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Optimization of a Curve Approximation Based on NURBS interpolation

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Abstract. In this paper, an approach is presented whereby optimal spatial positions and weights of a fixed number of NURBS control points are determined using a quasi-Newton optimization algorithm in order to approximate a general planar target curve. A method for constructing an adequate initial solution and a valid cost function based on interpolation error are introduced. Convergence of the iterative process is assessed, and the final interpolation error is related to prescribed manufacturing or analysis tolerances. The efficiency of the approach is demonstrated for actual wing profiles.

§1. Introduction

The problem of constructing a cost effective approximation of a general target curve is of great relevance in many engineering disciplines. This problem has been addressed quite thoroughly in the context of polynomial interpolation [1], but far less work has been published on rational approximation. Indeed, weights introduce another level of difficulty in the theoretical analysis of the approximation error. From a practical standpoint though, non-uniform rational B-Splines [4] (NURBS) provide more degrees of freedom for a given number of control points, which leads naturally to smoother curves.

The work presented here introduces a robust numerical approach for the determination of control point positions and weights of a NURBS curve; it can be used to construct an approximation to a general target planar curve. In the context of wing profile design, where this approach has been applied [2,3], very significant reductions in terms of data size and noise level have been observed.

In this paper, the approximation problem is first presented, and the method of computation of the approximation error discussed. Next, the optimization method itself is presented, including the choice of initial solutions. Finally, the performance of the method is evaluated for practical test cases, and conclusions are drawn.

§2. Approximation Problem

A NURBS curve is defined such that

$$\mathbf{A}(u) = \sum_{i=0}^n R_{i,p}(u) \mathbf{P}_i \quad (1)$$

with

$$R_{i,p}(u) = \frac{N_{i,p}(u)\omega_i}{\sum_{j=0}^n N_{j,p}(u)\omega_j}, \quad (2)$$

where $\mathbf{P}_i$ are the control point coordinates, ω_i their respective weights, $N_{i,p}$ the p -th degree B-spline basis functions and $\mathbf{A}(u)$ the position of a point on the curve. The basis functions are obtained through a *knot vector*, which defines the functions' break points, of the form

$$\underbrace{\{0, \dots, 0\}}_p, u_{p+1}, \dots, u_{m-p+1}, \underbrace{\{1, \dots, 1\}}_p.$$

Using these interpolation functions, the problem of approximating a general planar curve $\mathcal{C}(t)$ can be stated as follows: *find the set of control points $\mathbf{P}_i$ and weights ω_i such that $\|\mathbf{A}(u) - \mathcal{C}(t)\|$ is minimized in a suitable norm.*

Analytically, the L_2 norm would be a natural choice; numerically, though, for a completely general target curve, this norm can only be approximated through discretization. Numerical experiments have thus been carried out to develop and validate a robust computational approach for the determination of the approximation error. Consideration has been given to both the mean and maximum error, as well as to the level of continuity of the target curve. Three classes of target curves have been considered: curves only given as a set of points, piecewise linear curves, and C^1 or more continuous curves. In all cases, the mean error

$$\varepsilon_{mea} = \frac{1}{n} \sum_{k=1}^n d_k \quad (3)$$

is determined by summing the distance (d_k) of a set of points chosen on the target curve to their respective projection on the approximation curve, and the maximum error

$$\varepsilon_{max} = \max_{1 \leq k \leq n} d_k \quad (4)$$

gives the largest of these distances.

As a sample of these experiments, Fig. 1 illustrates the behaviour of the approximation error between two typical target curves, a piecewise linear (a) and a quadratically interpolated B-Spline curve (b), and their approximation constructed using a NURBS curve with 13 control points. Both target curves were specified using 143 control points. As can be observed from the graphs of Fig. 2, a very large number of evenly spaced discretisation points must be used to accurately compute both the mean and maximum approximation errors for the piecewise linear case. The behaviour for quadratic test cases and higher

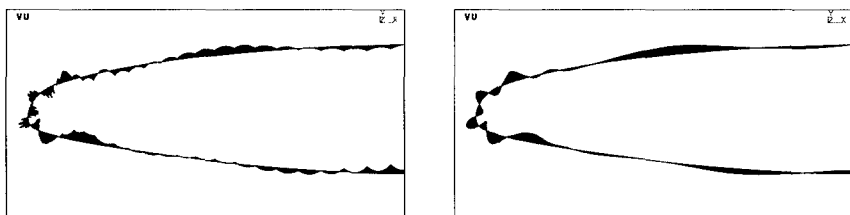


Fig. 1. Approximation error for piecewise linear (left) and quadratic (right) curve (magnified 100X).

