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Supercavitating Flows: Small Perturbation Theory and Matched Asymptotics

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Summary: The lecture discusses some applications of the theory of small perturbations as applied to supercavitating flows. In this context the linear theory is viewed as an outer expansion of a more complete nonlinear solution of the flow problem. In addition to comparing linear solutions for a supercavitating flat plate for different analogues of the cavity closure models, two examples are considered showing how to account for the presence of local flow regions where the perturbations are not small. In the first example a local asymptotic solution of the nonlinear flow problem in the vicinity of the leading edge is matched to the classical linear solution to provide a uniformly valid pressure distribution along a supercavitating flat plate. In the second example, the local nonlinear perturbation of the otherwise slightly perturbed flow is due to a spoiler fitted at the trailing edge of a flat plate.

1. Introduction

In a nonlinear steady 2-D problem formulation for a potential flow past a supercavitating body, one has to solve Laplace equation in the domain occupied by the fluid with the following conditions on the flow boundaries

- Slip condition on the wetted part of the body

$$\frac{\partial\varphi}{\partial x} = \frac{\mathrm{d}y}{\mathrm{d}x}\Big(1 + \frac{\partial\varphi}{\partial x}\Big), \qquad y = y_b(x), \tag{1}$$

  where φ is perturbation velocity potential, $y = y_b(x)$ is a function, describing the contour of the wetted part of the body in Cartesian coordinate system, x-axis being directed downstream. Note, that here all quantities and functions are non-dimensionalized with use of the characteristic length of the body and the velocity of the oncoming flow.

- Dynamic condition on the cavity

  The pressure on the boundary of the cavity is assumed constant wherefrom the corresponding pressure coefficient C_p should be taken equal in magnitude and opposite in sign to the cavitation number σ

$$C_p = C_{p_c} = 1 - (1 + \frac{\partial\varphi}{\partial x})^2 - \Big(\frac{\partial\varphi}{\partial y}\Big)^2 = -\sigma, \qquad y = y_c(x) \tag{2}$$

  where $y = y_c(x)$ describes the cavity contour, determined in the course of the problem solution.

- Kinematic condition on the cavity

  The cavity contour is a streamline, and, therefore, it should be subject to a slip condition, identical to (1).

- At the infinity the perturbation velocities should vanish.

In what follows we first consider a steady linearized flow problem for a supercavitating foil[1] using different linear analogues of the cavity closure schemes. Used in particular are the analogues of the *closed* cavity termination models (Riaboushinsky model, Efros-Gilbarg model, Tulin single-spiral vortex model), as well as those of Wu-Fabula and Tulin double-spiral vortex termination models. Then two examples are presented showing how the linearized (outer) solution can be supplemented by a nonlinear local (inner) solution in the flow regions when the perturbations are not necessarily small. These examples include: a uniformly valid solution of a flow problem for a flat plate at an angle of attack (zero cavitation number) and of that for a plate with a spoiler at the trailing edge. Both examples employ the method of matched asymptotic expansions (MAE), [2]. This technique consists in finding a local (inner) solution in the appropriately stretched coordinates, blending it smoothly to the outer (linear) solution, and, eventually obtaining a uniformly valid solution by additive composition of the inner and outer solutions.

[1]The term "supercavitating", as understood here, implies that the cavity extends beyond the trailing edge of the foil

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2. Linear solution of the problem of a supercavitating flow past a thin foil with different closure schemes

Consider a linear problem of a steady supercavitating flow past a flat plate with different closure schemes. In what follows all quantities and functions are rendered nondimensional with use of the chord c of the plate and the velocity of the incoming flow U_o. To avoid complicated derivations when explaining the essential points of the lecture, one assumes that the cavity detaches from a sharp leading edge of the supercavitating foil of zero thickness. In most of the examples, discussed herein, the foil is represented by a flat plate at at an angle of attack.
Assume that the flow perturbations are small. Expanding previous nonlinear formulation, one easily shows that the linear flow problem for the perturbation velocity potential $\varphi = \varphi(x, y)$ is governed by the following equations

- Laplace equation

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 0, \qquad (x, y) \in \mathcal{D}; \tag{3}$$

- Flow tangency condition on the wetted part of the foil

$$\frac{\partial\varphi}{\partial y} = \frac{\mathrm{d}y_o}{\mathrm{d}x}, \quad y = 0 - 0, \quad x \in (0, 1), \tag{4}$$

or, for a flat plate $y_o(x) = -\alpha x$,

$$\frac{\partial\varphi}{\partial y} = -\alpha, \quad y = 0 - 0, \quad x \in (0, 1),$$

where α is angle of attack (in radians);

- On the boundary of the cavity
As the linearized pressure coefficient is approximately equal to

$$C_p = \frac{2(p - p_o)}{\rho U_o^2} \approx -2\frac{\partial\varphi}{\partial x}, \tag{5}$$

and the cavitation number σ is defined as

$$\sigma = \frac{2(p_c - p_o)}{\rho U_o^2}, \tag{6}$$

where p_c and p_o are pressures in the cavity and at the upstream infinity. Then, the perturbed horizontal velocity on the boundary of the cavity should be

$$\frac{\partial\varphi}{\partial x} = \frac{1}{2}\sigma, \quad y = 0 \pm 0, \quad x \in (0, l) \quad \text{and} \quad x \in (1, l), \tag{7}$$

where l is the cavity length measured from the leading edge of the foil and related to the chord of the foil;

- Condition at infinity

$$\nabla\varphi = \vec{i}\,\frac{\partial\varphi}{\partial x} + \vec{j}\,\frac{\partial\varphi}{\partial y} \to 0, \quad \text{for} \quad x^2 + y^2 \to \infty, \tag{8}$$

where $\vec{i}$ and $\vec{j}$ are unit vectors of the axes x and y correspondingly.

It should be noted that the linearization implies that both α and σ tend to zero at the same speed, i.e. $\sigma = \mathrm{O}(\alpha)$. In order to use powerful methods of the complex functions' theory it is convenient to re-write the flow problem formulation, described previously, in terms of complex variables and functions. Instead of the perturbation velocity potential $\varphi = \varphi(x, y)$, introduce a complex potential $F = F(z) = \varphi(x, y) + \mathrm{i}\,\psi(x, y)$, where $\mathrm{i} = \sqrt{-1}$ is an imaginary unit, $z = x + \mathrm{i}\,y$ and $\psi(x, y)$ is a stream function. In this case the solution of the problem is reduced to finding an analytic function $\mathrm{d}F/\mathrm{d}z = w(z) = u(x, y) - \mathrm{i}\,v(x, y)$ in the domain $\mathcal{D}$ of the flow with the following boundary conditions on the flow boundaries

- On the wetted part of the plate

$$\Im w(z) = -v(x, y) = \alpha, \quad y = 0 - 0, \quad x \in (0, 1); \tag{9}$$

- On the cavity
$$\Re w(z) = u(x,y) = \frac{1}{2}\sigma, \quad y = 0 \pm 0, \quad x \in (0,l) \quad \text{and} \quad x \in (1,l); \tag{10}$$
- At infinity
$$w(z) \to 0, \quad \text{for} \quad z \to \infty. \tag{11}$$

2.1. Cavity closure models with a square root singularity

In the nonlinear formulation this type of the cavity closure models is characterized by presence of a stagnation point in the region of cavity termination, i.e. that of Riaboushinsky and Efros-Gilbarg (see Fig. 1(1) and Fig. 1 (2)). As discussed in [1] a linear analogue for all such models contains a square root singularity near the point of the cavity termination. Consider a corresponding flow problem solution following [1].
The linearized "physical" complex plane z of the flow past a foil with a trailing cavity of finite length is illustrated in Fig. 2. Note that in this plane *the plate plus the cavity* are represented by a slit $y = 0 \pm 0, x \in [0,l]$. To solve the problem, map the exterior of the slit in the complex plane z onto the upper half of the auxiliary complex plane $\zeta = \xi + \mathrm{i}\,\eta$ (see Fig. 2) with help of the function

$$\zeta = -\mathrm{i}a\sqrt{\frac{z}{z-l}}, \qquad a = \sqrt{l-1}. \tag{12}$$

The inverse function, i.e. $z = z(\zeta)$, can be written as

$$z = \frac{l\zeta^2}{\zeta^2 + a^2}. \tag{13}$$

The correspondence of the points in z and ζ planes can be seen from Fig. 2. Note that the point $z = l$, which represents the closure point of the cavity in the "physical" plane, passes into a point $\zeta = \infty$ in the auxiliary halfplane $\Im\zeta = \xi < 0$. In fact, for $z \to l$

$$\zeta \approx -\mathrm{i}a\sqrt{\frac{l}{z-l}} \to \infty. \tag{14}$$

On the other hand, the infinity of the "physical" plane, i.e. $z = \infty$ passes into the point $\zeta = -\mathrm{i}a$. Eventually, the wetted part of the slip coincides with the interval $\xi \in (0,1)$ whereas the upper and the lower boundary of the cavity in z plane are mapped onto the negative $\eta = 0-0, \xi < 0$ and positive $\eta = 0-0, \xi \in (1,\infty)$ parts of the real axis in the auxiliary plane.
In the context of the closure models it is important to specify the anticipated behavior of the complex conjugate velocity $w(\zeta)$ in the vicinity of the cavity closure point. It can be shown that the linear analogues of the nonlinear closure models, containing a stagnation point within the cavity termination zone, are characterized by a square root singularity of the perturbed velocity at the corresponding closure point (i.e. at $z = l$). Assuming that

$$\text{for} \quad z \to l \quad w(z) \approx \frac{\kappa}{\sqrt{z-l}}, \tag{15}$$

