{1}{a} \left[k^2 + \frac{1}{3} + \frac{1}{(1-e^{-D})^2} \left[1 - \frac{2}{D} (1 - e^{-D}) + \frac{1 - e^{-2D}}{2D} \right] \right.$$

$$\left. + k - \frac{2k}{1 - e^{-D}} \left(1 - \frac{1 - e^{-D}}{D} \right) - \frac{2}{1 - e^{-D}} \left[\frac{1}{2} + e^{-D} \left(\frac{1}{D} + \frac{1}{D^2} \right) - \frac{1}{D^2} \right] \right]$$

$$= \frac{1}{a} \left[\left(\frac{1}{3} + k + k^2 \right) + \frac{2k}{D} + \frac{2}{D^2} + \frac{1}{(1-e^{-D})^2} - \frac{(2 + 4k) \cdot D + 3 + 3e^{-D}}{2D (1 - e^{-D})} \right].$$

It is evident from physical considerations that the minimum of this expression occurs for a fairly large D. In fact the error curve was plotted for k = .5 (Figure 6) and the minimum is seen to be at about 7 or 8. With D this large the above expression is very nearly equal to

$$\phi = \frac{1}{2}\left[\left(\frac{1}{3} + k + k^2\right) + \frac{2k}{D} + \frac{2}{D^2} + 1 - \frac{(2 + 4k)\cdot D + 3}{2\,D}\right]$$

since e^{-D} is very small. To locate the minimum we have

$$\frac{d\phi}{dD} = -\frac{2k}{D^2} - \frac{4}{D^3} - \frac{2D\,(2 + 2k) - 2\,[(2 + 4k)\,D + 3]}{4\,D^2} = 0$$

whence

$$(6 - 8k)\,D = 16$$

$$D = \frac{8}{3 - 4k} \quad .$$

For $k = \frac{1}{2}$

$$D = 8 \quad .$$

Since the minimum is so flat (Figure 6) this formula is certainly close enough. However a second approximation may be found as follows: for x small $\frac{1}{1 - x} \cong 1 + x$. Using this in the exact expression to eliminate the denominators we get as a second approximation

$$\psi = \frac{1}{2}\left[\left(\frac{1}{3} + k + k^2\right) + \frac{2k}{D} + \frac{2}{D^2} + (1 + 2e^{-D})\right.$$

$$\left. - (1+k)\,(1+e^{-D}) - \frac{3}{2D}\,(1+e^{-D}) - \frac{3}{2D}\,(1+e^{-D})\,e^{-D}\right]$$

$$\frac{dy}{dD} = 0 = -8 + (3-4k)\,D + [6D\,(D+1) + 2D^3\,(k-1)]\,e^{-D} + 6D\,(D+1)\,e^{-2D}$$

Using the first approximation to obtain the values involving exponentials, a better value may be obtained. For $k = \frac{1}{2}$ the second approximation is $D = 8.03$. The first and second approximations are plotted in Figure 7.

With $k = \frac{1}{2}$ the curve $x(t)$ is plotted for an interval with the "best" D, in Figure 8. It will be noted that the curve is highly damped in comparison to the time between readings. The RMS error is then equal to

$$\frac{I}{b} = \sqrt{\frac{.053}{a}} = \frac{.23}{\sqrt{a}} \quad .$$

It is interesting to compare this with the RMS errors obtained under other conditions. If the device is not used at all, but a direct coupling made between the input and output, the RMS error between the step function and the linear interpolation between points $t_n + 0$ is

$$\left(\frac{I}{b}\right)^2 = \frac{1}{a}\int_0^a [0 - (-\frac{t}{a})^2]\,dt$$

$$\frac{I}{b} = \frac{1}{\sqrt{3a}} = \frac{.577}{\sqrt{a}}$$

so that the RMS error has been reduced to 40% of this value.