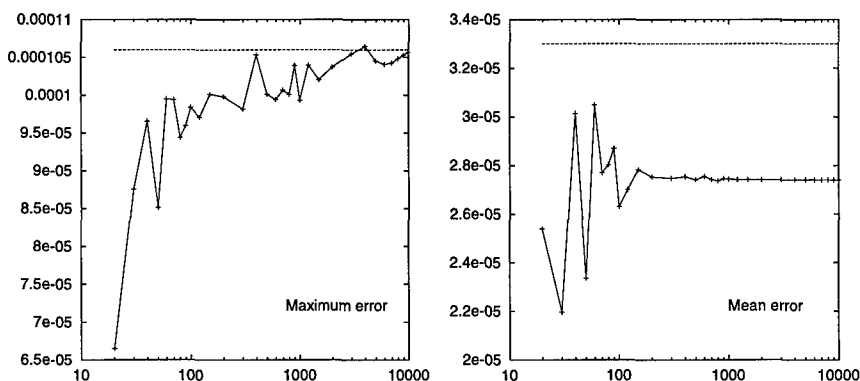


Fig. 2. Maximum (left) and mean(right) error as a function of the number of integration points for the piecewise linear target.

degree of continuity examples (not shown) are extremely similar. The same graphs of Fig. 2 also show, as a straight line, the error computed using only the definition control points of the target curve.

In light of these experiments, it was determined that the error computed using the control points constitutes an adequate bound on both the mean and maximum error of approximation, and it can be computed at a fraction of the cost of using evenly spaced discretization points. This method of computing the error also has the property that it includes naturally the case of target curves given as a discrete set of points, which is not a rare case in many practical applications.

§3. Optimization Method

Using these definitions and computational method of the approximation error, the optimization problem can be further specified by introducing a cost function of the form

$$F(\mathbf{X}) = 2 \times \varepsilon_{\text{mea}} + \varepsilon_{\text{max}},$$

where $\mathbf{X}$ is the vector of design variables, in this case the position and weights of the approximation curve: $\mathbf{X} = \{x_1, y_1, \omega_1, x_2, \dots, x_n, y_n, \omega_n\}$. This choice

of a cost function significantly accelerates convergence of the optimization process by including both the maximum error, which controls the quality of the final approximation, and the mean error, which globally compares the quality of different solutions.

Clearly, this is a non-linear optimization problem, and we will now examine the chosen solution process, including the choice of an initial solution.

Solution method

The primary solution method used was the second-order quasi-Newton method, which, given a reasonably close initial solution $\mathbf{X}_0$, will iteratively converge towards an optimal solution using the relation

$$\mathbf{X}_{k+1} = \mathbf{X}_k + \alpha_k \mathbf{S}_k,$$

where $\mathbf{S}_k = -\mathbf{H}_k \cdot \nabla F(\mathbf{X}_k)$ is the direction of descent vector, and α_k the distance of descent in direction $\mathbf{S}_k$. The descent vector is computed using the BFGS [6] algorithm, based on a second order approximation of the gradient of $F(\mathbf{X})$:

$$\nabla F(\mathbf{X}) \simeq \nabla F(\mathbf{X}_k) + \mathbf{H}(\mathbf{X}_k) \cdot \delta \mathbf{X},$$

where $\delta \mathbf{X} = \mathbf{X} - \mathbf{X}_k$ is used as the direction of descent vector ($\mathbf{S}_k$). Here $\mathbf{H}$, the approximate Hessian matrix, is initially set to identity and iteratively updated using the relation

$$\mathbf{H}_{k+1} = \mathbf{H}_k + \frac{\mathbf{Y}_k \otimes \mathbf{Y}_k}{\mathbf{Y}_k \cdot \mathbf{S}_k} - \frac{(\mathbf{H}_k \cdot \mathbf{S}_k) \otimes (\mathbf{H}_k \cdot \mathbf{S}_k)}{\mathbf{S}_k \cdot \mathbf{H}_k \cdot \mathbf{S}_k}$$