Accounting for the expression (14), it means that in the auxiliary plane

$$w[z(\zeta)] \approx -\frac{\kappa\zeta}{\mathrm{i}a\sqrt{l}} = \mathrm{i}A\zeta, \tag{16}$$

where A is a real constant to be determined.
It is convenient to solve the boundary problem for the function

$$w^*(z) = w(z) - \frac{1}{2}\sigma, \tag{17}$$

For this function $\Re w^*(z) = 0$ on the boundary of the cavity, i.e. on $\eta = 0-0$, $\xi < 0$ and $1 < \xi < \infty$. The latter fact simplifies the final expression for the solution. Note that the boundary conditions for $w^*(\zeta)$ on the real axis of the auxiliary complex plane ζ are *mixed*, prescribing $\Im w^*(\zeta)$ on one part of the axis and $\Re w^*(\zeta)$ on another part of the axis, see Fig. 2.
Assuming a quarter-root singularity $w(z) = \mathrm{O}(z^{-1/4})$ at the leading edge and square root zero $w(z) = \mathrm{O}[(1-z)^{1/2}]$ at the trailing edge, and using Keldysh-Sedov formula, one can derive the solution for the complex conjugate velocity in the form

$$w(\zeta) = \frac{1}{2}\sigma + \frac{\alpha}{\pi}\sqrt{\frac{1-\zeta}{\zeta}}\int_0^1 \sqrt{\frac{\xi}{1-\xi}}\frac{\mathrm{d}\xi}{\zeta-\xi}\mathrm{d}\xi + B\sqrt{\frac{1-\zeta}{\zeta}} + A\sqrt{\zeta(1-\zeta)}. \tag{18}$$

Note that B and A are real constants to be determined, and the last term in the formula is a permissible solution accounting for the required behaviour of the function $w(\zeta)$ for $\zeta \to iA\zeta$.
After integration, one can obtain

$$w(\zeta) = \frac{1}{2}\sigma + (B - \alpha)\sqrt{\frac{1-\zeta}{\zeta}} + A\sqrt{\zeta(1-\zeta)}. \tag{19}$$

The solution thus obtained contains three unknown (real) parameters a, B and A. The first relationship between these parameters can be obtained by applying the infinity condition (8). For $\zeta \to -ia$ the conjugate complex velocity should equal zero, i.e.

$$w(-ia) = 0, \tag{20}$$

The latter complex equation is equivalent to the two real equations

$$\Re w(-ia) = 0, \qquad \Im w(-ia) = 0. \tag{21}$$

The remaining relationship necessary to determine all three unknowns of the solution can be obtained from the closure condition, which, in other words, is a requirement that the cavity should have finite length. The corresponding equation can be written as

$$\oint_{\mathcal{L}_z} \Im[w(z)]\,\mathrm{d}z = \Im \oint_{\mathcal{L}_z} w(z)\,\mathrm{d}z = \Im \oint_{\mathcal{L}_\zeta} w(\zeta)\frac{\mathrm{d}z}{\mathrm{d}\zeta}\,\mathrm{d}\zeta = 0, \tag{22}$$

where contour $\mathcal{L}_z$ encloses the slit $y = 0 \pm 0$, $x \in (0, l)$ and is passed in the clockwise direction, and the contour $\mathcal{L}_\zeta$ encloses the point $\zeta = -ia$ and is passed in the clockwise direction. The derivative $\mathrm{d}z/\mathrm{d}\zeta$ can be derived from the expression (13) in the form

$$\frac{\mathrm{d}z}{\mathrm{d}\zeta} = \frac{2l\zeta}{\zeta^2 + a^2} - \frac{2l\zeta^3}{(\zeta^2 + a^2)^2}. \tag{23}$$

Treating the equation (22) is reduced to calculation of the imaginary part of the residue of the function $w(\zeta)\mathrm{d}z/\mathrm{d}\zeta$ at the point $\zeta = -ia$, i.e.

$$\Im \oint_{\mathcal{L}_\zeta} w(\zeta)\frac{\mathrm{d}z}{\mathrm{d}\zeta}\,\mathrm{d}\zeta = \Im\Big\{2\pi i \,\mathrm{Res}\Big[w(\zeta)\frac{\mathrm{d}z}{\mathrm{d}\zeta}\Big]\Big|_{\zeta=-ia}\Big\}. \tag{24}$$

Calculating the residue with use of the Taylor series of $w(\zeta)$ in the vicinity of $\zeta = -ia$

$$w(\zeta) = \frac{\mathrm{d}w}{\mathrm{d}\zeta}\Big|_{\zeta=-ia}(\zeta + ia) + \mathrm{O}[(\zeta + ia)^2] \tag{25}$$

and the expression for the derivative of the inverse mapping function $z = z(\zeta)$, one comes to the following form of the equation (22)

$$\Im\Big\{\frac{\mathrm{d}w}{\mathrm{d}\zeta}\Big|_{\zeta=-ia}\Big\} = 0. \tag{26}$$

The lift coefficient can be derived by integrating the pressure over the slit in the "physical" plane, so that

$$C_y = \frac{2Y}{\rho U_o^2 c} = 2\Re \oint_{\mathcal{L}_z} w(z)\,\mathrm{d}z = 2\Re \oint_{\mathcal{L}_\zeta} w(\zeta)\frac{\mathrm{d}z}{\mathrm{d}\zeta}\,\mathrm{d}\zeta. \tag{27}$$

Similarly to the previous derivations leading to the equation (22), the calculation of the lift coefficient is reduced to that of the real part of the residue of the function $w(\zeta)\mathrm{d}z/\mathrm{d}\zeta$, which yields

$$C_y = 2\pi la \Re\Big\{\frac{\mathrm{d}w}{\mathrm{d}\zeta}\Big|_{\zeta=-ia}\Big\} \tag{28}$$

The results obtained for the case of a flat plate at an angle of attack α have quite a simple form. After some algebra the expression for the lift coefficient becomes

$$Cy = \pi\alpha l\Big(\frac{\sqrt{l}}{\sqrt{l-1}} - 1\Big), \tag{29}$$

The relationship between cavitation number and length of the cavity is given by the formula

$$\bar{\sigma} = \frac{\sigma}{\alpha} = \frac{2}{a} = \frac{2}{\sqrt{l-1}}. \tag{30}$$

Note that both of the proceeding formulae imply that $l > 1$. Using the expressions, describing parametric dependences of the lift coefficient and the ratio σ/α on the nondimensional length of the cavity l, one can obtain the following formula

$$\frac{C_y}{\alpha} = \pi\Big(\sqrt{\frac{4}{\bar\sigma^2}+1}-\frac{2}{\bar\sigma}\Big)\Big(\frac{\bar\sigma}{2}+\frac{2}{\bar\sigma}\Big) \tag{31}$$

2.2. Wu-Fabula closure model

The solution, corresponding to Wu-Fabula cavity closure model (see Fig. 1(4)), can be easily derived from the previous solution by requiring a "smooth" termination at the point $z = l$, or $\zeta = \infty$. This is achieved by putting the real constant A equal to zero. The corresponding linearized "physical" plane is shown in Fig. 3. The resulting expressions for the lift coefficient and the ratio $\bar\sigma = \sigma/\alpha$ versus nondimensional length of the cavity are given by the formulae

$$\frac{C_y}{\alpha} = \frac{\pi\sqrt{l}}{\sqrt{l}+\sqrt{l-1}}, \tag{32}$$

$$\bar\sigma = \frac{\sigma}{\alpha} = \frac{2}{\sqrt{l}+\sqrt{l-1}}. \tag{33}$$

2.3. Tulin double spiral-vortex cavity closure model

In the double spiral-vortex model introduced by Tulin (Fig. 1(3)) it is assumed that the cavity termination is followed by a wake extending to the downstream infinity, [1]. Therewith, the cavity pressure turns abruptly into that of unperturbed flow. The corresponding linearized flow problem formulation is shown in Fig. 4. Note, that the linearization implies that the abscissas of the cavity termination points on the upper and the lower "banks" of the cut are identical. To formulate the relevant boundary problem for the perturbed conjugate complex velocity, map the exterior of the semi-infinite cut in the "physical" plane z onto the lower half plane $\Im\zeta = \xi < 0$ of the auxiliary plane ζ. This can be done by means of the function

$$\zeta = -\sqrt{z} \tag{34}$$

where the selected branch of the root transfers the point $z = 1 + \mathrm{i}(0-0)$ into the point $\zeta = -1 + \mathrm{i}(0+0)$. The correspondence of the points in z and ζ complex planes is indicated in Fig. 4. Note, that the termination points of the cavity $z = l + \mathrm{i}(0 \mp 0)$ are mapped respectingly into the points $\zeta = \pm b$, where $b = \sqrt{l}$. The boundary conditions for the conjugate complex velocity $w(\zeta)$ on the real axis $\Re\zeta = \xi$ of the auxiliary plane are shown in Fig. 4. Using Keldysh-Sedov formulae to solve this mixed boundary problem, one can derive the following expression

$$w(\zeta) = \frac{\alpha}{\pi}\sqrt{\frac{1-\zeta}{\zeta}}\int_0^1\sqrt{\frac{\xi}{1-\xi}}\frac{\mathrm{d}\xi}{\zeta-\xi}+$$

$$\frac{\sigma}{2\pi}\sqrt{\frac{1-\zeta}{\zeta}}\Big(\int_1^b+\int_{-b}^0\Big)\sqrt{\frac{\xi}{\xi-1}}\frac{\mathrm{d}\xi}{\zeta-\xi}+B\sqrt{\frac{1-\zeta}{\zeta}}, \tag{35}$$

where B is a real constant to be determined. Integrating, one obtains

$$w(\zeta) = (B-\alpha)\sqrt{\frac{1-\zeta}{\zeta}}+\frac{\sigma}{2\pi}\sqrt{\frac{1-\zeta}{\zeta}}\Big[\mathcal{F}(\zeta,\xi)\Big|_1^b+\mathcal{F}(\zeta,\xi)\Big|_{-b}^0\Big]+\mathrm{i}\alpha, \tag{36}$$

where

$$\mathcal{F}(\zeta,\xi) = \int\sqrt{\frac{\xi}{\xi-1}}\frac{\mathrm{d}\xi}{\zeta-\xi} = -2\ln\frac{\sqrt{\xi}+\sqrt{\xi-1}}{\sqrt{2}}+$$

$$2\sqrt{\frac{\zeta}{\zeta-1}}\ln\frac{\sqrt{\zeta(\xi-1)}+\sqrt{\xi(\zeta-1)}}{\sqrt{\zeta-\xi}} \tag{37}$$

Requiring that the perturbations vanish at the infinity, one obtains $B = 0$. After some additional algebra the expression for $w(\zeta)$ acquires the following form

$$w(\zeta) = \mathrm{i}\alpha\Big(1-\sqrt{\frac{\zeta-1}{\zeta}}\Big)-\frac{\mathrm{i}\sigma}{2\pi}\Big[2\sqrt{\frac{\zeta-1}{\zeta}}\ln\frac{\sqrt{b}+\sqrt{b-1}}{\sqrt{b}+\sqrt{b+1}}+$$

$$\ln\frac{\zeta+b}{\zeta-b}+2\ln\frac{\sqrt{\zeta(b-1)}+\sqrt{b(\zeta-1)}}{\sqrt{\zeta(b+1)}+\sqrt{b(\zeta-1)}}\Big] \tag{38}$$