In Figure 9, the output of the smoothing mechanism, x(t), is plotted for a certain forcing function e(t), using the "best" value of m. It may appear that the output is still far from smooth, and this is in a sense true, but it must be remembered that the variations in e(t) are here greatly exaggerated over what would be expected in practice.

Finally it should be pointed out that a very material improvement in operation could be obtained if the operator were trained to turn the handwheel to obtain a ratio $\frac{a}{b}$ nearer to zero than $\frac{1}{2}$. This, however, would probably be impractical.

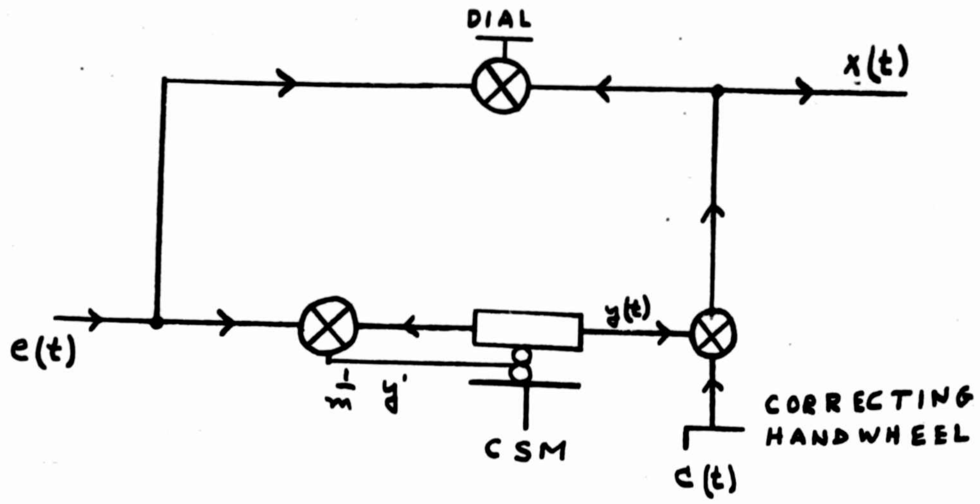

DIAL
x(t)
e(t)
y(t)
$\frac{1}{m}$ y'
CSM
CORRECTING HANDWHEEL
c(t)

FIG. 1

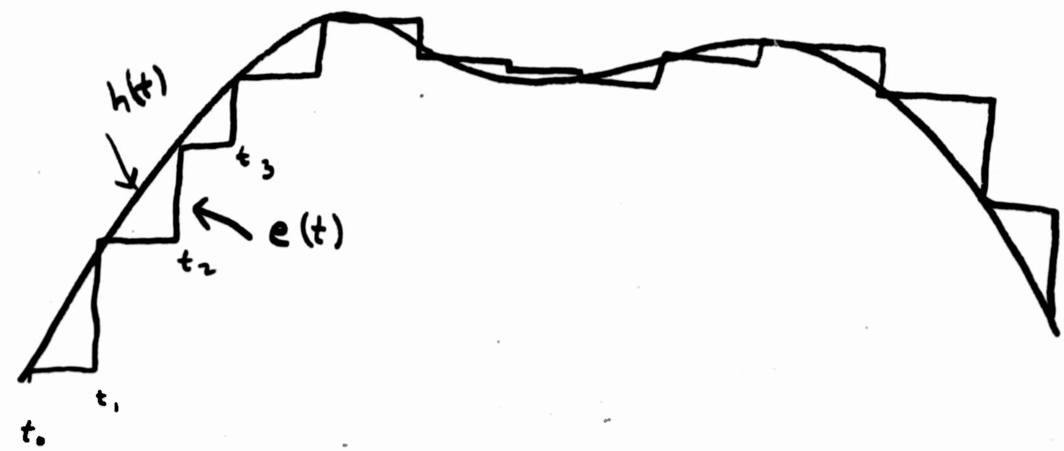

h(t)
e(t)
t_0
t_1
t_2
t_3

FIG. 2.

