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# ON LOCAL AND GLOBAL MINIMA IN STRUCTURAL OPTIMIZATION

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## 0. Summary

This paper deals with convexity properties in structural optimization, and with the closely related question of local versus global optima.

The problem we investigate is that of minimizing the structural weight subject to constraints on displacements, stresses and natural frequencies. It is assumed that the structure is described by a finite element model, and that the transverse sizes of the elements, e.g. thicknesses of membrane plates, are the design variables. This implies that both the objective function, i.e. the weight, and the structural stiffness matrix depend linearly on the design variables. The constraint functions, however, become nonlinear and they may in the general case give rise to a nonconvex feasible region in the design space. Then there is a risk that a local, but not global, minimum is attained when any of the various existing methods for numerically solving the problem is applied. This fact is illustrated by examples of nonconvex problems.

However, as shown in this paper, there are some special cases where the feasible region always becomes convex, so that, due to the linearity of the objective function, each local optimum is in fact also a global one. Three examples of constraints which are proved to possess such convexity properties are: i) a natural kind of "symmetric" displacement constraint, where the magnitude of the displacement is measured in the direction of the applied load, ii) a "global" displacement constraint, which may be interpreted as a lower bound of the smallest eigenvalue of the structural stiffness matrix, and iii) a lower bound on the lowest natural frequency.

## 1. Introduction

When an "optimal" solution of a structural optimization problem is obtained, by some numerical method, it is often very difficult to decide if it is the "global" optimal solution or just a "local" one. If, however, the considered problem is known to be convex, then some strong statements about the global nature of obtained solutions can be made. This is the main reason for the investigation of convexity properties in structural optimization accomplished in this paper. A second reason, which we do not go into details about, is that if the considered problem is known to be convex, then methods specially developed for convex problems could be used. Such methods might be more efficient than methods developed for more general nonlinear problems.

The problem we consider is to minimize the structural weight subject to constraints on displacements and stresses, under multiple statical load conditions, and on natural frequencies. It is assumed that the considered structure is described by a linearly elastic finite element model, and that the transverse sizes of the elements (cross section areas of bars, thicknesses of membrane plates etc.) are the design variables. The elements may be linked together in groups, so that the j:th design variable $x_j$ determines the sizes of all the elements in the j:th group. Also, some elements may have fixed sizes.

Under these assumptions the structural weight w is a linear function of the design variables: $w(x) = w_o + \sum w_j x_j$. The structural stiffness matrix K and the structural mass matrix M are also linear functions of x, i.e. $K(x) = K_o + \sum x_j K_j$ and $M(x) = M_o + \sum x_j M_j$, where $K_j$ and $M_j$ (j=0,1,..,n) are constant symmetric matrices. It is further assumed that each design variabel $x_j$ is restricted between given bounds: $x_j^{min} \leq x_j \leq x_j^{max}$, where $x_j^{min}$ and $x_j^{max}$ are constant positive real numbers. This may be written: $x \in X$, where $x = (x_1,..x_n)^T$ is the vector of variables and $X = \{x \in R^n | x_j^{min} \leq x_j \leq x_j^{max},\ j=1,..n\}$. For each $x \in X$, the structure is assumed to be non-degenerate, so that K(x) and M(x) are symmetric and positive definite.

It should be noted that since the weight w is a linear function in x, it is also a convex function in x. Thus, when investigating if the minimum weight problem is convex, it suffices to investigate if the imposed constraints (on displacements, etc.) give rise to a convex feasible domain in the design space. (This statement is made clear in section 2.)

We will occasionally use the notation $\|v\|$, to denote the euclidean norm of the vector v. If $v = (v_1,..v_n)^T \in R^n$, then $\|v\|^2 = \sum v_j^2$. We also use the notation S for the "unit sphere" (of suitable dimension apparent from the context), i.e. $S = \{v \mid \|v\| = 1\}$.

## 2. Convex optimization problems

In this section we bring together some basic definitions and results, concerning nonlinear optimization in general and convex problems in particular, which will be frequently referred to in the forthcoming sections. The results collected in this section have been known for several decades, and we therefore omit the proofs. A more exhaustive description of the matter may be found in refs [1] and [2].