Finally, the expression for the lift coefficient is obtained as

$$\frac{Cy}{\alpha} = \bar{\sigma} b\sqrt{b}(\sqrt{b+1} - \sqrt{b-1}), \quad \bar{\sigma} = \frac{\sigma}{\alpha} \tag{39}$$

where $b = \sqrt{l}$. The ratio $\bar{\sigma}$ versus l is described by the following formula

$$\frac{1}{\bar{\sigma}} = \frac{\alpha}{\sigma} = \frac{1}{\pi}\Big[\sqrt{b}(\sqrt{b-1} + \sqrt{b+1}) - \ln\frac{\sqrt{b} + \sqrt{b+1}}{\sqrt{b} + \sqrt{b-1}}\Big] \tag{40}$$

The previous two expressions can be viewed as a parametric dependence of the lift coefficient upon cavitation number and angle of attack, or rather as C_y/α as a function of σ/α for the linear supercavitating flow past a flat plate with a Tulin double spiral-vortex cavity closure scheme. Note that in order to find the relationship between the cavitation number σ and the length l of the cavity, it was assumed herein that the total drag of the cavity+wake system should be equal to zero. The latter statement can be formally expressed as

$$\Im \oint\limits_{\zeta=\infty} w^2(\zeta)\frac{\mathrm{d}z}{\mathrm{d}\zeta}\,\mathrm{d}\zeta = 0. \tag{41}$$

The corresponding calculation is reduced to finding the resudue of the integrand at the infinity in the auxiliary plane ζ.
Plotted in Fig. 5 are curves of lift coefficient related to angle of attack versus cavitation number related to angle of attack, i.e. $C_y/\alpha = f(\sigma/\alpha)$ for different linear analogues of the cavity closure scheme for the case of a flat plate. Presented in the Figure are the results:

1. For Tulin open closure scheme featuring two double-spiral vortices,
2. For a cavity scheme with a square-root singularity of perturbation velocity at the cavity termination points, and
3. Wu-Fabula cavity closure scheme (no singularity at the cavity termination point)

Plotted in the same Figure are experimental points, obtained by F.F. Bolotin and E.B. Anoufriev for a series of segment foils with different relative thickness 4.2, 5 and 6%, [3]. It should be noted that in the case of the developed cavitation starting from the leading edge, the flow past the segment foil is equivalent to that of the flat plate. It follows from the comparison of test data with the calculated results that the correlation is fair not only for the case of developed cavitation (long cavities), but also for the transitional regime when $1 \le l \le 1.5$. Tulin double-spiral vortex scheme provides satisfactory results for the lift coefficient of flat plate up to $\sigma/\alpha \approx 6$, i.e. practically up the boundary of supercavitation. The experimental points are seen to be located between the schemes 1) and 2). Wu-Fabula scheme allows to obtain the magnitudes of C_y/α only for $\sigma/\alpha < 2$, i.e. for sufficiently long cavities. For σ/α close to 2, Wu-Fabula scheme gives somewhat excessive magnitudes of the lift coefficient.

2.4. The case of zero cavitation number - analogy with a fully wetted foil.

The simplest albeit practical case corresponds to zero cavitation number. Therewith the cavity becomes semi-infinite. The "physical" flow domain in z-plane is transformed onto an auxiliary lower half-plane $\Im\zeta = \eta < 0$ with the mapping function $\zeta = -\sqrt{z}$, discussed previously. As before, one can easily construct the solution of the ensuing boundary problem for $w(\zeta)$ with use of the Keldysh-Sedov formula. Restricting the analysis to the case of a flat plate, one comes to the following simple formula

$$w(\zeta) = \alpha\Big(\mathrm{i} - \sqrt{\frac{1-\zeta}{\zeta}}\Big), \tag{42}$$

which satisfies the requirement that the perturbed velocity should vanish at the infinity. The latter property of (42) can be easily verified. In fact for large $z \to \infty, \zeta \to \infty$

$$\sqrt{\frac{1-\zeta}{\zeta}} = \mathrm{i}\Big[1 - \frac{1}{2\zeta} + \mathrm{O}\Big(\frac{1}{\zeta^2}\Big)\Big], \quad w(\zeta) \sim \frac{\mathrm{i}}{2\zeta} \to 0. \tag{43}$$

In particular, the perturbed velocity distribution on the wetted (lower) side of the plate ($\zeta \to \xi - \mathrm{i}0, \xi \in (0,1)$) becomes

$$u(\xi, 0-0) = -\alpha\sqrt{\frac{1-\xi}{\xi}}, \tag{44}$$

Referring to the classical theory of a thin foil, one can conclude that the above result coincides with the perturbed velocity distribution on the lower side of a flat plate placed in a uniform flow in ζ-plane. This suggests a concept of an equivalence, existing between the original supercavitating flow in z plane and a fictitious non-cavitating (fully wetted) flow in ζ-plane, which was first indicated by Tulin.
It should be noted that since the fluid domain has been mapped entirely into the lower part of the ζ-plane, the region $\Im\zeta = \eta > 0$ has no physical significance. Mathematically, it can be regarded as the second Riemann sheet of the z-plane, or as the domain of the cavity. Owing to the fact that on the boundary of the cavity in ζ plane the real part of the conjugate complex velocity $w(\zeta)$ is zero, one can use the Schwartz reflection principle to continue $w(\zeta)$ across the real axis $\Re\zeta = \xi$ (excluding the cut $\xi \in (0,1)$), provided u is an odd function of η. To ensure that $w(\zeta) = u(\xi,\eta) + \mathrm{i}v(\xi,\eta)$ is analytic in the entire ζ-plane, it follows from the Cauchy-Riemann equations that v should be an even function of η. Thus, the appropriate reflections into the upper half-plane of ζ are given by

$$u(\xi,\eta) = -u(\xi,-\eta), \tag{45}$$

$$v(\xi,\eta) = v(\xi,-\eta) \tag{46}$$

The reflected boundary-value problem shown is mathematically identical to the lifting problem of a fully wetted thin foil. In both cases a Kutta-Zhukovsky condition should be imposed at the trailing edge, and the perturbation velocity should vanish at the infinity. *Thus, the solution of the supercavitating flow problem ($\sigma = 0$) in the complex plane z written in auxiliary complex variable ζ is identical to that of a fully wetted foil in ζ-plane.* This analogy enables to obtain the coefficients of hydrodynamic forces and moments of the supercavitating foil from those of a fictitious fully wetted foil. For example, to obtain the lift coefficient C_y of the supecavitating flat plate, one has to integrate the pressure coefficient along the wetted part of the plate, i.e.

$$C_y = \frac{2Y}{\rho U^2 c} = \int_0^1 C_p(x, 0-0)\,\mathrm{d}x = -2\int_0^1 u(x, 0-0)\,\mathrm{d}x. \tag{47}$$

Passing over to the integration in ζ-plane ($x = \xi^2$), one derives from the previous line

$$C_y = -2\int_0^1 u(x,0-0)\,\mathrm{d}x = -2\int_0^1 u(\xi,0-0)\frac{\mathrm{d}x}{\mathrm{d}\xi}\,\mathrm{d}\xi == -4\int_0^1 u(\xi,0-0)\xi\,\mathrm{d}\xi \tag{48}$$

Accounting for the adopted reflection of u into the upper half-plane ζ, one can re-write (48) in the following way

$$C_y = -4\int_0^1 u(\xi,0-0)\xi\,\mathrm{d}\xi = 2\int_0^1 [u(\xi,0+0) - u(\xi,0-0)]\xi\,\mathrm{d}\xi =$$

$$\int_0^1 [C_p(\xi,0-0) - C_p(\xi,0+0)]\xi\,\mathrm{d}\xi = C_{m_\zeta}, \tag{49}$$

where C_{m_ζ} is a coefficient of the longitudinal hydrodynamic moment of a fictitious fully wetted foil of length $c_\zeta = \sqrt{c}$ in ζ-plane. This coefficient, calculated with respect to the leading edge of the fictitious plate, is defined as

$$C_{m_\zeta} = \frac{2M_\zeta}{\rho U^2 c_\zeta^2} = \frac{2M_\zeta}{\rho U^2\sqrt{c}\sqrt{c}}. \tag{50}$$

Similarly, one can show that the coefficient of the longitudinal hydrodynamic moment of a supercavitating foil ($\sigma = 0$) is equal to the coefficient of the third moment of the fictitious fully wetted foil.

$$C_m = \int^1 C_p(x,0-0)x\,\mathrm{d}x = \int_0^1 [C_p(\xi,0-0) - C_p(\xi,0+0)]\xi^3\,\mathrm{d}\xi \tag{51}$$

It can also be shown that the drag coefficient for the supercavitating foil can be found as

$$C_x = \frac{1}{8\pi}C_{y_\zeta}^2 \tag{52}$$

This useful equivalence was first established by Tulin and Burkart in 1955. For a supercavitating flat plate ($\sigma = 0$) at a given angle of attack α the corresponding calculations give

$$C_y = \frac{\pi\alpha}{2}, \quad C_m = \frac{5\pi\alpha}{32}, \quad C_x = \frac{\pi\alpha^2}{2} \tag{53}$$

It is interesting that the lift coefficient for the supercavitating flat plate is four times less than that of the fully wetted plate and two times less that for the case of a flat plate gliding on the surface. In comparison to the fully wetted flat plate, here the centre of pressure is shifted from the quarter-chord point to a position 5/16 of the chord downstream from the leading edge. Note that the drag coefficient of the supercavitating plate is seen to be equal to $C_y\alpha$, i.e. the total force is normal to the plate. This is different from the fully wetted case, where, according to D'Alembert paradox, the vector of the force, acting upon the foil, is strictly vertical due to the suction force at the leading edge and the resulting drag is zero. In the supercavitating case there is no suction force, since the square root singularity of the equivalent fully-wetted flow in the ζ-plane is reduced to a quarter-root singularity in the physical z-plane.

