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On an adaptive mesh algorithm with minimal distance control

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Abstract

In this paper, we present a new technique for generating error equidistributing meshes that satisfy both local quasi-uniformity and a preset minimal mesh spacing. This is firstly done in the one-dimensional case by extending the Kautsky and Nichols method [6] and then in the two-dimensional case by generalizing the tensor product methods to alternating curved line equidistributions. With the new meshing approach, we have achieved better accuracy in approximation using interpolatory radial basis functions (RBFs). Furthermore improved accuracy in numerical results have been obtained for a class of linear and non-homogeneous PDEs solved by the dual reciprocity method (DRM).

1   Introduction

The adaptive mesh algorithms have been widely used in the numerical solution of partial differential equations (PDEs) for boundary value problems [1, 13]. One undesirable feature of an error equidistributing mesh is that there is no guarantee of it being sufficiently smooth. For our applications of interpolation (using RBFs), the distance between points becoming too small can imply that the underlying interpolation matrix becomes ill-conditioned.

In this paper, we propose a method to deal with this problem in Section 2. Essentially our method consists of modifying the error monitor function in a suitable way and then equidistributing the new function so that the minimal mesh size constraint can be satisfied. We deal with the extension of adaptive mesh to two dimensions in Section 3. Finally, some numerical results will be given in Section 4.

2   An adaptive mesh with minimal mesh size control

In the 1D case, a typical adaptive mesh problem can be stated as follows: given a mesh (uniform or non-uniform) $t_0, t_1, \ldots, t_m$, and its corresponding error values (usually estimated from the numerical solution using a monitor function [5]) $f_0, f_1, \ldots, f_m$, we wish

to find a new mesh

$$\Pi : x_0, \ x_1, \ \ldots, \ x_n, \tag{2.1}$$

that is locally bounded with respect to a positive constant $k \geq 1$ such that $1/k \leq h_j/h_{j-1} \leq k, \ j = 1, 2, \ldots, n-1, \ h_j = x_{j+1} - x_j$, while the errors are equidistributed on mesh Π. One solution to this problem was given in [6] by replacing f_j by $\hat{f}_j$ followed by a standard equidistribution algorithm. $\hat{f}_j$ is referred to as the padded function and the main idea of replacing f_j is increasing the values of the function f, where too small, to prevent considerably large mesh sizes. We now propose a method of further modifying $\hat{f}_j$ in such a way that the resulting equidistribution mesh satisfies the preset minimal mesh size h_{min}. Before proceeding, we consider replacing the piecewise linear function $\hat{f}(x)$ (with endpoint values $\hat{f}_j = \hat{f}(t_j)$) by another piecewise linear function $Z(x)$ (with endpoint values $Z_j \equiv \hat{f}(x_j)$). This is a technical approximation to simplify the presentation; actually the proposed method may work without this step. Note that if we were to equidistribute $Z(x)$, the resulting mesh would not differ from x_j much; define the average value of the monitor function as

$$d' = d'(Z) = \frac{1}{n} \sum_{j=0}^{n-1} (Z_j + Z_{j+1}) \frac{h_j}{2}. \tag{2.2}$$

Our aim now is to modify some Z_j values so that the modified average value is the same as d' while the modified values ensure a preset minimal mesh size h_{min} is satisfied. To present our method, we note that insisting on $h_j \geq h_{min}$ implies $Z_j \leq \bar{Z}$ where

$$\bar{Z} h_{min} = d' \tag{2.3}$$

and $\bar{Z}$ is the critical constant to realize h_{min}. This points a way of modifying those large values of Z_j. However it is not obvious how to ensure the new and modified average values are the same, i.e. equidistribution is maintained for the same error constant. Suppose that among the current Z_j values, there are $M+1$ of them that are larger than $\bar{Z}$ (i.e. whose corresponding mesh size is less than h_{min}); denote these values by Z_{k_j} for $j = 0, 1, \ldots, M$. This means that $Z_{k_j} \leq \bar{Z}$ for $j = M+1, M+2, \ldots, n$. Here the sequence $k_0, k_1, \ldots, k_n$ represents a permutation of $0, 1, 2, \ldots, n$.

