APPLICATION OF THE BOOTSTRAP METHOD TO A MEASURE OF FORCE EFFECTIVENESS (AN EMPIRICAL CASE STUDY)

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ABSTRACT. A paper was presented at the ~~Twenty-Eighth~~ previous Conference on the Design of Experiments about estimating the variance of the loss exchange ratio (LER). The LER is a measure of force effectiveness that is often used in military analysis of combat. Two methods of estimation were discussed: (1) the method of error propagation, and (2) the application of Fieller's theorem. The discussion that followed the presentation and further references to the literature pointed to Fieller's method as the preferred methodology to use to estimate confidence intervals about this measure of force effectiveness. Professor Bradley Efron (Stanford University) presented an overview of bootstrap methods. Dutoit and Shannahan have applied bootstrap methods to data to compute an estimate of the LER. Confidence intervals were also determined. The distribution of LERs about the mean value derived from the bootstrap have been compared to results using error propagation and Fieller's theorem. The results of this comparison as well as the bootstrap sensitivity to different replication sizes are presented.

1. INTRODUCTION AND BACKGROUND.

a. Error Propagation and Fieller. As pointed out in reference (2), the LER is defined as the ratio of Red casualties (R) to Blue casualties (B):

$$LER = R/B. \qquad (1)$$

Usually the values of R and B are obtained by replicating a stochastic wargame model. The average LER ($\widehat{LER}$) is computed as:

$$\widehat{LER} = \bar{R}/\bar{B} \qquad (2)$$

Because the generators of these average values are the results of a stochastic wargame, it would be useful to determine a confidence interval around the measure for various forms of hypothesis testing. Using error propagation methods, reference (2) shows that the variance of the (LER) can be estimated as:

$$\mathrm{VAR}(\widehat{\mathrm{LER}}) = \frac{1}{n}\left[\left(\frac{1}{\bar{B}}\right)^2 S_R^2 + \left(\frac{-\bar{R}}{\bar{B}^2}\right)^2 S_B^2 + 2\left(\frac{1}{\bar{B}}\right)\left(\frac{-\bar{R}}{\bar{B}^2}\right) R \; S_R \; S_B\right] \tag{3}$$

The appropriate $100(1-\alpha)$ confidence interval (C.I.) for the LER would be calculated as:

$$100(1-\alpha)\ \mathrm{C.I.}\ (\mathrm{LER}) = \widehat{\mathrm{LER}} \pm t\sqrt{\mathrm{VAR}(\widehat{\mathrm{LER}})} \tag{4}$$

Similarly, reference (2) also shows that Fieller's theorem can be used to find the fiducial limits of the ratio of two means. In this case, the upper and lower limits ($R_{U,L}$) can be found as the solution of a quadratic equation and are:

$$R_{U,L} = \frac{\bar{B}\bar{R} - \dfrac{t^2 R S_B S_R}{n} \pm \sqrt{\left(\bar{B}\bar{R} - \dfrac{t^2 R S_B S_R}{n}\right)^2 - \left[\bar{B}^2 - t^2\left(\dfrac{S_B^2}{n}\right)\right]\left[\bar{R}^2 - t^2\left(\dfrac{S_R^2}{n}\right)\right]}}{\bar{B}^2 - t^2\left(\dfrac{S_B^2}{n}\right)} \tag{5}$$

In operations (2), (3), (4), and (5) the following notation is used:

(a) $\bar{R}$, $\bar{B}$ are the average number of Red and Blue casualties, respectively.

(b) n is the number of stochastic wargame replications. This is used to calculate $\bar{R}$, $\bar{B}$, S_B, S_R, and R.

(c) S_R, S_B are the sample standard deviations for Red and Blue casualties.

(d) R is the correlation between Red and Blue casualties based on n replications of the wargame.

(e) t is the two tailed value of the student's t with (n-1) degrees of freedom.

The discussion that followed the presentation of this paper and further references to the literature pointed to Fieller's method as the preferred way (compared to error propagation) to compute a confidence interval about a ratio although there was an indication that both error propagation and Fieller's method to give "reasonably" consistent results.

b. Bootstrap. The purpose of this paper is not to provide a detailed description of bootstrap methods. Reference (1), entitled "Computer-Intensive Methods in Statistics" is a readily available and clearly worded explanation of the bootstrap method co-written by one of the bootstrap inventors (Efron). Figure 1 below shows how the bootstrap method was applied to sets of data to compute estimates of the LER and the frequency distribution of these estimates.