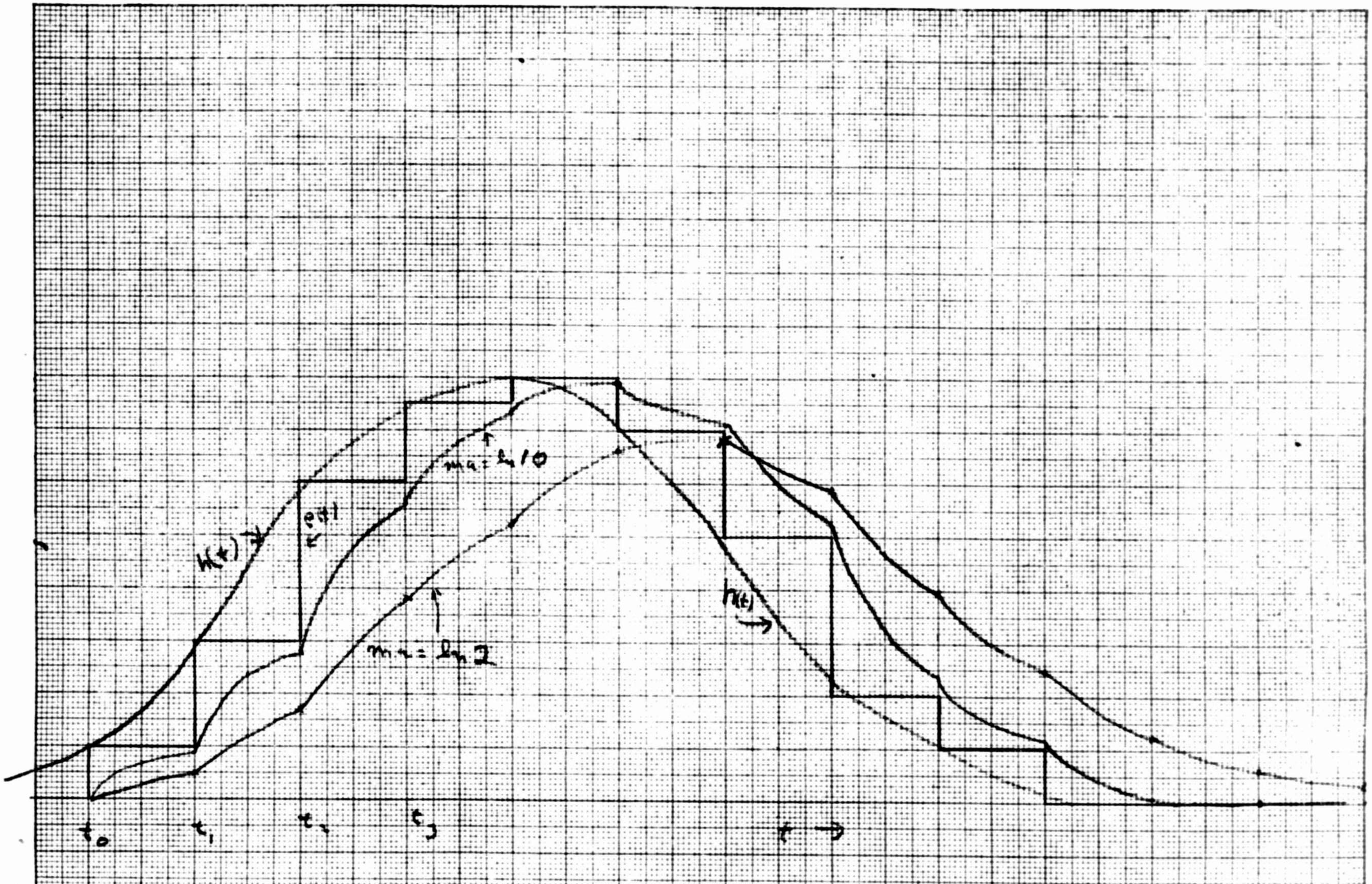


Fig. 3.