3. Uniformly valid asymptotic solution of the problem for the flow past a supercavitating flat plate

The linear theory reveals a quarter-root singularity of the perturbation velocity at the leading edge of a supercavitating foil. This defficiency can be corrected by a special consideration of the local region of the flow near the leading edge, as first done in [4].

3.1. Linear solution as a leading order outer expansion

Consider the perturbation velocity distribution along the wetted (lower) side of the supercavitating ($\sigma = 0$) flat plate, following from (44). In the auxiliary variable $\Re\zeta = \xi \in (0, 1)$ the horizontal component of w equals to

$$u(\xi, 0-0) = \Re w(\zeta) = -\alpha\sqrt{\frac{1-\xi}{\xi}} \tag{54}$$

As follows from the mapping function $\zeta = -\sqrt{z}$ the real variables x and ξ are related to each other as

$$\xi = \mp\sqrt{x}, \quad \text{for} \quad y = 0 \pm 0 \tag{55}$$

so that on the wetted side of the supercavitating plate $x \in (0, 1), y = 0-0$ the expression (54) can be re-written in terms of "physical" x coordinate

$$u(x, 0-0) = -\alpha\sqrt{\frac{1-\sqrt{x}}{\sqrt{x}}} \tag{56}$$

This expression shows explicitly the quarter-root singularity of the perturbation velocity (pressure) in the vicinity of the leading edge of the wetted side of the supercavitating flat plate. In fact, for $x \to 0+0$

$$u(x, 0-0) = -\frac{1}{2}C_p(x, 0-0) \sim -\alpha x^{-1/4} \tag{57}$$

Considering this linear theory solution as an *outer* expansion of a complete solution of a nonlinear problem for a flow past a supercavitating flat plate for $\alpha \to 0$, i.e.

$$u(x, 0-0) = u^o(x, \alpha) = \alpha u_1^o(x) + \mathrm{O}(\alpha^2), \quad u_1^o(x) = \mathrm{O}(1) \tag{58}$$

one can note that this asymptotic expansion loses uniform validity for sufficiently small x when the product $\alpha u_1^o(x)$ acquires the order of O(1). This takes place at distances from the leading edge of the order of $x = \mathrm{O}(\alpha^4)$. For the purpose of further analysis, one would need to have the expression for the ordinates of the upper boundary of the cavity from the linear theory. Recalling that for $z = x + \mathrm{i}(0+0), x > 0$ the ξ and x coordinates are related as $\xi = -\sqrt{x}, \xi < 0$, one arrives at the following *outer* expression for the vertical component of the conjugate complex perturbation velocity

$$v(x, 0+0) = \Im w(x) = v^o(x, \alpha) = -\alpha\Big(1 - \sqrt{\frac{1+\sqrt{x}}{\sqrt{x}}}\Big) =$$

$$-\alpha + \alpha v_1^o(x, 0+0) + \mathrm{O}(\alpha^2), \quad v_1^o = \mathrm{O}(1) \tag{59}$$

The *outer* expansion for the ordinates of the upper boundary of the cavity, measured with respect to the plate can be determined with use of the following formula

$$y_c^o(x, 0+0) = \alpha\int_0^x v_1^o(x, 0+0)\,\mathrm{d}x + \mathrm{O}(\alpha^2) = \alpha y_1^o(x, 0+0) + \mathrm{O}(\alpha^2). \tag{60}$$

Integrating this expression with account of the formula for $v_1^o(x, 0+0)$ gives the following expression for the *outer* contour of the upper boundary of the cavity

$$y_c^o(x, 0+0) \sim \alpha y_1^o(x, 0+0) = \frac{\alpha}{2}[(1+2\sqrt{x})\sqrt{x+\sqrt{x}} - \ln(\sqrt{1+\sqrt{x}} + \sqrt{\sqrt{x}})] \tag{61}$$

3.2. Inner expansion in the vicinity of the leading edge

As follows from the preceeding estimates, the linear solution is not valid in the vicinity of the leading edge of the supercavitating plate, having dimensions of the order of $O(\alpha^4)$. As a result of this nonuniformity the pressure on the wetted side at the leading edge becomes unrealistically infinite. This also means that within the initial characteristic scale of the order of the length of the plate the stagnation point, located close to the leading edge is invisible. In order to the flow near this edge in more detail, introduce local (stretched) coordinates

$$X = \frac{x}{\alpha^4}, \quad Y = \frac{y}{\alpha^4}, \qquad Z = X + \mathrm{i}Y = \frac{z}{\alpha^4} \tag{62}$$

In these *inner* variables for $\alpha \to 0$ the training edge of the plate recedes to the infinity and the *inner* flow becomes as shown in Fig. 6. For zero cavitation number this flow is completely characterized by the distance s of the stagnation point from the leading edge of the plate. The inner problem is solved with use of the velocity hodograph method and conformal mappings.
Introduce the hodograph variable

$$r = \frac{1}{w^i} = \frac{\exp \mathrm{i}\theta}{q}, \tag{63}$$

where $w^i(Z, \alpha)$ is the conjugate complex velocity (of relative fluid motion) in the inner region of the flow, $q = |w^i|$ and $\theta = -\arg w^i$. To be able to determine the flow pattern in the nearfield using hodograph complex plane, one requires additionally an estimation of the behaviour of the angle θ of the tangent to the (upper) boundary of the cavity at the downstream infinity. Essentially, this is equivalent to estimating of the one-term *inner* limit of the one-term *outer* expansion. Replacing the *outer* variables in the expression (61), by the *inner* variables, i.e. $x = \alpha^4 X$ and $y = \alpha^4 Y$, and expanding for $X = O(1)$ and $\alpha \to 0$, one obtains

$$y(x) = y(\alpha^4 X)\Big|_{\alpha \to 0} = y_c^{oi}(X, \alpha) =$$

$$\alpha^4 \Big[\frac{4}{3} X^{3/4} + \frac{2}{5}\alpha^2 X^{5/4} - \frac{1}{14}\alpha^4 X^{7/4} + O(\alpha^6)\Big] \tag{64}$$

wherefrom the one-term *inner* expansion of the cavity contour in the form

$$Y = \frac{y_c^{oi}(X, \alpha)}{\alpha^4} = \frac{4}{3} X^{3/4} + O(\alpha^2) \tag{65}$$

The inclination of this curve for $X \to \infty$ tends to zero, so that one can assume that $\theta = 0$ at the downstream infinity. The hodograph plane, shown in Fig. 7, is then transformed onto an auxiliary plane ζ in such a way that the images of the plate and the cavity be found on the real axis $\Im\zeta = \xi$, see Fig. 7.

$$\zeta = \frac{1}{2}\Big(r + \frac{1}{r}\Big). \tag{66}$$

To complete the procedure of obtaining the inner solution it is necessary to relate the complex potential *of relative motion* of the fluid

$$F^i = \varphi + \mathrm{i}\psi, \tag{67}$$

where φ and ψ are correspondingly the velocity potential and stream function (of relative motion) with the function w^i by means of an intermediate complex plane ζ. The complex potential plane F^i, shown in Fig. 7, is transformed onto ζ-plane with help of the function

$$F^i = \frac{A_i}{(\zeta - 1)^2}, \tag{68}$$

where A_i is a real constant. Using formulae (63) and (66), one comes to the following relationship

$$w^i = \zeta \pm \sqrt{\zeta^2 - 1}, \tag{69}$$

where the *plus* sign is used for $|\zeta| < 1$, and the *minus* sign should be taken for $|\zeta| \geq 1$.
The relationship between Z and w^i through complex variable ζ can be derived in the following way

$$\int_{Z_1}^{Z_2} \mathrm{d}Z = Z_2 - Z_1 = \int_{\zeta_1}^{\zeta_2} \frac{\mathrm{d}F^i}{\mathrm{d}\zeta} \frac{\mathrm{d}\zeta}{w^i}. \tag{70}$$

Using formulae (68) and (69) in combination with (70) and integrating from the leading edge to the stagnation point, one finds

$$A_i = \frac{12}{17}\,a \tag{71}$$

Making use of (70) one can find the following relationship between Z and $\zeta = \xi$ on the plate and the cavity in the following form

- On the plate, for $0 \le X \le a$,Y=0-0

$$\frac{X}{a} = \frac{24}{17}\Big[\frac{1}{2(\xi-1)^2} + \frac{1}{\xi-1} - \frac{1}{3}\frac{(\xi^2-1)^{3/2}}{(\xi-1)^3} + \frac{3}{8}\Big] \tag{72}$$

- On the plate, for $a \le X \le \infty$,Y=0-0

$$\frac{X}{a} = 1 + \frac{24}{17}\Big[\frac{1}{2(\xi-1)^2} + \frac{1}{\xi-1} + \frac{1}{3}\frac{(\xi^2-1)^{3/2}}{(\xi-1)^3} - \frac{1}{3}\Big] \tag{73}$$

- On the cavity, $-1 \le \xi \le 1$

$$\frac{Z}{a} = \frac{X}{a} + \mathrm{i}\frac{Y}{a} = \frac{24}{17}\Big[\frac{1}{2(\xi-1)^2} + \frac{1}{\xi-1} + \frac{3}{8} + \frac{\mathrm{i}}{3}\frac{(1-\xi^2)^{3/2}}{(1-\xi)^3}\Big] \tag{74}$$

The latter equation provides a parametric relationship between $Y = Y(\xi)$ and $X = X(\xi)$ on the cavity contour in inner variables.

