THE VORTEX SKELETON MODEL FOR THREE-DIMENSIONAL STEADY FLOWS

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SUMMARY

The work of Hornung and Perry (1982) is reviewed in which the essential concepts of two-dimensional separation were extended to three-dimensional steady flow, and the vortex skeleton model and electromagnetic analogy were introduced. The model is extended to give a simple topological rule by which the vortex skeleton of a flow can be established from the structure of the wall streamline pattern. The important question of the occurrence of smoothly starting separation without zeros in the wall shear stress, is examined in the light of a new local solution of the Navier-Stokes and continuity equations.

1. INTRODUCTION

Consider steady, incompressible, viscous flow over a simply connected body. Let the flow be uniform at large distances from the body, and let the Reynolds number be large. Vorticity is generated in such a flow by viscous friction mainly in the thin boundary layer at the body surface (wall). This vorticity can be carried away from the wall by viscous diffusion or by convection. At large Reynolds number the latter mechanism is the one that is responsible for carrying vorticity to distances of one ore more characteristic body lengths from the wall.

The distribution of vorticity in the flow field around a body is very important to the aerodynamicist. If it is known, the forces on the body may be determined fairly accurately by a relatively crude inviscid model of the flow. This model can be simplified further if the vorticity concentrates into "ropes" of high vorticity within a relatively small distance from the body. A very descriptive illustration of how this may occur behind a wing was given by Lanchester (1907), see Fig.1. If this occurs, the distribution of vorticity may be represented approximately by a relatively small number of discrete vortices, whose locations and strengths must, of course, be known.

It is not surprising therefore, that a great deal of effort has been invested in the study of the process by which vorticity is transported into the flow field ("separation") and how the vorticity behaves after leaving the surface. An excellent compendium of the work on this subject has been given by Peake and Tobak (1980) (see also Tobak and Peake, 1982). An example of sophisticated numerical work in the latter field is to be found in Saffman (1982) and a description of the field is given by Lugt (1979).

For a calculation of the forces on a body from the disposition of discrete vortices, it is necessary to know their strengths and shapes. An even more basic requirement is the knowledge of the number of vortices which occur and their topological structure. In particular, changes from one to another topological structure of the flow field around the same body are of considerable importance, for example, such as arise in the problems of missile aerodynamics. Such transitions cannot at present be accurately predicted theoretically, and only in relatively simple cases have they been mapped out experimentally. (An example for the case of thick delta wings is given by Szodruch, 1977, see also Szodruch 1980).

The aim of the present paper is not to study the exact geometric modelling of flow fields, but to find a model which will reproduce the topological structure of three-dimensional, steady flows. It should be emphasized that our concern is not merely with the topological structure of the *wall* streamlines (integral curves of the wall shear stress) but with that of the complete three-dimensional flow field. A terminology has been given by Hornung and Perry (1982) which allows the qualitative features of such flow fields to be described unambiguously. This work is briefly reviewed and extended to allow some of the features of the spatial topology of a flow field to be determined systematically from a knowledge of the qualitative features of the wall streamline field. In addition, the question of the occurrence of a gradually beginning three-dimensional separation (i.e. without points at which the wall shear stress is zero) is examined in some detail.

2. TWO- AND THREE-DIMENSIONAL SEPARATION

The essential feature of steady, two-dimensional separated flow is that there exist two half-saddle points in the streamline pattern. They occur on the wall and are joined by one and the same streamline, see Fig.2. The half-saddle points are called separation and reattachment point respectively and the special streamline joining them is sometimes referred to as a separatrix. It separates a region of closed streamlines from the remainder of the flow field. A few accompanying features are that the wall shear stress is zero at the two half-saddle points, and that backflow occurs at the wall between them.

Separation is by no means so easy to describe in three-dimensional flow. The questions that arise are: What is the equivalent of a saddle point in three dimensions? What is the meaning of backflow in three dimensions? The wall shear stress is now a vector. Which component needs to be zero at separation, if any? Finally, what is the equivalent of a separatrix?

The last of these questions gives a clue as to how one might start in extending the concepts of two-dimensional separation to the spatial situation. In two-dimensional flow a stream*line* is able to separate two regions of the space. In three dimensions a *surface* is needed for this purpose. To find the logical extension of the two-dimensional concept of a separatrix to three-dimensional flow, it is therefore necessary to look for a particular stream*surface*. A streamsurface in a given flow field is defined by those streamlines which pass through a given curve in the three-dimensional space. (The concept "streamsurface" is probably more familiar in the form "streamtube", where the defining curve is closed.)

A stream*surface* in a given three-dimensional flow field is only uniquely determined by its defining *curve* if it does not bifurcate anywhere, just as, analogously, a stream*line* in a given two-dimensional flow field may only be uniquely defined by a *point* if it does not bifurcate anywhere. Stream*lines* in two-dimensional flow bifurcate at saddle or half-saddle *points*, i.e. for example at the separation or reattachment point. The logical extension to three dimensions of the separation and reattachment points of two-dimensional flow is therefore to be sought in *streamsurface bifurcation lines* on the wall.

An example of how streamsurface bifurcations may occur in three-dimensional flow is shown qualitatively in Fig.3. From the bifurcation line PQ on the wall, the streamsurface S_1 emerges and rolls up into a vortex. At the same time the streamsurface S_2 divides at the streamsurface bifurcation line P'Q' on the wall. The streamsurfaces S_1 and S_2 respectively emerge from and flow into the three-dimensional analogues of half-saddle points (i.e. streamsurface bifurcation lines) and are therefore the logical extensions of the separatrices of two-dimensional flow. They are the free sheets of the streamsurface bifurcations. The significant difference from the two-dimensional case is that these special streamsurfaces do not in general, form closed regions of the space, i.e. S_1 and S_2 are not the same streamsurface.