It turns out that a suitable modification (from Z_j to $\hat{Z}_j$) is the following:

$$\begin{cases} \text{(i)} \quad \hat{Z}_{k_j} = \bar{Z} \quad \text{when} \quad Z_{k_j} > \bar{Z}, \quad \text{i.e. for } j = 0, 1, \ldots, M, \\[2ex] \text{(ii)} \quad \hat{Z}_{k_j} = Z_{k_j} + \dfrac{Z_{k_j}}{\displaystyle\sum_{l=M+1} Z_{k_l}} \left[\sum_{i=0}^{M} (Z_{k_i} - \bar{Z}) \bar{h}_{k_i} \right] \Big/ \bar{h}_{k_j} \\[4ex] \hspace{4cm} \text{for} \quad j = M+1, M+2, \ldots, n, \end{cases} \tag{2.4}$$

where

$$\bar{h}_{k_i} = \begin{cases} (h_{k_i} + h_{k_i - 1})/2 & \text{when} \quad k_i \neq 0, n, \\ h_0/2 & \text{when} \quad k_i = 0, \\ h_{n-1}/2 & \text{when} \quad k_i = n. \end{cases} \tag{2.5}$$

For a simple illustration, see the plot of Fig 3b. To prove that the above modification is suitable, we first present the following result for a simple case.

Theorem 2.1 *Let $x_0, x_1, \ldots, x_n$ be a non-uniform mesh with the mesh sizes $h_j = x_{j+1} - x_j$ and $Z_0, Z_1, \ldots, Z_n$ are the corresponding error values. If the critical constant value $\bar{Z}$ as in (2.3), and only one value $Z_1 > \bar{Z}$ (i.e. $M = 1$ and all others Z_j are less than or equal to $\bar{Z}$), the modification (2.4) takes the following form,*

$$\begin{cases} (i) & \hat{Z}_0 = Z_0, \qquad \hat{Z}_1 = \bar{Z}, \\ (ii) & \hat{Z}_j = Z_j + \dfrac{Z_j}{\sum_{i=2}^{n} Z_i} \left[(Z_1 - \bar{Z})(h_0 + h_1)/2 \right]/(h_j + h_{j-1})/2 \text{ for } j = 2, 3, \ldots, n. \end{cases}$$

Then the average value $d = d(\hat{Z})$ of the modified values $\hat{Z}_j$ is the same as $d' = d'(Z)$ in (2.2).

Note $M = 1$ here; in fact the results holds for any one value $Z_j > \bar{Z}$. Now we are ready to present the main result on equation (2.4) with regard to minimal mesh size control.

Theorem 2.2 *With the error function modified as in (2.4), the new mesh $\hat{h}_j$ resulting from equidistribution satisfies (i) the average error value remains as d'; (ii) $\hat{h}_j \geq h_{min}$.* Here h_{min} cannot be specified to be larger than $h = 1/n$ (the uniform mesh size); practically we found $h_{min} \in [h^2, h/2]$ is adequate. Full proofs to these results will be given in the full version of this paper [10].

   In the method in (2.4), the values of Z_{k_j} which are less than but close to $\bar{Z}$ may become unnecessarily larger (e.g. larger than $\bar{Z}$) and therefore we can propose a further refinement. We can keep some of the Z_{k_j} values which are between $\bar{Z}/2$ and $\bar{Z}$. In other words, we only modify the very large and very small values of Z_{k_j} (see plot of Fig 3b). Then our theorems are still valid but the proofs may need minor changes. Finally we summarise our adaptive method with minimal mesh size control as follows (see the plot of Fig 3b for an illustration).

Algorithm 2.3. (Numerical algorithm) *For given non-uniform mesh $a = t_0, t_1, \ldots, t_m = b$, the error values $f_0, f_1, \ldots, f_m$, values c and h_{min}:*

(1) *Does the locally bounded mesh algorithm converge to the new mesh $a = x_0 < x_1 < \cdots < x_n = b$ which is sub-equidistributing with respect to c and f, that is, for a sufficiently large value of the integer n such that $\int_a^b f \leq nc$, and the inequalities*

$$\int_{x_j}^{x_{j+1}} f \leq c, \quad j = 0, 1, \ldots, n-1$$

are satisfied.