Having determined the relationship $Z = Z(\zeta)$, described by (70), with account of (69) and (70) one can find an implicit expression for distribution of the velocity along the plate in the inner region

$$\frac{X}{a} = \frac{24}{17}\Big[\frac{2(u^i)^2}{(1-u^i)^4} + \frac{2u^i}{(1-u^i)^2} + \frac{1}{3}\Big(\frac{1+u^i}{1-u^i}\Big)^3 + \frac{3}{8}\Big], \quad \text{for} \quad 0 \le X \le a, \quad Y = 0-0; \tag{75}$$

and

$$\frac{X}{a} = 1 + \frac{24}{17}\Big[\frac{2(u^i)^2}{(1-u^i)^4} + \frac{2u^i}{(1-u^i)^2} + \frac{1}{3}\Big(\frac{1+u^i}{1-u^i}\Big)^3 - \frac{1}{3}\Big], \quad \text{for} \quad a \le X < \infty, \quad Y = 0-0. \tag{76}$$

3.3. Matching and additive composition

To determine the parameter a, entering the inner solution, one uses the *Van-Dyke asymptotic matching principle*, [2]. First of all, find a one-term inner representation of the two-term outer solution[2]

$$u^o \sim u^{oi} \sim 1 - \frac{\alpha}{x^{1/4}}. \tag{77}$$

On the other hand, take the outer representation of (75), (76) for u^i. Introducing $x = \alpha^4 X$, one can write

$$u^i \sim 1 - \alpha u_1^i + \mathrm{O}(\alpha^2). \tag{78}$$

Substituting this expression into (75) and (76) expanding it for $\alpha \to 0$ and x =fixed, one has

$$u^i \sim u^{io} \sim 1 - \alpha\Big(\frac{48a}{17x}\Big)^{1/4}. \tag{79}$$

Comparing formulae (77) and (79) gives

$$a = \frac{17}{48} \tag{80}$$

wherefrom one can see that the distance of the stagnation point from the leading edge is equal to $\alpha^4/4$.

Uniformly valid expression for the velocity u^c on the wetted side of the plate can be found by means of the additive composition of the outer (56) and inner (75),(76) solution, The latter procedure implies adding these solutions and substracting their common part (77), i.e.

$$u^c = u^i + u^o - u^{io} = u^i + u^o - u^{oi} = u^i - \frac{\alpha\sqrt{\sqrt{x}}}{1+\sqrt{1-\sqrt{x}}} \tag{81}$$

The uniformly valid distribution of the pressure coefficient is calculated by means of the formula

$$\bar{C}_p^2 = 1 - (u^c)^2 \tag{82}$$

Figures 8 and 9 illustrate comparison of the pressure distributions, calculated using the formula (82) and the exact solution for $\alpha = 10^o$. The results of the linear theory are plotted with dot-and-dash line. The upper cavity shape (in stretched coordinates) in the immediate vicinity of the trailing edge of the supercavitating plate is shown in Fig. 10.

[2] Adding preliminarily the velocity of the incoming flow

4. Supercavitating foil with a spoiler

As another example of how the matched asymptotics technique can be used to complement the outer (linear) expansion of the solution in a local region of large flow perturbations is that of a flow past a supercavitating plate with a spoiler, [5], [6]. The spoiler is represented by a plate of a small relative height mounted upon the lifting surface either at the trailing edge on the pressure side or at an appropriate station on the suction side. The spoiler is usually oriented normally to the oncoming stream. The presence of the spoiler results both in local pressure rise due to creation of the stagnation zone in its vicinity, and redistribution of pressures around the entire foil, and, eventually, the additional lift occurs. Experiments of recent years have shown that the spoiler is one of the most effective, yet relatively simple devicess to enhance the lifting capacity of the supercavitating hydrofoils as well as the thrust of the supercavitating screw propellers. It is worth mentionning that Professor Tulin marked spoilers on one of the branches of his famous family tree of supercavitating flow theory, [6].

In the example below, considered is the case of a two-dimensional flow past a supercavitating foil with a spoiler of a small relative width ε at a trailing edge. For the sake of illustrating the procedure, the simplest case of zero cavitation number is considered. In the outer region (far from the spoiler in terms of its length) appropriate linear solutions are used incorporating an admissible (square root) singularity of unknown strength at the trailing edge. In the inner region (in the vicinity of the spoiler) the problem is reduced to that of a Kirchhoff type separated flow past a symmetric wedge. Asymptotic matching of the outer and inner solutions permits to determine hydrodynamic characteristics of the supercavitating foil with the spoiler.

Let the supercavitating foil be slightly curved and oriented to the flow at a small angle of attack α. As per foregoing, assume that the cavitation number $\sigma = 0$. The spoiler has a small relative width $\varepsilon << 1$ and is oriented at an arbitrary angle β to the foil at the trailing edge.

4.1. Flow near the spoiler (inner problem)

In the region near the spoiler introduce stretching of the local independent variables

$$X = \frac{x-1}{\varepsilon}, \qquad Y = \frac{y}{\varepsilon}, \qquad Z = X + \mathrm{i}Y. \tag{83}$$

Assume that the distance from the flow boundaries is of the order of the chord, i.e. O(1). Then, the pattern of the local flow does not depend on the type or on the number of the boundaries. It does not depend either on the distance of the trailing edge from the flow boundaries, and represents a flow past a semi-infinite horisontal flat plate with a spoiler of a unit width. Analytic continuation of this flow into the upper half-plane leads to a problem for a separated flow of unit velocity past a symmetric wedge. The solution of this classical problem can be obtained by the methods of the jet theory in an ideal fluid. Following [7] find the conjugate complex velocity $w^i(Z)$, by mapping the complex potential plane $F^i(Z) = \varphi^i + \mathrm{i}\psi^{i3}$ and the logarithmic velocity hodograph plane $\omega = -\ln w = -\ln|w| + \mathrm{i}\theta$ upon auxiliary plane t (Fig. 11):

$$F(Z) = \varphi_o t^2, \qquad \omega = \frac{2\beta}{\pi}\ln\Big(\sqrt{1-\frac{1}{t^2}} + \frac{\mathrm{i}}{t}\Big), \tag{84}$$

where φ_o is a parameter related to the wedge angle β (in radians) by means of the following relationship

$$\varphi_o = \Big[2\int_0^1 \Big(\frac{\sqrt{1-t^2}+1}{t}\Big)^{2\beta/\pi} t\,\mathrm{d}t\Big]^{-1} \tag{85}$$

For better convergence of the integral in the denominator it is practical to to use it in the following alternative form

$$\int_0^1 \Big(\frac{\sqrt{1-t^2}+1}{t}\Big)^{2\beta/\pi} t\,\mathrm{d}t =$$

$$\int_0^1 t^{(1-2\beta/\pi)}\Big[\Big(\sqrt{1-t^2}+1\Big)^{2\beta/\pi} - 2^{2\beta/\pi}\Big]\,\mathrm{d}t + \frac{2^{2\beta/\pi-1}}{\pi-\beta/\pi} \tag{86}$$

The coordinates of the free boundary of the cavity, detaching from the spoiler (lower cheek of the wedge) can be calculated with use of the formula

$$Z = 2\varphi_o\int_{-1}^{t}\Big(\frac{\sqrt{1-t^2}+1}{t}\Big)^{2\beta/\pi} t\,\mathrm{d}t + \exp(-\mathrm{i}\beta) \tag{87}$$

[3] of relative fluid motion

In order to perfom the matching of the inner solution to the outer solution one has to obtain the asymptotics of the inner solution far the spoiler. With the purpose to perform the matching of the ordinates of the free boundary of the cavity[4] find the asymptotics of the free boundary for $X \to \infty$, $(t \to -\infty)$

$$Z(t) = X(t) + \mathrm{i}Y(t) \sim 2\varphi_o\Big(\frac{1}{2}t^2 + \frac{2\mathrm{i}\beta}{\pi}t\Big) + C^* \tag{88}$$

where

$$C^* = 2\varphi_o \int_1^\infty \Big[\Big(\sqrt{1-\frac{1}{\tau^2}} - \frac{\mathrm{i}}{\tau}\Big)^{2\beta/\pi} - 1\Big]\,\tau\,\mathrm{d}\tau \tag{89}$$

wherefrom

$$Y = \frac{y}{\varepsilon} \sim -\frac{4\beta}{\pi}\sqrt{\varphi_o X}, \quad \text{for} \quad X \to +\infty \tag{90}$$

Besides, estimating the outer limit of the inner solution, one can show that the outer description of the conjugate complex velocity has a square-root singularity at the point $x = 1$. In fact, expanding $w(t)$ and $Z(t)$ for large t, one can obtain

$$Z(t) \sim \varphi_o t^2 + \mathrm{O}(t), \tag{91}$$

and

$$w^i(t) = \exp\left[-\omega^i(t)\right] = \Big(\sqrt{1-\frac{1}{t^2}} + \frac{i}{t}\Big)^{-2\beta/\pi} \sim 1 - \frac{2\mathrm{i}\beta}{\pi t} + \mathrm{O}(\frac{1}{t^2}) \tag{92}$$

Excluding the auxiliary variable t from these two expressions, one can find

$$w^i(Z) = \frac{\mathrm{d}F^i}{\mathrm{d}Z} \sim 1 - \frac{2\mathrm{i}\beta\sqrt{\varphi_o}}{\pi\sqrt{Z}}, \qquad Z \to \infty \tag{93}$$

On the foil

$$Z = X = \frac{x-1}{\varepsilon} \tag{94}$$

Finally, one can evaluate the behaviour of w on the foil near the trailing edge

$$w \sim -\frac{2\beta\sqrt{\varphi_o}\sqrt{\varepsilon}}{\pi\sqrt{1-x}}, \quad \text{for} \quad x \to 1-0 \tag{95}$$

4.2. The outer (linearized) flow

For the simplest case of zero cavitation number ($\sigma = 0$) and angle of attack $\alpha << 1$ the flow field outside of the vicinity of the spoiler experiences small perturbations. The corresponding linear solution for the conjugate complex velocity can be written in the form

$$w(\zeta) = \alpha\Big(\mathrm{i} - \sqrt{\frac{1-\zeta}{\zeta}}\Big) - \frac{\mathrm{i}B_s}{\sqrt{\zeta(\zeta-1)}}, \tag{96}$$

Note that the foregoing expression differs from the solution for the foil without spoiler (42) by the last term, where B_s is a real constant. This latter term represents an admissible solution of the corresponding Riemann-Hilbert mixed boundary problem which

- has zero real part on $\zeta = \xi > 1$ and $\zeta = \xi < 0$, i.e. on the boundary of the cavity
- has zero imaginary part on $1 > \zeta = \xi > 0$, i.e. on the wetted part of the foil
- yields a square root singilarity of the type $1/\sqrt{x-1}$ at the point $\zeta = \xi = 1-0$, corresponding to the point $x = 1-0$. In other words, such a solution secures matching of the outer solution with the inner solution, discussed above

  In fact near that point ($\xi \to 1-0$, and $x \to 1-0$) the "spoiler" term behaves as

$$-\frac{\mathrm{i}B_s}{\sqrt{\xi(\xi-1)}} = -\frac{B_s}{\sqrt{\sqrt{x}\sqrt{1-\sqrt{x}}}} = -\frac{B_s\sqrt{1+\sqrt{x}}}{\sqrt{\sqrt{x}}\sqrt{1-x}} \sim -\frac{\sqrt{2}B_s}{\sqrt{1-x}}, \tag{97}$$

[4]Note that the matching can be performed with respect to other parameters of the flow, e.g. pressure or velocity on the foil

- yields a (proper) quarter-root singularity of the type $x^{-1/4}$ on the wetted side of the foil close to the leading edge.