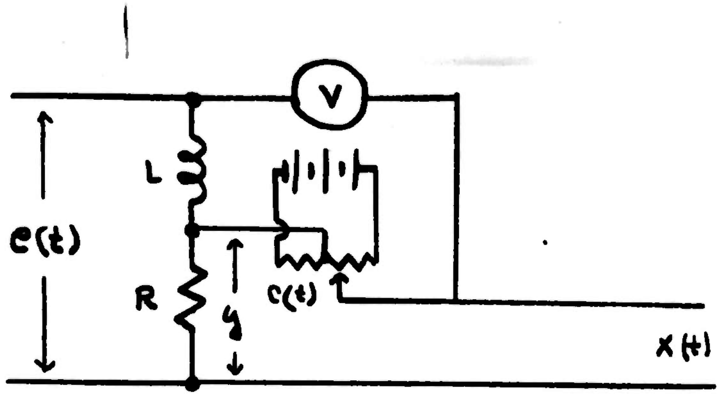
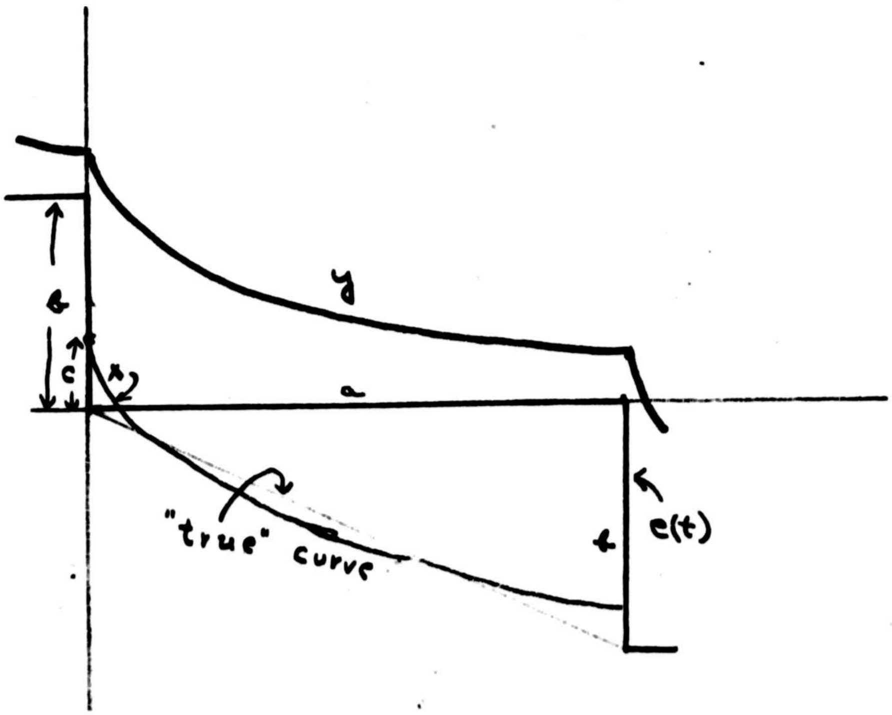


