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# THE RIEMANN NON-DIFFERENTIABLE FUNCTION AND IDENTITIES FOR THE GAUSS SUMS

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Riemann's example of a continuous, non-differentiable function is given by the sum  $\sum_{n=1}^{\infty} \sin(n^2 x)/n^2$ . This function is sufficiently irregular and its graph is fractal. Hardy proved <sup>1</sup> that Riemann's non-differentiable function is not differentiable at any irrational point because of a square root singularity at these points. Careful investigation of the differentiability of Riemann's non-differentiable function was carried out by Gerver, who showed <sup>2</sup> that this function has derivative equal to  $-1/2$  at every rational point of a special type (forming the orbit of the point 1 under the theta-modular group<sup>3</sup>). Different proofs of this surprising fact was given by other authors<sup>3,4,5</sup>, providing also a close relation between Riemann's non-differentiable function and classical  $\theta$ -function and Gauss sums. Duistermaat obtained <sup>3</sup> an exact functional equations for this function under transformations of the theta-modular group. In this article we use functional equations for Riemann's non-differentiable function under theta-modular transformations to derive functional equations on Gauss sums generalizing Genocchi-Schaar identity.

A Gauss sum is a sum of the form

$$S(p, q) = \sum_{n=0}^{q-1} \exp(\pi i n^2 p/q),$$

where  $p$  and  $q$  are relatively prime integers of opposite parity, i.e. one is odd and the other is even. The Genocchi-Schaar identity on Gauss sums is the following: for positive integers  $p$  and  $q$  of opposite parity,

$$\frac{1}{\sqrt{q}} \sum_{n=0}^{q-1} \exp(\pi i n^2 p/q) = \frac{\exp(\pi i/4)}{\sqrt{p}} \sum_{n=0}^{p-1} \exp(-\pi i n^2 q/p).$$

This identity can be interpreted as the transformation of the Gauss sum under the change  $\sigma : z \rightarrow -1/z$  where  $z = p/q$ .

The modular group  $\Gamma$  is a group of fractional linear transformations  $\gamma : z \rightarrow (az + b)/(cz + d)$  where  $a, b, c, d \in \mathbb{Z}$  and  $ad - bc = 1$ . Theta-modular group  $\Gamma_\theta$  is a sub-group of modular group generated by the following mappings:  $\tau : z \rightarrow z + 2$  and  $\sigma : z \rightarrow -1/z$ . For any element of the theta-modular group the following are valid:  $ab \equiv 0 \pmod{2}$  and  $cd \equiv 0 \pmod{2}$ . Every fractional transformation  $\gamma \in \Gamma_\theta$  has a simple pole at the point  $z = -d/c$ . The rational points  $x = p/q$  with integers  $p, q$  of opposite parity and infinity constitute theta-modular orbit of point 0.

Riemann's non-differentiable function has the following local estimations at the point  $x$ :  $f(x \pm \epsilon) = f(x) + R_\pm \epsilon^\delta$ , where  $0 < \delta \leq 1$ . Using the functional equations for  $f(x)$  it is possible to find functional equations on the functions  $R_\pm$ . These functions are known only at the rational points, where they coincide with the Gauss sums. The following theorems describe these functional equations:

**Theorem:** Let  $r$  and  $s$  be integers of which one is even and the other is odd, and  $s$  is positive,  $\gamma \in \Gamma_\theta$  be an element of the theta-modular group,  $\gamma : z \rightarrow (az + b)/(cz + d)$ , and suppose  $c$  is positive and  $cr + ds \neq 0$ . Define  $r' = ar + bs$  and  $s' = cr + ds$ . Then the following formula for Gauss sums is valid:

$$\frac{\exp(\pi i \text{Sign}(s')/4)}{\sqrt{|s'|}} S(r', s') \frac{1}{\sqrt{c}} S(-d, c) = \frac{1}{\sqrt{s}} S(r, s) \quad (1)$$

**Proof:** The proof is based on the differentiability properties of Riemann's non-differentiable function. Let us consider the following function  $\phi(z) = \sum_{n=1}^{\infty} \exp(\pi i n^2 z)/\pi i n^2$ . Using technique of papers<sup>4,5</sup> we calculate the following estimation for the function  $\phi(z)$  at the point  $x = u/v$ , where  $u$  and  $v$  are relatively prime integers of opposite parity,  $(u, v) = 1$  and  $uv \equiv 0 \pmod{2}$ :

$$\phi(x + h^2) - \phi(x - h^2) = h^2 \sum_{n=-\infty}^{+\infty} \exp(\pi i n^2 x) \varphi(nh) - h^2 = \quad (2)$$

$$h^2 \sum_{t=0}^{|v|-1} \exp(\pi i t^2 x) \sum_{k=-\infty}^{+\infty} \varphi(k|v|h + th) - h^2 = 2^{1/2} S(u, v) h/|v| + O(h^2).$$

Here  $\varphi(x) = \sin(\pi x^2)/\pi x^2$  if  $x \neq 0$  and  $\varphi(x) = 1$  if  $x = 0$ , and we write  $n = k|v| + t$ ,  $(0 \leq t \leq |v|)$  and use that  $\exp(\pi i n^2 x) = \exp(\pi i t^2 x)$ , since  $uv \equiv 0 \pmod{2}$ . The function  $\phi(z)$  obey a functional equation under the action of theta-modular group<sup>3</sup>. Let  $\gamma \in \Gamma_\theta$  be an element of theta modular group, then the function  $\psi(z) = \phi(z) - \gamma'(z)^{-1} \mu_\gamma(z) \phi(\gamma(z))$  is differentiable, and analytical function  $\mu_\gamma$  is given by:

$$\mu_\gamma(z) = e^{(\pi i/4)} c^{-1} (z + d/c)^{-1/2} S(-d, c). \quad (3)$$

Eq. (2) is valid for any point of theta-modular orbit of  $x = r/s$ , except infinity. Calculating estimation (2) for the differentiable function  $\psi(x)$  at the point  $x = r/s$  and supposing  $\gamma(r/s) \neq \infty$ , we find that

$$\frac{S(r, s)}{s} - \gamma'(r/s)^{-1/2} \mu_\gamma(r/s) \frac{S(ar + bs, cr + ds)}{|cr + ds|} = 0. \quad (4)$$

Note, that  $r' = ar + bs$  and  $s' = cr + ds$  are relatively prime integers. Substituting the expression of  $\mu_\gamma(z)$  in terms of Gauss sums (Eq. 3) we obtain the desired result.

Q.E.D

The Genocchi-Schaar identity corresponds to the case of  $a = 0$ ,  $b = -1$ ,  $c = 1$  and  $d = 0$ . Functional equation (Eq. 1) describe the theta-modular transformations of Gauss sums and can be used to derive the values of the Gauss sums.

## References

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