

UNCLASSIFIED

Defense Technical Information Center
Compilation Part Notice

ADP014628

TITLE: Evaluation of Performance Reliability Using Regression Models

DISTRIBUTION: Approved for public release, distribution unlimited

This paper is part of the following report:

TITLE: Proceedings of the Eighth Conference on the Design of
Experiments in Army Research Development and Testing

To order the complete compilation report, use: ADA419759

The component part is provided here to allow users access to individually authored sections of proceedings, annals, symposia, etc. However, the component should be considered within the context of the overall compilation report and not as a stand-alone technical report.

The following component part numbers comprise the compilation report:

ADP014598 thru ADP014630

UNCLASSIFIED

EVALUATION OF PERFORMANCE RELIABILITY USING REGRESSION MODELS

Seymour K. Einbinder, Picatinny Arsenal

and

Ingram Olkin, Stanford University

O. ABSTRACT. Performance reliability is the probability that a weapon will perform its prescribed function under given conditions of environment at some particular time. Performance reliability models are defined for continuous performance response variables. Procedures are then described for the evaluation of reliability with emphasis on the application of univariate and multivariate regression analysis to single and multiple continuous response variables, respectively. Point and confidence interval estimation methods for performance reliability are discussed, and a sample problem is presented illustrating some of the basic concepts and results.

1. INTRODUCTION. A major problem during the research and development of a weapon or warhead is the assurance of high functioning reliability of the final prototype design. The reliability concepts and evaluation methods to be described are general and are applicable to a wide variety of systems and components.

A weapon during its lifetime may be subjected to many environmental factors or stresses such as temperature, vibration, acceleration, rough handling, etc. In addition, the stresses may be encountered singly, simultaneously or in sequence. The problem of testing and estimating reliability is of importance to the weapon developer in order to assure the user of a reliable weapon for use in any potential combat situation.

The establishment of high reliability with a high level of confidence generally requires the testing of larger numbers of items than are usually available during a development program for a complex and expensive item. Thus, it is generally necessary to obtain the most information with a minimum number of samples and tests. Improved and more efficient statistical methods are required in many cases to solve the reliability estimation problem.

Before solutions to a problem can be obtained, it is important to delineate the problem so that a representative mathematical model can be developed. Obviously, any solutions obtained can be no better than the underlying mathematical model which is assumed as a prototype of the problem.

In a previous paper [3], the emphasis was on test-to-failure and stress vs. strength analyses for single and multiple environments (stresses). The present paper is concerned mainly with reliability in the case of a continuous regression response surface, and thereby is an extension of [3]. Methods of point and interval estimation for the univariate and multivariate regression problems are discussed.

2. RELIABILITY CONCEPTS. Reliability of a weapon may be defined as the probability of a successful functioning under required conditions of the environment at some particular time. Successful functioning might require the successful operation of several or all components of a system. The outputs or responses of each component may be attribute or continuous. In the case of the continuous response, success may require that the response lie within certain limits (possibly specification limits),

To illustrate these concepts, a hypothetical shaped charge warhead section for a missile will be used as an example. Successful functioning of the warhead section may require that the S and A (Safety and Arming Device) must arm and detonate the warhead on target impact, and the warhead must then penetrate at least a specified distance into an armor plate target. This example could be further complicated by specifying arming limits for the S and A. Failure of the warhead section may occur in two fundamental ways:

- (1) a complete dud or catastrophic failure may result such that no warhead detonation takes place, or,
- (2) the warhead explosive train may be initiated but the armor penetration requirement may not be met.

The reliability of the warhead section is given by

$$R = (1 - P_D) R_{WHD},$$

where R is the overall warhead section reliability, P_D is the probability of a dud or catastrophic failure, and R_{WHD} is the conditional probability that the warhead exceeds the specified performance requirement. The latter will be referred to as the performance reliability and is of prime concern in this paper. The dud probability can be broken down

further according to various components. In general it is necessary to evaluate the dud or catastrophic failures separately from the performance failures since they do not have the same distribution and are mutually exclusive. Dud failures, being attribute, normally require larger sample sizes for evaluation with the same precision and confidence levels as would performance reliability based on continuous variables.

The remainder of this paper will be concerned with the evaluation of the performance reliability, R_{WHD} .

3. MATHEMATICAL MODELS. In this section we define the mathematical models upon which subsequent analysis is based. Univariate and multivariate responses and single and multiple stresses are considered. Estimation procedures are described in the subsequent sections.

3.1 Univariate Response. We have defined performance reliability as the probability that a continuous performance variable lies within certain specified limits. Thus, it may be required that the arming time for an S and A Device be greater than some minimum time required for safety. The performance variable or response may be thought of as a dependent variable which is a function of one or more environments or stresses which are the independent variables. In general, this functional relationship is unknown; however, we can approximate the response function over small regions of the function space by linear regression methods. Generally, we are concerned with the reliability under some critical stress conditions. These conditions will be referred to as a critical point or critical reliability boundary. The regression experiment is designed to provide the best information in the vicinity of this point. Figure 1 illustrates the experimental design in general terms for the univariate case. $(x_1, \dots, x_m)$ represent the applied stresses or environments such as temperature, vibration, etc., and the elements of the design matrix represent the levels of each of these stresses. For example, x_{12} represents the second level of the stress x_1 , etc. The column titled response vector represents the observed response obtained with each treatment combination. The response, y , is a continuous variable such as arming time in the case of the fuze, or possibly depth of penetration in the case of a shaped charge warhead; the response, y_i , for the i th treatment combination may be expressed as a linear combination of the treatment levels plus some random error. The regression model and underlying assumptions are:

$$y = X' \beta + u,$$

y : $n \times 1$ is the observation vector,
 u : $n \times 1$ is the vector of random errors, and is normally distributed with mean vector O and covariance matrix $\sigma^2 I$, i. e.,
 $E(u) = O$, $E(u u') = \sigma^2 I$,

X : $m \times n$ is the design matrix of rank r , $r \leq m \leq n$,
 β : $m \times 1$ is the vector of regression coefficients.