We consider here the following general optimization problem P:

$$P: \quad \min f(x)\ ,\ x \in R^n$$
$$\text{subject to } g_i(x) \leq \alpha_i,\ i=1,..m$$
$$x_j^{min} \leq x_j \leq x_j^{max},\ j=1,..n$$

f and $g_i$ are real-valued functions. $\alpha_i$, $x_j^{min}$ and $x_j^{max}$ are constant real numbers. To shorten the notation we will also write:

$$P: \quad \min f(x)$$
$$\text{subject to } x \in \Omega$$

where $\Omega$, the feasible set, is defined by $\Omega = \{x \in X | g_i(x) \leq \alpha_i,\ i \in I\}$, where $X = \{x \in R^n | x_j^{min} \leq x_j \leq x_j^{max},\ j=1,..n\}$ and $I = \{1,..m\}$.

**Definition 2.1**: A point $x^* \in \Omega$ is said to be a global minimum point of P if $f(x^*) \leq f(x)$ for all $x \in \Omega$.

A global minimum point is what we normally are searching for. However, algorithms for numerically solving nonlinear problems of realistic sizes can not, in the general case, be expected to find anything better than a local minimum point:

**Definition 2.2**: A point $x^* \in \Omega$ is said to be a local minimum point of P if there is an $\varepsilon > 0$ such that $f(x^*) \leq f(x)$ for all $x \in \Omega$ such that $\|x-x^*\| < \varepsilon$.

If $x^*$ is a global minimum point then $x^*$ is obviously also a local minimum point, but the converse is in general not true.

**Definition 2.3**: A set C in $R^n$ is said to be convex if $\mu x + (1-\mu)y \in C$, for every $x, y \in C$ and every real number $\mu$ such that $0 < \mu < 1$.

The following property is fundamental for convex sets:

**Lemma 2.4**: Let $C_\nu$, $\nu \in V$, be a collection of convex sets in $R^n$. Then the intersection set $C = \bigcap_{\nu \in V} C_\nu = \{x \in R^n \mid x \in C_\nu \text{ for all } \nu \in V\}$ is convex. (If C is empty then C is by definition convex.)

V may be a finite or infinite index set. The lemma is illustrated in fig 2.1.

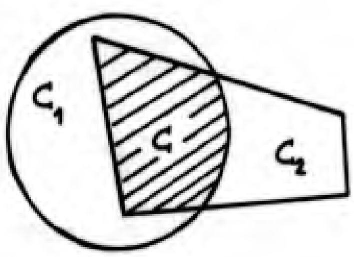


Fig 2.1 $C = C_1 \cap C_2$ is convex if $C_1$ and $C_2$ are convex.

**Definition 2.5**: A real-valued function $\varphi$, defined on a convex set C, is said to be convex over C if $\varphi(\mu x + (1-\mu)y) \leq \mu\varphi(x) + (1-\mu)\varphi(y)$, for every $x, y \in C$ and every $\mu$ such that $0 < \mu < 1$.

For twice continuously differentiable functions there is an alternative characterization of convexity:

**Lemma 2.6**: Assume that the function $\varphi$ has continuous second partial derivatives on a convex set C in $R^n$. Then $\varphi$ is convex over C if and only if the Hessian matrix $\Phi$ (= the matrix of second derivatives) of $\varphi$ is positive semidefinite throughout C, i.e. if and only if:

$$h^T\Phi(x)h = \sum\sum \frac{\partial^2\varphi}{\partial x_j \partial x_k} h_j h_k \geq 0 \text{ for all } x \in C \text{ and every}$$

vector $h \in R^n$.

The following property is fundamental for convex functions:

**Lemma 2.7**: Let $g_\nu$, $\nu \in V$, be a collection of functions which are convex over the convex set C. Assume that, for each $x \in C$, $\max_{\nu \in V} \{g_\nu(x)\}$ exists. Then the "pointwise maximum function" g, defined by $g(x) = \max_{\nu \in V} \{g_\nu(x)\}$, is convex over C.

V may be a finite or infinite index set. The lemma is illustrated in fig 2.2.

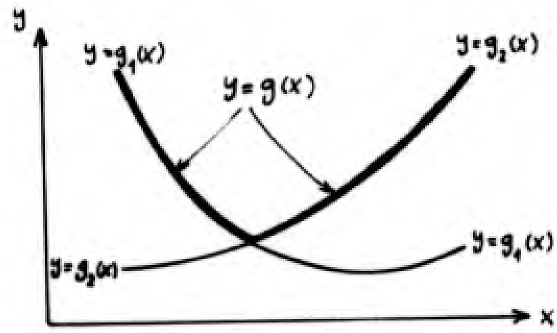


Fig 2.2 $g(x) = \max\{g_1(x), g_2(x)\}$. g is convex if $g_1$ and $g_2$ are convex.

