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Image Charge Forces Inside Conducting Boundaries

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Abstract. Evaluation of the force on a charge particle due to the surface charge distributions it induces in nearby conductors is generally a challenging problem. We show these forces can be evaluated explicitly and fairly simply in a number of elementary, but important cases.

INTRODUCTION

The force on a charged particle due to the charges it induces on nearby conducting surfaces is a factor in many scientific measurements. The effects of these forces are particularly important for non-neutral plasmas. Until now, general techniques for quantifying image charge forces have not been available. An important advance was made in a recent paper by Fine and Driscoll [1] which addressed the lowest order solutions to this problem for infinite cylindrical geometry including finite charge length effects. Here, we generalize those results considerably.

IMAGE CHARGE PSEUDOPOTENTIAL

The measurable physical variable associated with induced surface charge is the force it produces on the point charge. We would like to express this force as the gradient of a function of the point charge's position. There is a subtlety involved in this potential function that is best illustrated by a simple example using the method of images.

Motivating example

If a point charge of strength q is at a position $(x, y, z) = (x_0, 0, 0)$, ($x_0 > 0$) with a grounded conductor in the y - z plane, we know from the method of images[2] that the electrostatic potential for $x > 0$ can be expressed as the sum of the potentials from the point charge and a fictitious image charge of strength $-q$ located at $(x, y, z) = (-x_0, 0, 0)$:

$$\phi(x, y, z) = \frac{q}{4\pi\epsilon_0} \left(\frac{1}{\sqrt{[(x-x_0)^2 + y^2 + z^2]}} - \frac{1}{\sqrt{[(x+x_0)^2 + y^2 + z^2]}} \right) \quad (1)$$

for $x > 0$. This is the potential that determines the force on an infinitesimal point charge, which is to say a point charge whose own induced surface charges produce negligible forces. Can we use it to find the force on the charge q ? We must first drop the potential produced by the charge, since it cannot exert a force on itself. This leaves the potential of the image charge, which evaluated along the x axis is just

$$\Phi_I(x, 0, 0) = \frac{-q}{4\pi\epsilon_0} \frac{1}{x+x_0}, \quad (2)$$

for $x > 0$. Two courses of action present themselves for finding the x -component of the force: take $\partial\Phi_I/\partial x$ and evaluate at $x = x_0$, or evaluate Φ_I at $x = x_0$ and take $\partial/\partial x_0$ of the result. The results differ by a factor of two:

$$\left. \frac{\partial\Phi_I}{\partial x} \right|_{x=x_0} = \frac{q}{4\pi\epsilon_0} \frac{1}{4x_0^2}, \quad (3)$$

and

$$\frac{\partial[\Phi_I|_{x=x_0}]}{\partial x_0} = \frac{q}{4\pi\epsilon_0} \frac{1}{2x_0^2}. \quad (4)$$

The force can be found directly from Coulomb's Law using the image charge[2]:

$$\vec{F} = \frac{-q^2}{4\pi\epsilon_0} \frac{1}{4x_0^2} \hat{x}, \quad (5)$$

so at least in this case, we have a choice of formulae for the force in terms of the potential:

$$\vec{F}(x_0) = -q \left. \frac{\partial\Phi_I}{\partial x} \right|_{x=x_0} \hat{x} \quad (6)$$

or

$$\vec{F}(x_0) = \frac{-q}{2} \frac{\partial[\Phi_I|_{x=x_0}]}{\partial x_0} \hat{x}. \quad (7)$$

The first choice seems like the natural one, but it requires us to keep a function of two position coordinates (x and x_0). The second choice contains an unfamiliar factor of $1/2$, but the force is reconstructed from a simple function of one position coordinate (x_0). This is the useful form, which we will now derive more generally.

General derivation

Next we find a general expression for the force on a point charge q in the presence of both fixed charges and conductors at fixed potentials.

$$\vec{F} = -q\nabla\phi - \frac{1}{2}q\nabla\Phi, \quad (8)$$

where ϕ is the ordinary electrostatic potential due to the fixed charges and potentials, and Φ is the potential due to charges induced by the point charge, evaluated at the point charge coordinate.

From the definition of the electrostatic potential ϕ , the work required to bring a point charge q in to position $\vec{x}$ from infinity (where $\phi = 0$) while holding the charges producing ϕ fixed in place is $W = q\phi(\vec{x})$. The force on the particle is the negative of the gradient with respect to its position coordinate of the *total system energy* under the circumstances considered. If the charges producing $\phi(\vec{x})$ are indeed fixed and there are no other energy terms involving the position of the point charge, then the force is $\vec{F} = -q\nabla\phi$. This is the familiar result for an infinitesimal test charge, but it is a special case. It is not the definition of ϕ .