A geometrical interpretation of this regression model and its relation to performance reliability is shown in Figure 2 [Tables and figures can be found at the end of the article.] for the univariate response case. $(x_1, \dots, x_m)$ are the stress variables, $(c_1, \dots, c_m)$ are the components of the critical reliability boundary vector c . The points shown in the $(x_1, \dots, x_m)$ plane represent the treatment combinations for the regression experiment, and the average response, y , is represented by the regression surface shown above the points. The regression equation provides estimates of the response for any point in the $(x_1, \dots, x_m)$ space. The response y at some point such as the critical reliability boundary c : $m \times 1$ is denoted by $y^{(c)}$ and is distributed according to $N(c' \beta, \sigma^2)$. The lower performance limit which the response $y^{(c)}$ is required to exceed is denoted by $y^{(O)}$, and consequently, the performance reliability R is given by:

$$(1) R(c) = P\{Y^{(c)} \geq y^{(O)}\} = \int_{y^{(O)}}^{\infty} n(y^{(c)} | c' \beta, \sigma^2) dy^{(c)} \equiv g_1(c' \beta, \sigma^2; c),$$

where $n(x | \mu, \sigma^2)$ is the density function for a normal population with mean μ and variance σ^2 . This expression represents the shaded area under the normal curve shown in Figure 2.

Thus, our problem is to estimate g which is a function of the unknown parameters β , σ^2 , and the fixed point c , based upon a sample of size n treated as a single or multiple regression experiment.

3.2 Multivariate Response. The univariate model will now be extended to include cases where more than one continuous response may be observed on a single experimental unit such as S and A arming time, functioning time, and self-destruct time; also, the responses may be correlated. Multivariate analysis techniques permit the correlation between responses to be investigated. As before, the problem is best illustrated by examining the table shown in Figure 3. The design matrix X is exactly the same as for the univariate case; $(x_1, \dots, x_m)$ is still the vector of applied stresses. However, instead of a single response vector of y 's, we now have p responses $(y_1, \dots, y_p)$. Thus, for each treatment combination we observe p responses so that our response vector for the univariate case has now become a response matrix where the column vectors may be correlated, and the rows which represent independent response vectors are uncorrelated. The multivariate model and assumptions are:

$$Y = X' B + U,$$

Y : $n \times p$ is the response matrix

X : $m \times n$ is the design matrix of rank $r \leq m < n$, $p \leq n-r$,

B : $m \times p$ is the matrix of regression coefficients,

U : $n \times p$ is the error matrix,

$u_j, j=1, \dots, n$ are the rows of U and are independently and identically distributed, each having a p -variate normal distribution with mean vector O and positive definite covariance matrix Σ .

From the multivariate model, a $p \times 1$ response vector $y^{(c)}$ is obtained for the response at the critical reliability boundary vector c : $m \times 1$.

The response vector $y^{(c)}$ is distributed according to $N(c'B, \Sigma)$, in which the $p \times p$ covariance matrix Σ takes into account any correlations between responses.

The performance reliability R for the multiple response case is given by:

$$(2) \quad R(c) = P\{Y_1^{(c)} \geq y_1^{(0)}, \dots, Y_p^{(c)} \geq y_p^{(0)}\}$$

$$= \int_{y_1^{(0)}}^{\infty} \dots \int_{y_p^{(0)}}^{\infty} n(y^{(c)} | c' B, \Sigma) dy_1^{(c)} \dots dy_p^{(c)}$$

$$= g_p(c' B, \Sigma; c) .$$

A graphical representation of the performance reliability in two dimensions is shown in Figure 4 for the multivariate case. The above integral represents the volume of the multivariate normal density function over the shaded quadrant whose vertex $y^{(0)}$ is the vector of specification limits.

Thus, the general problem may be summarized as follows: Based upon the results of a suitable experimental design with a sample of size n , it is required to estimate the g function for the univariate and multivariate cases, both by point estimation and confidence limits.

4. POINT ESTIMATION. The general problem is to estimate the performance reliability functions defined for the univariate and multivariate responses both by a point estimate and confidence limits based upon responses observed on a sample of size n subjected to various stress treatments in accordance with a suitable experimental design. The experimental designs used for exploring response surfaces [1, 2] are generally suitable for exploring the region around the critical reliability boundary.

4.1 Univariate Point Estimates. The g or R functions to be estimated may be written as follows for the univariate case:

$$R(\beta, \sigma^2) \equiv g_1(c' \beta, \sigma^2; c) = \int_{(y^{(0)} - c' \beta)/\sigma}^{\infty} (2\pi)^{-1/2} \exp(-t^2/2) dt,$$

where c and β are $m \times 1$ vectors.

We consider three types of point estimates. Suppose we write

$$K(\beta, \sigma) = (y^{(0)} - c'\beta)/\sigma,$$

then

$$(3) \quad R(\beta, \sigma^2) = \int_{K(\beta, \sigma)}^{\infty} (2\pi)^{-1/2} \exp(-t^2/2) dt.$$

The first estimate of R is based on using $K(\hat{\beta}, \hat{\sigma})$, where $\hat{\beta}$ and $\hat{\sigma}$ are appropriate estimates of β and σ .