FIG. 4



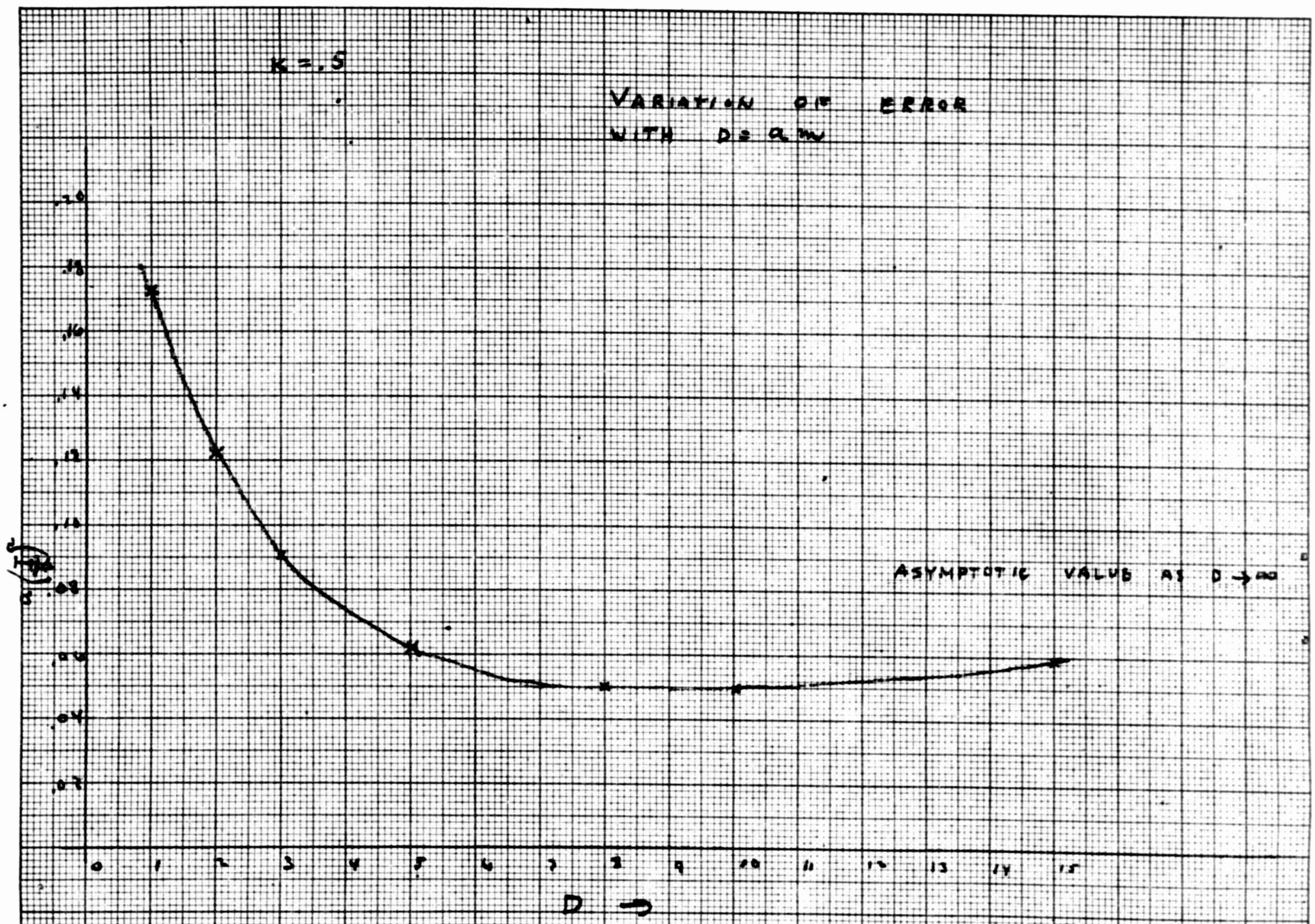

K = .5
VARIATION OF ERROR
WITH D = a w
ASYMPTOTIC VALUE AS D → ∞
.20
.18
.16
.14
.12
.10
.08
.06
.04
.02
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
D →