  In fact, near the latter point one has

$$-\frac{\mathrm{i}B_s}{\sqrt{\xi(\xi-1)}} = -\frac{B_s}{\sqrt{\sqrt{x}\sqrt{1-\sqrt{x}}}} \sim -\frac{B_s}{x^{1/4}} \tag{98}$$

- complies with the requirement of the decay of the perturbation velocity at the infinity

The real constant B_s characterizes the strength of the square-root singularity at the point $x = 1$ and has to be determined from the matching procedure. Matching of the linearized outer solution and the inner solution can be performed in terms of the perturbation velocities, or in terms of the ordinates of the lower free boundary of the cavity detaching form the spoiler. Using the former option and employing the matching principle, one should equate the outer expansion w^{io} of the inner solution of with the inner expansion w^{oi} of the outer solution[5]. In other words, one should equate the expressions (95) and (97). As a result, the constant B_s is obtained in the form

$$B_s = \frac{\beta\sqrt{2\varphi_o}}{\pi}\sqrt{\varepsilon} \tag{99}$$

It is easy to verify that the matching could have been done in terms of the ordinates of the lower free boundary of the cavity. Find the asymptotics of the outer description of this boundary[6] near the spoiler, i.e. for $x \to 1-0$. Using the kinematic condition on the cavity $u = 0$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = v = -\Im w = \frac{B_s}{\sqrt{\xi(1-\xi)}}, \quad (\xi = \sqrt{x}, x > 1, \xi > 1)$$

Integrating this expression one obtains

$$y = \int_1^x v\,\mathrm{d}x = \int_1^{\sqrt{x}} v(\xi)\frac{\mathrm{d}x}{\mathrm{d}\xi}\,\mathrm{d}\xi = 2\int_1^{\sqrt{x}} v(\xi)\xi\,\mathrm{d}\xi =$$

$$-B_s\{2\sqrt{\sqrt{x}(\sqrt{x}-1)} + \ln[2\sqrt{x} - 1 + 2\sqrt{\sqrt{x}(\sqrt{x}-1)}]\}. \tag{100}$$

Asymptotic representation of (100) for $x \to 1-0$ is with use of the previously obtained magnitude of the constant B

$$y \sim -2\sqrt{2}B_s\sqrt{x-1} = -\frac{4\beta\sqrt{\varphi_o\varepsilon}}{\pi}\sqrt{x-1} \tag{101}$$

Re-writing (90) in terms of the outer variable x one can verify complete coincidence of the resulting expression with (101). This fact proves correctness of the matching procedure.
The additional lift coefficient due to presence of the spoiler

$$C_{y_s} = -2\Re\oint w_s(z)\,\mathrm{d}z = -\int_0^1 u_s(x)\,\mathrm{d}x = 2B\pi = 2\sqrt{2\varphi_o}\beta\sqrt{\varepsilon} \tag{102}$$

The drag coefficient due to presence of th spoiler in the case of flat plate and zero angle of attack is

$$C_{x_s} = -\Im\oint w^2(z)\,\mathrm{d}z = -\Im\oint w^2(\zeta)\frac{\mathrm{d}z}{\mathrm{d}\zeta}\,\mathrm{d}\zeta = 2\pi B_s^2 = \frac{4\varphi_o\beta^2}{\pi}\varepsilon \tag{103}$$

Some calculated results reflecting dependence of the spoiler contributions to the lift and drag coefficients on the spoiler installation angle β are presented in Fig. 12. For the case of non-zero angle of attack $\alpha \neq 0$ the resulting lift and drag coefficients can be determined with use of the known superposition rules

$$C_y = C_{y_\alpha} + C_{y_s} \tag{104}$$

$$C_x = (\sqrt{C_{x_\alpha}} + \sqrt{C_{x_s}})^2 \tag{105}$$

For the case of the flat plate the concrete expressions for these coefficients become

$$C_y = \frac{\pi\alpha}{2} + 2\sqrt{2\varphi_o}\beta\sqrt{\varepsilon} \tag{106}$$

[5] In the overlap regions
[6] Based on the possibility of linear superposition, it is sufficient to take the case of zero incidence

$$C_x = \Big(\sqrt{\frac{\pi}{2}}\alpha + 2\sqrt{\frac{\varphi_o}{\pi}}\beta\sqrt{\varepsilon}\Big)^2 \tag{107}$$

The spoiler contribution to the upper boundary of the cavity can be calculated by integrating the corresponding "upwash" $\Im w = v$ for $z = x + \mathrm{i}0, \zeta = \xi, \xi = -\sqrt{x}$, $x > 0$. Using (96), one has

$$y = \int_0^x v(x)\,\mathrm{d}x = \int_0^{-\sqrt{x}} v(\xi)\frac{\mathrm{d}x}{\mathrm{d}\xi}\,\mathrm{d}\xi = 2\int_0^{-\sqrt{x}} v(\xi)\xi\,\mathrm{d}\xi =$$

$$2B_s[\sqrt{x+\sqrt{x}} + \ln(\sqrt{1+\sqrt{x}} - \sqrt{\sqrt{x}})] \tag{108}$$

The structure of asymptotic solutions can be used for some useful estimates. One can evaluate e.g. optimal ratio angle of attack and relative length of the spoiler. Utilizing expressions (106) and (107) one can write the following formula

$$\frac{L}{D}\sqrt{\varepsilon} = \frac{C_y}{C_x} = \frac{a_\alpha\kappa + b_\varepsilon}{(c_\alpha\kappa + d_\varepsilon)^2 + c_f/\varepsilon} \tag{109}$$

where the coefficients $a_\alpha, c_\alpha, b_\varepsilon, d_\varepsilon$ can be easily determined comparing (106) and (107). Parameter κ is defined as

$$\kappa = \frac{\alpha}{\sqrt{\varepsilon}}, \tag{110}$$

and $c_f = c_f(Re)$ is a friction coefficient, which can be calculated as a function of Reynolds number with use of an appropriate formula. For example, assuming fulll turbulent regime of the flow past the wetted part of the cavitating plate, one can use the formula

$$c_f = \frac{0.455}{(\log Re)^{2.58}} \tag{111}$$

Differentiating (109) with respect to κ and equating the result to zero, one obtains the following *optimal* magnitude of κ

$$\kappa_{opt} = \frac{-c_\alpha b_\varepsilon + \sqrt{(c_\alpha b_\varepsilon - a_\alpha d_\varepsilon)^2 + c_f/\varepsilon}}{a_\alpha c_\alpha} \tag{112}$$

Note that for $c_f/\varepsilon \to 0$ one can assume that

$$\kappa_{opt} \to -\frac{d_\varepsilon}{c_\alpha} = -\frac{2\sqrt{2\varphi_o(\beta)}\beta}{\pi}, \tag{113}$$

so that the optimal ratio $\alpha/\sqrt{\varepsilon}$ is negative. The latter results means that to secure maximum lift-to-drag ratio, one has to provide a negative angle of attack, for example for a spoiler normal to the plate ($\beta = \pi/2$), $\alpha_{opt} = -0.748\sqrt{\varepsilon}$.

The value of κ_{opt} can be employed to calculate the maximum lift-to-drag ratio of a flat plate with a spoiler at the trailing edge. However, there exits a somewhat contradicting requirement of *univalence* which, in simple words, means that the upper side of the cavity generated by the optimal combination of the spoiler and (negative) angle of attack *should not intersect the plate*[7]. One can find the domain of univalence by considering the thickness of the cavity with respect to the plate. Using, the corresponding expressions (61) and (108) for the contributions to the ordinates of the upper side of the cavity due to angle of attack and presence of spoiler, one finds the aforementionned cavity thickness in the form

$$y_t(x) = y_\alpha^u + y_\varepsilon = \alpha f_\alpha(x) + \frac{\beta\sqrt{2\varphi_o(\beta)}}{\pi}\sqrt{\varepsilon} f_\varepsilon(x) \tag{114}$$

where

$$f_\alpha(x) = \frac{1}{2}[(1+2\sqrt{x})\sqrt{x+\sqrt{x}} - \ln(\sqrt{1+\sqrt{x}} + \sqrt{\sqrt{x}})] \tag{115}$$

$$f_\varepsilon(x) = 2[\sqrt{x+\sqrt{x}} + \ln(\sqrt{1+\sqrt{x}} - \sqrt{\sqrt{x}})] \tag{116}$$

Requiring that the cavity thickness be non-negative at the trailing edge of the plate ($y_t(1) \le 0$), one obtains the expression for the domain of univalence in terms of of the relationship between angle of attack and length of the spoiler

$$\kappa = \frac{\alpha}{\sqrt{\varepsilon}} \le -\frac{f_\varepsilon(1)}{f_\alpha(1)} = -N_o\beta\sqrt{\varphi_o(\beta)}, \tag{117}$$

[7] Condition of univalence for this case was first found by A.S. Achkinadze and G.M. Fridman

where

$$N_o = \frac{4\sqrt{2}}{\pi} \frac{\sqrt{2} + \ln(\sqrt{2} - 1)}{3\sqrt{2} - \ln(\sqrt{2} + 1)} \approx 0.285 \tag{118}$$

The optimal and "univalent"ratios $\alpha/\sqrt{\varepsilon}$ are plotted versus the spoiler installation angle β in Fig. 13. This Figure clearly shows that in a certain range of β the optimal ratio of the angle of attack and the length of the spoiler cannot be achieved in reality. The ("pedestrian") maximum of lift-to-drag ratio derived through optimisation of the ratio $\alpha/\sqrt{\varepsilon}$ is plotted in Fig. 14 together with the lift-to-drag ratio attainable within the regime of univalence.