The least squares estimate of β is given by

$$(4) \quad \hat{\beta} = (XX')^{-1}X'y,$$

where $X: m \times n$, of rank $m \leq n$, is the design matrix, and $y: n \times 1$ is the response vector. An unbiased estimator of σ^2 is

$$(5) \quad \hat{\sigma}^2 = \frac{(y - X'\hat{\beta})'(y - X'\hat{\beta})}{n - m}.$$

Thus, we may use the estimate

$$(6) \quad K(\hat{\beta}, \hat{\sigma}) = (y^{(0)} - c'\hat{\beta})/\hat{\sigma},$$

from which we obtain

$$(7) \quad R(\hat{\beta}, \hat{\sigma}^2) = \int_{K(\hat{\beta}, \hat{\sigma})}^{\infty} (2\pi)^{-1/2} \exp(-t^2/2) dt.$$

A second estimate is based on the UMVU estimate of K , namely

$$(8) \quad \tilde{K}(\beta, \sigma) = \sqrt{\frac{2}{f}} \frac{\Gamma(\frac{f}{2})}{\Gamma(\frac{f-1}{2})} \frac{y^{(0)} - c'\hat{\beta}}{\hat{\sigma}} = \sqrt{\frac{2}{f}} \frac{\Gamma(\frac{f}{2})}{\Gamma(\frac{f-1}{2})} K(\hat{\beta}, \hat{\sigma}),$$

where $f = n - m$, from which we may use

$$(9) \quad \tilde{R}(\beta, \sigma^2) = \int_{\tilde{K}(\beta, \sigma)}^{\infty} (2\pi)^{-1/2} \exp(-t^2/2) dt$$

as an estimate at $R(\beta, \sigma^2)$.

Although $\tilde{K}$ is a UMVU estimate of K , it is not the case that $\tilde{R}$ is a UMVU estimate of R . Consequently, a third procedure is based on the UMVU estimate of R , [4], and is given by

$$(10) \quad \bar{R}(\beta, \sigma^2) = \int_0^{\max[0, \eta]} \frac{t^{\left[\frac{f-1}{2} - 1\right]} (1-t)^{\left[\frac{f-1}{2} - 1\right]}}{B\left(\frac{f-1}{2}, \frac{f-1}{2}\right)} dt,$$

$$\text{where } \eta = \frac{1}{2} - \frac{(y^{(10)} - c'\beta)}{2\hat{\sigma} \sqrt{f(1-c'(XX')^{-1}c)}} = \frac{1}{2} - \frac{K(\hat{\beta}, \hat{\sigma})}{2 \sqrt{f(1-c'(XX')^{-1}c)}}.$$

Note that $\bar{R}(\beta, \sigma^2) = 1$ if $\eta > 1$, and that the estimate is valid for critical vectors c such that $c'(XX')^{-1}c < 1$.

Unfortunately, comparisons of the risk of these estimators are unavailable, since the determination of the variance is quite complicated, and was not attempted in this paper.

4.2 Multivariate Case. In the multivariate case, we have

$$R(B, \Sigma) = \int_{y_1(0)}^{\infty} \int_{y_p(0)}^{\infty} \frac{e^{-1/2 \operatorname{tr} \Sigma^{-1} (Y - X'B)'(Y - X'B)}}{|\Sigma|^{pn/2} (2\pi)^{pn/2}} dY,$$

where $Y : n \times p$, $X : m \times n$, $B : m \times p$. As in the univariate case, we can consider $R(\hat{B}, \hat{\Sigma})$ as an estimate of $R(B, \Sigma)$, where $\hat{B} = (XX')^{-1}XY$, and $\hat{\Sigma} = (Y - X'\hat{B})'(Y - X'\hat{B}) / [p(n-m)]$. The problem, however, is still to evaluate the multivariate normal distribution over an orthant. In fact, whether we use this estimation procedure or another, the difficulty of carrying out such an integration still remains. However, for any particular problem, one can employ numerical procedures to yield an answer. Another possibility which has not been considered in the literature is to obtain a lower bound for $R(B, \Sigma)$ in terms of known functions. Further work in this area is required.

5. CONFIDENCE INTERVALS. The problem of obtaining confidence intervals for the g or R functions is considered next. The general method is discussed in [3], and is now extended to the regression model. In Section 4, three estimates were presented. For only the second procedure is the distribution theory known, so that exact confidence intervals can be obtained. However, the first procedure does lead to approximate or asymptotic intervals based on the normal distribution.

5.1 Exact Confidence Intervals. Since $R(\beta, \sigma^2)$ is a monotone function of $K(\beta, \sigma)$, if we can find a confidence interval (K_1, K_2) for K , we will then have a confidence interval (R_1, R_2) for R , where

$$R_i = \int_{K_i}^{\infty} (2\pi)^{-1/2} \exp(-y^2/2) dy.$$

It is shown in Appendix A that $K(\hat{\beta}, \hat{\sigma})/\|a\| \equiv t(f, \delta)$, where $\|a\|^2 = c'(XX')^{-1}c$, has a non-central t -distribution with $f = n - m$ degrees of freedom and non-centrality parameter.

$$\delta = (y^{(0)} - c'\beta)/(\sigma \|a\|) = K(\beta, \sigma)/\|a\| .$$

Thus, a lower and upper confidence limit with confidence coefficient $1 - \alpha$ may be obtained by finding the values of δ_i for which

$$(11) \quad P \left\{ t > \frac{K(\hat{\beta}, \hat{\sigma})}{\|a\|} \mid f, \delta_i \right\} = \alpha_i, \quad i = 1, 2, \quad \text{where } \alpha_1 = 1 - \alpha/2, \text{ and}$$

$\alpha_2 = \alpha/2$. Table IV in [6] may be used to obtain δ_i for seventeen values of ϵ .

The tabulation by Resnikoff and Lieberman [6] of the percentage points of the non-central t-statistic may be conveniently used to obtain the limits δ_1 and δ_2 that satisfy (11). The entries in the table give the values of x_1 such that

$$P \left\{ \frac{t}{\sqrt{f}} > x \right\} = \epsilon .$$

The table should be entered for the degrees of freedom $f = n - m$, the probability ϵ_i corresponding to α_i , and $x = K(\hat{\beta}, \hat{\sigma})/(\|a\|\sqrt{f})$. The required non-centrality value $\delta_i = \sqrt{f+1} K_p$, where K_p is the standardized normal random variable exceeded with probability p . The present concern was with one sided tails (one sided specification limits) for both the univariate and multivariate cases. A review of available point and confidence methods for two sided tails is given in [3].