These concepts and their implications are discussed at greater length and with mathematical support by Hornung and Perry (1982). Among other results, their work shows that the concept "strength of separation" becomes meaningful in three-dimensional flow, and that complicated topological structures of steady, three-dimensional flow fields may be described unambiguously by the types and disposition of their bifurcation lines and free sheets. They proceed to classify a number of frequently occurring flow configurations.

One of the questions that arose is whether it is necessary for a streamsurface bifurcation line to start at a point where the wall shear stress is zero, or whether it can start gradually, without a zero-shear point. An often-observed example of the former case is shown in the sketch of Fig.4 in which a negative streamsurface bifurcation starts from a saddle in the wall streamline pattern. This pattern also shows a positive bifurcation line starting smoothly, without a zero in the wall shear stress. (The notation "positive" means that the streamsurface divides into two in the direction of flow and vice versa.) Experiments certainly suggest that both smooth and zero-shear beginnings of bifurcation lines are possible. This question is examined in more detail in section 4.

3. THE VORTEX SKELETON MODEL

While the qualitative features of the structure of a steady three-dimensional flow may be described unambiguously by the bifurcating streamsurfaces of the flow, such descriptions become extremely cumbersome except for the simplest configurations. If only the qualitative features are to be described, an equally effective and much simpler method consists of representing the flow field by a finite number of discrete vortices in a uniform flow. The validity of this model depends on the assumption that any distributed vorticity that may be present does not alter the spatial topology of the flow.

In order to illustrate the two methods, a number of frequently observed flow situations are presented in Fig.'s 5,6,7 and 8 taken from Hornung and Perry (1982). Fig.5 shows simple U-shaped separation by means of the bifurcating streamsurfaces that occur. Figures 6,7 and 8 show the flow structure that sometimes occurs at sting-body junctions using both methods. This last example makes the superiority of the vortex skeleton method particularly clear. Only six discrete vortices are needed (see Fig.8) to present the topology of the flow field, in which a total of 18 free sheets of 12 streamsurface bifurcations occur. (Bifurcations that occur in the flow field, i.e. not at the wall, generally have four free sheets.) Clearly, it becomes extremely difficult to present the latter in a sketch, see Fig.7, while the presentation of the vortex skeleton of Fig.8 is relatively simple and provides corresponding information.

The assumption that all the vorticity of the flow is concentrated into discrete vortices is, of course, incorrect. However, the aim here is to represent only the topology of the flow correctly. This means, for example, that the number and connectivity of the nodes, saddles and bifurcation lines of the wall streamline pattern are to be correctly reproduced. It turns out that, whenever a pair of such critical points (node and saddle) occurs on the wall, it is associated with a concentration of vorticity in the vicinity. For example, in the U-shaped separation of Fig.5, one such node-saddle pair occurs in the wall streamline pattern (A, F). This is associated with the vortex formed by the rolling up of the streamsurface S_1. The converse is not necessarily the case, i.e. a vortex may be so weak and distant from the wall that it does not cause a saddle-node pair to occur on the wall. However, since our interest is in the vorticity that is spilt into the flow from the boundary layer, such vortices need not concern us. A more important way in which a vortex might occur without a saddle-node pair on the wall is through a gradually starting negative streamsurface bifurcation, similar to that proposed schematically in Fig.3. Supposing for the present (see also section 4) that such smoothly starting bifurcations are possible, it is a relatively simple matter to establish the following topological rule:

The minimum number of discrete vortices required to represent the spatial topology of uniform flow over a simply connected body is given by the equality limit of the inequality

$$V \geqslant P + S, \qquad (1)$$

where V is the number of vortices, P is the number of saddle-node pairs in the wall streamline pattern, and S is the number of those negative streamsurface bifurcations in the wall streamline pattern which do not start at a saddle point.

Consider, for example, the flow of Fig.6. The wall streamline pattern contains six saddle-node pairs and all negative streamsurface bifurcations on the wall start from saddle points. Thus, P = 6, S = 0, so that six vortices are needed to present the topological structure of the flow, see Fig.8. The fact that the body is chopped off at front and back means only that, were it closed, two additional nodes (e.g. front and rear stagnation point) would occur in the wall streamline pattern. The vortex skeleton model of a particular flow, for which the wall streamline pattern is known, can thus be established systematically by arranging the number of vortices prescribed by the rule (1) appropriately.

A large amount of distributed vorticity always resides in the boundary layer near the wall. However, the presence of this distributed vorticity does not introduce topological features such as nodes, saddles or bifurcation lines in the wall streamline pattern except through the formation of a vortex, i.e. by distributed vorticity concentrating into discrete bundles. Thus, the topological structure of the wall streamlines is not affected by the distributed vorticity in the boundary layer, but is correctly reproduced by the vortex skeleton model with the assumption that the flow outside the vortex cores is irrotational. The exact shape of the wall streamlines is, of course, influenced by the distributed vorticity, but their topological structure is not.

Such a model can be simulated directly by the electromagnetic analogy, in which the vortices are replaced by wires carrying electric current, so that the induced magnetic field corresponds to the velocity. The uniform flow field can be simulated by a current-carrying solenoid coil generating a uniform magnetic field within itself. Solid surfaces, at which the velocity must be tangential in the fluid mechanical analogue, can be simulated for a plane wall by the method of images, i.e. by introducing a mirror image of the vortex system on the opposite side of the wall". In order to show the wall streamlines, for example, iron filings may be scattered on it and exposed to the magnetic field. The electric current carried by a wire corresponds to the strength of the simulated vortex.