FIG. 6

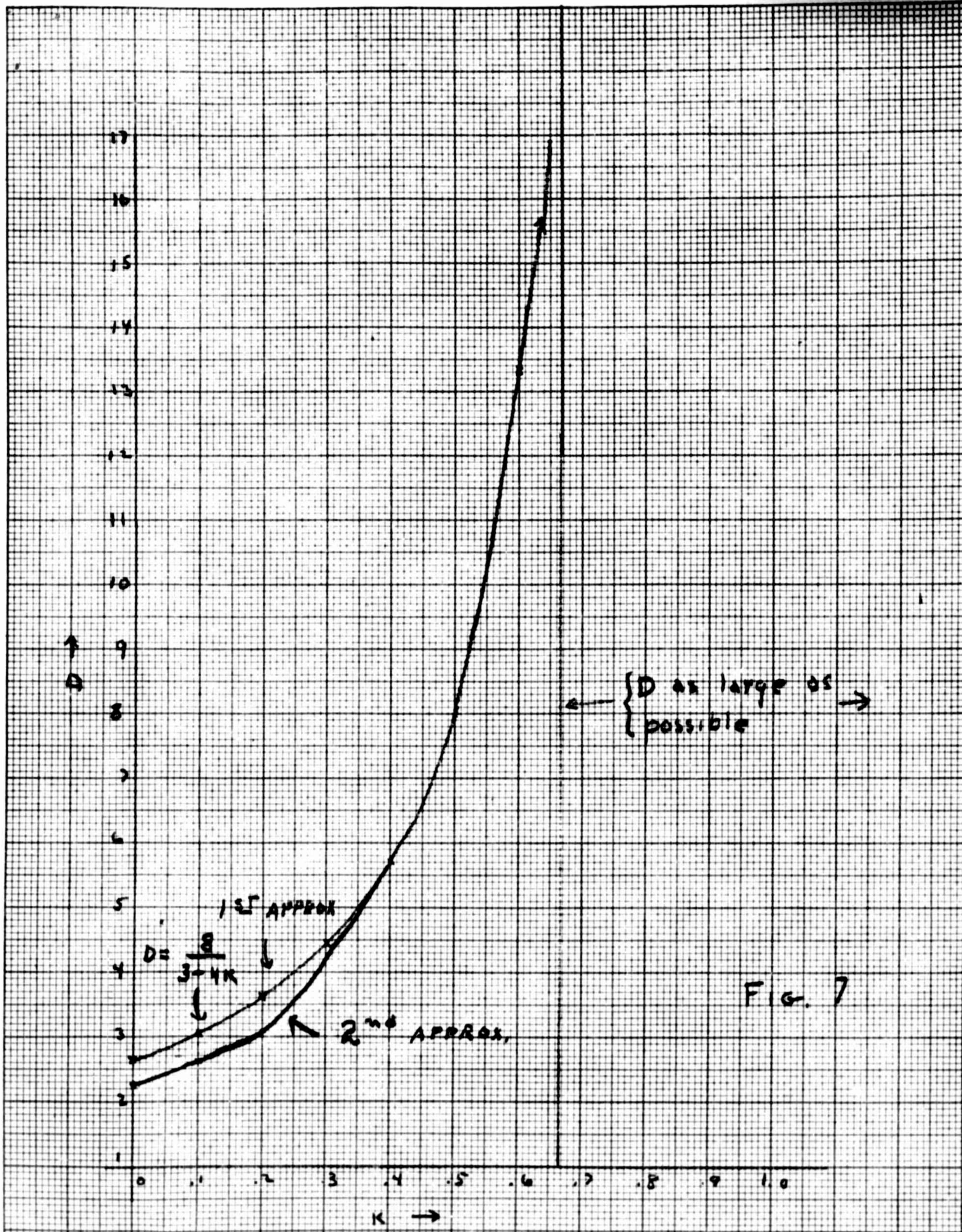


Fig. 7

Value of D = ma to minimize squared error vs. $K = \frac{c}{b}$.