We now come to what, in this paper, will be meant by a convex problem:

**Definition 2.8**: If, in problem P, the feasible set $\Omega$ is convex and the objective function f is convex over $\Omega$, then P is said to be a convex (optimization) problem.

**Lemma 2.9**: A sufficient condition for P to be a convex problem is that the objective function f and the constraint functions $g_i$, $i \in I$, are convex over X.

The following theorem is of obvious practical importance:

**Theorem 2.10**: Assume that P is a convex problem and that $x^*$ is a local minimum point of P. Then $x^*$ is also a global minimum point of P.

Most of the methods commonly used in structural optimization, see e.g. ref [3], do not explicitly use definitions 2.1 and 2.2 when searching for a minimum point. Instead they search for, and usually end up with, a point which satisfies the so called KT-conditions (= Kuhn-Tucker conditions).

**Definition 2.11**: Assume that f and $g_i$ are differentiable functions. A point $x^* \in \Omega$ is said to be satisfying the KT-conditions of problem P if there are non-negative real numbers $\lambda_i$, $i \in I$, such that the following two conditions are satisfied:

$$\text{i)} \quad \frac{\partial f}{\partial x_j} + \sum_{i \in I} \lambda_i \frac{\partial g_i}{\partial x_j} \begin{cases} \geq 0 \text{ if } x_j^* = x_j^{min} \\ = 0 \text{ if } x_j^{min} < x_j^* < x_j^{max} \\ \leq 0 \text{ if } x_j^* = x_j^{max} \end{cases}$$

for all $j \in \{1,..,n\}$, where the derivatives are evaluated at $x = x^*$.

ii) $\lambda_i(g_i(x^*)-\alpha_i) = 0$ for all $i \in I$.

In the general case, such a "KT-point" $x^*$ may fail to be even a local minimum point of P. In the convex case, however, the situation is completely satisfactory:

Theorem 2.12: Assume that P is a convex problem and that $x^*$ satisfies the KT-conditions of P. Then $x^*$ is a global minimum point of P.

Convex problems possess a variety of other interesting and useful properties, see e.g. ref [4], but we believe that the above theorems, 2.10 and 2.12, suffice to point out that convex problems are well-behaved compared to nonconvex ones.

## 3. Displacement- and stress constraints

Displacement constraints are usually of the form $q^T u \le \alpha$, where u is the nodal displacement vector, q is a given constant vector and $\alpha$ is a given real number. u is obtained from the system $Ku = p$, where p is the load vector and $K = K_0 + \sum x_j K_j$ is the structural stiffness matrix. Thus $u = K^{-1}p$, and a given displacement constraint may be written:

$$d(x) = q^T K^{-1} p \le \alpha$$

q is in the literature often considered to be a "virtual load" vector, a convention we will follow in this paper. Note that q and p are both assumed to be given constant vectors, while $K^{-1}$ depends on x. We will therefore occasionally (when we want to emphasize this dependence) write $q^T K^{-1}(x) p$ instead of the shorter $q^T K^{-1} p$. The meaning is however always the same.

Unfortunately, displacement constraints are not always convex:

Proposition 3.1: Displacement- and stress constraints may give rise to a nonconvex feasible set.

We prove this statement by presenting an example of a nonconvex problem, where the chosen displacement constraint is equivalent to a stress constraint. Consider the two-dimensional truss structure shown in fig 3.1.

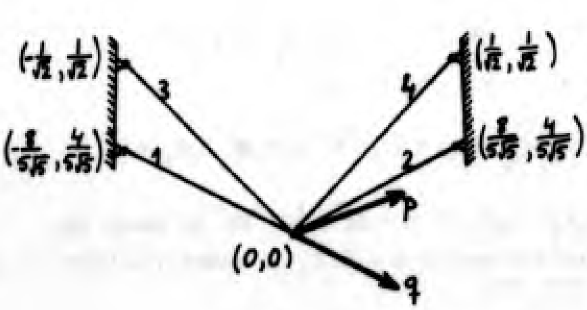


Fig 3.1 Two-dimensional 4-bar truss.