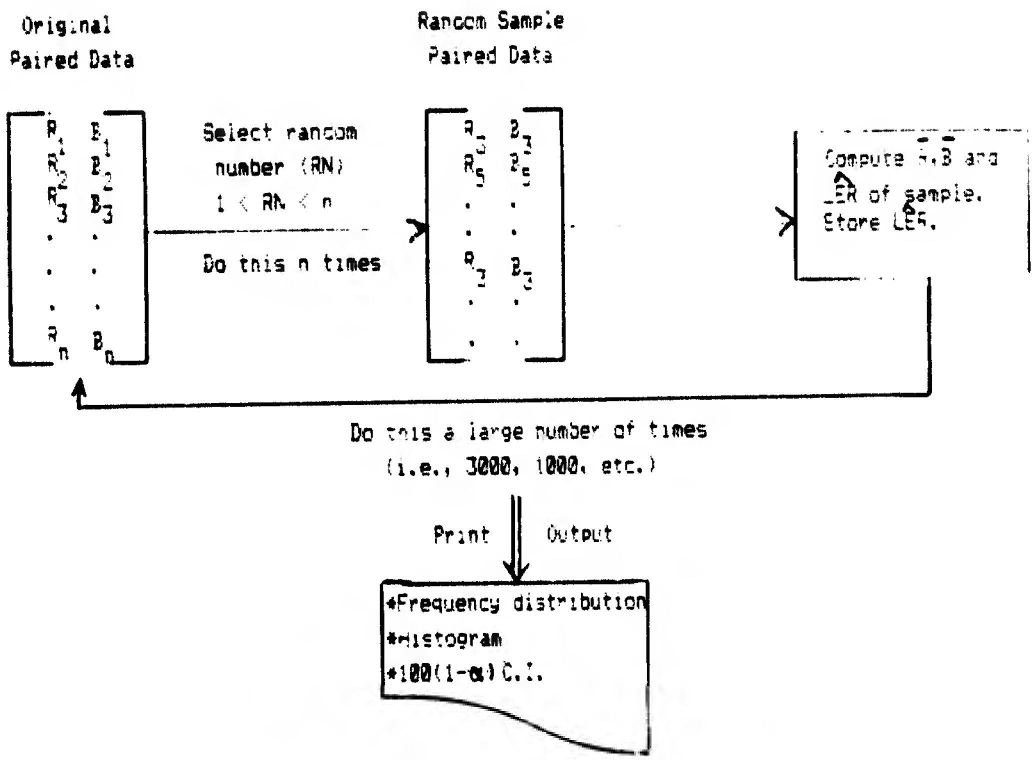

FIGURE 1. APPLICATION OF THE BOOTSTRAP METHOD TO ESTIMATE THE LOSS EXCHANGE RATIO (LER)

Each replication (1,2,3,...,n) of the stochastic wargame provides a set of paired data (i.e., Red and Blue casualties). Through random selection (with replacement), another set of data of n paired observations is selected from the original data set. From this additional sample, the values of $\bar{R}$ and $\bar{B}$ are obtained and the value $\widehat{LER}$ is computed. This value of $\widehat{LER}$ is stored in the computer memory. This bootstrap process is done a large number of times (3000, 1000, etc.) and the sample $\widehat{LER}$ is stored for each additional sample. At the completion of a large number of bootstrap runs, the frequency distribution is printed and the average LER, as well as the appropriate confidence limits, are determined from this empirical distribution. These LER estimates, and the confidence limits derived from the bootstrap, were compared to results using error propagation and Fieller's theorem. The results of this comparison as well as the bootstrap sensitivity to different replication sizes (3000, 1000, 750 500, 250, 100) was studied.

2. ASSUMPTIONS AND CONSTRAINTS. The following assumptions and constraints apply to this study.

a. This is a case study based on actual data obtained from the CARMONETTE stochastic wargame. The findings or observations should be interpreted as emerging trends with respect to LERs within the constraints of the forces and systems modeled using this wargame. Perhaps this paper will serve as a catalyst for some additional theoretical studies using bootstrap methods to estimate measures of force effectiveness.

b. It is assumed that the Radio Shack TRS-80 Modoel II (64K) system random number generator produces a statistically valid stream of 36,000 random numbers (minimum).

c. The emerging findings or trends apply to 99, 95, 90 and 85% confidence intervals.

d. Estimates of the average value of the LER($\widehat{LER}$) are carried out to the nearest tenth. This measure of force effectiveness is a rough indicator and estimates made with any greater precision are not considered to be operationally meaningful.