(2) *Check the minimal mesh size and compare it with the h_{min}. If it is less than h_{min}, go to the Step 3 otherwise stop.*

(3) *Approximate the padding values $Z_j = \hat{f}(x_j)$ corresponding to the new mesh by using piecewise linear interpolation of $\hat{f}_i$ values and calculate the average value*

$$d = \frac{1}{n} \sum_{j=0}^{n-1} (Z_j + Z_{j+1}) \frac{h_j}{2}, \quad \text{where} \quad h_j = x_{j+1} - x_j,$$

and $\bar{Z}$ according to $\bar{Z}h_{min} = d$.

(4) *Obtain the decreasing arrangement of Z_j, Z_{k_j} by ordering them.*

(5) *Modify the Z_{k_j} values as follows,*

$$
\left\{
\begin{array}{l}
\text{(i)} \quad \hat{Z}_{k_j} = \bar{Z} \quad when \quad Z_{k_j} > \bar{Z}, \\[2mm]
\quad\quad assuming,\ that\ for \quad j = 0, 1, \ldots, M \quad Z_{k_j} > \bar{Z}, \\[1mm]
\text{(ii)} \quad \hat{Z}_{k_j} = Z_{k_j} \quad when \quad \bar{Z}/2 \le Z_{k_j} \le \bar{Z}, \\[2mm]
\quad\quad assuming,\ that\ for \quad j = M+1, M+2, \ldots, N, \quad \bar{Z}/2 \le Z_{k_j} \le \bar{Z}, \\[1mm]
\text{(iii)} \quad \hat{Z}_{k_j} = Z_{k_j} + \frac{Z_{k_j}}{\sum_{l=N+1}^{n} Z_{k_l}} \left[\sum_{i=0}^{M} (Z_{k_i} - \bar{Z}) \bar{h}_{k_i} \right] / \bar{h}_{k_j}, \\[1mm]
\quad\quad for \quad j = N+1, N+2, \ldots, n,
\end{array}
\right.
$$

where $\bar{h}_{k_i}$ was introduced in (2.5).

(6) *Check the modified values $\hat{Z}_{k_j}$ in the stage (iii) of the Step 5. If $\hat{Z}_{k_j} \le \bar{Z}/2$ for all j, go to Step 7 otherwise repeat Step 5.*

(7) *Perform the equidistribution procedure for the modified values $\hat{Z}_{k_j}$ and obtain the new adapting mesh.*

3 Extension to two dimensions

The concept of adapting mesh in one dimension is well known (see e.g. [5, 3]). Extension of this idea to two dimensions is not straightforward. For a given function $f(x, y)$ and 2D domain Ω, an obvious extension is dividing the domain Ω into some subdomains Ω_i in such a way that

$$
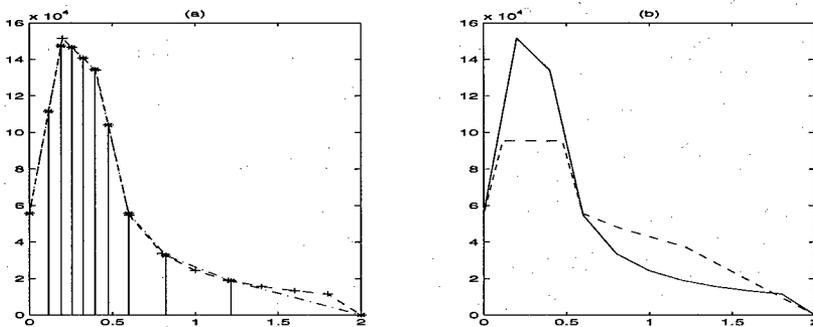\int\int_{\Omega_i} f(x, y) = \text{constant}. \tag{3.1}
$$

FIG. 1. In Fig (a) the monitor values corresponding to the new mesh are represented by '*', the linear interpolation for these values is shown by '-.' and in Fig (b) the modified values of the padded function, represented by dash line, are compared with the original values.
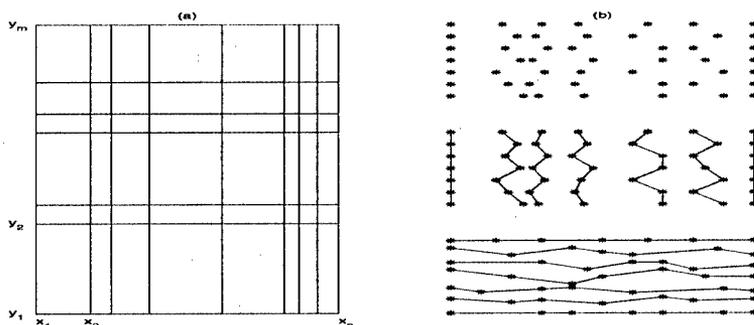
FIG. 2. In Fig (a) equidistribution of slabs in the two coordinate direction and in Fig (b) three stages of the new method are shown.