The structure has only got 2 degrees of freedom, namely the horizontal and vertical displacements of the single non-fixed node. The elements are linked together two by two, so that we have the following variables:

$x_1$ = cross section area of elements 1 and 2.

$x_2$ = cross section area of elements 3 and 4.

The lower and upper bounds on the variables are $x_1^{min} = x_2^{min} = 0.1$ and $x_1^{max} = x_2^{max} = 1.5$

There is only one displacement constraint. The corresponding load vector p and "virtual load" vector q, which are shown in fig 3.1, are:

$$p = \frac{1}{\sqrt{10}}\begin{pmatrix}3\\1\end{pmatrix} \quad \text{and} \quad q = \frac{1}{\sqrt{5}}\begin{pmatrix}2\\-1\end{pmatrix}$$

Note that, since q is parallel to element 1, the considered displacement constraint is equivalent to a constraint on the tensile stress in element 1.

After some calculations we obtain the following structural stiffness matrix:

$$K = E \cdot \begin{pmatrix} 2x_1 + x_2 & 0 \\ 0 & \frac{x_1}{2} + x_2 \end{pmatrix}$$

where E = Young's modulus of elasticity. We then get the following displacement constraint:

$$d(x) = q^T K^{-1} p = \frac{1}{\sqrt{50}\,E}\left(\frac{6}{2x_1+x_2} - \frac{2}{x_1+2x_2}\right) \le \alpha$$

Assume, for simplicity, that $\alpha = \frac{4}{3\sqrt{50}\,E}$. The constraint then becomes:

$$\frac{3}{2x_1+x_2} - \frac{1}{x_1+2x_2} \le \frac{2}{3}$$

The feasible set implied by this constraint, together with the lower and upper bounds on the variables, is shown in fig 3.2. It is clearly nonconvex.

To see what consequences this might lead to, assume that $x_1$ has been fixed to the value 1.0. The feasible set is then reduced to the two line segments:

$$0.1 \le x_2 \le 0.25 \text{ and } 1.0 \le x_2 \le 1.5$$

(see the dotted lines in fig 3.2). The global minimum weight solution of this, restricted, problem is obviously $x_2 = 0.1$ (and $x_1 = 1.0$), but it is also clear that $x_2 = 1.0$ is a local minimum solution. If a starting point with $x_2 \ge 1.0$ is chosen, then this latter solution is the one which most likely will be found by a standard method for solving the problem.

It should be noted that this undesirable existence of a non-global local minimum can not be avoided by any transformation of the design variables. If the problem is formulated in e.g. the reciprocal variables $\xi_1 = 1/x_1$ and $\xi_2 = 1/x_2$ we get the situation illustrated in fig 3.3.

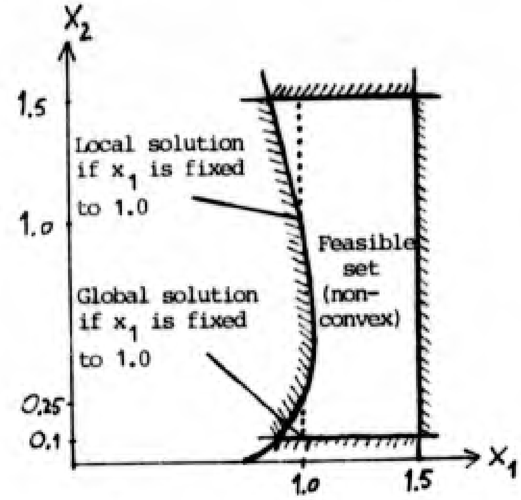


Fig 3.2 Feasible set of the 4-bar problem.