5.2 Approximate Confidence Intervals. If we expand $R(\hat{\beta}, \hat{\sigma}^2)$ about $R(\beta, \sigma^2)$, we obtain the result that

$$R(\hat{\beta}, \hat{\sigma}^2) - R(\beta, \sigma^2) \sim N(0, V_{\infty}(\beta, \sigma^2)) ,$$

where

$$V_{\infty}(\beta, \sigma) \equiv \sigma^2 [n(y^{(0)} | c'\beta, \sigma^2)]^2 \left\{ c'(XX')^{-1} c + \frac{(y^{(0)} - c'\beta)^2}{2\sigma^2 f} \right\} .$$

Consequently (see Appendix B),

$$\frac{R(\hat{\beta}, \hat{\sigma}^2) - R(\beta, \sigma^2)}{\sqrt{V_{\infty}(\hat{\beta}, \hat{\sigma})}} \sim N(0, 1) ,$$

from which we obtain the confidence interval

$$[R(\hat{\beta}, \hat{\sigma}^2) + z_{\alpha/2} \sqrt{V_{\infty}(\hat{\beta}, \hat{\sigma})} , R(\hat{\beta}, \hat{\sigma}^2) + z_{1-\alpha/2} \sqrt{V_{\infty}(\hat{\beta}, \hat{\sigma})}] ,$$

where z_{α} is the 100 α % point of the $N(0, 1)$ distribution.

6. SAMPLE PROBLEM. In order to illustrate the results of the previous sections, a sample problem will be solved. A model representing the performance of a hypothetical shaped charge warhead section for a missile will be described, and the performance reliability will be evaluated based upon a Monte Carlo simulation of test results. The point estimates and confidence intervals obtained using the methods previously described will be compared with the true value of the reliability. Only the univariate or independent response cases are considered.

6.1 Performance Model. The warhead section to be evaluated consists of a shaped charge warhead and a Safety and Arming Device. It is assumed that the warhead is required to penetrate at least 10 inches into an armor plate target and that the minimum arming time of the S and A is 0.5 seconds. The warhead section is expected to meet these performance requirements under all possible combinations of vibration and temperature shock that may be encountered. To facilitate the illustration, only two stresses are considered in this problem, but the procedure is easily extended to more than two stress variables.

The two stresses, vibration in g's and temperature shock in standard cycles, are denoted by X_1 and X_2 , respectively. Coded levels of the stresses are used throughout this problem to facilitate the analysis and simulation of test results. The relationship between the coded and actual stress units is of no importance with regard to illustrating the reliability evaluation methods and will be disregarded.

The critical reliability boundary is defined by the vector $c' = (c_0, c_1, c_2)$ where c_1, c_2 are the upper stress limits specified for vibration and temperature shock, respectively and c_0 is a dummy variable required to make the vector c consistent with the design matrix X and is equal to 1. The coded variables, in this example, are centered on the critical reliability boundary so that $c' = (1, 0, 0)$.

The warhead performance is measured in terms of depth of penetration t_w into monolithic armor, and S and A performance is measured by arming time t_f . The distribution of warhead penetration t_w and arming time t_f for the S and A is each distributed according to $N[\beta_0 + \beta_{1x1} + \beta_{2x2}, \sigma^2]$. The true values of the parameters are

	β_0	β_1	β_2	σ
Warhead	13"	-0.6	-0.4	1.5"
S and A	0.6 sec.	0.07	0.03	.033 sec.

These models thus assume that the average penetration decreases linearly with increasing stress and that the average arming time increases very slowly with increased stress within the region of interest. Thus, by (1), we see that the performance reliability for the warhead and S and A, respectively, are

$$\begin{aligned}
 (12) \quad R_{WHD} &= P\{t_w^{(c)} \geq 10''\} = \int_{10''}^{\infty} n(t_w^{(c)} | c'\beta, \sigma^2) dt_w^{(c)} \\
 &= g_1(c'\beta, \sigma; c) \equiv g_1(13'', 1.5''; c) = 0.977
 \end{aligned}$$

$$\begin{aligned}
 (13) \quad R_{S \text{ and } A} &= P\{t_f^{(c)} \geq 0.5\} = \int_{0.5}^{\infty} n(t_f^{(c)} | c'\beta, \sigma^2) dt_f^{(c)} \\
 &= g_1(c'\beta, \sigma; c) \equiv g_1(0.6, .033; c) = 0.999.
 \end{aligned}$$

The performance reliability of the warhead section is thus

$$R = R_{\text{WHD}} \cdot R_{S \text{ and } A} = .976.$$

Dud probabilities were not considered in this model. The evaluation of dud rates requires attribute test methods which are not as efficient as the variables plans and require much larger sample sizes. In conducting the type of test program described herein an estimate of the dud reliability may be made by noting the number of dud failures. However, useful interval estimation with these results may not be possible with reasonable confidence coefficients. When a dud occurs, it is desirable to repeat the appropriate test under the same conditions in order to avoid or minimize having to work with missing data in the test plan.

6.2. Multiple Regression Analysis. Multiple linear regression experimental designs of the type used in exploring response surfaces were used to evaluate the performance reliability of the warhead and S and A based upon the stated performance model. In particular, central composite rotatable experimental designs [3] were used. The experiments were conducted with sample sizes (n) of 8 and 30 for both the warhead and S and A. The treatment combinations and the responses generated by Monte Carlo simulation of the performance models are shown in Tables 1 to 4.

Least Squares estimates of the regression coefficients and error variance were made for the test results, and goodness-of-fit tests were conducted. In all four cases, a linear regression model was found to represent the data adequately. The least squares estimates of the regression coefficients obtained for each case are as follows:

Item	n	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_2$
Warhead	8	13.99	-.175	.675
	30	12.75	-.975	.215
S and A	8	.59	.055	.020
	30	.60	.0705	.0205

Tests of significance at the .05 level performed for the regression coefficients gave the following results. The estimates of β_1 and β_2 for the warhead based on $n = 8$ were not significantly different from zero. For $n = 30$, $\hat{\beta}_2$ was not significantly different from zero, but $\hat{\beta}_1$ which corresponds to the effect of the vibration stress X_1 was found to be significantly different from zero, which corresponds to the true situation for the model. In the case of the S and A, $\hat{\beta}_2$ (temperature shock) was not significantly different from zero, and $\hat{\beta}_1$ (vibration) was significantly different from zero for $n=8$ and 30.