3. THE SOURCE OF DATA USED IN THIS STUDY. The data used in this case study were obtained from a force-on-force evaluation of several medium antiarmor systems which were employed within an Infantry force and scenario. Twelve medium antiarmor concepts were examined (denoted as case A, B, C, ..., L). All medium antiarmor concepts were inserted in the same force and fought against the same threat on the same terrain. All other factors were held constant, therefore the differences in average Red and Blue casualties are attributed to the performance factors and synergistic influence of the different antiarmor systems. Table 1 below shows the input to this case study for each of the twelve antiarmor systems (cases A through L). The Red and Blue casualties are given in the format (xx/xx). Therefore, case A, replication 1 had 112 Red casualties and 24 Blue casualties. The other variable notation has been defined earlier in this paper. This represents the total input required to do the bootstrap experiment and compute the confidence interval estimates using error propagation and Fieller's theorem.

4. RESULTS. Tables 2, 3, 4 and 5 show the results of the bootstrap experiment and the error propagation and Fieller's theorem results for 99%, 95%, 90% and 85% confidence intervals, respectively. The results of the bootstrap method are based on 3000, 1000, ..., 100 replications. The upper limit (UL) and lower limit (LL) are given for the stated level of confidence for all estimates. The average value of the LER($\widehat{LER}$) is given for each bootstrap replication size in addition to the estimates obtained from error propagation and Fieller's theorem. The width of the confidence interval is given as the difference between UL and LL. For example, refer to Table 2. The case A 99% confidence statement of the bootstrap estimate based on 3000 replications is 5.2 for the LER. The upper and lower 99% confidence limits are 6.1 and 4.6, respectively. The width of the confidence interval is 1.5. Fieller's theorem gives upper and lower 99% confidence limits of 6.5 and 4.3 with an interval width of 2.2. Error propagation statistics were 5.2 for the estimate of the LER and 6.2 and 4.1 for the 99% confidence limits.

5. EMERGING TRENDS. The following emerging trends are based on the results shown in tables 2 through 5. These trends should be interpreted with respect to LERs appropriate to the forces and systems modeled using this wargame.

a. The upper and lower confidence limits and the LER estimate ($\widehat{LER}$) are relatively insensitive to the replication size (from 3000 to 100) for the four levels of α examined in this study. This was true for all 12 cases (A through L) for the 95, 90 and 85% confidence levels and true for about 3/4 of the cases at the 99% confidence level.

b. The bootstrap confidence interval is consistently shorter than intervals generated by either the error propagation or Fieller's theorem method.

c. Regarding the 99% and 95% confidence intervals, the bootstrap and Fieller's theorem interval estimates tend to yield LER distributions with positive skews. This effect is slightly stronger for the 99% confidence interval than for the 95% confidence interval. This same effect is also true for 90% and 85% interval estimates but not to the same degree as for the 99% and 95% intervals. In fact, the effect is relatively negligible for these two cases.

d. Regarding the 99% confidence interval. the bootstrap lower limit is better approximated by the Fieller's theorem estimates and the bootstrap upper limit is better approximated by the error propagation estimate. Although these findings are relatively consistent across all 12 cases, the degree of agreement is not always good.

e. Regarding the 95% confidence interval, neither the error propagation or Fieller's method has a strong advantage in approximating the bootstrap interval estimates. However, when the error propagation results do a better job in approximating the bootstrap estimates, it generally better approximates the upper confidence limit. The Fieller's theorem method most often approximates the bootstrap lower confidence limit. These 95% findings are consistent with the findings for the 99% confidence interval.

f. Regarding the 90% and 85% confidence intervals, the error propagation and Fieller's theorem estimates are, for the most part, good approximations to the bootstrap results.