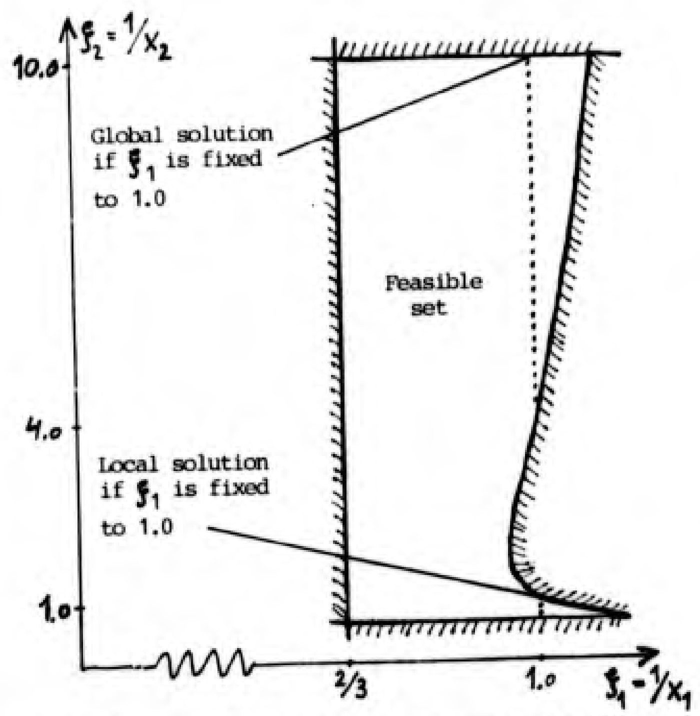


Fig 3.3 Feasible set in the reciprocal variables.

## 4. Symmetric displacement constraints

When defining a displacement constraint, it is sometimes natural to let the "virtual load" vector q be parallel to the load vector p. We will call such a displacement constraint "symmetric":

Definition 4.1: The constraint $d(x) = q^T K^{-1} p \leq \alpha$ is said to be a symmetric displacement constraint if $q = \gamma p$ for some positive real number $\gamma$.

As a typical example, consider the famous 10-bar cantilever truss in fig 4.1. Assume that one of the loadcases consists of a single vertical force in node 6. A reasonable constraint is then to place a limit on the vertical displacement of node 6, under this loadcase. This clearly becomes a symmetric displacement contraint.

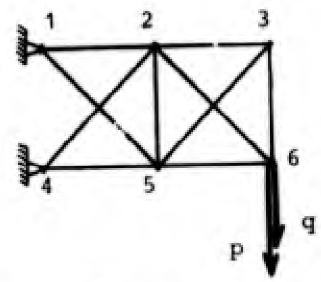


Fig 4.1 Example of a symmetric displacement constraint.

Symmetric displacement constraints can be given an alternative interpretation. Since $q = \gamma p$ and $p = Ku$, we have:

$d(x) = q^T K^{-1} p = \gamma p^T K^{-1} p = \gamma u^T K u = 2\gamma U$, where $U = \frac{1}{2} u^T K u$

is the strain energy in the structure when the given load p has been applied. A symmetric displacement constraint may thus be formulated: "For a given load p, the resulting strain energy must not exceed a given quantity".

We now prove that symmetric displacement constraints possess an attractive convexity property:

Theorem 4.2: Let $d(x) = q^T K^{-1} p$ and assume that $q = \gamma p$ for some real number $\gamma > 0$. Then d is a convex function over X.

Proof: We prove this theorem by showing that the Hessian matrix of d is positive semidefinite for all $x \in X$, cf. lemma 2.6. We also use the formula:

$\frac{\partial K^{-1}}{\partial x_j} = - K^{-1} \frac{\partial K}{\partial x_j} K^{-1}$, which is immediately obtained by differentiating the identity $K^{-1} K \equiv$ the unity matrix. If we differentiate $d = q^T K^{-1} p$ twice, we get:

$$\frac{\partial d}{\partial x_j} = - q^T K^{-1} \frac{\partial K}{\partial x_j} K^{-1} p = (\text{using } K = K_0 + \textstyle\sum x_j K_j) =$$

$$= - q^T K^{-1} K_j K^{-1} p$$

$$\frac{\partial^2 d}{\partial x_j \partial x_k} = q^T K^{-1} \frac{\partial K}{\partial x_k} K^{-1} K_j K^{-1} p + q^T K^{-1} K_j K^{-1} \frac{\partial K}{\partial x_k} K^{-1} p =$$

$$= q^T K^{-1} (K_k K^{-1} K_j + K_j K^{-1} K_k) K^{-1} p.$$

We must prove that $\sum \sum \frac{\partial^2 d}{\partial x_j \partial x_k} h_j h_k \geq 0$

for all $h = (h_1, \ldots h_n)^T$. But

$$\sum \sum \frac{\partial^2 d}{\partial x_j \partial x_k} h_j h_k = \sum \sum q^T K^{-1} (h_k K_k K^{-1} h_j K_j +$$

$+ h_j K_j K^{-1} h_k K_k) K^{-1} p = 2 q^T K^{-1} H K^{-1} H K^{-1} p$, where the introduced matrix $H = \sum h_j K_j$ is symmetric. With $q = \gamma p$ we then get:

$$\sum \sum \frac{\partial^2 d}{\partial x_j \partial x_k} h_j h_k = 2\gamma p^T K^{-1} H K^{-1} H K^{-1} p =$$

$= 2\gamma (HK^{-1} p)^T K^{-1} (HK^{-1} p) \geq 0$, since $K^{-1}$ is positive definite.