But, such a partition is not unique and furthermore satisfying condition (3.1) properly is not simple. Consequently, this condition has to be replaced. Among the methods given to satisfy the condition (3.1) as much as possible, two well known methods are transformation and dimension reduction. Transformation methods are based on mapping the physical domain into a simple domain with a uniform mesh and ultimately applying the equidistribution condition to obtain an adapting mesh in the physical domain [4, 12]. These methods are generally costly and complicated in theory. In this work we first consider the latter method which is easier and cheaper than the former method. We then present a new technique to generate a 2D mesh.

3.1   Dimensions reduction

We assume that Ω is a rectangle in the form $\Omega = \{(x,y),\ a \leq x \leq b,\ c \leq y \leq d\}$. A simple idea is to produce the mesh,

$$a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b,$$

$$c = y_0 < y_1 < \ldots < y_{m-1} < y_m = d,$$

such that

$$\int_{x_i}^{x_{i+1}} \int_{y_0}^{y_m} f_x(x,y)\, dy\, dx = \text{constant}, \tag{3.2}$$

and

$$\int_{y_j}^{y_{j+1}} \int_{x_0}^{x_n} f_y(x,y)\, dx\, dy = \text{constant}, \tag{3.3}$$

where $f_x(x,y)$ and $f_y(x,y)$ are the monitors in the x and y directions respectively (see Fig 3.1a). Obviously the generated mesh by this method is much different from an equi-distributing mesh that one expects from (3.1). Another method which leads to a non-rectangular grid is dimensional splitting [11]. We now describe a new method of type dimension reduction.
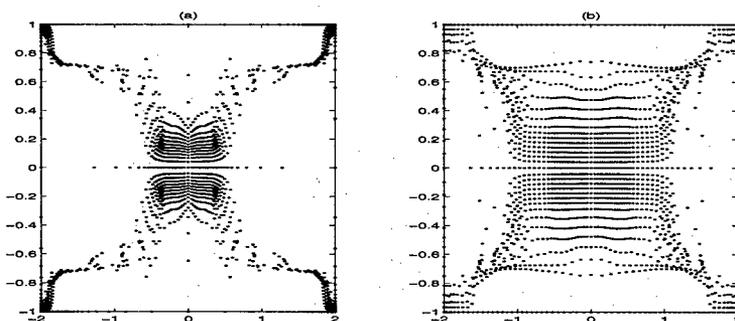
FIG. 3. In Fig (a) the mesh generated by the new method for function in (3.6) and in Fig (b) the resulting mesh when restricting the minimal mesh size as $h_{min} = h/2$ for the same function are shown.

3.2 A new approach for a 2D mesh

The idea is based on the tensor product method and therefore a non-rectangular grid. We start with a uniform mesh in a rectangular region Ω and perform the method in three stages. In the first stage, the error equidistributing is performed for each line in the horizontal direction (see the first part of Fig 3.1b), that is,

$$\int_{x_j}^{x_{j+1}} f_x(x, y_i) \, dx = \text{constant for } i = 0, 1, \ldots, m. \tag{3.4}$$

In the next stage, the mesh is redistributed in the vertical direction along the new grid lines (see the second part of Fig 3.1b), that is,

$$\int_{s_i}^{s_{i+1}} f_y(x_j, y) \, dy = \text{constant for } j = 0, 1, \ldots, n, \tag{3.5}$$

where $s_{i+1} - s_i$ is the distance between two consecutive points (x_j, y_i) and (x_j, y_{i+1}) along the new lines. In the final stage, equidistributing is repeated in the horizontal direction along the grid lines (the last part of Fig 3.1b). One can observe that repeating this procedure usually leads to a convergent mesh. According to our experiments, the number of iterations to achieve convergence is at most five. The resulting mesh by this procedure for function

$$u(x, y) = e^{\left(4 - x^2 - 4y^2\right)^2} \tag{3.6}$$

when applying the arc-length monitor is shown in Figure 3a. The idea of controlling the mesh size can also be applied in this technique. The generated mesh for the same function when the mesh sizes are restricted to $h_{min} = h/2$, where h is the mesh size in the case of uniform mesh, is given in Figure 3b.