with $\mathbf{Y}_k = \nabla F(\mathbf{X}_{k+1}) - \nabla F(\mathbf{X}_k)$. The distance of descent is computed using Armijo's rule [5], where $\alpha_k = (\frac{1}{2})^m$ and m is the smallest integer such that the relationship

$$F(\mathbf{X}_k + \alpha_k \mathbf{S}_k) \leq F(\mathbf{X}_k) + \sigma \alpha_k \nabla F(\mathbf{X}_k) \cdot \mathbf{S}_k$$

with σ the sufficient descent criterion, which must be chosen between 0 and $\frac{1}{2}$ (usually set to 10^{-4}).

Initial solution

In most cases, the optimization method described above will find a solution, but in the case of highly non-linear cost functions such as the one used in this problem, it is impossible to determine whether the minimum found is the global minimum or only a local one. The only way to circumvent this difficulty is to proceed with many optimizations, and select the best minimum as the solution. While this approach could be unaffordable if no clue were available about the solution, it can be implemented relatively cheaply in the context of curve approximation, where many good initial guesses can be constructed.

Specifically, a set of initial solutions is constructed by discretizing the target curve using a fixed number of points and by varying the concentration of

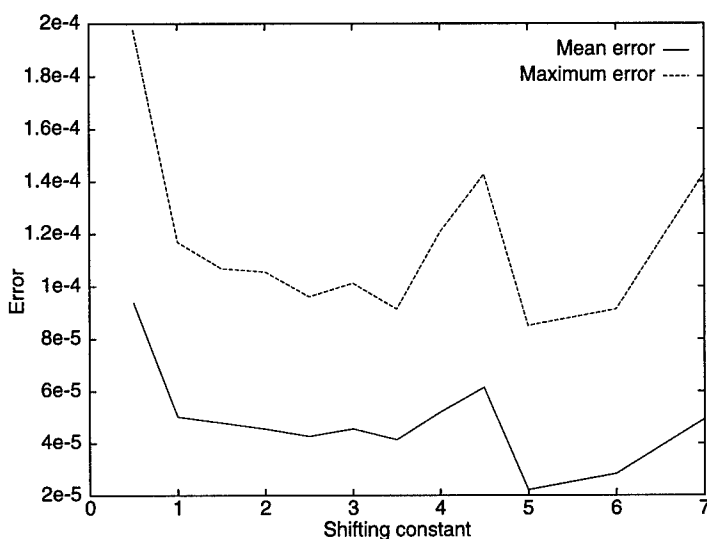


Fig. 3. Optimal Approximation error for various shifting constant values.

points along the curve. Basically, points are gathered closer together in regions of high curvature, and a shifting constant is introduced to construct various concentration laws. For target curves of continuity less than C^2 , curvature is approximated using centered differencing. The concentration law is evaluated using

$$\mathcal{F}(u) = \frac{1}{K} \int_0^1 C(v) + D \, dv,$$

where $C(v)$ is the true or approximated curvature of the target curve, and D the shifting constant.

When the shifting constant becomes large, the concentration law becomes almost uniform. In practice, sets of 8 to 10 initial solutions are constructed by varying D typically between 1.0 and 10, and each initial solution is then optimized. Fig. 3 shows the final approximation error for a run where D took the values $\{0.5, 1.0, 2.5, 3.0, 3.5, 4.5, 5.0, 6.0, 7.0\}$. The target curve for this problem is a standard NACA 2412 wing profile, and 9 control points are used for the approximation, which leads to a 21 parameter optimization problem (the two endpoints are fixed). Initial weights are all set to 1.0.

Fig. 3 vividly illustrates the high non-linearity of the problem, where small variations in the initial solution lead to completely different optimal solutions, as expressed, for example, by the steep variation in final error for $D = 4.5$ and $D = 5.0$.