Point estimates of the performance reliability $R(\beta, \sigma^2)$ at $c'=(1, 0, 0)$ were made using the UMVU estimate $\tilde{K}(\beta, \sigma)$ of $K(\beta, \sigma)$. Exact one sided lower confidence limits using the non-central t-distribution were also obtained. A summary of these results is tabulated below, and a sample computation is given in Appendix C for one case. Estimates of $R(\beta, \sigma^2)$ based on the estimates $K(\hat{\beta}, \hat{\sigma})$ and $\tilde{K}(\hat{\beta}, \hat{\sigma})$ are also included in the appendix.

Case	Item	n	$\tilde{R}$	$R(.95)$	True R
1	Warhead	8	.974	.832	.977
2		30	.967	.905	
3	S and A	8	.983	.891	.999
4		30	.997	.981	

where $\tilde{R}$ is the Estimate of Performance Reliability based on the UMVU estimate $\tilde{K}(\beta, \sigma)$ or $K(b, \sigma)$ and $R(.95)$ is the one-sided lower 95% confidence limit for Performance Reliability.

A point estimate of the warhead section reliability is given by

$$\tilde{R} = \tilde{R}_{\text{WHD}} \cdot \tilde{R}_{\text{S and A}}$$

Conservative .90 confidence intervals for the warhead section reliability are obtained by multiplying the lower .95 confidence limits for the warhead and S and A. This result is easily proven by applying the Bonferroni inequality to obtain a conservative simultaneous confidence region T for R_{WHD} and $R_{\text{S and A}}$ and by making use of the fact that the product is monotone in each variable. Thus, we obtain the following results for the warhead section reliability.

<u>n</u>	<u>$\tilde{R}$</u>	<u>$R(\geq .90)$</u>	<u>True R</u>
8	.957	.741	.976
30	.964	.888	

REFERENCES

1. Box, G. E. P. and Draper, N. R., "A Basis for the Selection of a Response Surface Design", J. Amer. Statist. Assoc., 54 (1959) 622-654.
2. Box, G. E. P. and Hunter, J. S., "Multifactor Experimental Designs for Exploring Response Surfaces", Ann. Math. Statist., 28 (1957) 195-241.
3. Einbinder, S. K. and Olkin, I., "Reliability Testing and Estimation for Single and Multiple Environments (Preliminary Report)", Proceedings of the Seventh Conference on the Design of Experiments in Army Research Development and Testing, U. S. Army Research Office, Report No. 62-2, August, 1962, 261-291.
4. Ghurye, S. G. and Olkin, I., "Unbiased Estimation of Probability Densities", Abstract, Annals. Math. Statist., 32 (1961) 1346.
5. Johnson, N. L. and Welch, B. L., "Applications of the Non-central t-Distribution", Biometrika, 31 (1940) 362-389.
6. Resnikoff, George J. and Lieberman, Gerald J., Tables of the Non-central t-Distribution, Stanford University Press, Stanford, California, 1957.

TABLE I
WARHEAD

EXPERIMENTAL DESIGN N=8

<u>Sample</u>	<u>x₁</u>	<u>x₂</u>	<u>t_w</u>
1	-1	-1	13.3
2	+1	-1	14.9
3	-1	+1	16.6
4	+1	+1	14.3
5	0	0	12.6
6	0	0	12.7
7	0	0	12.2
8	0	0	15.3

x₁ = Vibration

x₂ = Temperature Shock

t_w = Penetration (inches)

TABLE 2

WARHEAD EXPERIMENTAL DESIGN N=30

<u>Sample</u>	<u>x₁</u>	<u>x₂</u>	<u>t_w</u>
1	-1	-1	13.8
2	+1	-1	12.1
3	-1	+1	13.8
4	+1	+1	12.7
5	0	0	12.1
6	0	0	10.9
7	0	0	12.5
8	0	0	12.2
9	-1	-1	13.7
10	+1	-1	10.5
11	-1	+1	12.7
12	+1	+1	11.0
13	0	0	14.8
14	-1	-1	13.7
15	+1	-1	9.5
16	-1	+1	11.5
17	+1	+1	13.0
18	0	0	13.3
19	-1	-1	12.0
20	+1	-1	12.5
21	-1	+1	14.4
22	+1	+1	11.4
23	0	0	13.9
24	-1	-1	14.8
25	+1	-1	11.5
26	-1	+1	15.6
27	+1	+1	14.3
28	0	0	12.0
29	0	0	16.0
30	0	0	12.3

 x_1 = Vibration x_2 = Temperature Shock t_w = Penetration (inches)

TABLE 3

S & A EXPERIMENTAL DESIGN N=8

<u>Sample</u>	<u>x_1</u>	<u>x_2</u>	<u>Arming Time (seconds)</u>
1	-1	-1	.48
2	+1	-1	.63
3	-1	+1	.56
4	+1	+1	.63
5	0	0	.61
6	0	0	.56
7	0	0	.60
8	0	0	.64

x_1 = Vibration

x_2 = Temperature Shock

TABLE 4

S & A EXPERIMENTAL DESIGN N=30

<u>Sample</u>	<u>x₁</u>	<u>x₂</u>	<u>Arming Time (seconds)</u>
1	-1	-1	.55
2	+1	-1	.63
3	-1	+1	.56
4	+1	+1	.71
5	0	0	.57
6	0	0	.68
7	0	0	.67
8	0	0	.60
9	-1	-1	.50
10	+1	-1	.66
11	-1	+1	.53
12	+1	+1	.73
13	0	0	.57
14	-1	-1	.58
15	+1	-1	.66
16	-1	+1	.57
17	+1	+1	.72
18	0	0	.55
19	-1	-1	.46
20	+1	-1	.64
21	-1	+1	.54
22	+1	+1	.65
23	0	0	.59
24	-1	-1	.51
25	+1	-1	.67
26	-1	+1	.56
27	+1	+1	.70
28	0	0	.53
29	0	0	.61
30	0	0	.60