TABLE 1. INPUT DATA REQUIRED FOR CASE STUDY
(Based on Force-on-Force Model)

Rep #	CASES: A	B	C	D	E	F	G	H	I	J	K	L
1	112/24	68/34	79/24	99/21	112/15	128/23	99/21	103/26	98/20	56/72	64/22	57/29
2	108/15	72/24	103/21	108/26	105/22	110/21	77/30	120/16	100/17	75/69	70/24	80/24
3	110/17	93/24	95/23	119/18	93/21	85/23	86/28	118/24	105/21	71/55	67/25	88/19
4	103/21	83/30	99/22	104/24	112/23	91/23	88/28	112/24	96/18	73/57	121/21	59/24
5	109/20	56/29	92/23	102/21	92/25	101/23	83/25	116/13	109/20		51/21	74/25
6	112/25	68/31		105/27	98/25	111/17					66/30	77/26
7	100/22	96/23		110/18		105/22					66/26	70/23
8	108/23	70/30		99/25							70/27	53/27
9		74/34		108/17							52/25	71/29
10											68/29	
11											78/25	
12											56/28	
N	8	9	5	9	6	7	5	5	5	4	12	9
$\bar{R}$	107.75	75.56	93.60	106.00	102.00	104.43	86.60	113.80	101.60	68.75	69.08	69.89
s_R	4.23	12.85	9.15	6.20	9.01	14.14	8.08	6.72	5.32	8.66	18.13	11.56
$\bar{B}$	20.88	28.78	22.60	21.90	21.83	21.71	26.40	20.60	19.20	63.25	25.25	25.11
s_B	3.44	4.21	1.14	3.80	3.71	2.21	3.51	5.73	1.64	8.50	2.96	3.14
R	.11	-.58	-.93	-.47	-.56	-.27	-.84	-.62	.55	-.55	-.31	-.63
$\widehat{LER}$	5.16	2.63	4.14	4.84	4.67	4.81	3.28	5.52	5.29	1.09	2.74	2.78
$t_{.99}$	3.499	3.355	4.604	3.355	4.032	3.707	4.604	4.604	4.604	5.841	3.106	3.355
$t_{.95}$	2.365	2.306	2.776	2.306	2.571	2.447	2.776	2.776	2.776	3.182	2.201	2.306
$t_{.90}$	1.895	1.860	2.132	1.860	2.015	1.943	2.132	2.132	2.132	2.353	1.796	1.860
$t_{.85}$	1.617	1.592	1.779	1.592	1.699	1.628	1.779	1.779	1.779	1.925	1.549	1.592

TABLE 2. 99% CONFIDENCE INTERVALS FOR LER