As a consequence of this theorem we get:

Corollary 4.3: The minimum weight problem subject to a collection of symmetric displacement constraints ( e.g. one for each loadcase ) is a convex problem.

## 5. Global displacement constraints

Consider a displacement constraint $q^T K^{-1} p \le \alpha$. We may "normalize" this constraint by normalizing the vectors p and q:

$$\left(\frac{q}{\|q\|}\right)^T K^{-1} \left(\frac{p}{\|p\|}\right) \le \frac{\alpha}{\|p\|\,\|q\|}$$

An arbitrary displacement constraint may thus, without any loss of generality, be written:

$q^T K^{-1} p \le \alpha$, where $\|p\| = \|q\| = 1$. Such a (normalized) displacement constraint is symmetric if and only if $q = p$.

Now, assume that, for a given value of $\alpha$, we require that $q^T K^{-1} p \le \alpha$ for all vectors p and q such that $\|p\| = \|q\| = 1$. This infinite set of constraints may equivalently be written:

$$\max_{p,q \in S} \{q^T K^{-1}(x) p\} \le \alpha$$

where S is the unit sphere, so that, for each $x \in X$, the maximum on the left hand side is taken over all vectors p and q such that $\|p\| = \|q\| = 1$. (This maximum clearly exists since, for each $x \in X$, $q^T K^{-1}(x) p$ is continuous in p and q, and the set over which the maximum is taken in compact.) We will call this constraint a "global displacement constraint":

Definition 5.1: The constraint $d_G(x) \le \alpha$, where $\alpha$ is a given real number and the function $d_G$ is defined by $d_G(x) = \max_{p,q \in S} \{q^T K^{-1}(x) p\}$, is called a global displacement constraint.

A global displacement constraint implies e.g. that: for any unit load applied on the structure, the displacement of any node in any direction does not exceed $\alpha$.

Using some well-known rules for vector and matrix calculus, see e.g. ref [5], we get:

$$d_G(x) = \max_{p \in S} \{\max_{q \in S} \{q^T K^{-1} p\}\} = \max_{p \in S} \left\{\frac{(K^{-1}p)^T K^{-1} p}{\|K^{-1} p\|}\right\} =$$

$$= \max_{p \in S} \{\|K^{-1} p\|\} = \|K^{-1}\| = 1/\lambda_K(x)$$

where $\lambda_K(x)$ is the smallest eigenvalue of the stiffness matrix K. It also follows that

$$d_G(x) = \max_{p \in S} \{p^T K^{-1} p\}.$$

A global displacement constraint $d_G(x) \le \alpha$ is thus equivalent to a lower limit on the smallest eigenvalue of K: $\lambda_K(x) \ge \lambda_K^{min}$ where $\lambda_K^{min} = 1/\alpha$, but also to the infinite set of symmetric displacement constraints: $p^T K^{-1} p \le \alpha$ for all p such that $\|p\| = 1$.

Theorem 5.2: The global displacement constraint function $d_G$ is convex over X.

Proof: From theorem 4.2 we know that $p^T K^{-1}(x) p$ is convex (in x) over X, for every vector p, and according to lemma 2.7 the pointwise maximum function of a collection of convex functions is convex. Since $d_G(x) = \max_{p \in S} \{p^T K^{-1}(x) p\}$, it thus follows that $d_G$ is convex over X.

As a consequence of this theorem we get:

Corollary 5.3: The set $\{x \in X | d_G(x) \le \alpha\}$, which alternatively can be expressed as $\{x \in X | \lambda_K(x) \ge \lambda_K^{min}\}$, is convex.

The convexity of the set $\{x \in X | \lambda_K(x) \ge \lambda_K^{min}\}$ alternatively follows from the following theorem:

Theorem 5.4: Let $\lambda_K(x)$ be the smallest eigenvalue of the stiffness matrix K. Then $\lambda_K$ is a concave function over X.
(Note: a function $\varphi$ is concave if and only if $-\varphi$ is convex.)