x₁ = Vibration

x₂ = Temperature Shock

UNIVARIATE RESPONSE

No.	Response Vector Y	<u>Design Matrix</u>		
		x_1	x_2	x_m
1	y_1	x_{11}	x_{21}	x_{m1}
2	y_2	x_{12}	x_{22}	x_{m2}
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
l	y_l	x_{1l}	x_{2l}	x_{ml}
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
n	y_n	x_{1n}	x_{2n}	x_{mn}

$$\Omega: y_l = \beta_1 x_{1l} + \beta_2 x_{2l} + \dots + \beta_m x_{ml} + u_l, l = 1, \dots, n$$

$\{u_1, \dots, u_n\}$ are independently and identically distributed with mean 0 and variance σ^2 .

Figure 1

PERFORMANCE RELIABILITY
UNIVARIATE RESPONSE

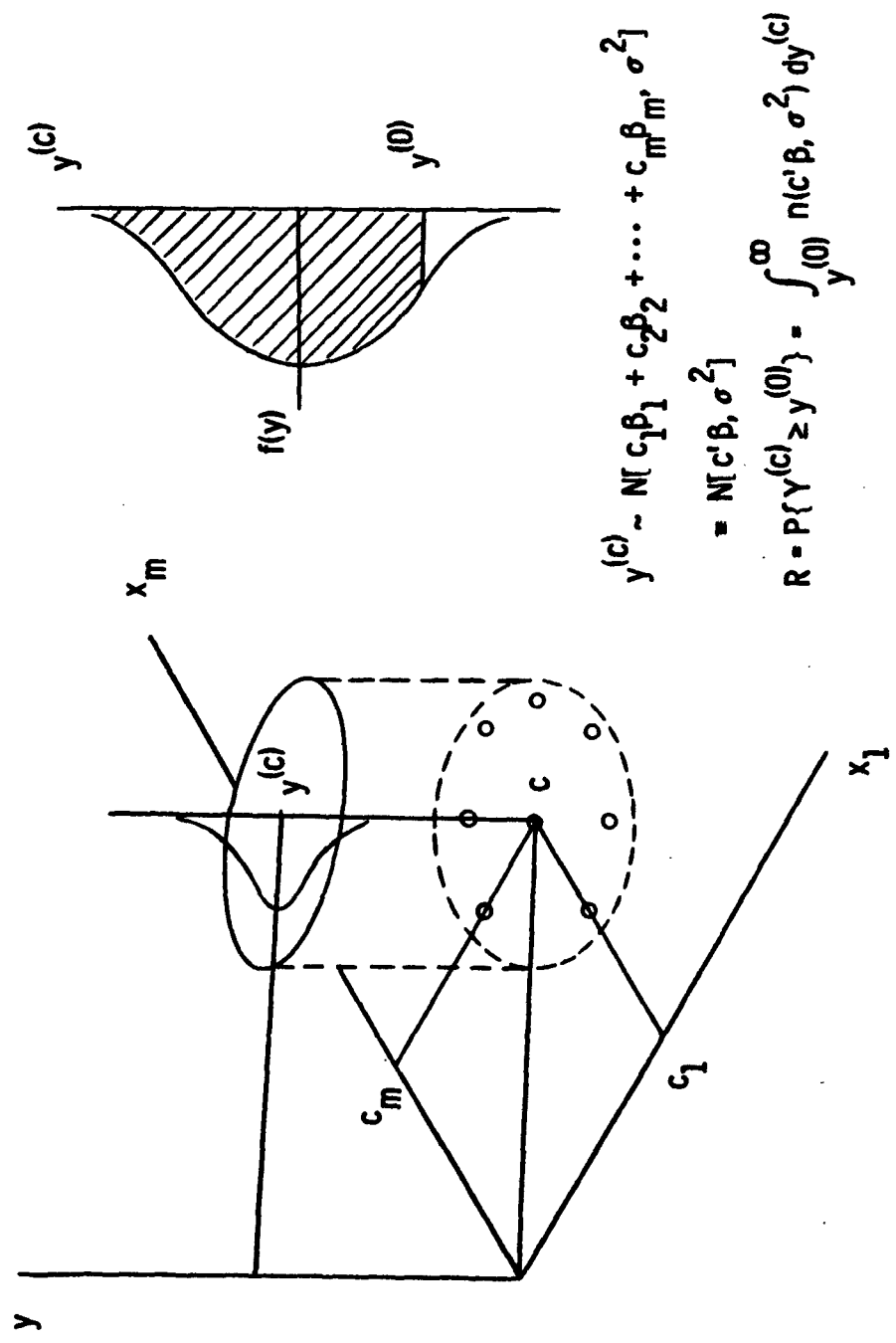


Figure 2

MULTIVARIATE RESPONSE

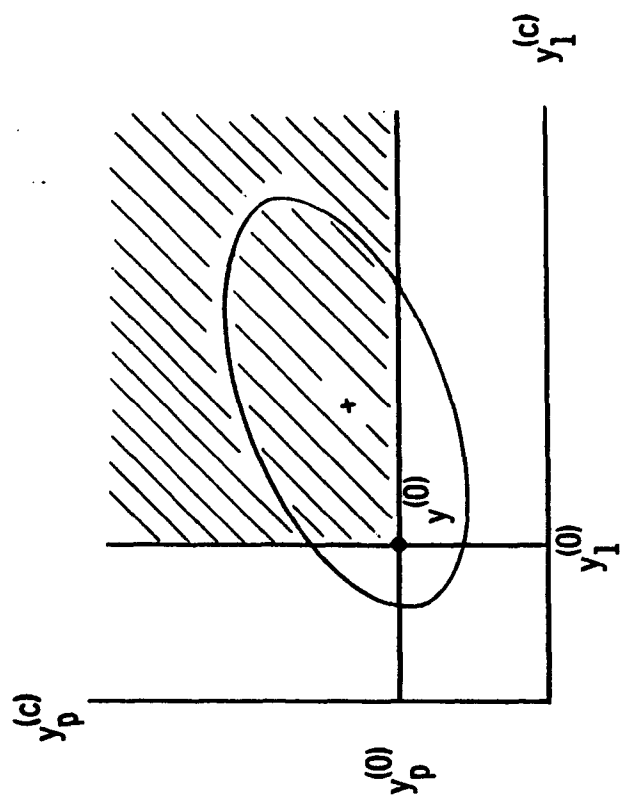
No	<u>Response Matrix</u>				<u>Design Matrix</u>			
	y_1	y_2	.	y_p	x_1	x_2	.	x_m
1	y_{11}	y_{21}	.	y_{p1}	x_{11}	x_{21}	.	x_{m1}
2	y_{12}	y_{22}	.	y_{p2}	x_{12}	x_{22}	.	x_{m2}
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
l	y_{1l}	y_{2l}	.	y_{pl}	x_{1l}	x_{2l}	.	x_{ml}
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
n	y_{1n}	y_{2n}	.	y_{pn}	x_{1n}	x_{2n}	.	x_{mn}

$$\Omega : Y = X' B + U ,$$

$\{u_1^i, \dots, u_n^i\}$ are independently identically distributed with mean vector 0 and common $p \times p$ positive definite covariance matrix Σ .

Figure 3

PERFORMANCE RELIABILITY
MULTIVARIATE RESPONSE (VECTOR)



$$y^{(c)} \sim N[c' B, \Sigma],$$

$$R = P\{Y_1^{(c)} \geq y_1^{(0)}, \dots, Y_p^{(c)} \geq y_p^{(0)}\}$$

$$= \int_{y_1^{(0)}}^{\infty} \dots \int_{y_p^{(0)}}^{\infty} n(c' B, \Sigma) \phi y^{(c)}$$

Figure 4

APPENDIX A

Define $K = (y^{(0)} - c'\beta)/\sigma$ and $K(\hat{\beta}, \hat{\sigma}) = (y^{(0)} - c'\hat{\beta})/\hat{\sigma}$. We first note that $\hat{\beta}$ and $\hat{\sigma}$ are independently distributed. Since $E\hat{\beta} = \beta$, we have that $E(y^{(0)} - c'\hat{\beta}) = (y^{(0)} - c'\beta)$. Also $v \equiv f\hat{\sigma}^2/\sigma^2$ has a χ_f^2 distribution, $f = n - m$, so that

$$Ev^{-1/2} = \frac{\Gamma(\frac{f-1}{2})}{\sqrt{2} \Gamma(\frac{f}{2})}.$$