In order to illustrate the use of the method, Fig.9 shows sketches of six types of symmetrical wall streamline patterns occurring in separated flows, together with their simulation by the electromagnetic analogy. In each case the topological rule, equation (1), is satisfied. In Fig.9f the two vortices have to disperse into "boundary-layer vorticity" at their leading end. In the electromagnetic analogy this has been achieved by joining the two wires with a copper plate parallel to and near the surface. It could also be achieved by joining the two wires at their upstream end with a wire which is far from the wall. Such connections (far from the surface or with distributed currents) are disregarded in counting the number of vortices for the purpose of equation (1). I.e. V = 2 for Fig.9f.

In some of these simulations the experiment was embedded in an overall field which converges near the wall in order to simulate the situation on the lee side of a slender body. This was achieved by two additional vortices flanking the region of interest. Equation (1) is not violated by this step because of the inequality sign which allows for such weaker or more distant vortices.

4. GRADUALLY STARTING SEPARATION

In this section the question is examined as to whether it is possible for a negative streamsurface bifurcation line to start forming gradually at a wall, without the wall shear stress becoming zero anywhere. It is necessary to discuss this question because this kind of separation is certainly observed experimentally (see e.g. Kreplin, Vollmers, and Meier, 1982) as well as numerically (see e.g. Wang, 1982) while at the same time it has been the cause for some dispute (see e.g. Wang, 1981).

In order to illustrate the fact that experiment certainly suggests that this is so, consider the two surface oil flow pictures of Fig.10. Fig.10a shows two negative bifurcation lines issuing from two saddle points of the wall streamline pattern, i.e. from points where the wall shear stress is zero. On the other hand, Fig.10b shows two negative bifurcation lines which start smoothly, without a saddle. Similar effects may, of course, also be produced by the electromagnetic analogy (see Fig.9).

The transition from the flow type of Fig.10a to that of Fig.10b may be imagined as follows: Consider a vortical node and a saddle in the wall streamline pattern to be embedded in a region, in which the wall shear stress falls smoothly in the flow direction. Let them be separated by a distance d. Consider the magnitude T of the wall shear stress. Fig.11a gives a sketch of the wall streamlines in such a situation. Fig.'s 11b and c show how this pattern is expected to change as the distance d is reduced to zero. At the same time these figures show profiles of T along the lines joining the critical points. Fig.11d shows how this process may be extrapolated to the case where the shear stress no longer quite reaches the value zero and the bifurcation line starts smoothly.

A method by which the flow in the vicinity of critical points or bifurcation lines may be studied theoretically, is to seek local solutions of the Navier-Stokes and continuity equations by expanding the velocity vector in a Taylor series in terms of the position vector, see Perry and Fairlie (1974), Hornung and Perry (1982). Usually, only the first term in this series is retained, as the number of terms in each component increases rapidly with each new vector term. Such solutions give asymptotic approximations which converge within a region of validity $r < R$ around the point $r = 0$, see Fig.12, the error being of the next higher order in $r = \sqrt{x^2 + y^2 + z^2}$ than the last retained term.

Recently Perry (1983) obtained such a solution to a particular degenerate case which has the following form:

$$\begin{aligned} u &= - exy^2z + 0(r^5) \\ v &= fyz + gz^2 + ex^2yz + 0(r^5) \\ w &= - fz^2/2 - ex^2z^2/2 + ey^2z^2/2 + 0(r^5) \\ p/\mu &= - ex^2z + ey^2z + 2gy - fz + p_o/\mu + 0(r^4), \end{aligned} \qquad (2)$$

where u, v, w are the x, y, z - components of velocity p and μ are pressure and viscosity and e, f, g and p_o are constants. For a particular choice of the constants e, f and g, namely $e > 0$, $f < 0$, $g = 0$, this solution is sketched in Fig.13. As can be seen this has the character of almost two-dimensional separation. A feature of equation (2) is that the inertia terms in the Navier-Stokes equation are unimportant to this approximation, the pressure being determined entirely by the viscous terms.

At first glance, the flow of Fig.13 does not seem to have any relevance to our problem. However, it may be seen with a little imagination, that superposition of a plane shear flow of the form

$$u = kz \qquad (3)$$

will produce a wall streamline pattern similar in character to that to be expected near the beginning of a negative bifurcation line. Naturally, one cannot just superpose solutions in this nonlinear system of differential equations. Indeed, it turns out that the term in (3) causes an additional term in u to become necessary, which, this time, arises from the inertia terms. The new form of u is thus:

$$u = - exy^2z + kz - kfz^4/(24\nu) + 0(r^5), \qquad (4)$$

where ν is the kinematic viscosity. The other components of velocity and the pressure are not affected by these additional terms. The solution (2) with u replaced by (4) represents an asymptotic approximation satisfying the Navier-Stokes and continuity equation and the no-slip condition at $z = 0$, to the stated accuracy.

Choosing $f < 0$, $e > 0$, $k > 0$ and setting $g = 0$, $a^2 = - f/e$, $b^3 = k/e$, the equation for the wall streamlines becomes ($z = 0$)

$$\frac{dy}{dx} = \frac{v}{u} = \frac{y(x^2 - a^2)}{b^3 - xy^2}. \qquad (5)$$

This pattern ... two critical points (which are vortical nodes in the parameter range of interest here, namely if $b^3/a^3 < 16$) at

$$(x, y) = (a, \pm\sqrt{b^3/a}). \qquad (6)$$

These lie outside the range of validity of the solution. The solution of equation (5) with $a = 0.5$, $b = 0.5$, is presented in Fig.14. As can be seen, the nature of this solution is like that expected for a gradually beginning bifurcation. Of course, this local solution does not only give the wall streamlines, but the whole three-dimensional flow field near the beginning bifurcation. It gives mathematical support to the supposition made in section 3, that negative streamsurface bifurcations may begin smoothly. The wall shear stress is not zero anywhere in this solution.