Proof: $\lambda_K(x) = \min_{s \in S} \{s^T K(x) s\} = \min_{s \in S} \{s^T K_o s + \sum x_j s^T K_j s\}$

$$\Rightarrow -\lambda_K(x) = \max_{s \in S}\{-s^T K_o s - \sum x_j s^T K_j s\}$$

For a fixed vector s, the function within the brackets is linear, and thus convex, in x. From lemma 2.7 it then follows that $-\lambda_K$ is convex, i.e. that $\lambda_K$ is concave, over X.

## 6. Semiglobal displacement constraints

Assume that, for a given value of $\alpha$ and a given load vector p, we require that $q^T K^{-1} p \le \alpha$ for all virtual load vectors q such that $\|q\| = 1$. This is a sort of "compromise" between an ordinary and a global displacement constraint, and we therefore use the name "semiglobal":

Definition 6.1: The constraint $\max_{q \in S} \{q^T K^{-1} p\} \le \alpha$, where $\alpha$ is a given real number and p is a given vector, is called a semiglobal displacement constraint.

The maximizing $q \in S$ in definition 6.1 is easily seen to be $q = \frac{K^{-1}p}{\|K^{-1}p\|}$. The semiglobal displacement constraint may thus be written: $\|K^{-1}p\| \le \alpha$ or, equivalently, $\|u\| \le \alpha$. (As before, u is the nodal displacement vector.)

Since $\|u\|^2 = u^T u = p^T K^{-2} p$, and since $p^T K^{-1} p$ is convex over X (as was shown in section 4), one might guess that also $\|u\|^2$ is convex over X. Then the set $\{x \in X \mid \|u\|^2 \le \alpha^2\} = \{x \in X \mid \|u\| \le \alpha\}$ would be convex. However, as will be shown by the example to follow, this is not always true. Instead, we can state the following:

Proposition 6.2: Semiglobal displacement constraints may give rise to a nonconvex feasible set. Also, $\|u\|$ and $\|u\|^2$, i.e. $\|K^{-1}(x)p\|$ and $p^T K^{-2}(x) p$, may be nonconvex functions in x.

Consider the two-dimensional 3-bar truss shown in fig 6.1.
There are two variables:

$x_1$ = cross section area of elements 1 and 2 (not elements 1 and 3!).

$x_2$ = cross section area of element 3.

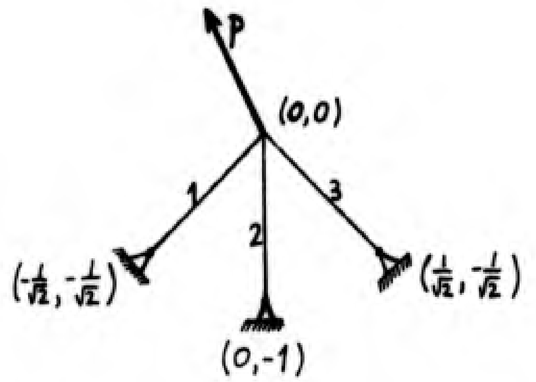


Fig 6.1 Two-dimensional 3-bar truss.

The load vector, applied in the non-fixed node, is $p = (-1\ ,\ 2)^T$. After some calculations we obtain the following displacement vector:

$$u = \frac{1}{E(x_1^2+3x_1x_2)} \cdot \begin{pmatrix} x_2-5x_1 \\ x_2+3x_1 \end{pmatrix}$$

(The two components of u are respectively the horizontal and vertical displacements of the nonfixed node.)

Assuming (for simplicity) that $\alpha = \frac{1}{E}$ , the semiglobal displacement constraint $\|u\|^2 \leq \alpha^2$ becomes:

$$(x_2-5x_1)^2 + (x_2+3x_1)^2 \leq (x_1^2+3x_1x_2)^2$$

The feasible set implied by this constraint is shown in fig 6.2. By direct calculations it is seen that the two points:

$$\left(\frac{\sqrt{74}}{3}\ ,\ \frac{\sqrt{74}}{9}\right) \text{ and } \left(\frac{\sqrt{410}}{5}\ ,\ \frac{\sqrt{410}}{35}\right)$$

are feasible, i.e. satisfy the above constraint. However, their midpoint is, also by direct calculations, seen to be unfeasible. The feasible set is thus nonconvex.