5. Bibliography

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Figures to "Supercavitating flows: small perturbation theory and matched asymptotics"

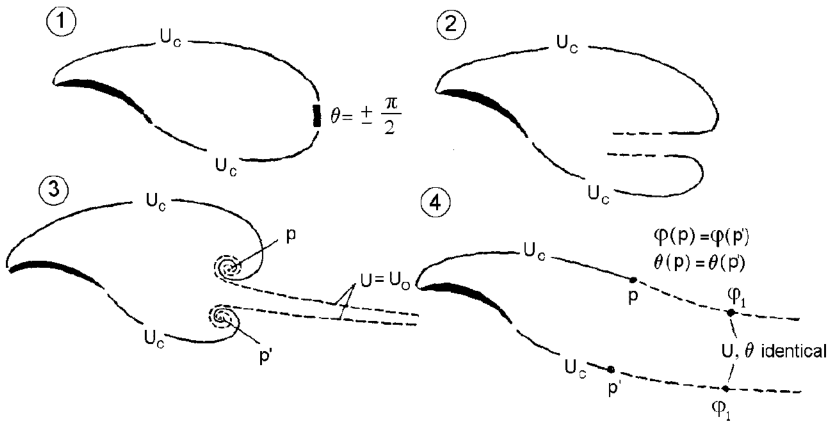

Fig. 1 Selected nonlinear cavity closure models (1 – Riaboushinsky model, 2 – Efros-Gilbarg model, 3 – Tulin double-spiral vortex model, 4 – Wu-Fabula model)

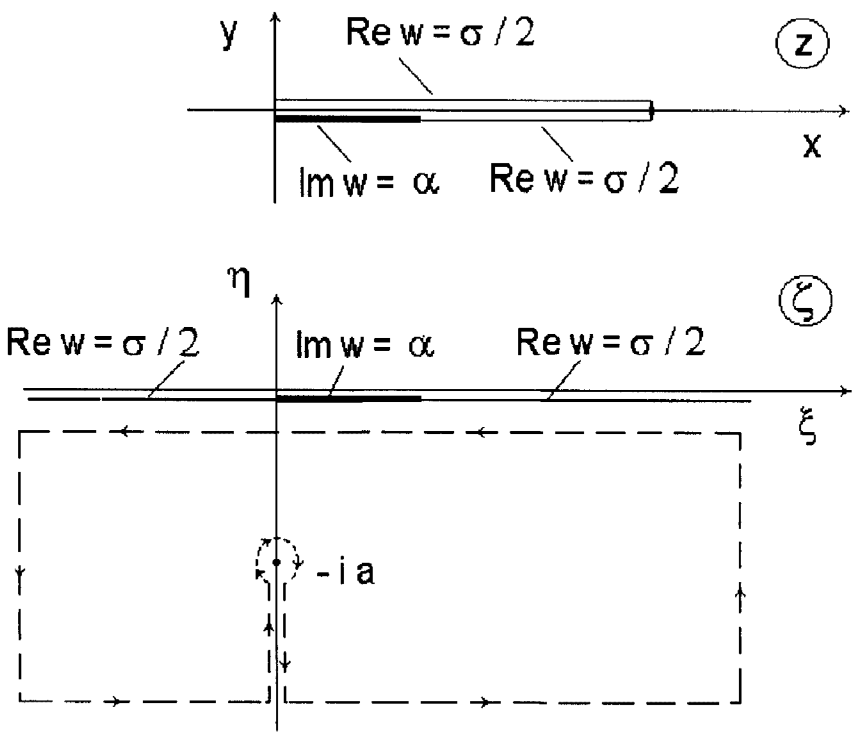

Fig. 2 Linearized "physical" complex plane z and auxiliary complex plane ζ for the case of closed cavity with singular termination. the ground

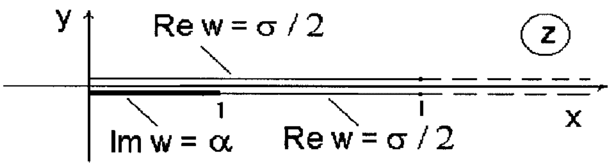

Fig. 3 Linearized "physical" complex plane z for the case of smooth cavity-wake transition (Wu-Fabula cavity closure scheme)

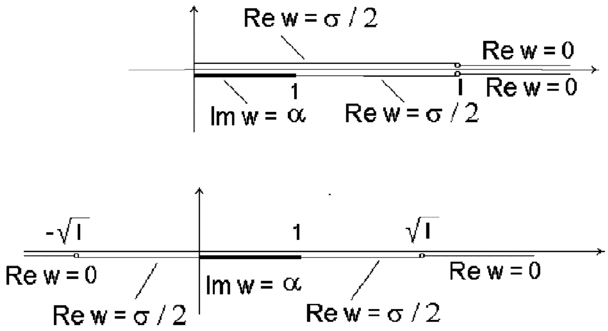

Fig. 4 Linearized "physical" complex plane z and auxiliary complex plane ζ for the case of cavity termination on Tulin double-spiral vortices

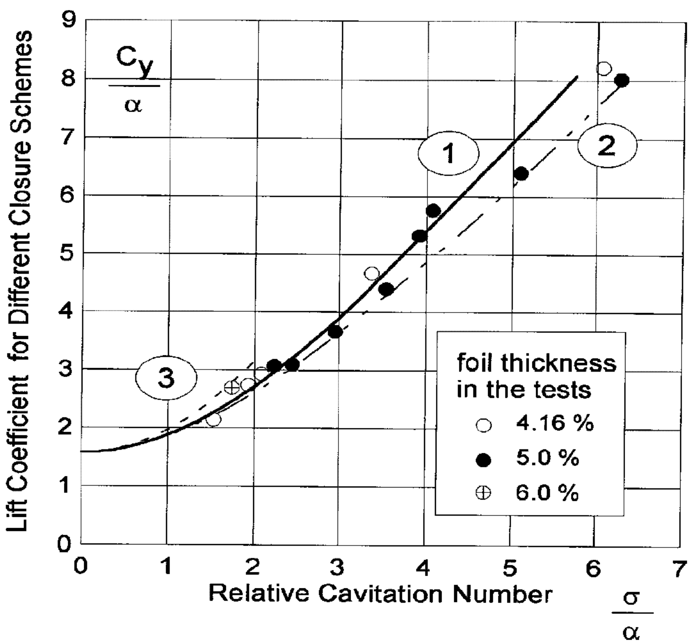

Fig. 5 Lift coefficient (C_y/α) of a supercavitating flat plate versus relative cavitation number σ/α for different linear analogues of the cavity closure models (1 – Tulin double-spiral vortex scheme, 2 – closed cavity scheme, 3 – Wu-Fabula scheme)

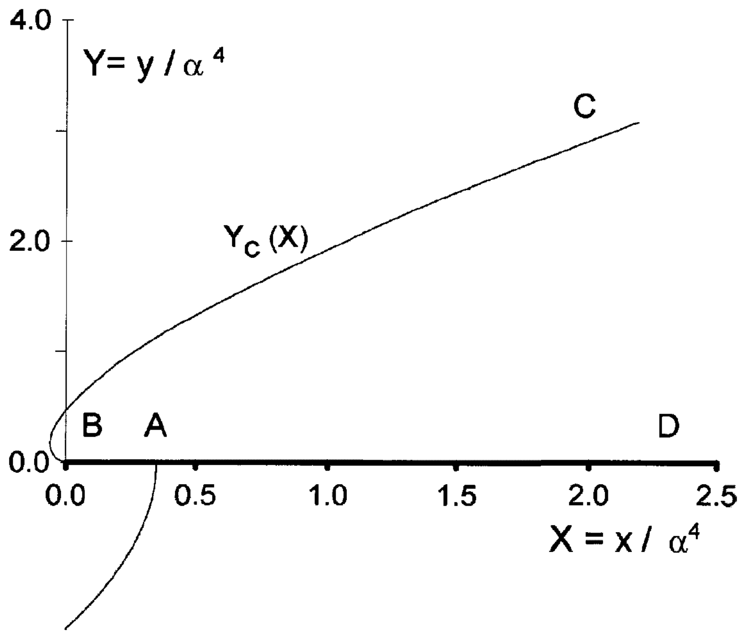

Fig. 6 Flow near the leading edge of supercavitating flat plate (inner region)

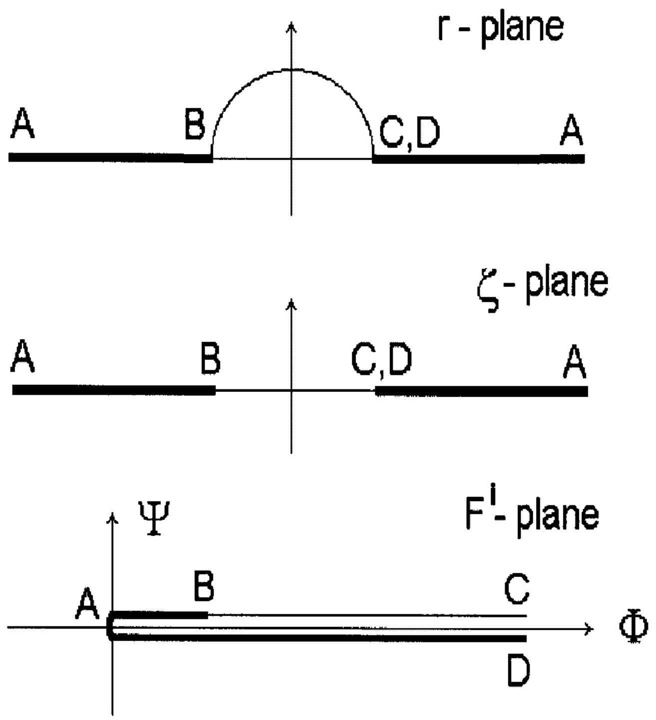

Fig. 7 Complex planes for the hodograph solution of the problem for the flow in the vicinity of the leading edge