5. CONCLUSIONS

The work of Hornung and Perry (1982) has been reviewed, in which the essential concepts of two-dimensional separation had been extended logically to three-dimensional steady flow, and the vortex skeleton model and electromagnetic analogy had been introduced A simple topological rule has been given, by which the minimum number of vortices necessary to represent a flow over a simply connected body with a given wall streamline pattern may be determined. It has been shown that the electromagnetic analogy is usually sufficiently accurate to represent the topological structure of a flow pattern, distributed vorticity being unable to introduce topological features such as nodes, saddles or bifurcations. Finally a local solution of the Navier-Stokes and continuity equations has been obtained, by extending a solution given by Perry (1983), which has all the features of a gradually beginning streamsurface bifurcation, with everywhere finite wall shear stress.

6. ACKNOWLEDGEMENTS

Most of this work has resulted from joint work and discussions with Dr.A.E. Perry of the University of Melbourne. The only reason why he is not a co-author is that my production of the manuscript has been too slow for us to communicate about it between Germany and Australia. I would also like to thank Dr. Bippes, Dr. Dallmann and Professor E. Becker for many fruitful discussions.

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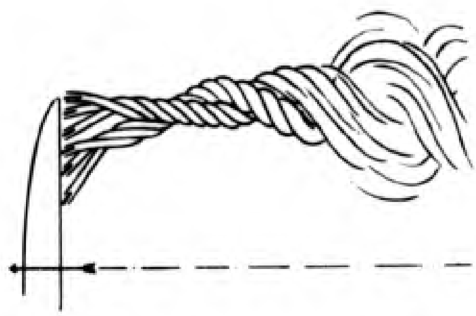

Fig.1: Lanchester's (1907) sketch of vorticity being concentrated into a vortex rope behind a wing.

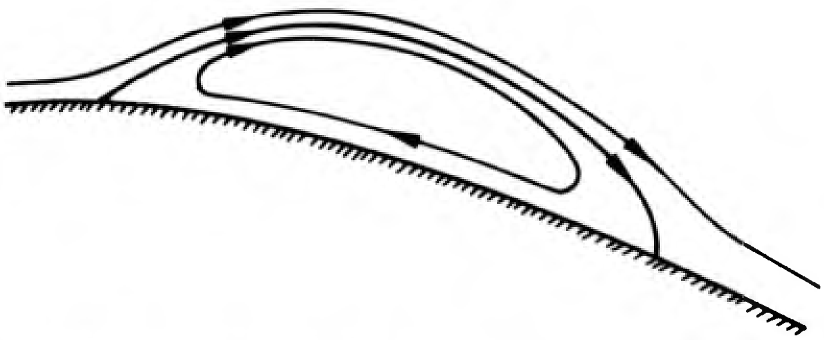

Fig.2: In two-dimensional, steady-flow separation, two half-saddle points occur at the wall. They are joined by a streamline which separates a region of closed streamlines from the rest of the flow field.

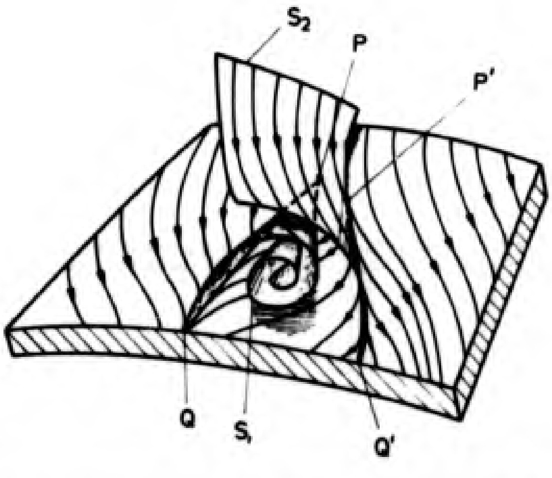

Fig.3: Example of three-dimensional separation. The free sheet S_1 of the negative streamsurface bifurcation line PQ is not the same as the free sheet S_2 of the accompanying positive bifurcation line P'Q'.

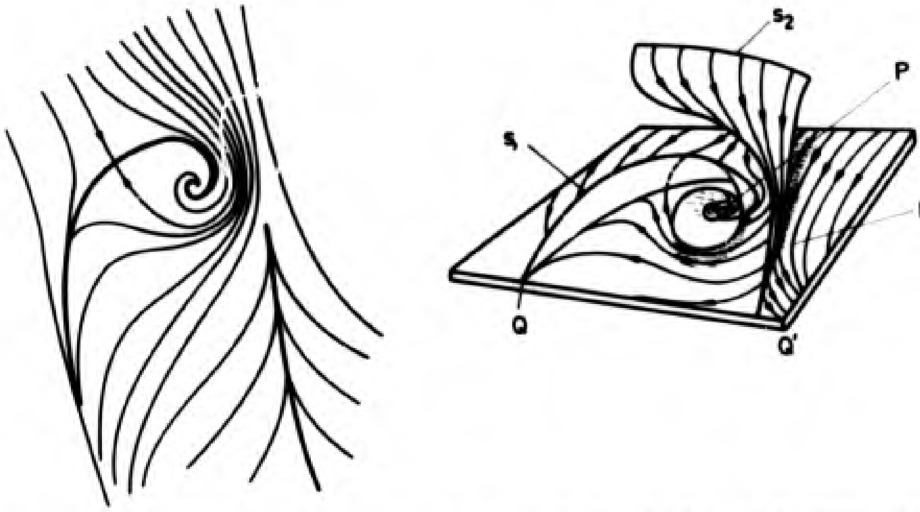

Fig.4: Example of three-dimensional separation in which the negative streamsurface bifurcation starts from a saddle point in the wall streamline pattern. At the same time the accompanying positive bifurcation line P'Q' starts without a critical point a) wall streamline pattern, b) perspective sketch.