4 Numerical examples

In this part the affect of adapting the mesh on the accuracy of interpolation and the DRM is considered. In the following examples, the infinity norm has been used to measure the
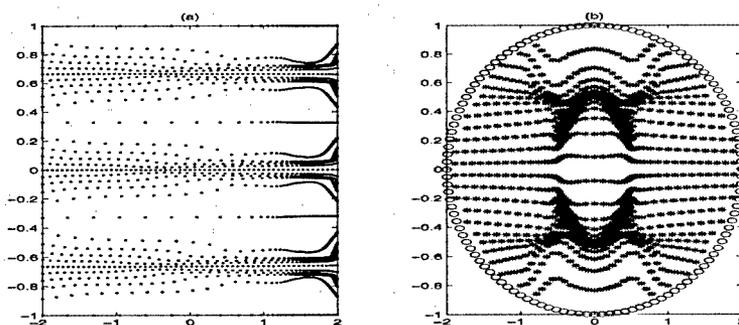
FIG. 4. The resulting mesh when using the new method for function in Examples 1 and 2 are shown in Figures (a) and (b) respectively.

Method	stage	Function (E1)	Derivative	Function (E2)	Derivative
uniform mesh	—	5.1E-2	9.5E-1	1.3E-2	2.2E-1
Adaptive mesh with control	first	5.4E-3	1.6E-1	2.5E-3	1.3E-2
	second	5.4E-3	3.0E-1	2.1E-3	1.0E-1
	third	3.8E-3	3.0E-1	3.7E-3	1.0E-1
Adaptive mesh without control	first	1.4E-2	9.9E-2	2.5E-3	1.5-2
	second	2.2E-2	7.5E-1	2.1E-3	1.0E-1
	third	1.8E-2	6.0E-1	4.5E-3	1.2E-1

TAB. 1. The interpolation error for Examples $1 - 2$ using adaptive mesh with and without control the mesh sizes.

accuracy, that is, if u and $\bar{u}$ are the exact and approximate values respectively then the error is calculated as

$$e_u = ||u(x) - \bar{u}(x)||_\infty = \max_{x \in D} |u(x) - \bar{u}(x)|.$$

A polynomial RBF, $1 + r^3$, has been employed in this work.

Example 4.1 We check the interpolation in terms of the RBFs for the function,

$$u(x, y) = \left(1 - e^{3\,x-3}\right) \sin(1.5\,\pi\,y), \qquad (4.1)$$

in a rectangular domain. The generated mesh for this function is shown in Figure 4a.

Table 4 shows the affect of adapting mesh on the interpolation accuracy with and without controlling the mesh sizes. As one can observe, using the adapting mesh considerably improves the accuracy in comparison with the case of uniform mesh. Moreover, the result in the case of controlling the minimal mesh size is better.

Example 4.2 In this example we first check the function $f_2(x, y) = 0.5 - 0.5 \tanh(-4 + 16x^2 + 16y^2)$ and then solve the linear PDE: $\nabla^2 u + y \frac{\partial u}{\partial x} + x \frac{\partial u}{\partial y} + xyu = d$, with the Dirichlet boundary condition over the elliptic domain $x^2 + 4y^2 = 4$, where d is a known function such that the exact solution is $u(x, y) = f_2(x, y)$.

Again from Table 4, we see improved approximation. We apply the DRM method [7] for solution, where the domain integrals are approximated by using RBF interpolation. The adaptive mesh for this function is given in Fig. 4b and has been observed to give rise to improved DRM solution.

5 Conclusions

We considered a new algorithm for producing a locally bounded mesh with a preset minimal mesh size. Such a mesh is used to overcome the ill-conditioning problems associated with radial basis function interpolation. Extension of the idea to the 2D case is also considered. Some preliminary and improved numerical results are given.

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