When a charge moves toward a conducting surface, the induced surface charges must bunch together, against their mutual repulsion, to maintain an equipotential. This adds a term to the total electrostatic energy of the system that acts to reduce the attractive force felt by the point charge, by exactly a factor of two. This is easily derived from the well-known expression for the total electrostatic energy (excluding self-energy) of n discrete charges:

$$W = \frac{1}{2} \sum_{j=1}^n q_j \Phi_j, \quad (9)$$

where Φ_j is the potential at charge q_j due to all the other charges.[3] If $j = 1$ denotes the point charge, and the others are the charges induced on a grounded conductor, then Φ_j is zero for all $j \neq 1$ (the induced charges) regardless of the position of the point charge, and the total electrostatic energy is

$$W = \frac{1}{2} q \Phi, \quad (10)$$

where $\Phi = \Phi_1$ is the potential at the point charge due to the induced surface charges. The force on the point charge is thus

$$\vec{F} = -\frac{1}{2} q \nabla \Phi, \quad (11)$$

where the gradient is taken with respect to the position of the point charge. If the conductor is at some potential other than ground, then we must add the familiar term $-q\nabla\phi$ to this thereby recovering eq.(8).

We refer to $\Phi(\vec{x})$ as the image charge pseudopotential, to emphasize its differences from ordinary potentials. Unlike $\phi(\vec{x})$, $\Phi(\vec{x})$ is not simply a solution of Laplace's or Poisson's equation, because its source term (the induced charge distribution) is a complicated functional of $\vec{x}$, the point charge location. Further, $\Phi(\vec{x})$ is only meaningful at the particular location of the charge. As we describe below, $\Phi(\vec{x})$ is proportional to the nonsingular part of $G(\vec{x}, \vec{x}')$, evaluated at $\vec{x} = \vec{x}'$. Thus, $\Phi(\vec{x})$ may be extracted from a family of solutions to Laplace's equation, but is not itself a solution. Numerous methods, include the classic "Method of Images" for solving these problems are discussed in Tinkle and Barlow.[4]

RESULTS

Generic limiting form

As a smooth region of a conductor is approached closely, the pseudopotential approaches that of an infinite flat plate. From the method of images, $\Phi \rightarrow -q/4\pi\epsilon_0 2d$ as $d \rightarrow 0$, where d is the distance to the surface. Thus if ξ_0 is the distance to a conductor along the ξ axis of a rectilinear coordinate system and the surface is perpendicular to the ξ axis where they intersect,

$$\Phi \rightarrow -\frac{q}{4\pi\epsilon_0\xi_0} \frac{1}{2(1-\xi)}, \quad (12)$$

as $\bar{\xi} \equiv \xi/\xi_0 \rightarrow 1$. If the geometry is symmetric in ξ , we can account for the limiting behaviors at both walls with the form

$$\Phi \rightarrow -\frac{q}{4\pi\epsilon_0\xi_0} \frac{1}{1-\bar{\xi}^2}, \quad (13)$$

as $|\bar{\xi}| \rightarrow 1$. The complete solution for Φ along the ξ axis will be the sum of this divergent term and a finite term. Expressed as a power series expansion about the origin, it will have the form

$$\Phi = -\frac{q}{4\pi\epsilon_0\xi_0} \sum_{n=0}^{\infty} C_n \bar{\xi}^n, \quad (14)$$

with $C_n = 0$ for odd n , due to symmetry. The divergent term by itself has $C_n = 1$ for all even n . The results we obtain for specific geometries can all be expressed in this form, with different values for the coefficients. Tinkle and Barlow[4] present solutions of this form for a number of important cases including the spherical shell, parallel plates and the rectangular box.

Cylindrical Shell

By way of example, we show here the results for a cylindrical box. The calculations are rather complex,[4] however, solutions of the form

$$\Phi(\bar{\rho}, \bar{z}) = -\frac{q}{4\pi\epsilon_0 z_0} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} C_{m,n}(\alpha) \bar{\rho}^{2m} \bar{z}^{2n}, \quad (15)$$

can be found. Here α is the ratio of the cylinder's length to diameter—the "aspect ratio." The limiting values listed in Table 1 are in line with our generic discussion above. These coefficient terms are shown plotted in Figure 1.