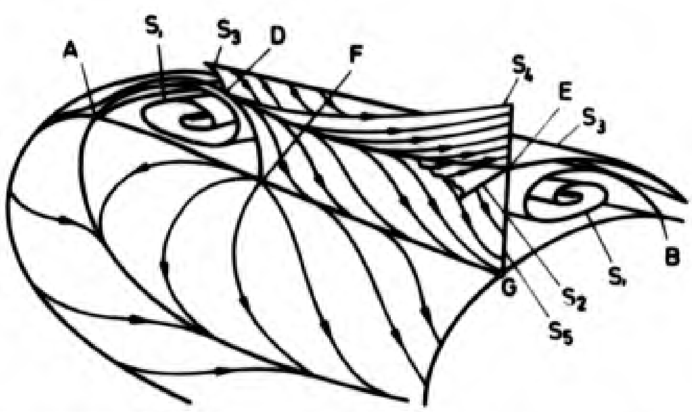

Fig.5: Simple U-shaped separation, port side showing wall streamline pattern, starboard side showing perspective sketch. Apart from a negative bifurcation AB, a double free bifurcation DE and a positive bifurcation FG occur. Of the five free sheets S_1 is the only one that rolls up into a vortex.

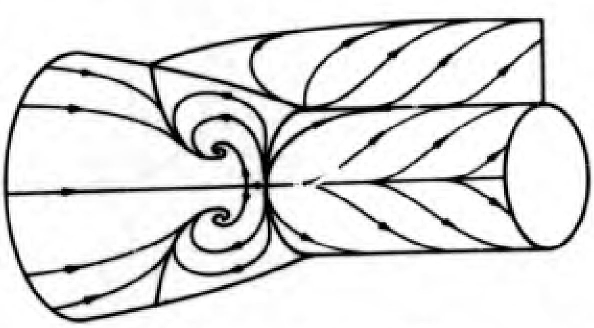

Fig.6: Wall streamline pattern and one of the free sheets of the separation pattern sometimes observed at sting-body junctions of axisymmetric bodies at zero incidence.

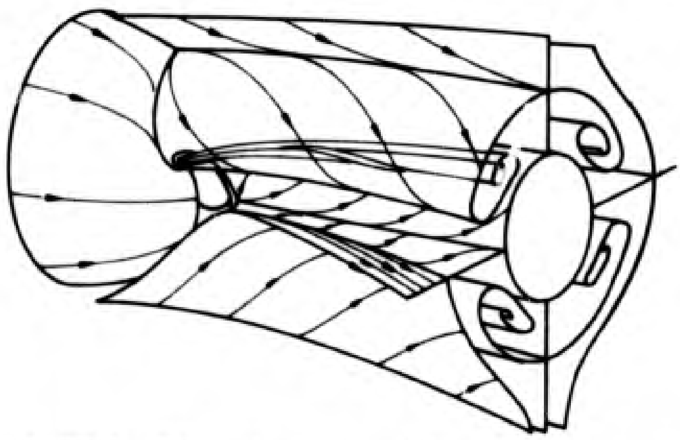

Fig.7: It becomes extremely difficult to sketch the 12 bifurcation lines and their 18 free sheets in the case of "sting-body" separation.

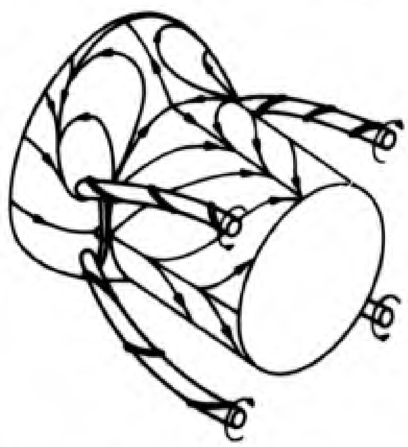

Fig.8: Equivalent information as that of Fig.7 can be conveyed more easily and more effectively with a presentation of the vortex skeleton of the flow, shown here together with the wall streamline pattern.

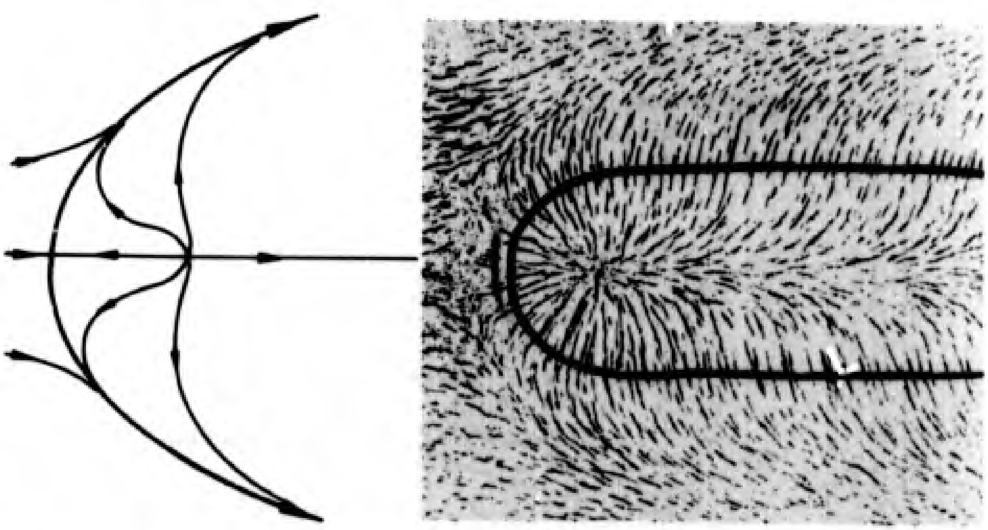

Fig.9a: Wall streamline pattern for simple U-shaped separation, sketch and electromagnetic simulation. One saddle-node pair occurs, all negative bifurcations start at saddles, i.e. P=1, S=0. Hence $V \geqslant 1$. One vortex only is needed in the skeleton.