CASE		BOOTSTRAP 3000	1000	750	500	250	100	FIELLER'S THEOREM	ERROR PROP.
A	UL	6.1	6.2	6.1	6.1	6.0	5.8	6.5	6.2
	LER	5.2	5.2	5.2	5.1	5.2	5.1	(5.2)	5.2
	LL	4.6	4.6	4.6	4.6	4.6	4.7	4.3	4.1
	Width	1.5	1.6	1.5	1.5	1.4	1.1	2.2	2.1
B	UL	3.4	3.3	3.4	3.6	3.4	3.4	3.6	3.5
	LER	2.6	2.6	2.6	2.6	2.7	2.7	(2.6)	2.6
	LL	2.1	2.2	2.1	2.2	2.2	2.2	1.9	1.8
	Width	1.3	1.1	1.3	1.4	1.2	1.2	1.7	1.7
C	UL	4.7	4.7	4.7	4.8	4.8	4.7	5.5	5.4
	LER	4.1	4.1	4.1	4.2	4.2	4.1	(4.1)	4.1
	LL	3.6	3.5	3.6	3.5	3.3	3.5	3.0	2.9
	Width	1.2	1.2	1.1	1.3	1.5	1.2	2.5	2.5
D	UL	5.8	5.8	5.9	5.8	5.9	5.7	6.2	6.0
	LER	4.9	4.9	4.9	4.9	4.9	4.8	(4.8)	4.8
	LL	4.2	4.1	4.2	4.2	4.2	4.3	3.9	3.7
	Width	1.6	1.7	1.7	1.6	1.7	1.4	2.3	2.3
E	UL	6.2	6.1	6.3	5.9	6.1	6.3	7.1	6.4
	LER	4.7	4.7	4.7	4.7	4.7	4.7	(4.7)	4.7
	LL	3.9	3.9	3.8	3.9	3.9	3.9	3.3	2.9
	Width	2.3	2.2	2.5	2.0	2.2	2.4	3.8	3.5
F	UL	5.7	5.7	5.7	5.8	5.8	5.4	6.2	6.1
	LER	4.8	4.8	4.8	4.8	4.8	4.8	(4.8)	4.8
	LL	4.0	4.0	4.1	4.0	4.1	4.2	3.6	3.5
	Width	1.7	1.7	1.6	1.8	1.7	1.2	2.6	2.6
G	UL	4.3	4.4	4.1	4.4	4.0	4.4	5.3	4.7
	LER	3.3	3.3	3.3	3.3	3.3	3.4	(3.4)	3.4
	LL	2.7	2.7	2.7	2.7	2.7	2.7	2.1	1.8
	Width	1.6	1.7	1.4	1.7	1.3	1.7	3.2	2.9
H	UL	8.3	8.0	7.6	8.4	8.0	7.2	13.9	9.1
	LER	5.6	5.6	5.6	5.7	5.6	5.5	(5.5)	5.5
	LL	4.2	4.2	4.3	4.2	4.3	4.1	3.2	1.9
	Width	4.1	3.8	3.3	4.2	3.7	3.1	10.7	7.2
I	UL	5.7	5.8	5.7	5.7	5.7	5.8	6.2	6.1
	LER	5.3	5.3	5.3	5.3	5.3	5.3	(5.3)	5.3
	LL	5.0	4.9	4.9	4.9	5.0	5.0	4.6	4.5
	Width	.7	.9	.8	.8	.7	.8	1.6	1.6
J	UL	1.3	1.3	1.3	1.3	1.3	1.3	2.2	1.8
	LER	1.1	1.1	1.1	1.1	1.1	1.1	(1.1)	1.1
	LL	.8	.8	.8	.9	.9	.9	.5	.4
	Width	.5	.5	.5	.4	.4	.4	1.7	1.4