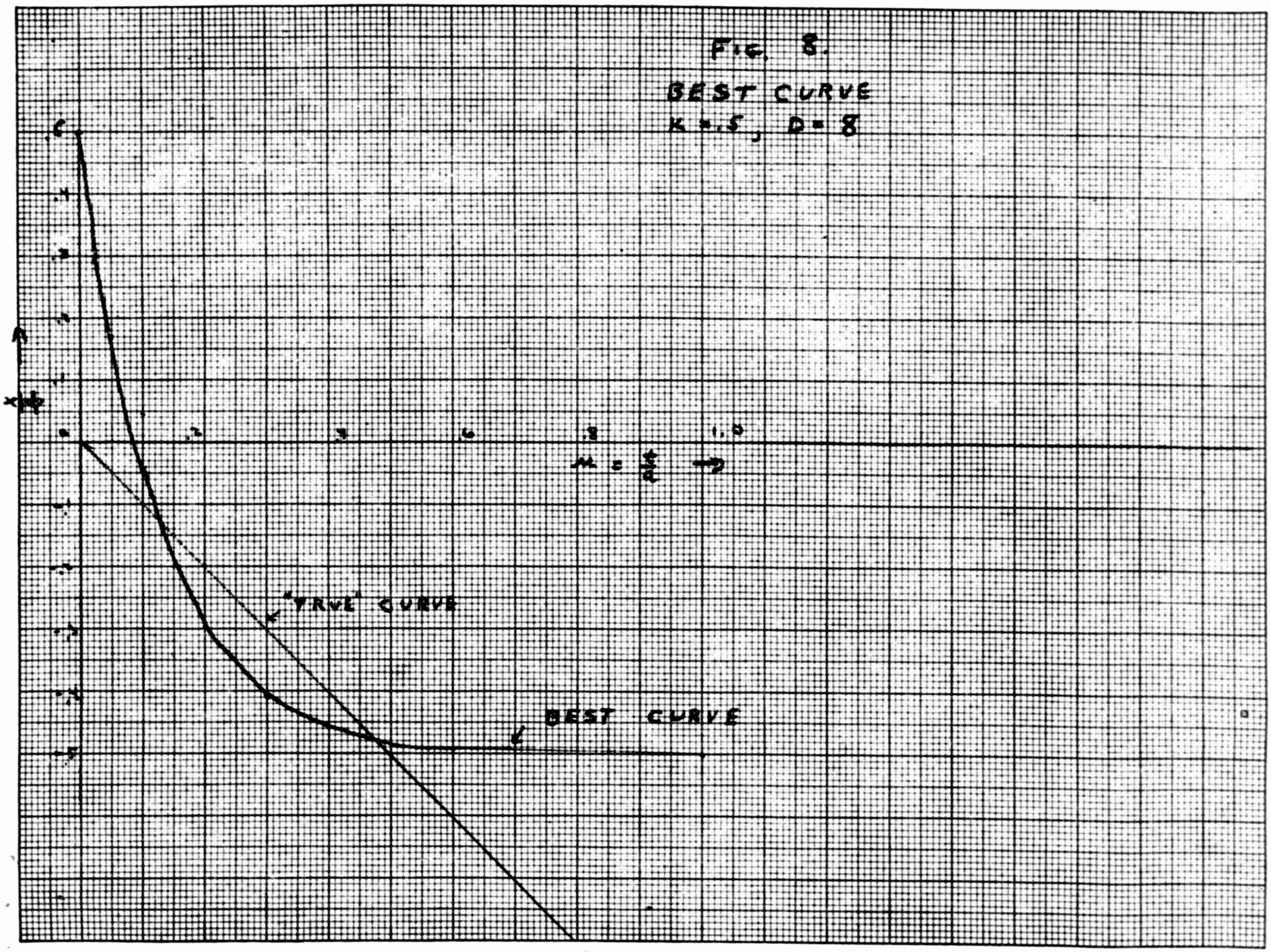


FIG. 8.
BEST CURVE
K = .5, D = 8
.2
.4
.6
.8
1.0
"TRUE" CURVE