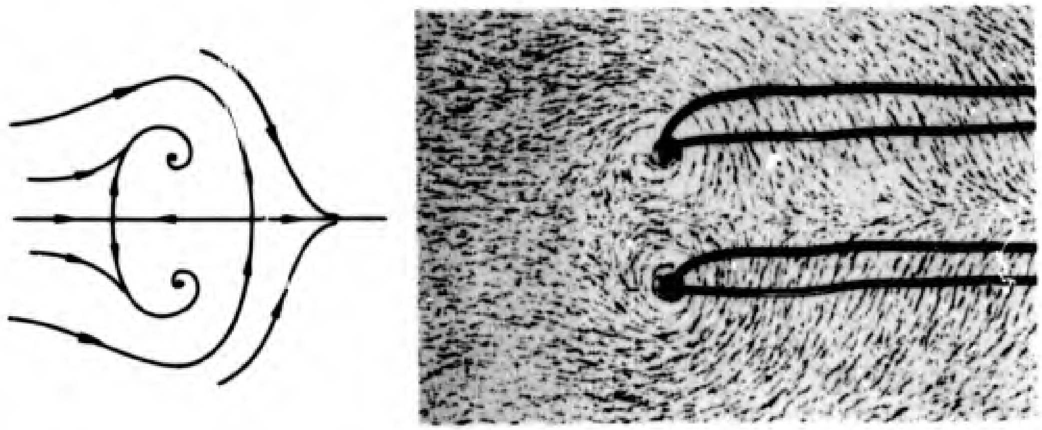

Fig.9b: Owl-face pattern of the first kind. P=2, S=0 so that V> 2. The two vortex-wires cast shadows onto the wall. Only two wires were used.

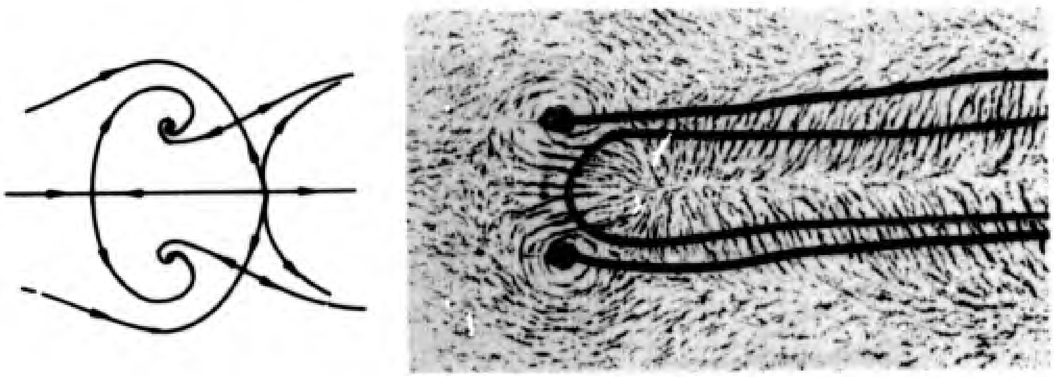

Fig.9c: Owl-face pattern of the second kind. P=3, S=0. so that V> 3.

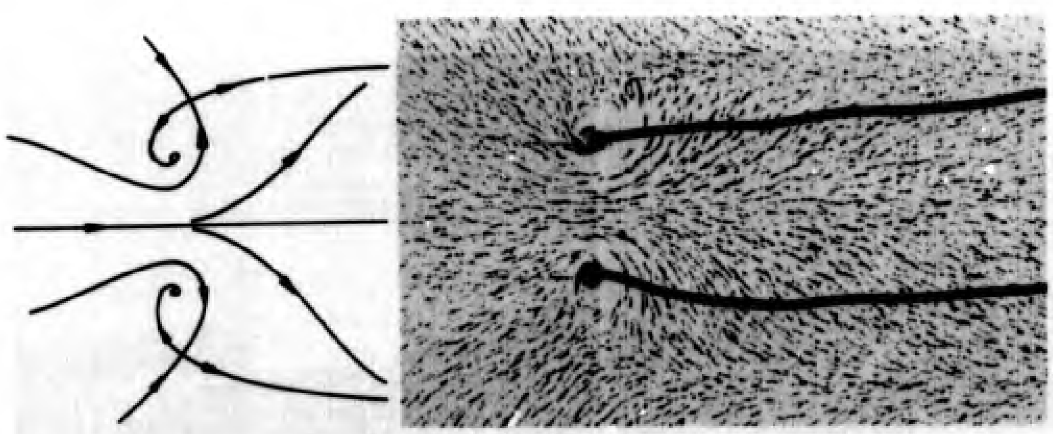

Fig.9d: Owl-face pattern of the third kind (Werlé-Legendre separation). P=2, S=0, so that V>2.