TABLE 2. 99% CONFIDENCE INTERVALS FOR LER (CONT'D)

CASE		BOOTSTRAP 3000	1000	750	500	250	100	FIELLER'S THEOREM	ERROR PROP.
K	UL	3.5	3.5	3.5	3.5	3.6	3.4	3.6	3.5
	LER	2.7	2.7	2.7	2.8	2.7	2.7	(2.7)	2.7
	LL	2.3	2.3	2.3	2.4	2.3	2.3	2.0	2.0
	Width	1.2	1.2	1.2	1.1	1.3	.9	1.6	1.5
L	UL	3.5	3.5	3.5	3.5	3.5	3.5	3.7	3.6
	LER	2.8	2.8	2.8	2.8	2.8	2.8	(2.8)	2.8
	LL	2.3	2.3	2.3	2.3	2.3	2.4	2.1	2.0
	Width	1.2	1.2	1.2	1.2	1.2	1.1	1.6	1.6

TABLE 3. 95% CONFIDENCE INTERVALS FOR LER

CASE		BOOTSTRAP 3000	1000	750	500	250	100	FIELLER'S THEOREM	ERROR PROP.
A	UL	5.9	5.8	5.8	5.9	5.8	5.8	5.9	5.9
	LER	5.2	5.2	5.2	5.2	5.2	5.2	(5.2)	5.2
	LL	4.7	4.7	4.7	4.7	4.7	4.7	4.5	4.4
	Width	1.2	1.1	1.1	1.2	1.1	1.1	1.4	1.5
B	UL	3.2	3.1	3.1	3.1	3.2	3.2	3.3	3.2
	LER	2.6	2.6	2.6	2.6	2.6	2.7	(2.6)	2.6
	LL	2.2	2.2	2.2	2.2	2.2	2.3	2.1	2.1
	Width	1.0	.9	.9	.9	1.0	.9	1.2	1.1
C	UL	4.6	4.6	4.6	4.6	4.6	4.6	4.9	4.9
	LER	4.2	4.1	4.1	4.1	4.1	4.2	(4.1)	4.1
	LL	3.7	3.7	3.7	3.7	3.7	3.8	3.4	3.4
	Width	.9	.9	.9	.9	.9	.8	1.5	1.5
D	UL	5.6	5.5	5.6	5.6	5.4	5.4	5.7	5.6
	LER	4.9	4.9	4.9	4.9	4.8	4.9	(4.8)	4.8
	LL	4.3	4.3	4.3	4.3	4.3	4.4	4.2	4.1
	Width	1.3	1.2	1.3	1.3	1.1	1.0	1.5	1.5
E	UL	5.7	5.7	5.6	5.7	5.7	5.8	6.0	5.8
	LER	4.7	4.7	4.7	4.7	4.7	4.6	(4.7)	4.7
	LL	4.0	4.0	4.1	4.0	4.0	4.0	3.7	3.5
	Width	1.7	1.7	1.5	1.7	1.7	1.8	2.3	2.3
F	UL	5.5	5.5	5.5	5.4	5.5	5.4	5.7	5.7
	LER	4.8	4.8	4.8	4.8	4.8	4.8	(4.8)	4.8
	LL	4.2	4.2	4.3	4.2	4.2	4.2	4.0	4.0
	Width	1.3	1.3	1.2	1.2	1.3	1.2	1.7	1.7
G	UL	4.0	3.9	4.0	4.0	4.0	4.0	4.3	4.2
	LER	3.3	3.3	3.3	3.3	3.3	3.3	(3.3)	3.3
	LL	2.8	2.8	2.8	2.8	2.8	2.8	2.5	2.4
	Width	1.2	1.1	1.2	1.2	1.2	1.2	1.9	1.7
H	UL	7.3	7.3	7.4	7.2	7.4	7.4	8.8	7.7
	LER	5.6	5.6	5.6	5.5	5.6	5.5	(5.5)	5.5
	LL	4.4	4.5	4.3	4.3	4.3	4.4	3.9	3.3
	Width	2.9	2.8	3.1	2.9	3.1	3.0	4.9	4.4
I	UL	5.6	5.6	5.6	5.6	5.6	5.7	5.8	5.8
	LER	5.3	5.3	5.3	5.3	5.3	5.3	(5.3)	5.3
	LL	5.0	5.0	5.0	5.0	5.0	5.1	4.9	4.8
	Width	.6	.6	.6	.6	.6	.6	.9	1.0
J	UL	1.3	1.3	1.3	1.3	1.3	1.2	1.6	1.5
	LER	1.1	1.1	1.1	1.1	1.1	1.1	(1.1)	1.1
	LL	.9	.9	.9	.9	.9	.9	.8	.7
	Width	.4	.4	.4	.4	.4	.3	.8	.8