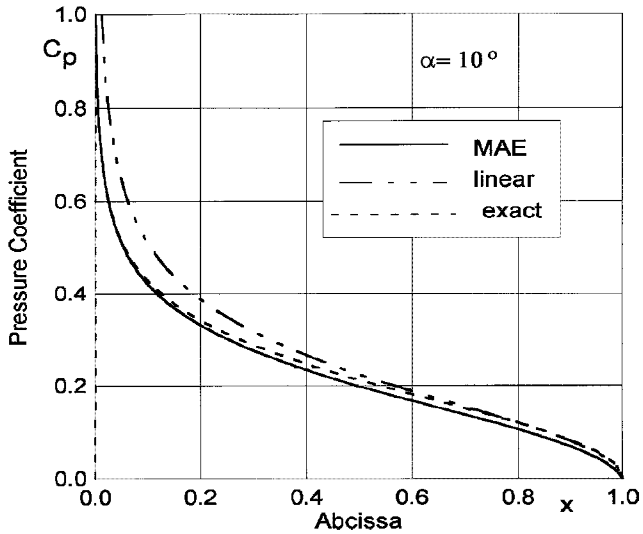

Fig. 8 Pressure distribution along supercavitating flat plate ($\alpha^o = 10^o$) (solid line – uniformly valid MAE, dashed line – exact solution, dot-and-dash line – linear theory)

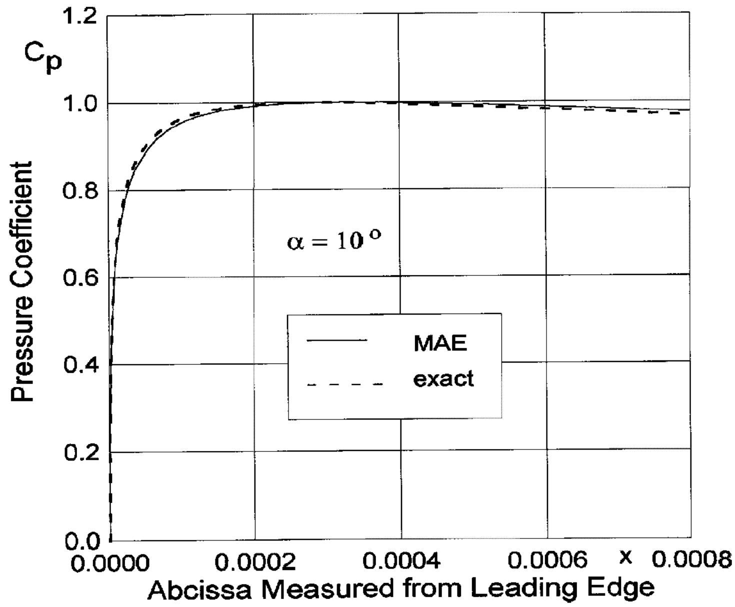

Fig. 9 Pressure distribution in the vicinity of the leading edge of a supercavitating flat plate for $\alpha^o = 10^o$ (solid line – MAE, dashed line – exact solution)

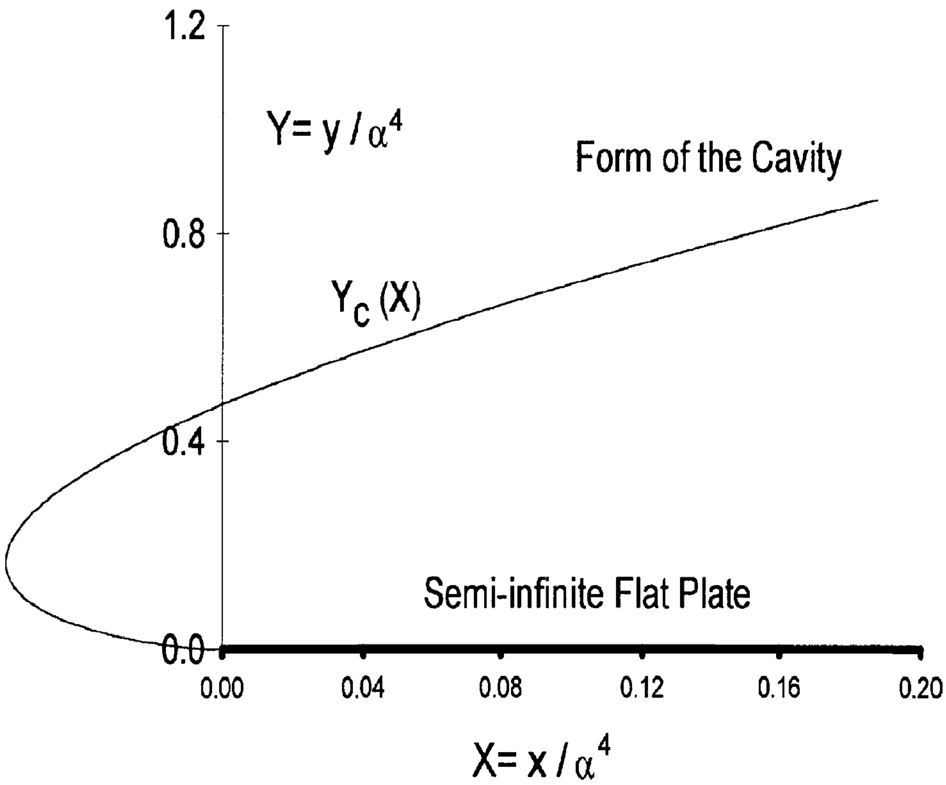

Fig. 10 Upper cavity shape in stretched coordinates in the immediate vicinity of the leading edge of supercavitating flat plate

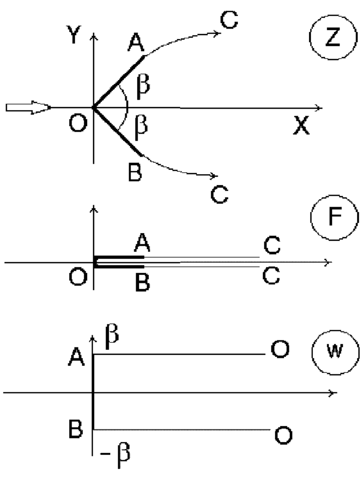

Fig. 11 "Physical" plane, complex potential plane and hodograph plane for the flow problem solution near the spoiler

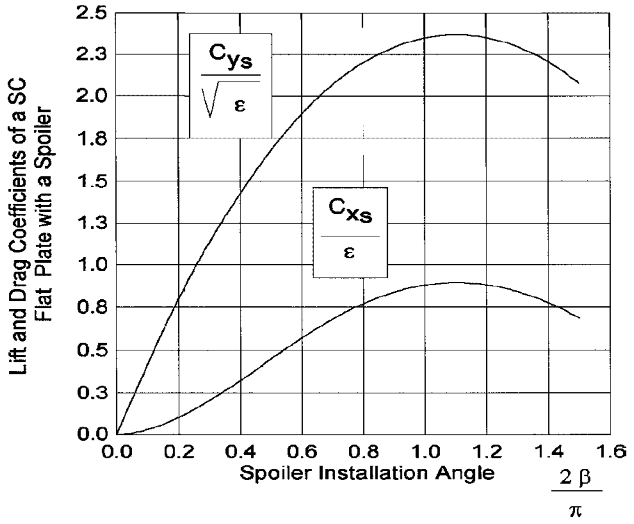

Fig. 12 Lift and drag coefficients of a supercavitating flat plate with a spoiler versus spoiler installation angle

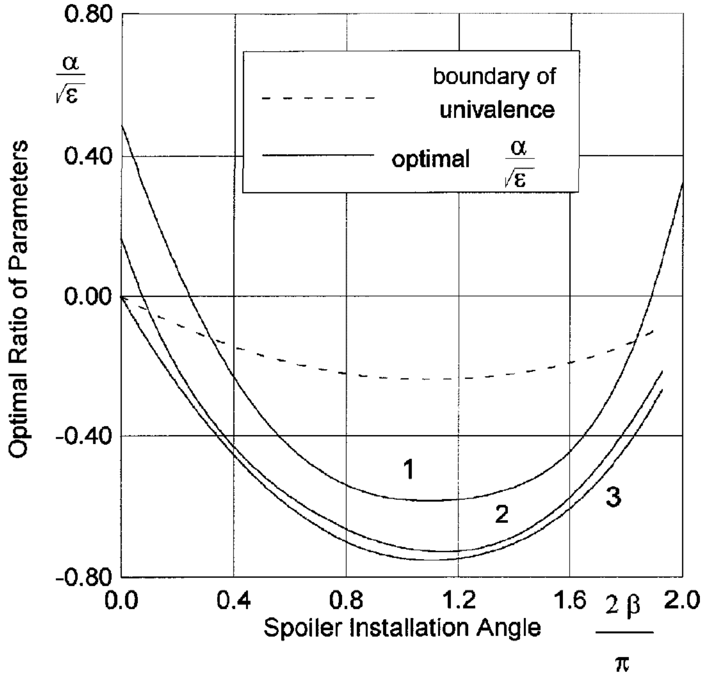

Fig. 13 Optimal and univalent ratios $\alpha/\sqrt{\varepsilon}$ versus the spoiler installation angle for a supercavitating flat plate with a spoiler at the trailing edge

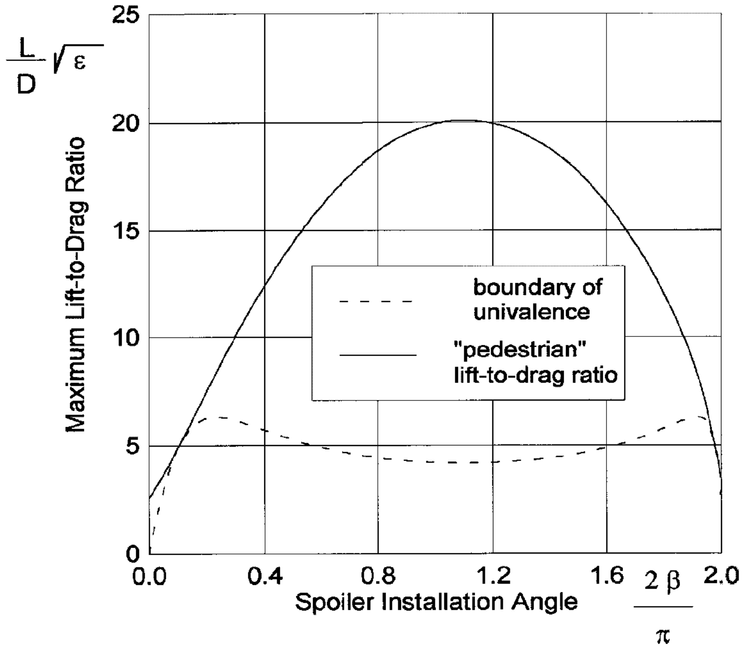

Fig. 14 Maximum "pedestrian" and "univalent" lift-to-drag ratios of a supercavitating flat plate with a spoiler versus the spoiler installation angle