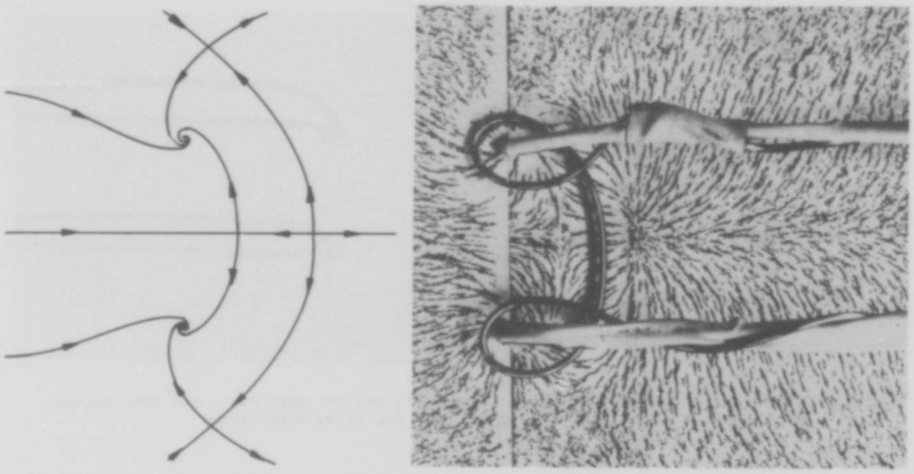

Fig.9e: Owl-face pattern of the fourth kind. P=3, S=0, so that V ⩾ 3.

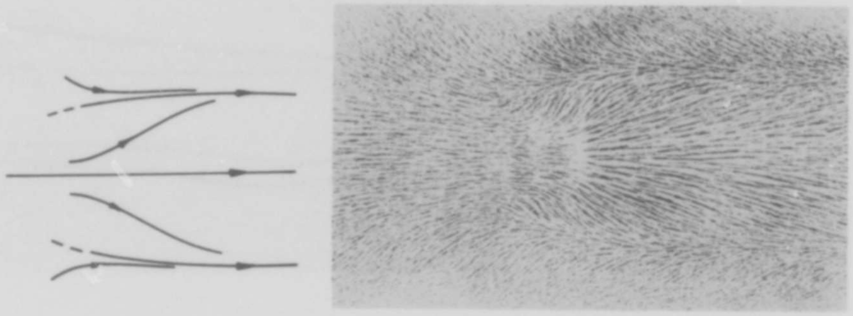

Fig.9f: Separation with two smoothly-beginning negative bifurcation lines. No critical points occur on the bifurcation lines. P=0, S=2, i.e. V ⩾ 2.

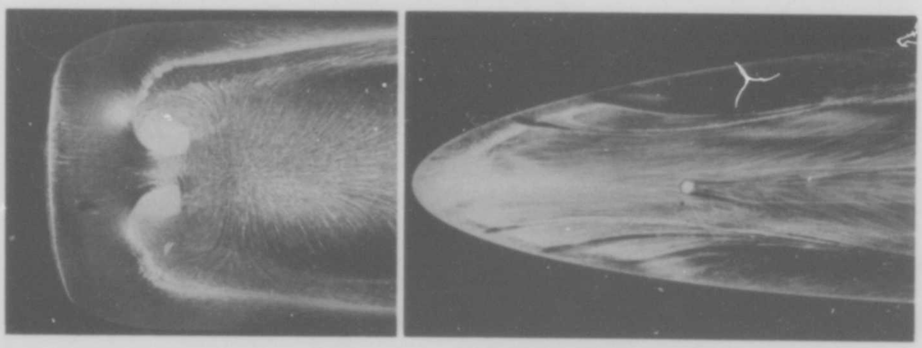

Fig.10: Examples of surface oil flow visualization photographs showing bifurcation lines starting with and without a point of zero wall shear stress. From Bippes (1983), and Kreplin, Vollmers and Meier (1982) respectively.