TABLE 3. 95% CONFIDENCE INTERVALS FOR LER (CONT'D)

CASE		BOOTSTRAP 3000	1000	750	500	250	100	FIELLER'S THEOREM	ERROR PROP.
K	UL	3.3	3.3	3.3	3.3	3.3	3.2	3.3	3.3
	LER	2.7	2.7	2.7	2.7	2.7	2.7	(2.7)	2.7
	LL	2.4	2.4	2.4	2.4	2.4	2.4	2.2	2.2
	Width	.9	.9	.9	.9	.9	.8	1.1	1.1
L	UL	3.3	3.3	3.3	3.3	3.3	3.2	3.4	3.3
	LER	2.8	2.8	2.8	2.8	2.8	2.8	(2.8)	2.8
	LL	2.4	2.3	2.4	2.4	2.3	2.5	2.3	2.2
	Width	.9	1.0	.9	.9	1.0	.7	1.1	1.1

TABLE 4. 90% CONFIDENCE INTERVALS FOR LER

CASE		BOOTSTRAP 3000	1000	750	500	250	100	FIELLER'S THEOREM	ERROR PROP.
A	UL	5.7	5.7	5.7	5.7	5.7	5.7	5.8	5.7
	LER	5.2	5.2	5.2	5.2	5.2	5.2	(5.2)	5.2
	LL	4.8	4.8	4.8	4.8	4.7	4.8	4.6	4.6
	Width	.9	.9	.9	.9	1.0	.9	1.2	1.1
B	UL	3.1	3.0	3.1	3.1	3.0	3.1	3.1	3.1
	LER	2.6	2.6	2.6	2.6	2.6	2.7	(2.6)	2.6
	LL	2.3	2.3	2.3	2.3	2.3	2.3	2.2	2.2
	Width	.8	.7	.8	.8	.7	.8	.9	.9
C	UL	4.5	4.6	4.5	4.5	4.6	4.6	4.7	4.7
	LER	4.1	4.1	4.1	4.1	4.2	4.1	(4.1)	4.1
	LL	3.7	3.7	3.7	3.7	3.8	3.8	3.6	3.6
	Width	.8	.9	.8	.8	.8	.8	1.1	1.1
D	UL	5.4	5.4	5.5	5.5	5.4	5.3	5.5	5.5
	LER	4.9	4.9	4.9	4.8	4.8	4.9	(4.8)	4.8
	LL	4.4	4.4	4.4	4.4	4.4	4.4	4.3	4.2
	Width	1.0	1.0	1.1	1.1	1.0	.9	1.2	1.3
E	UL	5.5	5.4	5.4	5.3	5.5	5.5	5.7	5.6
	LER	4.7	4.7	4.7	4.7	4.7	4.7	(4.7)	4.7
	LL	4.1	4.1	4.1	4.1	4.1	4.2	3.9	3.8
	Width	1.4	1.3	1.3	1.2	1.4	1.3	1.8	1.8
F	UL	5.4	5.4	5.4	5.4	5.3	5.3	5.5	5.5
	LER	4.8	4.8	4.8	4.8	4.8	4.8	(4.8)	4.8
	LL	4.3	4.3	4.3	4.3	4.3	4.3	4.2	4.1
	Width	1.1	1.1	1.1	1.1	1.0	1.0	1.3	1.4
G	UL	3.8	3.8	3.8	4.0	3.8	3.9	4.0	4.0
	LER	3.3	3.3	3.3	3.3	3.3	3.3	(3.3)	3.3
	LL	2.9	2.9	2.9	2.9	2.9	2.9	2.7	2.6
	Width	.9	.9	.9	1.1	.9	1.0	1.3	1.4
H	UL	7.1	7.1	7.1	7.1	6.9	7.1	7.8	7.2
	LER	5.6	5.6	5.6	5.6	5.5	5.6	(5.6)	5.6
	LL	4.5	4.5	4.5	4.6	4.5	4.5	4.2	3.8
	Width	2.6	2.6	2.6	2.6	2.4	2.6	3.6	3.4
I	UL	5.6	5.6	5.6	5.6	5.6	5.6	5.7	5.7
	LER	5.3	5.3	5.3	5.3	5.3	5.3	(5.3)	5.3
	LL	5.1	5.1	5.1	5.1	5.1	5.1	5.0	4.9
	Width	.5	.5	.5	.5	.5	.5	.7	.8
J	UL	1.3	1.2	1.3	1.2	1.3	1.3	1.4	1.4
	LER	1.1	1.1	1.1	1.1	1.1	1.1	(1.1)	1.1
	LL	.9	.9	.9	.9	.9	.9	.8	.8
	Width	.4	.3	.4	.3	.4	.4	.6	.6

TABLE 4. 90% CONFIDENCE INTERVALS FOR LER (CONT'D)

CASE		BOOTSTRAP 3000	1000	750	500	250	100	FIELLER'S THEOREM	ERROR PROP.
K	UL	3.2	3.2	3.2	3.2	3.2	3.2	3.2	3.2
	LER	2.8	2.8	2.7	2.7	2.7	2.7	(2.7)	2.7
	LL	2.4	2.4	2.4	2.4	2.4	2.4	2.3	2.3
	Width	.8	.8	.8	.8	.8	.8	.9	.9
L	UL	3.2	3.2	3.2	3.2	3.1	3.2	3.3	3.2
	LER	2.8	2.8	2.8	2.8	2.8	2.8	(2.8)	2.8
	LL	2.4	2.4	2.4	2.4	2.4	2.4	2.4	2.3
	Width	.8	.8	.8	.8	.7	.8	.9	.9