BEST CURVE

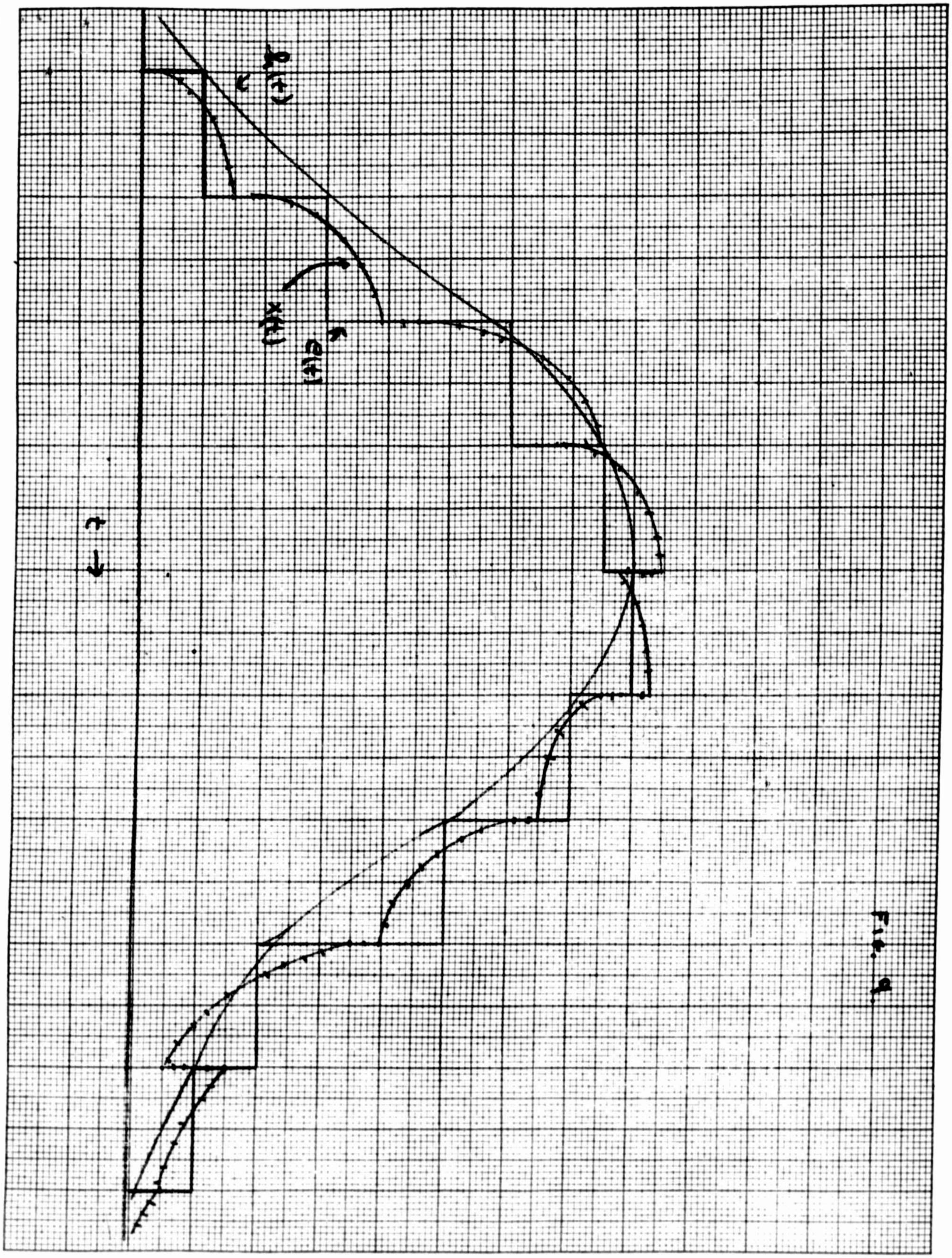


Fig. 9

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