§4. Application

We will now look at how this approximation method performs in the context of a practical application, both in terms of data reduction and approximation

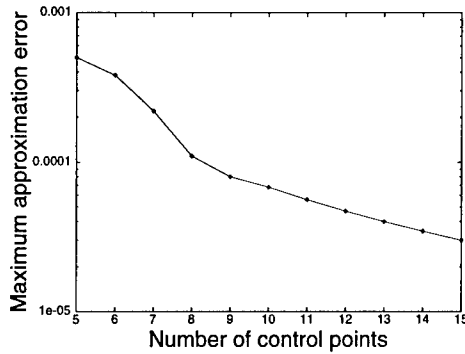


Fig. 4. Precision of the approximation as the number of control points increases.

characteristics. Two aspects of the approximation are of particular interest: precision and noise level.

The application consists in approximating wing profiles, specified either as analytical functions or experimental sets of points. In this context, study [3] of the combined precision levels dictated by both manufacturing tolerances and precision for analysis purposes indicates that a precision of the order of 8×10^{-5} is sufficient.

Precision

Fig. 4 illustrates the evolution of ε_{max} as the number of control points of the approximation curve is increased. Again the target curve is the NACA 2412 profile. As can be observed, the increase in precision of the approximation is very regular when 8 control points or more are used. In this case, the required precision of 8×10^{-5} is obtained with only 9 control points. Extensive experiments [3] involving numerous types of wing profiles have shown that the required level of precision can almost always be attained with 13 control points or less. These numbers have to be compared with the number of points needed to discretely represent a profile with the same precision, which can be shown to be of the order of 150. The approximation method thus offers excellent control over the precision of the resulting curve, while reducing by more than an order of magnitude the amount of data used for representation.

Experiments have also been carried out in order to determine whether the introduction of weights in the formulation had an impact on precision. Similar precision tests carried out using B-Splines instead of NURBS have shown that exactly 1.5 more control points were required for B-Splines to obtain comparable levels of precision. This increase in the number of control points, however, has a significant impact on noise level.

Noise

For many engineering applications, such as airplane wing design, noise level is often a bigger concern than absolute precision level. In that respect, the NURBS approximation method performs remarkably well, mainly because

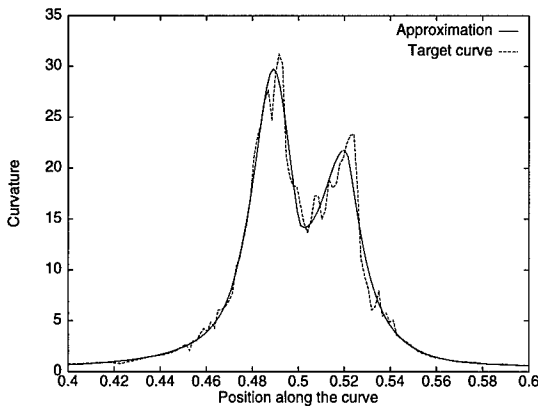


Fig. 5. Curvature of the Boeing A8 profile and of its approximation near the leading edge.

of the small number of control points needed. Noise appears as small fluctuations in the curvature of a curve, particularly when control points are gathered closely together in regions of high curvature. As shown in Fig. 5, the proposed approximation method can significantly reduce noise in cases where the target curve presents important fluctuations. Of course, this reduction of the noise level can only be accomplished as a trade-off to the precision of the approximation. For example, the precision of the 13 control point approximation of the Boeing A8 profile of Fig. 5 is 9.2×10^{-5} , which is slightly above the usual tolerance level for this application; better precision could be obtained by including a few more control points, but this would inevitably introduce more noise.

§5. Conclusion

We have presented a method of approximation for a general planar curve that permits a significant reduction in the size of data and guarantees a desired level of precision. The main advantages of this approach are

- generality,
- full automation,
- low noise.

By varying a single parameter, the shifting constant in the construction of the concentration law, as many initial solutions as needed are generated for a given number of points. Each solution is optimized independently, and the solution with minimum error is kept. Because of the typically small number of control points required – of the order of 10 to 15 – very smooth curves are obtained, which is a very important characteristic for many engineering applications.

Because of the significant reduction in the number of free parameters used to represent a curve, the approximation method is now being used as

a first step in a shape optimization procedure of wing profiles. This work is also currently being extended to three dimensional cases, where the method is now used to approximate wing surfaces.

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