TABLE 5. 85% CONFIDENCE INTERVALS FOR LER

CASE		BOOTSTRAP 3000	1000	750	500	250	100	FIELLER'S THEOREM	ERROR PROP.
A	UL	5.6	5.6	5.6	5.6	5.7	5.6	5.7	5.6
	LER	5.2	5.2	5.2	5.2	5.2	5.2	(5.2)	5.2
	LL	4.8	4.8	4.8	4.8	4.8	4.9	4.7	4.7
	Width	.8	.8	.8	.8	.9	.7	1.0	.9
B	UL	3.0	3.0	3.0	3.0	3.0	3.0	3.0	3.0
	LER	2.6	2.6	2.6	2.7	2.7	2.6	(2.6)	2.6
	LL	2.3	2.3	2.3	2.3	2.3	2.3	2.3	2.2
	Width	.7	.7	.7	.7	.7	.7	.7	.8
C	UL	4.5	4.5	4.5	4.5	4.5	4.5	4.6	4.6
	LER	4.2	4.2	4.1	4.1	4.2	4.1	(4.1)	4.1
	LL	3.8	3.8	3.8	3.8	3.8	3.8	3.7	3.7
	Width	.7	.7	.7	.7	.7	.7	.9	.9
D	UL	5.3	5.3	5.3	5.4	5.3	5.3	5.4	5.4
	LER	4.9	4.9	4.9	4.9	4.9	4.9	(4.8)	4.8
	LL	4.4	4.4	4.4	4.4	4.5	4.5	4.4	4.3
	Width	.9	.9	.9	1.0	.8	.8	1.0	1.1
E	UL	5.3	5.3	5.3	5.4	5.3	5.4	5.5	5.4
	LER	4.7	4.7	4.7	4.7	4.7	4.7	(4.7)	4.7
	LL	4.2	4.1	4.1	4.2	4.1	4.2	4.0	3.9
	Width	1.1	1.2	1.2	1.2	1.2	1.2	1.5	1.5
F	UL	5.3	5.3	5.3	5.3	5.2	5.3	5.4	5.4
	LER	4.8	4.8	4.8	4.8	4.8	4.8	(4.8)	4.8
	LL	4.4	4.3	4.4	4.4	4.4	4.3	4.3	4.2
	Width	.9	1.0	.9	.9	.8	1.0	1.1	1.2
G	UL	3.7	3.7	3.7	3.7	3.7	3.7	3.9	3.8
	LER	3.3	3.3	3.3	3.3	3.3	3.3	(3.3)	3.3
	LL	2.9	2.9	2.9	2.9	2.9	2.9	2.8	2.7
	Width	.8	.8	.8	.8	.8	.8	1.1	1.1
H	UL	6.7	6.9	6.9	6.6	6.9	6.6	7.3	6.9
	LER	5.6	5.6	5.6	5.6	5.7	5.5	(5.5)	5.5
	LL	4.7	4.7	4.7	4.6	4.7	4.5	4.1	4.1
	Width	2.0	2.2	2.2	2.0	2.2	2.1	2.9	2.8
I	UL	5.5	5.5	5.6	5.5	5.5	5.5	5.6	5.6
	LER	5.3	5.3	5.3	5.3	5.3	5.3	(5.3)	5.3
	LL	5.1	5.1	5.1	5.1	5.1	5.1	5.0	5.0
	Width	.4	.4	.5	.4	.4	.4	.6	.6
J	UL	1.2	1.2	1.2	1.2	1.2	1.2	1.4	1.3
	LER	1.1	1.1	1.1	1.1	1.1	1.1	(1.1)	1.1
	LL	.9	.9	.9	1.0	1.0	.9	.9	.9
	Width	.3	.3	.3	.2	.2	.3	.5	.4

TABLE 5. 85% CONFIDENCE INTERVALS FOR LER (CONT'D)

CASE		BOOTSTRAP						FIELLER'S THEOREM	ERROR PROP.
		3000	1000	750	500	250	100		
K	UL	3.1	3.1	3.1	3.1	3.1	3.1	3.1	3.1
	LER	2.7	2.7	2.7	2.7	2.7	2.7	(2.7)	2.7
	LL	2.4	2.4	2.4	2.4	2.4	2.4	2.4	2.3
	Width	.7	.7	.7	.7	.7	.7	.7	.8
L	UL	3.1	3.1	3.1	3.1	3.1	3.1	3.2	3.2
	LER	2.8	2.8	2.8	2.8	2.8	2.8	(2.8)	2.8
	LL	2.5	2.5	2.5	2.5	2.5	2.5	2.4	2.4
	Width	.6	.6	.6	.6	.6	.6	.8	.8

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Subject: Unusual Data Sets

The following results were obtained from a force-on-force evaluation.

Replication	Red Casualties	Blue Casualties
1	.9687	0
2	.5069	0
3	.5086	0
4	.1362	.3274
5	0	0
6	.1405	0
7	0	0

Note that the number of Blue casualties is zero six times out of seven and both Red and Blue casualties are zero two times out of seven. The application of equation (5), Fieller's theorem, yields 90% confidence limits of -7.41 and .62 . The estimate of the LER (equation 2) is 6.93. In this case, the upper and lower confidence limits do not include the point estimate of the LER. The results of the bootstrap are also seemingly anomalous. The mean LER value is about 1200 across different replication sizes ranging from 250 through 3000 and the upper and lower 90% confidence limits average about 7000 and 2.5, respectively. This observation does not negate the use of the bootstrap and Fieller's method, but does indicate that some unusual data sets (i.e., containing a preponderance of zeros and numbers less than one) should not be analyzed in this fashion. More theoretical work needs to be done concerning the make-up of the data before subjecting them to analysis. Of course, this is true for any statistical procedure.

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