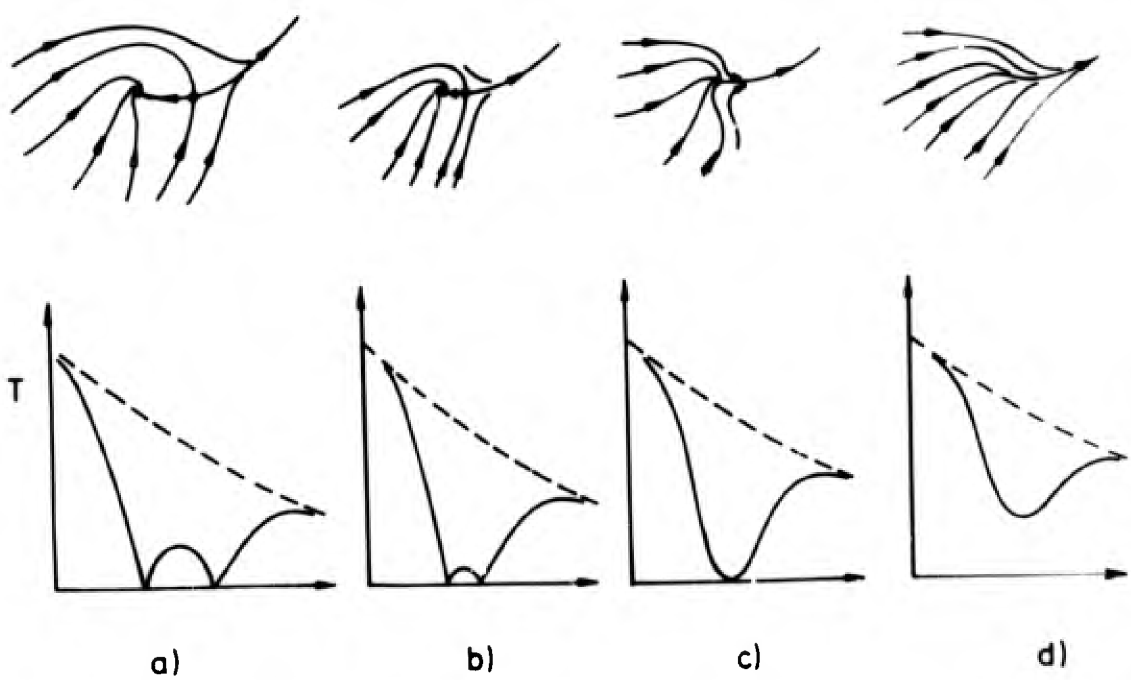

Fig.11: Bifurcation line starting from a saddle-node pair: As the distance between the saddle and node is reduced to zero (a-c) the degenerate case c) results, where the magnitude of the wall shear stress, T, just reaches zero. Fig.11d) shows that, when this process is extrapolated, the smoothly starting bifurcation line results, in which T does not reach zero any more.

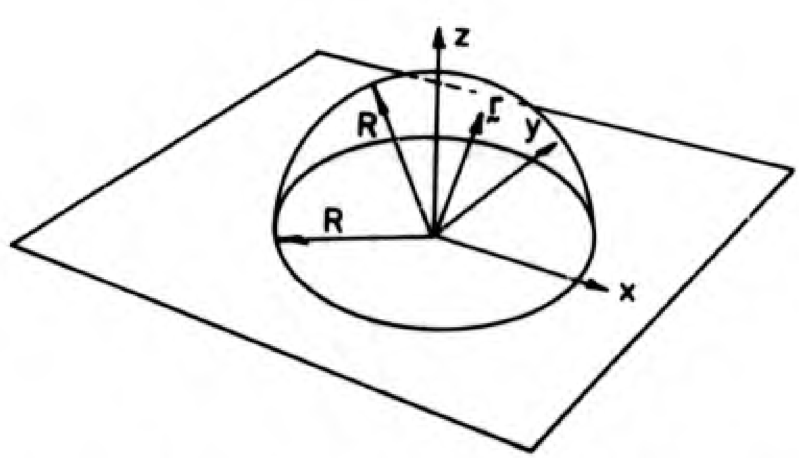

Fig.12: Local cartesian coordinate system illustrating region of convergence.

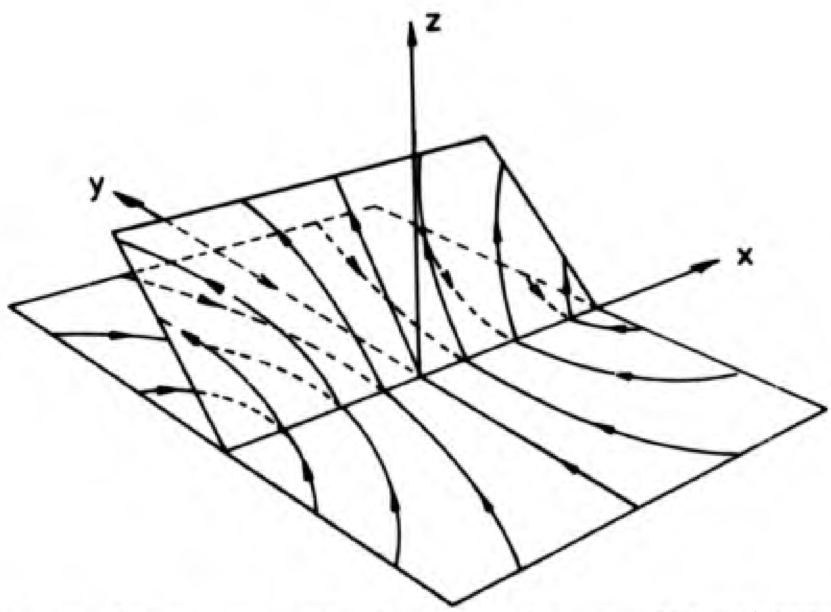

Fig.13: Perry't (1983) solution equation (2), for almost two-dimensional separation. The plane z=0 is the wall. The streamlines in the plane of symmetry (x=0) are those of two-dimensional separation.

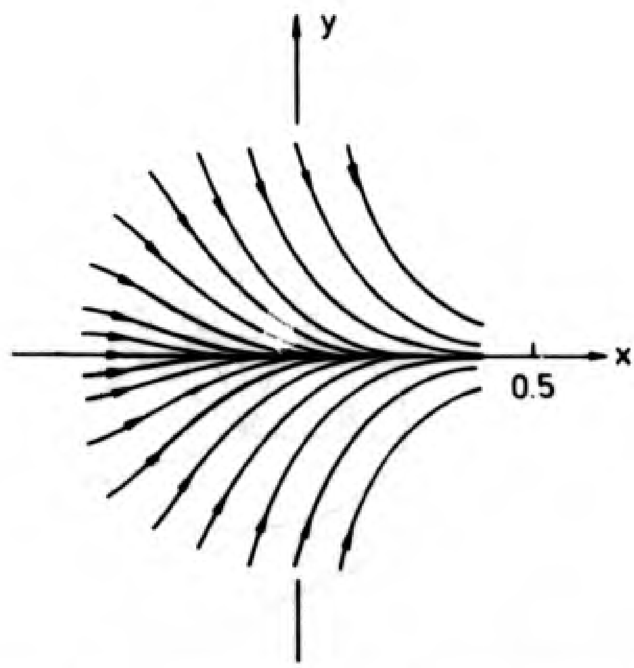

Fig.14: Wall streamlines near the beginning of a smoothly starting negative bifurcation line; numerical integration of equation (5) with a=0.5, b=0.5.