Hence,

$$E \frac{\sqrt{2}}{\sqrt{f}} \frac{\Gamma(\frac{f}{2})}{\Gamma(\frac{f-1}{2})} \frac{1}{\hat{\sigma}} = \frac{1}{\sigma}$$

which proves that $\tilde{K}(\beta, \sigma) = \frac{\sqrt{2}}{\sqrt{f}} \frac{\Gamma(\frac{f}{2})}{\Gamma(\frac{f-1}{2})} K(\hat{\beta}, \hat{\sigma})$ is an unbiased estimator of $K(\beta, \sigma)$. By completeness, it then follows that $\tilde{K}$ is the unique such estimator, and hence is UMVU.

An alternative approach is also useful, namely, that $K(\hat{\beta}, \hat{\sigma})/\|a\| \equiv t(f, \delta)$ has a non-central t -distribution with f degrees of freedom and non-centrality parameter

$$\delta = \frac{y^{(0)} - c'\beta}{\sigma\|a\|} = \frac{K(\beta, \sigma)}{\|a\|},$$

where $\|a\|^2 = c'(XX')^{-1}c$. To see this, we write

$$K(\hat{\beta}, \hat{\sigma}^2) = \frac{\frac{y^{(0)} - c'\hat{\beta}}{\sqrt{\text{Var}(c'\hat{\beta})}} - \frac{c'\hat{\beta} - c'\beta}{\sqrt{\text{Var}(c'\hat{\beta})}}}{\frac{\hat{\sigma}}{\sqrt{\text{Var}(c'\hat{\beta})}}}.$$

But $c'\hat{\beta} = c'(XX')^{-1}Xy$, and hence $\text{var}(c'\hat{\beta}) = \sigma^2 c'(XX')^{-1}c \equiv \sigma^2 \|a\|^2$.

Thus,

$$\frac{K(\hat{\beta}, \hat{\sigma}^2)}{\|a\|} = \frac{\frac{y^{(0)} - c'\beta}{\sigma\|a\|} - \frac{c'\hat{\beta} - c'\beta}{\sigma\|a\|}}{\frac{\hat{\sigma}}{\sigma}} = t(f, \delta).$$

By noting that $E t(f, \delta) = \delta \sqrt{f/2} \Gamma(\frac{f-1}{2})/\Gamma(\frac{f}{2})$, we can also obtain

$$E \hat{K}(\beta, \sigma) = K(\beta, \sigma).$$

APPENDIX B

Since $R(\hat{\beta}, \hat{\sigma}^2)$ is a function of the sample moments, it follows that $R(\hat{\beta}, \hat{\sigma}^2)$ is asymptotically normal with mean $R(\beta, \sigma^2)$ and variance

$$\sum_{i,j} \left(\frac{\partial R}{\partial \beta_i} \bigg|_{\beta, \sigma^2} \right) \left(\frac{\partial R}{\partial \beta_j} \bigg|_{\beta, \sigma^2} \right) \text{Cov}(\hat{\beta}_i, \hat{\beta}_j) + \left(\frac{\partial R}{\partial \sigma^2} \bigg|_{\beta, \sigma^2} \right) \text{Var}(\hat{\sigma}^2).$$