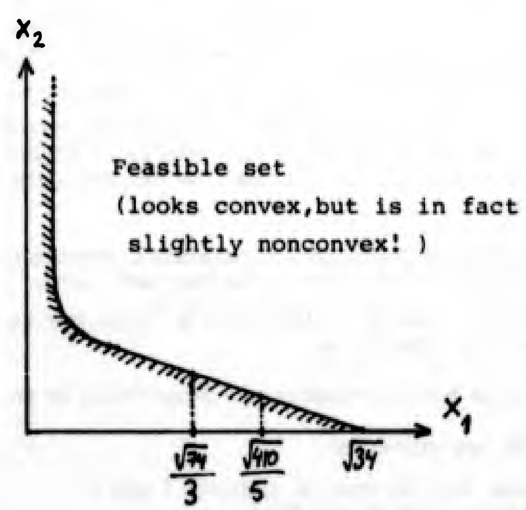


Fig 6.2 Feasible set of the 3-bar problem.

## 7. Natural frequency constraints

The natural frequencies $\omega_i$ of a structure are obtained from the generalized eigenvalue problem:

$$Ky = \omega^2 My$$

, where $K = K_o + \sum x_j K_j$ is the structural stiffness matrix and $M = M_o + \sum x_j M_j$ is the structural mass matrix.

An important example of a frequency constraint, the only one that will be considered here, is that no natural frequency should be less than a given number $\omega^{min}$. This may be written $\omega(x) \geq \omega^{min}$, where $\omega(x)$ is the lowest natural frequency of the structure (i.e. the smallest number $\omega$ for which there exists a vector $y \neq 0$ such that $Ky = \omega^2 My$).

In general, constraints on natural frequencies may give rise to a nonconvex feasible set. The constraint mentioned above, however, possesses the following attractive property:

Theorem 7.1: Let $\omega(x)$ be the lowest natural frequency and let $\omega^{min}$ be a given positive real number. Then the set $\{x \in X | \omega(x) \geq \omega^{min}\}$ is convex.

To prove this theorem we need the following well-known result, which may be found in e.g. chapter 3 of ref [5].

Lemma 7.2: If $\lambda$ is the smallest solution of the eigenvalue problem $Ky = \lambda My$, where K and M are symmetric and positive definite, then:

$$\lambda = \min_{y \neq 0} \left\{ \frac{y^T K y}{y^T M y} \right\}$$

Proof of theorem 7.1: According to the above lemma we have:

$$\omega^2(x) = \min_{y \neq 0} \left\{ \frac{y^T K y}{y^T M y} \right\} = \min_{y \neq 0} \left\{ \frac{k(x,y)}{m(x,y)} \right\}$$

, where the introduced functions $k(x,y) = y^T K_o y + \sum_j x_j y^T K_j y$ and $m(x,y) =$

$= y^T M_o y + \sum x_j y^T M_j y$ are linear in x and quadratic in y. We also have $k(x,y) > 0$ and $m(x,y) > 0$ for all $x \in X$ and $y \neq 0$.

Now, consider $\Omega = \{x \in X | \omega(x) \geq \omega^{min}\} =$

$= \{x \in X | \omega^2(x) \geq \lambda^{min}\}$, where $\lambda^{min} = (\omega^{min})^2$, $=$

$= \{x \in X | \frac{k(x,y)}{m(x,y)} \geq \lambda^{min}$ for all $y \neq 0\} =$

$= \{x \in X | k(x,y) \geq \lambda^{min} m(x,y)$ for all $y \neq 0\} =$

$$= \bigcap_{y \neq 0} \{x \in X | k(x,y) \geq \lambda^{min} m(x,y)\}.$$

For any fixed $y \neq 0$, the set $\{x \in X | k(x,y) \geq \lambda^{min} m(x,y)\}$ is convex, since k and m are linear functions in x. But according to lemma 2.4 the intersection of any collection of convex sets is convex, and it thus follows that $\Omega$ is a convex set.

## 8. Conclusions

In this paper an investigation concerning convexity properties in structural optimization has been accomplished. The investigation is not claimed to be exhaustive. It has been shown that, although the problems in general may be nonconvex, there are some nontrivial special cases where the convexity of the problem can be proved, e.g. when the structural weight should be minimized subject to a collection of symmetric displacement constraints and/or a global displacement constraint and/or a lower bound on the lowest natural frequency.

It is reasonable to believe that there are other special cases, than the ones discussed in this paper, which also possess important convexity properties. Further research concerning these questions is thus recommended.

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