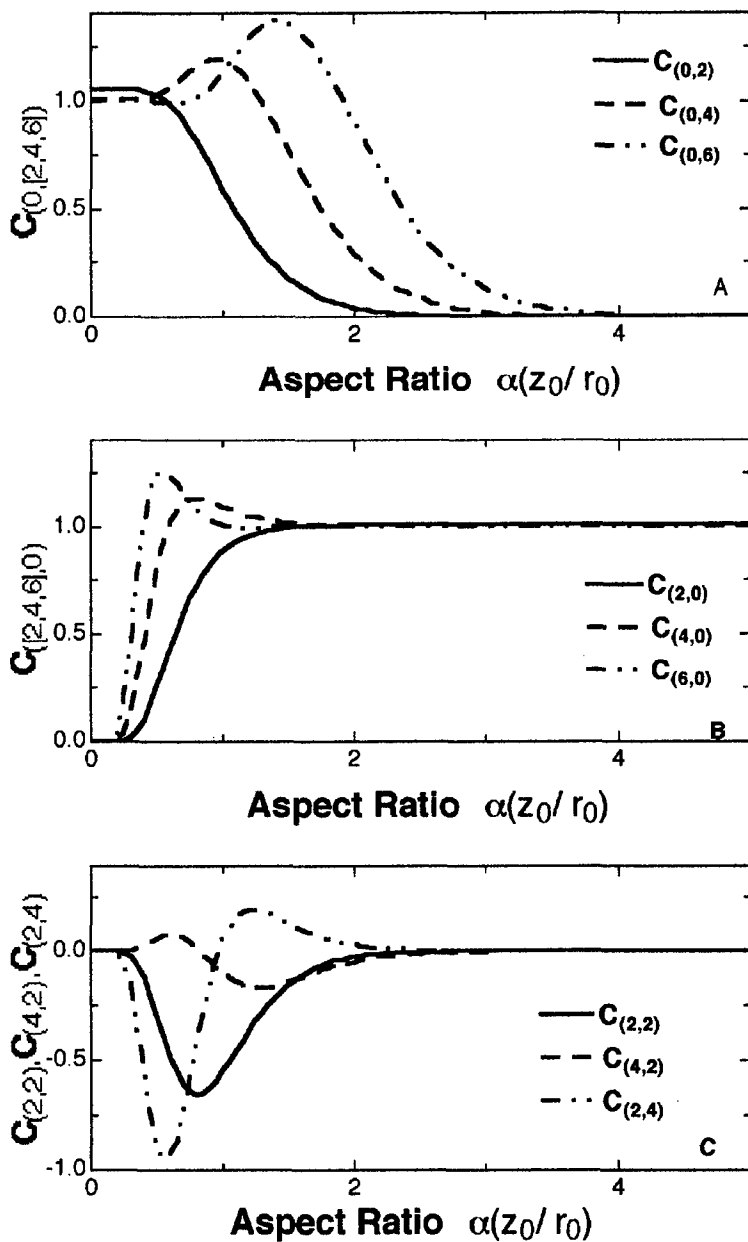


FIGURE 1. Plots of the the first six terms of the power series expansion of the image charge pseudopotential for a cylindrical box as functions of the aspect ratio. (A) The pure "axial" terms-powers of z only. (B) The pure "radial" terms-powers of r only. (C) The crossed "radial" and "axial" terms-powers of r and z .

TABLE 1. Asymptotic limits of image charge coefficients for cylindrical box, see text for explanation. *agrees with results of Fine and Driscoll [ref 1, eq(32)]

$C_{st,2s}$	$\alpha \rightarrow 0$ parallel plate	4 th Order Exp'l. Trap	$\alpha = 1$	$\alpha \rightarrow 0$ long tube
$C_{0,2}$	1.0518	0.7755	0.5880	0
$C_{0,4}$	1.0045	1.1669	1.1845	0
$C_{0,6}$	1.0005	1.0180	1.1246	0
$C_{2,0}$	0	0.7534	0.8792	1.0027*
$C_{4,0}$	0	1.1243	1.0884	1.0009
$C_{6,0}$	0	1.0572	1.0020	1.0003
$C_{2,2}$	0	-0.9157	-1.0309	0
$C_{4,2}$	0	-0.3470	-0.4359	0
$C_{2,4}$	0	0.1822	0.1884	0

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