The cross-product terms involving $\hat{\beta}_1$ and $\hat{\sigma}^2$ drop out because of the independence of $\hat{\beta}$ and $\hat{\sigma}^2$. From

$$R(b, s^2) = \int_{(y^{(0)} - c'b)/s}^{\infty} (2\pi)^{-1/2} \exp(-1/2 t^2) dt,$$

we obtain

$$\frac{\partial R}{\partial b_1} = \frac{c_1}{\sqrt{2\pi} s} \exp\left[-\frac{1}{2} \frac{(y^{(0)} - c'b)^2}{s^2}\right], \quad \frac{\partial R}{\partial s^2} = \frac{(y^{(0)} - c'b)}{2s^2} \frac{1}{\sqrt{2\pi} s} \exp\left[-\frac{1}{2} \frac{(y^{(0)} - c'b)^2}{s^2}\right]$$

Also, $\text{Cov}(\hat{\beta}_1, \hat{\beta}_j) = \sigma^2 a_{1j}$, where $A = (XX')^{-1}$, $\text{Var}(\hat{\sigma}^2) = 2\sigma^4/f$, and hence the asymptotic variance is

$$V_{\infty}(\beta, \sigma^2) = \sigma^2 [n(y^{(0)} | c'\beta, \sigma^2)]^2 \{c'(XX')^{-1}c + \frac{(y^{(0)} - c'\beta)^2}{2\sigma^4 f}\}.$$

But, $V_{\infty}(\hat{\beta}, \hat{\sigma}^2)$ is a rational function of the sample moments, so that, by Slutsky's Theorem, $V_{\infty}(\hat{\beta}, \hat{\sigma}^2)$ converges in probability to $V_{\infty}(\beta, \sigma^2)$, and hence

$$\frac{R(\hat{\beta}, \hat{\sigma}^2) - R(\beta, \sigma^2)}{\sqrt{V_{\infty}(\hat{\beta}, \hat{\sigma}^2)}} \rightarrow N(0, 1).$$

APPENDIX C

The computation of the point and confidence interval estimates for the performance reliability of the warhead for the sample size $n = 8$ is described in this appendix. From the test data in Table I, we obtain the following results:

Point Estimation

$$\hat{\sigma}^2 = 2.958, \quad c' = (1, 0, 0), \quad f = n - m = 8 - 3 = 5$$

$$K(\hat{\beta}, \hat{\sigma}) = \frac{y^{(0)} - c'\hat{\beta}}{\hat{\sigma}} = \frac{10 - 13.99}{1.720} = -2.320$$

$$\tilde{K}(\hat{\beta}, \hat{\sigma}) = K(\hat{\beta}, \hat{\sigma}) \sqrt{\frac{f}{2}} \frac{\Gamma(\frac{f}{2})}{\Gamma(\frac{f-1}{2})} = -1.95$$

Substituting, these two estimates of $K(\beta, \sigma)$ in

$$R(\beta, \sigma^2) = \int_{K(\beta, \sigma)}^{\infty} (2\pi)^{-1/2} e^{-y^2/2} dy$$

gives the two estimates of reliability $R(\hat{\beta}, \hat{\sigma})$ and $\tilde{R}(\beta, \sigma)$, respectively. Thus,

$$R(\hat{\beta}, \hat{\sigma}) = \int_{-2.320}^{\infty} (2\pi)^{-1/2} e^{-y^2/2} dy = .9898, \text{ and}$$

$$\tilde{R}(\beta, \sigma) = \int_{-1.95}^{\infty} (2\pi)^{-1/2} e^{-y^2/2} dy = .974.$$

Confidence Intervals

$$\|a\|^2 = c'(XX')^{-1}c = \frac{1}{n} \text{ for } c' = (1, 0, 0).$$

Following the notation of Resnikoff and Lieberman, a confidence interval may be obtained using the non-central t-tables in [6]. The percentage points of t are denoted by $x(f, \delta, \epsilon)$ where x is the value such that $P\left(\frac{t}{\sqrt{f}} > x \mid f, \delta\right) = \epsilon$.

$$x = \frac{K(\hat{\beta}, \hat{\sigma})}{\sqrt{f} \|a\|} = \sqrt{\frac{n}{f}} K(\hat{\beta}, \hat{\sigma}) = \sqrt{\frac{8}{5}} (-2.320) = -2.935.$$

The one sided lower .95 confidence limit for R is obtained by finding the corresponding limit for $K(\beta, \sigma)$ because of the monotone relation between R and $K(\beta, \sigma)$. The $1-\alpha$ confidence limit for $K(\beta, \sigma)$ is obtained by solving

$$x(n - m, \delta_{1-\alpha}, 1-\alpha) = \frac{K(\beta, \sigma)}{\sqrt{f} \|a\|},$$

$$x(5, \delta_{.95}, .95) = -2.935.$$

Making use of the relation $x(f, \delta, \epsilon) = -x(f, -\delta, 1-\epsilon)$, we obtain

$$x(5, -\delta_{.95}, .05) = 2.935.$$

From the Resnikoff-Lieberman table of percentage points of t , we obtain a non-centrality value $\delta = \sqrt{f} + 1 K_p = \sqrt{6} (1.107) = 2.712$ by interpolating on K_p . Since $\delta = \frac{K(\beta, \sigma)}{\|a\|}$, the .95 lower confidence limit for $K(\beta, \sigma)$ is

$$K(\beta, \sigma)_{.95} = \|a\| \delta_{.95} = \frac{\delta_{.95}}{\sqrt{n}} = \frac{-2.712}{\sqrt{8}} = -.959.$$

Finally, the .95 confidence limit for R is

$$R(\beta, \sigma^2)_{.95} = \int_{K(\beta, \sigma)_{.95}}^{\infty} (2\pi)^{-1/2} e^{-y^2/2} dy = .832.$$

$\delta_{.95}$ may also be computed using the Johnson-Welch table IV and following the procedure on page 372 of [5].

This Page Intentionally
Left Blank

502